MATH 421/510, 2019WT2

Homework set 7 – Solution

Problem 1. (i) That C is open follows from the openness of A, B. $0 \in C$ since $a_0 \in A, b_0 \in B$. $x_0 \in C$ iff $a = b$, which is a contradiction with $A \cap B = \emptyset$. Finally, let $x, y \in C$ and $\lambda \in [0,1]$. There are $a_x, a_y \in A, b_x, b_y \in B$ such that $x = a_x - b_x + x_0, y = a_y - b_y + x_0$. But then

$$\lambda x + (1-\lambda)y = (\lambda a_x + (1-\lambda)a_y) - (\lambda b_x + (1-\lambda)b_y) + x_0 \in A - B + x_0,$$

showing that C is convex whenever A, B are convex.
(ii) Since C is open and $0 \in C$, there is $r > 0$ such that $B_r(0) \in C$. Moreover, for any $x \in V$, $x \in B_{2\|x\|}(0) = B_{\lambda r}(0) \subset \lambda C$ for $\lambda = 2r^{-1}\|x\|$. Hence $p(x) \leq 2r^{-1}\|x\|$. Moreover, $x \in C$ implies $1 \in \{\lambda > 0 : x \in \lambda C\}$, namely $p(x) \leq 1$. Since C is open, $C \subset \{x \in V : p(x) < 1\}$.
(iii) First of all, $p(\alpha x) = \inf\{\lambda > 0 : \alpha x \in \lambda C\} = \inf\{\lambda > 0 : x \in \lambda\alpha^{-1} C\} = \inf\{\alpha\mu > 0 : x \in \mu C\} = \alpha p(x)$ for any $\alpha > 0$. Let $x, y \in V$ and $\lambda \in [0,1]$. Hence there are $\mu > 0, \nu > 0$ and $x_0, y_0 \in C$ such that $x = \mu x_0$ and $y = \nu y_0$. Therefore,

$$p(\lambda x + (1-\lambda)y) = (\lambda\mu + (1-\lambda)\nu)\, p\left(\frac{\lambda\mu x_0 + (1-\lambda)\nu y_0}{\lambda\mu + (1-\lambda)\nu}\right) \leq \lambda\mu + (1-\lambda)\nu$$

because the argument of p belongs to C since C is convex. The claim follows by taking the infimum over μ, ν.
(iv) Since $x_0 \notin C$, we have that $p(x_0) \geq 1$. Let f be as in the hint. Then

$$t \geq 0 : \ f(tx_0) = t \leq tp(x_0) = p(tx_0) \qquad t < 0 : \ f(tx_0) = t < 0 \leq p(tx_0).$$

By Hahn-Banach, there is a real-linear functional ℓ such that $\ell(tx_0) = f(tx_0) = t$ for all $t \in \mathbb{R}$ and $\ell(x) \leq p(x)$ for all $x \in V$. Moreover,

$$|\ell(x)| = \max\{\ell(x), \ell(-x)\} = \max\{p(x), p(-x)\} \leq 2r^{-1}\|x\|$$

so that $\ell \in V^*$. Finally, for any $a \in A, b \in B$,

$$\ell(a) - \ell(b) = \ell(a - b + x_0) - \ell(x_0) < 0$$

since $\ell(x_0) = 1$ and $\ell(a - b + x_0) \leq p(a - b + x_0) < 1$ since $a - b + x_0 \in C$.
(v) By Urysohn's lemma, see HW 3, Problem 2(ii), there is $r > 0$ such that $B_r(x) \subset V \setminus B$ for all $x \in A$. It remains to apply the result above to the open set $A^r = \{x \in V : \mathrm{dist}(x, A) < r\}$ and B to obtain

$$\max\{\ell(a) : a \in A\} < \sup\{\ell(x) : x \in A^r\} \leq \inf\{\ell(b) : b \in B\},$$

as we had set to prove.

Problem 2. The identity map $I : (V, \|\cdot\|_2) \to (V, \|\cdot\|_1)$ given by $I(x) = x$ for all $x \in V$ is bijective and bounded since $\|I(x)\|_1 = \|x\|_1 \leq c\|x\|_2$. By the open mapping theorem, I^{-1} is bounded, namely $\|x\|_2 = \|I^{-1}(x)\|_2 \leq C\|x\|_1$.

Problem 3. Since $\mathrm{Ran}(T)$ is closed, it is complete. Assume by contradiction that there is a sequence $(x_n)_{n\in\mathbb{N}}$ in V such that

$$\|x_n\|_V = 1, \qquad \|Tx_n\|_W + \|x_n\| < \frac{1}{n}.$$

We apply the open mapping theorem to the surjective $T : V \to \mathrm{Ran}(T)$, where V is equipped with the norm $\|\cdot\|_V$. There is $\delta > 0$ such that $B^Y_\delta \subset T(B^X_1)$, and hence $B^Y_{1/n} \subset T(B^X_{1/(n\delta)})$ by linearity. Since $\|Tx_n\|_W < 1/n$, it follows that there exists $\tilde{x}_n \in V$ such that

$$\|\tilde{x}_n\|_V < \frac{1}{n\delta}, \qquad Tx_n = T\tilde{x}_n.$$

Let $z_n = x_n - \tilde{x}_n \in \mathrm{Ker}(T)$. On the one hand, $\|\tilde{x}_n\|_V \to 0$ as $n \to \infty$ implies that $\|z_n\|_V \to 1$. On the other hand, $\|x_n\| < 1/n$ implies $\|z_n\| < 1/n + \|\tilde{x}_n\| \leq 1/n + M\|\tilde{x}_n\|_V$ (by the assumption that $\|\cdot\|$ is weaker than $\|\cdot\|_V$) which vanishes as $n \to \infty$. Since $\mathrm{Ker}(T)$ is a finite dimensional vector space, the two norms $\|\cdot\|_V$ and $\|\cdot\|$ are equivalent on $\mathrm{Ker}(T)$, and the two claims above are a contradiction.

Problem 4. (i) That $V \neq \ell^1$ follows from the fact that $(1/n^2)_{n\in\mathbb{N}}$ is in ℓ^1 but not in V. Density follows by truncation. Let $z \in \ell^1$. We claim that the sequence $(w^n)_{n\in\mathbb{N}}$ in V given by

$$(w^n)_j = \begin{cases} z_j & \text{if } j \leq n \\ 0 & \text{otherwise} \end{cases}$$

converges to z in the $\|\cdot\|_1$ norm. Indeed, for $\epsilon > 0$, there is $N \in \mathbb{N}$ such that $\sum_{j=N}^\infty |z_j| < \epsilon$, and hence $\|w^n - z\|_1 \leq \sum_{j=N}^\infty |z_j| < \epsilon$ for all $n > N$.
(ii) We consider the sequence $(w^n)_{n\in\mathbb{N}}$ in V given by $(w^n)_j = \delta_{n,j}$. Then $\|w^n\|_1 = 1$ for all $n \in \mathbb{N}$ and $\|Tw^n\|_1 = n$, showing that T is unbounded. It is however closed: Let $(z^n, Tz^n)_{n\in\mathbb{N}}$ be a sequence in $\Gamma(T)$ that converges in $V \times \ell^1$ to (z, w). $z^n \to z$ reads $\lim_{n\to\infty} \sum_{j=1}^\infty |z^n_j - z_j| = 0$, which implies that $z^n_j \to z_j$ for any $j \in \mathbb{N}$ as $n \to \infty$. Similarly, $Tz^n \to w$ implies that $jz^n_j \to w_j$ for any $j \in \mathbb{N}$ as $n \to \infty$. Together, it follows that $jz_j = w_j$ for any $j \in \mathbb{N}$, namely $w = Tz$ indeed.
(iii) The map $S : \ell^1 \to V$ given by $(Sz)_n = z_n/n$ is well-defined and bounded since $\|Sz\|_1 = \sum_{j=1}^\infty |z_n|/n < \sum_{j=1}^\infty |z_n| = \|z\|_1$. If $z \in V$, then by definition $z = STz$ so that S is surjective. It follows that if S were open, then $T = S^{-1}$ would be bounded, but we have just proved the contrary. Hence S is not open.

Problem 5. Let $x \in V$. There is a sequence $(x_n)_{n\in\mathbb{N}}$ in D such that $x_n \to x$. Let $\epsilon > 0$. Since $(x_n)_{n\in\mathbb{N}}$ is Cauchy, there is $N \in \mathbb{N}$ such that $\|x_n - x_m\| < \epsilon/\|T\|$ for all $n, m \geq N$ and hence $\|Tx_n - Tx_m\| \leq \epsilon$ proving that $(Tx_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in W. Let y be its limit, and we define $y = \tilde{T}x$. Clearly, $\tilde{T}x = Tx$ for $x \in D$ by continuity. $\tilde{T}$ is linear by the linearity of the limit. Moreover, the definition is independent of the sequence: if $\tilde{x}_n \to x$, the alternating sequence $(x_1, \tilde{x}_1, x_2, \tilde{x}_2, \ldots)$ also converges to x, and the above argument yields a limiting $\tilde{y}$. But then $y = \tilde{y}$ by uniqueness of the limit since $(x_n)_{n\in\mathbb{N}}, (\tilde{x}_n)_{n\in\mathbb{N}}$ are subsequences. $Tx_n \to \tilde{T}x$ implies that $\|Tx\| = \lim_{n\to\infty} \|Tx_n\| \leq \lim_{n\to\infty} \|T\|\|x_n\| = \|T\|\|x\|$, proving that $\tilde{T}$ is bounded. Finally, if T' is another continuous extension of T to V, then $T'x = \lim_{n\to\infty} Tx_n$ for any $x_n \to x$ by continuity, proving that $T' = \tilde{T}$, again by uniqueness of the limit.

Problem 6. (i) Assume by contradiction that for any nonzero $v \in V$, there is $j \in \{1, \ldots, n\}$ such that $\ell_j(v) \neq 0$. Then the map $L : V \to \mathbb{C}^n$ defined by $L(v)_j = \ell_j(v)$ is injective so that $\dim(V) = \dim(\mathrm{Ran}(L)) \leq n$, which is a contradition with the assumption that V is infinite dimensional.
(ii) We denote the unit sphere S and the closed unit ball B. Let $v \in V, \|v\| < 1$ and let $N = N_v(\ell_1, \cdots \ell_n, \epsilon) = \{x \in V : |\ell_j(x) - \ell_j(v)| < \epsilon \,\forall j = 1, \ldots, n\}$ be a weakly open neighbourhood of v. Let now v_0 be as in (i). Then for any $t \in \mathbb{R}$, $\ell_j(v + tv_0) = \ell_j(v)$ so that $v + tv_0 \in N$. The function $f(t) = \|v + tv_0\|$ is continuous from $[0,\infty) \to [0,\infty)$ with $f(0) = \|v\| < 1$ and $\lim_{t\to\infty} f(t) = \infty$. Hence there is $t_0 \in (0,\infty)$ such that $f(t_0) = 1$, namely $v + t_0 v_0 \in S \cap N$. In other words, $N \cap S \neq \emptyset$, which implies that $v \in \overline{S}$ and hence $S \subset B \subset \overline{S}$. Since $\overline{S}$ is the smallest closed set containing S, the proof is complete if we show that B is weakly closed. But Hahn-Banach yields that $v \in B$ if and only if $\sup\{|\ell(v)| : \ell \in V^*, \|\ell\| \leq 1\} \leq 1$ so that

$$B = \bigcap_{\ell\in V^*:\|\ell\|\leq 1} \{v \in V : |\ell(v)| \leq 1\} = \Bigg(\bigcup_{\ell\in V^*:\|\ell\|\leq 1} \{v \in V : |\ell(v)| > 1\}\Bigg)^c.$$

By definition of the weak topology, all sets on the right are weakly open, so that B is weakly closed indeed.
(iii) If the open unit ball b is weakly open, then b^c is weakly closed, and so is $S = B \cap b^c$, which contradicts (ii).