MATH 421/510, 2019WT2

Homework set 8 – Solution

Problem 1. Assume first that $z^n \rightharpoonup z$. Then z^n is bounded. Let δ^j be the sequence defined by ${\delta^j}_k = \delta_{j,k}$. Clearly, $\delta^j \in \ell^q$ for all $j \in \mathbb{N}$. Moreover, weak convergence implies that for any $j \in \mathbb{N}$,

$$z_j = \sum_{k=1}^{\infty} {\delta^j}_k z_k = \lim_{n\to\infty} \sum_{k=1}^{\infty} {\delta^j}_k z_k^n = \lim_{n\to\infty} z_j^n,$$

concluding the proof. Reciprocally, assume that $z_j^n \to z_j$ for all $j \in \mathbb{N}$. Let w be a sequence with a finite number of nonzero components. Then

$$\Big| \sum_{j=1}^{N} w_j (z_j^n - z_j) \Big| \leq \sup\{|z_j^n - z_j| : j \leq N\} \sum_{j=1}^{N} |w_j| \to 0 \qquad (n \to \infty),$$

by pointwise convergence. If $(z^n)_{n\in\mathbb{N}}$ is bounded in ℓ^p, this proves weak convergence since the compactly supported sequences are dense in ℓ^q.

Problem 2. Let H be the *convex hull* of $\{v_n : n \in \mathbb{N}\}$, namely the set of all finite convex combinations of elements in $\{v_n : n \in \mathbb{N}\}$. It is a convex set. Indeed, let $a = \sum_{j=1}^N \alpha_j v_j$ and $b = \sum_{j=1}^N \beta_j v_j$, where some of the coefficients in these convex combinations may be zero. Then $\lambda a + (1-\lambda)b = \sum_{j=1}^N (\lambda\alpha_j + (1-\lambda)\beta_j) v_j$ is an element in H since $\sum_{j=1}^N (\lambda\alpha_j + (1-\lambda)\beta_j) = \lambda \sum_{j=1}^N \alpha_j + (1-\lambda)\sum_{j=1}^N \beta_j = 1$. Hence $\overline{H}^w = \overline{H}$. By weak convergence, $v \in \overline{H}^w = \overline{H}$. We conclude that there is a $(w_j)_{j\in\mathbb{N}}$ in H converging to v in the norm topology.

Problem 3. Let $(v_n)_{n\in\mathbb{N}}$ be a sequence in X with $v_n \rightharpoonup v$ and let $F_0 = \liminf_{n\to\infty} F(v_n)$. By definition of the $\liminf$, there is a subsequence such that $F(v_{n_k}) \to F_0$ as $k \to \infty$. Let $(w_j)_{j\in\mathbb{N}}$ be the strongly convergent sequence given by Mazur's theorem, namely $w_j \to v$ as $j \to \infty$, where

$$w_j = \sum_{k=1}^{j} \alpha_k^j v_{n_k}, \qquad \alpha_k^j \geq 0,\ \sum_{k=1}^{j} \alpha_k^j = 1.$$

Since X is convex, $(w_j)_{j\in\mathbb{N}}$ is a sequence in X. For any $k_0 \in \mathbb{N}$, the above continues to hold for the truncated sequence $(v_{n_k})_{k\in\mathbb{N},k\geq k_0}$, with X replaced by the closure of the convex hull of $\{v_k : k \geq k_0\}$. Hence, fixing k_0, we can assume that $\alpha_k^j = 0$ for all $k < k_0$. F being convex,

$$F(w_j) \leq \sum_{k=k_0}^{j} \alpha_k^j F(v_{n_k}) \leq \sup\{F(v_{n_k}) : k \geq k_0\}.$$

Since F is strongly continuous, this yields $F(v) = \lim_{j\to\infty} F(w_j) \leq \sup\{F(v_{n_k}) : k \geq k_0\}$, and hence $F(v) \leq \limsup_{k\to\infty} F(v_{n_k}) = F_0$ by taking the limit $k_0 \to \infty$.

Problem 4. Clearly, $|T_\epsilon f| \leq \sup\{|f(x)| : 0 \leq x \leq \epsilon\} \leq \|f\|_\infty$ so that $\|T_\epsilon\| \leq 1$ for any $0 < \epsilon \leq 1$. Let now $(\epsilon_n)_{n\in\mathbb{N}}$ be a sequence converging to 0 and we assume that $T_{\epsilon_n} \overset{*}{\rightharpoonup} T$ as $n \to \infty$. By going to a subsequence, we assume that

$$1 > \frac{\epsilon_{n+1}}{\epsilon_n} \to 0$$

as $n \to \infty$. Let now $f = \sum_{n=1}^{\infty} (-1)^n \chi_{[\epsilon_{n+1}, \epsilon_n)}(x)$, which is such that $\|f\|_\infty = 1$. But

$$T_{\epsilon_n} f = (-1)^n \frac{\epsilon_n - \epsilon_{n+1}}{\epsilon_n} + \frac{1}{\epsilon_n} \int_0^{\epsilon_{n+1}} f dx$$

namely

$$|T_{\epsilon_n} f - (-1)^n| \leq \frac{\epsilon_{n+1}}{\epsilon_n} + \frac{1}{\epsilon_n} \int_0^{\epsilon_{n+1}} |f| dx \leq 2 \frac{\epsilon_{n+1}}{\epsilon_n} \to 0$$

as $n \to \infty$. Hence $(T_{\epsilon_n} f)_{n \in \mathbb{N}}$ accumulates both at (-1) and 1 and therefore does not converge, which contradicts the assumption. This proves that bounded set $\{T_\epsilon : 0 < \epsilon \leq 1\}$ is not weakly-* sequentially compact in $(L^\infty)^*$. In view of the Banach-Alaoglu theorem, this shows that L^∞ is not reflexive.

Problem 5. Hanner's inequality with $g = f_n$ yields

$$\limsup_{n \to \infty} \left((\|f + f_n\|_p + \|f - f_n\|_p)^p + |\|f + f_n\|_p - \|f - f_n\|_p|^p \right) \leq 2^{p+1} \|f\|_p^p \tag{1}$$

since $\|f_n\|_p \to \|f\|_p$ as $n \to \infty$. Now, $f_n + f \rightharpoonup 2f$ and by the sequential lower semicontinuity of norms,

$$\liminf_{n \to \infty} \|f_n + f\|_p \geq 2\|f\|_p.$$

On the other hand, Minkowski's inequality yields

$$\limsup_{n \to \infty} \|f_n + f\|_p \leq \limsup_{n \to \infty} (\|f_n\|_p + \|f\|_p) = 2\|f\|_p.$$

Hence, $\lim_{n \to \infty} \|f_n + f\|_p = 2\|f\|_p$ and the left hand side of (1) is equal to $\limsup_{n \to \infty} (J(F + t_n) + J(F - t_n))$, where $J(t) = |t|^p$ and $F = 2\|f\|_p, t_n = \|f - f_n\|_p$. Since J is strictly convex for $p > 1$, we conclude that if $\|f - f_n\|_p$ does not converge to zero, then for n large enough,

$$2^{p+1} \|f\|_p^p = 2J(F) < \limsup_{n \to \infty} (J(F + t_n) + J(F - t_n)) \leq 2^{p+1} \|f\|_p^p$$

which is a contradiction. Hence $\lim_{n \to \infty} \|f - f_n\|_p = 0$ indeed.