Homework set 9 – Solution

Problem 1. Assume that f is analytic at z_0. Then

$$\left|\frac{\ell(f(z))-\ell(f(z_0))}{z-z_0}-\ell(f'(z_0))\right| \leq \|\ell\| \left\|\frac{f(z)-f(z_0)}{z-z_0}-f'(z_0)\right\|$$

vanishes as $z\to z_0$, proving weak analyticity. Reciprocally, assume that f is weakly analytic. Let $\ell\in V^*$. The Cauchy integral formula for F_ℓ yields

$$\begin{aligned}\ell\left(\frac{f(z)-f(z_0)}{z-z_0}-\frac{f(w)-f(z_0)}{w-z_0}\right) &= \frac{F_\ell(z)-F_\ell(z_0)}{z-z_0}-\frac{F_\ell(w)-F_\ell(z_0)}{w-z_0}\\ &= \frac{1}{2\pi \mathrm{i}}\oint_\gamma \left(\frac{1}{z-z_0}(\frac{1}{\zeta-z}-\frac{1}{\zeta-z_0})-\frac{1}{w-z_0}(\frac{1}{\zeta-w}-\frac{1}{\zeta-z_0})\right)\ell(f(\zeta))d\zeta\\ &= \frac{1}{2\pi \mathrm{i}}\oint_\gamma \frac{z-w}{(\zeta-z)(\zeta-z_0)(\zeta-w)}\ell(f(\zeta))d\zeta\end{aligned}$$

where γ is the circle or radius r around z_0 oriented positively containing both z,w. Since $\ell(f(\zeta))$ is continuous on the compact γ, it is bounded, namely $|\ell(f(\zeta))|<C_\ell$. Let $\mathcal{F}=\{\mathcal{I}(f(\zeta)):\zeta\in\gamma\}$ be a family of bounded linear functionals on V^{**}, where $\mathcal{I}$ is the canonical isomorphism $V\to V^{**}$. By the above, $\{|\ell(f(\zeta))|:\zeta\in\gamma\}$ is bounded for any $\ell\in V^*$. Buy the principle of uniform boundedness, $\sup\{\|\mathcal{I}(f(\zeta))\|_{V^{**}}:\zeta\in\gamma\}$ is finite, namely there is C such that

$$\sup\{\ell(\mathcal{I}(f(\zeta))):\zeta\in\gamma\}\leq C\|\ell\|.$$

Hence, for $|z-z_0|,|w-z_0|<r/2$

$$\left|\ell\left(\frac{f(z)-f(z_0)}{z-z_0}-\frac{f(w)-f(z_0)}{w-z_0}\right)\right| \leq \frac{2\pi r}{2\pi}\frac{|z-w|}{r^3/4}C\|\ell\| = \frac{4C\|\ell\|}{r^2}|z-w|$$

and hence

$$\left\|\frac{f(z)-f(z_0)}{z-z_0}-\frac{f(w)-f(z_0)}{w-z_0}\right\| \leq \frac{4C}{r^2}|z-w|$$

Let now $(z_n)_{n\in\mathbb{N}}$ be a sequence converging to z_0. Plugging $z_n=z, z_m=w$ above shows that quotient forms a Cauchy sequence and it therefore convergent in V.

(ii) If f is analytic in Ω, then for any $\ell\in V^*$, $F_\ell(z)$ is analytic in Ω hence bounded on the compact $K\subset\Omega$. It follows that $\|f(z)\|=\sup\{|F_\ell(z)|/\|\ell\|:\ell\in V^*\}<\infty$ by the principle of uniform boundedness.

(iii) For any $\ell\in V^*$, the Cauchy integral formula yields

$$\ell(f(z)) = F_\ell(w) = \frac{1}{2\pi\mathrm{i}}\oint_\gamma \frac{\ell(f(z))}{z-w}dz = \frac{1}{2\pi\mathrm{i}}\ell\left(\oint_\gamma \frac{f(z)}{z-w}dz\right)$$

which yields the claim since linear functionals separate.

(iv) The claim follows from $f(z)=\frac{1}{2\pi\mathrm{i}}\oint\frac{f(\zeta)}{\zeta-z}d\zeta$ as in ordinary complex analysis by picking a circle γ of radius $r>0$ centred at z_0 containing z in its interior and using

$$\frac{1}{\zeta-z}=\frac{1}{\zeta-z_0}\left(1-\frac{z-z_0}{\zeta-z_0}\right)^{-1}=\frac{1}{\zeta-z_0}\sum_{j=0}^\infty\left(\frac{z-z_0}{\zeta-z_0}\right)^j.$$

The series $\sum_{j=0}^{\infty} \frac{f(\zeta)}{\zeta - z_0} \left(\frac{z-z_0}{\zeta-z_0}\right)^j$ converges in norm, uniformly in ζ since $\left|\frac{z-z_0}{\zeta-z_0}\right| = \frac{|z-z_0|}{r} < 1$. Hence it commutes with the integral and yields

$$A_j = \frac{1}{2\pi \mathrm{i}} \oint_\gamma \frac{f(\zeta)}{(\zeta - z_0)^{j+1}} d\zeta.$$

Problem 2. (i) Follows from the fact that the set of invertible operators is open, since $\|(\lambda 1 - T) - (\mu 1 - T)\| = |\lambda - \mu|$.
(ii) If $\lambda \in \rho(T)$, then $(\lambda 1 - T)$ is invertible with bounded inverse. Then so is $(\lambda - \mu)1 - T$ for $|\mu| < \|(\lambda 1 - T)^{-1}\|^{-1}$, and

$$((\lambda - \mu)1 - T)^{-1} = \sum_{j=0}^{\infty} ((\lambda 1 - T)^{-1}\mu)^j (\lambda 1 - T)^{-1} = \sum_{j=0}^{\infty} (\lambda 1 - T)^{-j-1} \mu^j$$

proving that $\lambda \mapsto \lambda 1 - T$ has a convergent power series expansion around any $\lambda \in \rho(T)$.
(iii) Let $\lambda \in \rho(T)$. From the above, we conclude that $|\mu| < \|(\lambda 1 - T)^{-1}\|^{-1}$ implies $(\lambda - \mu) \in \rho(T)$ and hence $(\lambda - \mu) \in \sigma(T)$ implies $|\mu|^{-1} \leq \|(\lambda 1 - T)^{-1}\|$.
(iv) The spectrum is closed as the complement of the resolvent set which is open. Moreover, $\lambda 1 - T = \lambda(1 - \lambda^{-1} T)$ is invertible whenever $|\lambda^{-1}| \|T\| < 1$, proving that $\sigma(T) \subset \overline{B_{\|T\|}(0)}$. Moreover, if $|\lambda| > \|T\|$

$$(\lambda 1 - T)^{-1} = \sum_{n=0}^{\infty} T^n \lambda^{-n-1} \tag{1}$$

is the Laurent series of $\lambda \mapsto \lambda 1 - T$ around ∞, and the coefficient of λ^{-1} is 1. Hence $-\oint (z1 - T)^{-1} dz = 2\pi \mathrm{i}\, 1$ (as an operator equality). On the other hand, if the spectrum was empty, then $\lambda \mapsto \lambda 1 - T$ would be analytic in all of $\mathbb{C}$, in which case the integral above would vanish. Contradiction.
(v) Recall that $r(T) = \lim_{n\to\infty} \|T^n\|^{1/n}$ and let $R(T) = \max\{|\lambda| : \lambda \in \sigma(T)\} \leq \|T\|$. From (1),

$$\frac{1}{2\pi \mathrm{i}} \oint \zeta^k (\zeta 1 - T)^{-1} d\zeta = \sum_{n=0}^{\infty} T^n \frac{1}{2\pi \mathrm{i}} \oint \zeta^{-n+k-1} d\zeta = T^k \tag{2}$$

where as above, the contour is a large circle of $r = R(T) + \delta$ for some $\delta > 0$. $(\zeta 1 - T)^{-1}$ being uniformly bounded, we conclude that $\|T^k\| \leq C(R(T) + \delta)^{k+1}$ and hence

$$\limsup_{k\to\infty} \|T^k\|^{1/k} \leq \limsup_{k\to\infty} C^{1/k} (R(T) + \delta)^{1+1/k} = R(T) + \delta$$

Since δ is arbitrary, we conclude that $r(T) = \limsup_{k\to\infty} \|T^k\|^{1/k} \leq R(T)$. Finally, we note that in this calculation, $\|T\|$ could be replaced by $\max\{|\lambda| : \lambda \in \sigma(T)\}$. Reciprocally, let $k \in \mathbb{N}$ be fixed and let $n = kl + m$. Then

$$\left|\sum_{n=0}^{\infty} \frac{T^n}{\lambda^{n+1}}\right| \leq \sum_{m=0}^{k-1} \frac{|T^m|}{|\lambda|^{m+1}} \sum_{l=0}^{\infty} \left(\frac{\|T^k\|}{|\lambda|^k}\right)^l,$$

and the series is convergent for $\|T^k\| < |\lambda|^k$. Hence, $|\lambda| > \|T^k\|^{1/k}$ implies $\lambda \in \rho(T)$. Hence, $R(T) \leq \|T^k\|^{1/k}$ for any k and in particular $R(T) \leq \inf\{\|T^k\|^{1/k} : k \in \mathbb{N}\} = r(T)$.