THE EICHLER TRACE OF $\mathbb{Z}_p$ ACTIONS ON RIEMANN SURFACES

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1. INTRODUCTION

In this paper we study group actions of $\mathbb{Z}_p$, the cyclic group of odd prime order p, on compact connected Riemann surfaces S. If the genus of S is g then the vector space V of holomorphic differentials on S has dimension g and any action of $\mathbb{Z}_p$ on S determines a representation $\rho\colon \mathbb{Z}_p \to GL(V)$. If T is a preferred generator of $\mathbb{Z}_p$ then this representation yields a matrix $\rho(T) \in GL(V)$. The trace of this matrix, $\chi = tr(T)$, is referred to as the Eichler trace. It is an element of the ring of integers $\mathbb{Z}[\zeta]$, where $\zeta = e^{\frac{2\pi i}{p}}$.

One of the goals of this paper is to determine how much information about the action of $\mathbb{Z}_p$ is captured by the Eichler trace. There are actually two questions here.

Question (1): What elements $\chi \in \mathbb{Z}[\zeta]$ can be realized as the trace of some action?
Question (2): What is the relationship between two actions, not necessarily on the same surface, if they have the same trace?

We give complete answers to both questions. To explain our results we need to develop some notation. Let T be an automorphism of order p on a compact connected Riemann surface S. Suppose there are t fixed points $P_1, \dots, P_t$. In a sufficiently small neighbourhood of a fixed point P_j the automorphism will have the form $T\colon z \to \zeta^{k_j} z$ for some integer k_j, $1 \le k_j \le p-1$. This integer is defined to be the rotation number at P_j. The Eichler Trace Formula then determines the trace of T as

$$\chi = 1 + \sum_{j=1}^{t} \frac{1}{\zeta^{k_j} - 1}. \tag{1}$$

See [9] for a proof of this result.

Let A denote the set of all Eichler traces of all possible actions, that is

$$A = \{\chi \in \mathbb{Z}[\zeta] \mid \chi = tr(T)\}, \tag{2}$$

where T is any automorphism of order p on any compact connected Riemann surface S. A simple calculation with the Eichler Trace Formula (1) shows that $\chi + \bar{\chi} = 2 - t$ for any $\chi \in A$, where $\bar{\chi}$ denotes the complex conjugate of χ. Thus $A \subset B$, where

$$B = \{\chi \in \mathbb{Z}[\zeta] \mid \chi + \bar{\chi} \in \mathbb{Z}\}. \tag{3}$$

Research supported by NSERC grant A7218.

In Section (3) we show that B is a free abelian subgroup of $\mathbb{Z}[\zeta]$ of rank $(p+1)/2$ and determine a basis. Thus a reasonable first step in describing A is to determine the "index" of A in B. Unfortunately, it turns out that A is not a subgroup of B, so this does not make sense. On the other hand, the quotient set $\widehat{A} = A/\mathbb{Z}$, that is the elements of A modulo the integers, is a group, in fact a subgroup of $\widehat{B} = B/\mathbb{Z}$. We prove that $\widehat{B}$ is a free abelian group of rank $(p-1)/2$ and that the index of $\widehat{A}$ in $\widehat{B}$ is finite.

Theorem 1. *The index of $\widehat{A}$ in $\widehat{B}$ is h_1, the first factor of the class number h of $\mathbb{Z}[\zeta]$.*

This theorem gives a partial answer to Question (1). In Section (4) we find free generators of $\widehat{A}$, thereby answering completely Question (1). See Theorems (4), (5) and Corollary (1) in this section.

Implicit in these theorems is an answer to Question (2). To an automorphism $T\colon S \to S$ of order p we associate a "vector" $[g; k_1, \dots, k_t]$, where g is the genus of the orbit surface $S/\mathbb{Z}_p$, t is the number of fixed points, and the k_j are the rotation numbers. The rotation numbers are unique modulo p, but their order is not determined. From the Eichler Trace Formula (1) it is clear that $\chi = tr(T)$ does not depend on g or on the order of the k_j. If a cancelling pair $\{k, p-k\}$, where $1 \le k \le p-1$, appears amongst the set of rotation numbers $\{k_1, \cdots, k_t\}$, then an easy calculation shows that their contribution to the Eichler trace is

$$\frac{1}{\zeta^k - 1} + \frac{1}{\zeta^{p-k} - 1} = -1.$$

Thus we can replace the cancelling pair $\{k, p-k\}$ by any other cancelling pair $\{l, p-l\}$ and not change the Eichler trace.

Given two such automorphisms

$$T_1\colon S_1 \to S_1,\ T_2\colon S_2 \to S_2$$

we have two "vectors" $[g; k_1, \dots, k_t]$, $[h; l_1, \dots, l_u]$. Let χ_1 and χ_2 denote the respective Eichler traces.

Theorem 2. *$\chi_1 = \chi_2$ if, and only if, $t = u$ and the set of rotation numbers $\{k_1, \dots, k_t\}$ agrees with $\{l_1, \dots, l_u\}$ up to permutations and replacements of cancelling pairs.*

As far as the set of Eichler traces is concerned there is no loss of generality in assuming that the orbit surface $S/\mathbb{Z}_p$ is the extended complex plane $\widehat{\mathbb{C}}$. In other words, if χ is the Eichler trace of some action with orbit genus $g > 0$ (the orbit genus is defined to be the genus of $S/\mathbb{Z}_p$) then there will be some other action, on a different Riemann surface, with the same Eichler trace χ and orbit genus $g = 0$. There is also no loss of generality in considering actions up to topological conjugacy since an easy consequence of the Eichler Trace Formula (1) is that conjugate actions have the same Eichler trace.

In Section (2) we show that $\mathbb{Z}_p$ actions with orbit genus $g = 0$ can be parametrized up to topological conjugacy by sequences

$$[a_1, \dots, a_t],\ \textit{where}\ 1 \le a_1 \le \dots \le a_t \le p-1,\ \textit{and}\ \sum_{j=1}^{t} a_j \equiv 0 \pmod{p}. \tag{4}$$

Theorem 3. *There is a one-to-one correspondence between the set of topological conjugacy classes of automorphisms $T\colon S \to S$ of order p and orbit genus 0, where S is an arbitrary compact connected Riemann surface, and sequences satisfying the conditions in (4). The integer t is the number of fixed points and the rotation numbers k_j are determined by the equations $k_j a_j \equiv 1 \ (mod\ p),\ 1 \leq j \leq t$.*

We can also parametrize these actions by sequences $[a_1, \dots, a_t]$, where $1 \leq a_j \leq p-1$ for $1 \leq j \leq t$, and $\sum_{j=1}^{t} a_j \equiv 0 \ (mod\ p)$. Two such sequences $[a_1, \dots, a_t]$, $[b_1, \dots, b_u]$ are the same if $t = u$ and they agree up to a permutation.

Now consider the infinitely generated free abelian group $\mathcal{F}$ generated by all sequences $[a_1, \dots, a_t]$ as in (4). The goal is to make the Eichler trace into a group isomorphism, so we introduce some relations.

Definition 1. *Let $\mathcal{A}$ denote the abelian group $\mathcal{F}/\mathcal{R}$, where $\mathcal{R}$ is the subgroup of $\mathcal{F}$ generated by the following relations:*

(i) $[a_1, \dots, a_t] + [b_1, \dots, b_u] = [a_1, \dots, a_t, b_1, \dots, b_u]$.
(ii) $[\dots, a, \dots, p-a, \dots] = [\dots, \hat{a}, \dots, \widehat{p-a}, \dots]$.

In relation (i) the sequences $[a_1, \dots, a_t]$, $[b_1, \dots, b_u]$ are arbitrary sequences satisfying condition (4), and in relation (ii) the sequence $[\dots, a, \dots, p-a, \dots]$ is any sequence satisfying condition (4) and having a cancelling pair $\{a, p-a\}$. The first relation says that addition in $\mathcal{A}$ is, up to rearrangement, concatenation of sequences, and the second relation allows us to delete a cancelling pair $\{a, p-a\}$. It follows that the identity element of $\mathcal{A}$ is represented by the empty sequence [ ], or by any sequence consisting entirely of cancelling pairs. The inverse of $[a_1, \dots, a_t]$ is represented by $[p-a_1, \dots, p-a_t]$, up to rearrangement.

In Section (4) we prove the following two theorems.

Theorem 4. *The Eichler trace determines a natural group isomorphism $\eta\colon \mathcal{A} \to \widehat{A}$.*

Theorem 5. *The abelian group $\mathcal{A}$ is free of rank $(p-1)/2$. A free basis is given by the triples $[1, r, s]$, where $1 \leq r \leq s \leq p-1$ and $1 + r + s \equiv 0 \ (mod\ p)$.*

It follows from these theorems that $\widehat{A}$ is a free abelian group of rank of $(p-1)/2$. In the next corollary we give a basis, thereby completely answering Question (1).

Corollary 1. *$\widehat{A}$ is a free abelian group of rank $(p-1)/2$. It is freely generated by the mod $\mathbb{Z}$ representatives of the $(p-1)/2$ elements:*

$$\chi_{r,s} = \frac{1}{\zeta - 1} + \frac{1}{\zeta^r - 1} + \frac{1}{\zeta^s - 1}, \ \textit{where } 1 \leq r \leq s \leq p-1 \textit{ and } 1 + r + s \equiv 0 \ (mod\ p).$$

Then in Section (5) we give Theorems (4) and (5) geometric content by relating equivariant cobordism of $\mathbb{Z}_p$ actions on compact connected Riemann surfaces to $\widehat{A}$. To explain this let Ω denote the group of equivariant cobordism classes of $\mathbb{Z}_p$ actions. In the definition of Ω we do not assume the orbit genus is zero. We show that the Eichler trace induces a natural group homomorphism $\phi\colon \mathcal{A} \to \Omega$.

Theorem 6. *$\phi\colon \mathcal{A} \to \Omega$ is a group isomorphism.*

Corollary 2. *Ω is a free abelian group of rank $(p-1)/2$. A free basis is given by those cobordism classes of automorphisms $T\colon S \to S$ having order p, orbit genus 0, and three fixed points, at least one of which has rotation number one. If the other two rotation numbers are k_2, k_3 then the only restriction is that $1 + a_2 + a_3 \equiv 0 \ (mod p)$, where $k_2 a_2 \equiv k_3 a_3 \equiv 1 \ (mod p)$.*

The results in this paper were motivated by the papers [7] and [8] of Ewing. The first paper is quite technical. It contains our Theorem (1), but stated in terms of Witt classes and G-signatures. The second paper is an elegant exposition of the first.

Some of our results overlap earlier papers, in particular [5], [6], and [16]. However, the approach that we take is new.

2. PRELIMINARIES

In this section we collect some of the preliminaries needed for later sections, and prove Theorem (3). First we describe how all group actions on Riemann surfaces occur and then we specialize to the case of the group $\mathbb{Z}_p$.

We will use the notation $Aut(S)$ for the group of analytic automorphisms of a Riemann surface S. Throughout the paper all surfaces will be connected, orientable and without boundary. By the uniformization theorem the universal covering space $\mathbb{U}$ of S is one of three possibilities: the extended complex plane $\widehat{\mathbb{C}}$, the complex plane $\mathbb{C}$, or the upper half plane $\mathbb{H}$. The letter $\mathbb{U}$ will always denote one of these three.

If G is a finite group acting topologically on a surface S by orientation preserving homeomorphisms then the positive solution of the Nielsen Realization Problem guarantees that there exists a complex analytic structure on S for which the action of G is by analytic automorphisms (see [14], [11], [10] or [4]). Thus there is no loss of generality in assuming that the action of G is complex analytic to begin with, and we will tacitly do so.

The orbit space $\bar{S} = S/G$ of the action of G is also a Riemann surface and the orbit map $\pi\colon S \to \bar{S}$ is a branched covering, with all branching occurring at fixed points of the action. If $x \in \bar{S}$ is a branch point then each point in $\pi^{-1}(x)$ has a non-trivial stabilizer subgroup in G.

To any action of G on S we associate a short exact sequence of groups

$$1 \to \Pi \to \Gamma \xrightarrow{\theta} G \to 1, \qquad (5)$$

with Γ being a discrete subgroup of $Aut(\mathbb{U})$ and Π a torsion free normal subgroup of Γ, as follows. Let $\pi\colon \mathbb{U} \to S$ denote the covering map. Then Γ is defined by

$$\Gamma = \left\{\gamma \in Aut(\mathbb{U}) \middle| \ \pi \circ \gamma = g \circ \pi,\ g \in G\right\}. \qquad (6)$$

In other words Γ consists of all lifts $\gamma\colon \mathbb{U} \to \mathbb{U}$ of all automorphisms $g\colon S \to S$, $g \in G$. The subgroup Γ is unique up to conjugation in $Aut(\mathbb{U})$. See the commutative diagram below.

$$\begin{array}{ccc} \mathbb{U} & \xrightarrow{\gamma} & \mathbb{U} \\ \downarrow & & \downarrow \\ S & \xrightarrow{g} & S \end{array}$$

The epimorphism $\theta\colon \Gamma \to G$ is defined by $\theta(\gamma) = g$, where γ and g are as in (6). The kernel of $\theta\colon \Gamma \to G$ is Π, the fundamental group of S, and is therefore torsion free. The Riemann surface $S = \mathbb{U}/\Pi$ and the action of G on $\mathbb{U}/\Pi$ is given by $g[z]_\Pi = [\gamma(z)]_\Pi$, where $z \in \mathbb{U}$, $g \in G$, and $\gamma \in \Gamma$ is any element such that $\theta(\gamma) = g$. Here the square brackets denote the orbits under the action of Π. The orbit surface $\bar{S} = \mathbb{U}/\Gamma$, and the branched covering $\pi\colon S \to \bar{S}$ is just the natural map $\mathbb{U}/\Pi \to \mathbb{U}/\Gamma$, $[z]_\Pi \mapsto [z]_\Gamma$.

Conversely, suppose $1 \to \Pi \to \Gamma \xrightarrow{\theta} G \to 1$ is a given short exact sequence of groups, where Γ is a discrete subgroup of $Aut(\mathbb{U})$ and Π is torsion free. Then this short exact sequence corresponds to the one arising from the action of G on the Riemann surface $S = \mathbb{U}/\Pi$ defined above.

Thus there is a one-to-one correspondence between analytic conjugacy classes of analytic actions by the finite group G on compact connected Riemann surfaces and short exact sequences (5), where Γ is a discrete subgroup of $Aut(\mathbb{U})$, unique only up to conjugation in $Aut(\mathbb{U})$, and Π is a torsion free subgroup of Γ.

Now suppose G is the cyclic group $\mathbb{Z}_p$ and $T \in \mathbb{Z}_p$ denotes a fixed generator. Actions of $\mathbb{Z}_p$ on Riemann surfaces correspond to short exact sequences $1 \to \Pi \to \Gamma \xrightarrow{\theta} \mathbb{Z}_p \to 1$. Since the kernel of θ is torsion free the signature of Γ must have the form $(g; \overbrace{p, \ldots, p}^{t \text{ times}})$, where g and t are non-negative integers. As an abstract group Γ has the following presentation

(i) $t + 2g$ generators $A_1, \ldots, A_t, X_1, Y_1, \ldots, X_g, Y_g$.
(ii) $t + 1$ relations $A_1^p = \cdots = A_t^p = A_1 \cdots A_t [X_1, Y_1] \cdots [X_g, Y_g] = 1$.

We denote this group by $\Gamma = \Gamma(g; \overbrace{p, \ldots, p}^{t \text{ times}})$. Any such group can be embedded in $Aut(\mathbb{U})$ as a discrete subgroup in many different ways up to conjugation. In fact the set of conjugacy classes of embeddings is a cell of dimension

$$d(\Gamma) = 6g - 6 + 2t \text{ so long as } 6g - 6 + 2t \geq 0.$$

See [1] and [3]. The genus of the orbit surface $S/\mathbb{Z}_p$ is g and the number of fixed points is t.

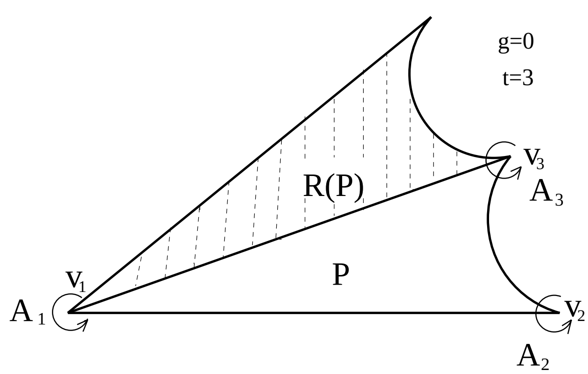

FIGURE 1. Fundamental Domain

Figure (1) illustrates a fundamental domain for a particular embedding when $g = 0$ and $t = 3$. It consists of a regular 3-gon P, all of whose angles are π/p, together with a copy of P obtained by reflection in one of its sides. The generators A_1, A_2, A_3 are the rotations by $2\pi/p$ about consecutive vertices, ordered in the counterclockwise sense. In this case the cell dimension is $d(\Gamma) = 6g - 6 + 2t = 0$, in other words, up to conjugacy in $Aut(\mathbb{U})$ there is a unique subgroup of signature $(0; p, p, p)$.

In a similar manner, when $g = 0$ and $t > 3$, a fundamental domain for a particular Fuchsian group Γ of signature $(0; \overbrace{p, \dots, p}^{t \text{ times}})$ is given by $P \cup R(P)$, where P is a regular t-gon all of whose angles are π/p and R is a reflection in one of its sides. In this case Γ is the Fuchsian group generated by the rotations $A_1, \dots, A_t$ through $2\pi/p$ about consecutive vertices. The dimension of the cell is $d(\Gamma) = 6g - 6 + 2t = -6 + 2t > 0$. Thus the embedding is not unique up to conjugacy in $Aut(\mathbb{U})$.

Let Γ be any Fuchsian group of signature $(0; \overbrace{p, \dots, p}^{t \text{ times}})$. Then an epimorphism $\theta \colon \Gamma \to \mathbb{Z}_p$ is determined by the images of the generators. The relations in Γ must be preserved and the kernel of θ must be torsion free, so θ is determined by the equations

$$\theta(A_j) = T^{a_j},\ 1 \le j \le t;\ \theta(X_k) = T^{b_k},\ \theta(Y_k) = T^{c_k}, 1 \le k \le g.$$

The following restrictions must hold:

(i) The a_j are integers such that $1 \le a_j \le p - 1$ and $\sum_{j=1}^{t} a_j \equiv 0 \pmod p$.
(ii) The b_k, c_k are arbitrary integers mod p, except that at least one of them is non-zero if $t = 0$ (this guarantees that θ is an epimorphism).

It follows from the first restriction that the only possible values of t are $t = 0, 2, 3, \dots$.

Conversely, given integers a_j, b_k, c_k satisfying conditions (i) and (ii), there is an epimorphism $\theta \colon \Gamma \to \mathbb{Z}_p$ with torsion free kernel Π and a corresponding $\mathbb{Z}_p$ action $T \colon S \to S$, where $S = \mathbb{U}/\Pi$.

The integer t equals the number of fixed points of $T \colon S \to S$ and g is the genus of the orbit surface $S/\mathbb{Z}_p$. A well known result of Nielsen [13] says that the topological conjugacy class of $T \colon S \to S$ is completely determined by g and the unordered sequence $(a_1, \dots, a_t)$. We use the notation $[g \mid a_1, \dots, a_t]$ to denote the topological conjugacy class of the homeomorphism $T \colon S \to S$ determined by this data. If $g = 0$ we use the notation $[a_1, \dots, a_t]$, and usually order the a_j so that $1 \le a_1 \le \dots \le a_t \le p - 1$.

Of particular interest is the case $g = 0$. Then the orbit surface $S/\mathbb{Z}_p$ is the extended complex plane $\widehat{\mathbb{C}}$ and Γ has the presentation

(i) t generators $A_1, \dots, A_t$.
(ii) $t + 1$ relations $A_1^p = \dots = A_t^p = A_1 \cdots A_t = 1$.

The epimorphism θ is given by the equations

$$\theta(A_j) = T^{a_j},\ \textit{where}\ 1 \le a_1 \le \dots \le a_t \le p - 1,\ \textit{and}\ \sum_{j=1}^{t} a_j \equiv 0\ (\textit{mod}\ p). \tag{7}$$

Now we complete the proof of Theorem (3).

Proof: It follows from the above that we can associate to an automorphism $T\colon S \to S$ of order p, where S is any compact connected Riemann surface such that the genus of $S/\mathbb{Z}_p$ is 0, a sequence $[a_1, \dots, a_t]$ satisfying the conditions in (4). According to the results of Nielsen two such automorphisms are topologically conjugate if, and only if, the associated sequences are identical.

Conversely, given any sequence $[a_1, \dots, a_t]$ satisfying (4) we can construct an automorphism $T\colon S \to S$ of order p and orbit genus 0 as follows. Let Γ be any discrete subgroup of $Aut(\mathbb{U})$ of signature $(0; \overbrace{p, \dots, p}^{t \text{ times}})$. Then Equation (7) defines an epimorphism $\theta\colon \Gamma \to \mathbb{Z}_p$ with a torsion free kernel Π, and this in turn determines an automorphism T of order p on $S = \mathbb{U}/\Pi$. The topological conjugacy class of T does not depend on the embedding of Γ, only on the signature and the sequence $[a_1, \dots, a_t]$. Thus the correspondence is one-to-one on the level of topological conjugacy.

A particular embedding of Γ in $Aut(\mathbb{U})$ is the one indicated above; that is, Γ is the subgroup generated by $A_1, \dots, A_t$, where the A_j are rotations by $2\pi/p$ about the vertices of a regular t-gon P, all of whose angles are π/p. See Figure (1) for the case where $t = 3$. The fixed points of this action correspond to the orbits of the vertices, and thus there are t of them, $P_1, \dots, P_t$, where P_j is the orbit of the vertex of rotation for the generator A_j. The epimorphism θ satisfies $\theta(A_j) = T^{a_j}$, and therefore $\theta(A_j^{k_j}) = T$, where the k_j are defined by the equations $k_j a_j \equiv 1 \ (mod\ p),\ 1 \le j \le t$. This implies that the automorphism $T\colon S \to S$ in a small neighbourhood of P_j is represented by $A_j^{k_j}$, a rotation about P_j by an angle of $2k_j\pi/p$. In other words the rotation numbers are the k_j for this particular embedding. This completes the proof of Theorem (3) since the number of fixed points and their rotation numbers are invariants of topological conjugacy. ∎

3. THE EICHLER TRACE

In this section we prove Theorems (1) and (2). Recall that the class number h of the ring of integers $\mathbb{Z}[\zeta]$ is the number of equivalence classes of non-zero integral ideals I in $\mathbb{Z}[\zeta]$, where the equivalence relation is fractional equivalence:

$$I \sim J \text{ if there exist non-zero elements } r, s \in \mathbb{Z}[\zeta], \text{ such that } rI = sJ.$$

In fact the collection of equivalence classes of integral ideals forms a finite abelian group $\mathcal{C}$ of order h, where the group structure is given by multiplication of ideals. See [12]. If I is an ideal then so is its complex conjugate $\bar{I}$ and it is easy to see that

$$\mathcal{C}_1 = \{I \mid I\bar{I} \text{ is a principal ideal}\}$$

is a subgroup of $\mathcal{C}$. The order of this subgroup is by definition the first factor h_1 of the class number.

We begin by observing that the set A is not a subgroup of $\mathbb{Z}[\zeta]$. To see this suppose that $\chi \in A$, that is

$$\chi = 1 + \sum_{j=1}^{t} \frac{1}{\zeta^{k_j} - 1}$$

is the Eichler trace of some automorphism $T\colon S \to S$. The possible values for the number of fixed points are $t = 0,\ 2,\ 3, \dots$, and therefore the possible values of $\chi + \bar{\chi} = 2 - t$ are $2,\ 0,\ -1,\ -2, \dots$

We also have $\bar{\chi} \in A$ since

$$\bar{\chi} = 1 + \sum_{j=1}^{t} \frac{1}{\zeta^{-k_j} - 1}$$

is the trace of $T^{-1}\colon S \to S$. Therefore, if A were a subgroup we would have $\chi + \bar{\chi} = 2 - t \in A$, and hence $\mathbb{Z}$ would be a subgroup of A. But if $n \in A$ is an integer, $n \geq 2$, then $n + \bar{n} = 2n \geq 4$ is not of the form $2 - t$ for an admissible t. Therefore A is not a subgroup.

Recall that $\widehat{A}$ is the set of realizable Eichler traces modulo $\mathbb{Z}$.

Proposition 1. *$\widehat{A}$ is a subgroup of $\widehat{\mathbb{Z}[\zeta]}$.*

Proof: Suppose χ_1 and χ_2 are in A, say

$$\chi_1 = 1 + \sum_{j=1}^{t} \frac{1}{\zeta^{k_j} - 1},\ \chi_2 = 1 + \sum_{j=1}^{u} \frac{1}{\zeta^{l_j} - 1}.$$

Therefore $\widehat{\chi_1} + \widehat{\chi_2} = \widehat{\chi}$, where $\chi = 1 + \sum_{j=1}^{t} \frac{1}{\zeta^{k_j}-1} + \sum_{j=1}^{u} \frac{1}{\zeta^{l_j}-1}$. If χ_1 and χ_2 are represented by $T_1\colon S_1 \to S_1$ and $T_2\colon S_2 \to S_2$ respectively, then χ can be represented by the equivariant connected sum of T_1 and T_2. Namely, for $j = 1, 2$ find discs D_j in S_j such that $D_j, T_j(D_j), \ldots, T_j^{p-1}(D_j)$ are mutually disjoint. Excise all discs $T^k(D_j)$, $k = 0, 1, \ldots, p-1$, from S_j, $j = 1, 2$, and then take the connected sum by matching $\partial(T^k(D_1))$ to $\partial(T^k(D_2))$ for $k = 0, 1, \ldots, p-1$. The resulting surface S has p tubes joining S_1 and S_2. The automorphisms T_1, T_2 can be extended to an automorphism $T\colon S \to S$ by permuting the tubes. The Eichler trace of T is χ. Thus $\widehat{A}$ is closed under sums.

If $\chi \in A$ then also $\bar{\chi} \in A$ and $\chi + \bar{\chi} = 2 - t$. Therefore $\bar{\chi}$ is the inverse of χ once we reduce modulo the integers. The identity element of $\widehat{A}$ is represented by any fixed point free action. ∎

To determine the index of $\widehat{A}$ in $\widehat{B}$ we need a basis for $\widehat{B}$, but first we find a basis for B. Let $m = (p-1)/2$.

Definition 2. Define elements $\theta_1,\ \theta_2,\ \ldots, \theta_m$ in $\mathbb{Z}[\zeta]$ by $\theta_1 = \zeta + \sum_{j=m+1}^{p-2} \zeta^j$ and $\theta_k = \zeta^k - \zeta^{-k}$, $2 \leq k \leq m$.

Proposition 2. *A basis of B is given by the $m+1$ elements $1,\ \theta_1,\ \theta_2, \ldots,\ \theta_m$.*

Proof: Suppose $\chi = \sum_{j=0}^{p-2} a_j \zeta^j \in \mathbb{Z}[\zeta]$. Then a short calculation gives

$$\chi + \bar{\chi} = 2a_0 - a_1 + \sum_{j=2}^{p-2} (a_j + a_{p-j} - a_1)\,\zeta^j,$$

and therefore $\chi \in B$ if, and only if, $a_j + a_{p-j} = a_1,\ 2 \leq j \leq p-2$. Solving for $a_{m+1}, \ldots, a_{p-2}$ in terms of $a_1, \ldots, a_m$ and substituting into χ gives

$$\chi = a_0 + a_1\theta_1 + a_2\theta_2 + \cdots + a_m\theta_m.$$

Thus the elements $1,\ \theta_1,\ \theta_2, \ldots,\ \theta_m$ form a spanning set for B.

Now suppose some linear combination is zero, say $a_0 + a_1\theta_1 + a_2\theta_2 + \cdots + a_m\theta_m = 0$. It is easy to see that this is equivalent to

$$a_0 + a_1\zeta + \cdots + a_m\zeta^m + (a_1 - a_m)\zeta^{m+1} + \cdots + (a_1 - a_2)\zeta^{p-2} = 0.$$

Thus we get $a_0 = a_1 = a_2 = \cdots = a_m = 0$, that is the elements are linearly independent. ■

Remark . Every integer $n \in B$ since $\theta_1 + \overline{\theta}_1 = -1$. We also have $\zeta - \zeta^{-1} \in B$; in fact

$$\zeta - \zeta^{-1} = 1 + 2\theta_1 + \theta_2 + \cdots + \theta_m.$$

It follows that the elements $1,\ \zeta - \zeta^{-1},\ \zeta^2 - \zeta^{-2}, \ldots, \zeta^m - \zeta^{-m}$ form a basis for an index 2 subgroup of B.

An immediate corollary of Proposition (2) is

Corollary 3. *$\widehat{B}$ is a free abelian group of rank $(p-1)/2$. A basis is given by the elements*

$$\widehat{\theta_1},\ \widehat{\theta_2},\ \ldots,\ \widehat{\theta_m}.$$

Before completing the calculation of the index of $\widehat{A}$ in $\widehat{B}$ we first consider Question (2) from Section (1). Thus suppose two elements from A have the same Eichler trace, say

$$1 + \sum_{j=1}^{t} \frac{1}{\zeta^{k_j} - 1} = 1 + \sum_{j=1}^{u} \frac{1}{\zeta^{l_j} - 1}.$$

This leads us into consideration of when certain linear combinations of the elements $\frac{1}{\zeta^k - 1}$ are zero, that is we want to solve the equation $\sum_{k=1}^{p-1} \frac{x_k}{\zeta^k - 1} = 0$ for the integers x_k.

If s is any integer relatively prime to p then let $R(s)$ denote that integer q such that $1 \le q \le p-1$ and $q \equiv s (mod\ p)$, that is, $s = [s/p]p + R(s)$. In what follows $\sum_{jk\equiv n}$ denotes the sum over all ordered pairs (j,k) such that $jk \equiv n (mod\ p)$ and $1 \le j \le p-1$.

Lemma 1.

$$\sum_{k=1}^{p-1} \frac{x_k}{\zeta^k - 1} = -\frac{1}{p} \sum_{jk\equiv -1} jx_k + \frac{1}{p} \sum_{n=1}^{p-2} \left(\sum_{jk\equiv n} jx_k - \sum_{jk\equiv -1} jx_k \right) \zeta^n.$$

Proof: We use the identity $\frac{1}{\zeta^k-1} = \frac{1}{p}\sum_{j=1}^{p} j\zeta^{k(j-1)}$ and get

$$\begin{aligned}
\sum_{k=1}^{p-1}\frac{x_k}{\zeta^k-1} &= \frac{1}{p}\sum_{j=1}^{p}\sum_{k=1}^{p-1} jx_k\zeta^{k(j-1)} \\
&= \frac{1}{p}(x_1+\cdots+x_{p-1}) + \frac{1}{p}\sum_{j=2}^{p}\sum_{k=1}^{p-1} jx_k\zeta^{k(j-1)} \\
&= \frac{1}{p}(x_1+\cdots+x_{p-1}) + \frac{1}{p}\sum_{n=1}^{p-1}\left(\sum_{jk\equiv n}(j+1)x_k\right)\zeta^n \\
&= \frac{1}{p}(x_1+\cdots+x_{p-1}) + \frac{1}{p}\sum_{n=1}^{p-2}\left(\sum_{jk\equiv n}(j+1)x_k\right)\zeta^n + \frac{1}{p}\left(\sum_{jk\equiv -1}(j+1)x_k\right)\zeta^{p-1}.
\end{aligned}$$

Now substitute $\zeta^{p-1} = -1-\zeta-\cdots-\zeta^{p-2}$ into the last term to see that

$$\begin{aligned}
\sum_{k=1}^{p-1}\frac{x_k}{\zeta^k-1} &= \frac{1}{p}(x_1+\cdots+x_{p-1}) + \frac{1}{p}\sum_{n=1}^{p-2}\left(\sum_{jk\equiv n}(j+1)x_k - \sum_{jk\equiv -1}(j+1)x_k\right)\zeta^n \\
&- \frac{1}{p}\sum_{jk\equiv -1}(j+1)x_k \\
&= -\frac{1}{p}\sum_{jk\equiv -1} jx_k + \frac{1}{p}\sum_{n=1}^{p-2}\left(\sum_{jk\equiv n} jx_k - \sum_{jk\equiv -1} jx_k\right)\zeta^n.
\end{aligned}$$

■

As a corollary we get

Corollary 4. $\sum_{k=1}^{p-1}\frac{x_k}{\zeta^k-1} = 0$ *if, and only if,* $\sum_{jk\equiv n} jx_k = 0,$ *for* $1\le n\le p-1$.

Now it is convenient to change the variables $x_1,\ldots,x_{p-1}$ to new variables $y_1,\ldots,y_{p-1}$ according to the equation

(8) $$y_l = x_k, where\ kl\equiv 1 (mod\ p).$$

Then Corollary (4) becomes

Corollary 5. $\sum_{k=1}^{p-1}\frac{x_k}{\zeta^k-1} = 0$ *if, and only if,* $\sum_{k=1}^{p-1} R(nk)y_k = 0,$ *for* $1\le n\le p-1$.

The coefficient matrix of this linear system is the $(p-1)\times(p-1)$ matrix M whose (i,j) entry is $M_{(i,j)} = R(ij)$. To solve this system of $p-1$ equations in $p-1$ unknowns y_k we apply a sequence of row and column operations to the matrix M. We use the fact that $R(ij)+R((p-i)j) = p$. Recall that $m = (p-1)/2$.

1. Adding the i^{th} row to the $(p-i)^{th}$ row, $1\le i\le m$, yields the matrix

$$M_1 = \begin{bmatrix} 1 & 2 & \dots & m & m+1 & m+2 & \dots & p-1 \\ 2 & 4 & \dots & 2m & 1 & 3 & \dots & p-2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ i & R(2i) & \dots & R(mi) & R((m+1)i) & R((m+2)i) & \dots & R((p-1)i) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ m & R(2m) & \dots & R(m^2) & R((m+1)m) & R((m+2)m) & \dots & R((p-1)m) \\ p & p & \dots & p & p & p & \dots & p \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ p & p & \dots & p & p & p & \dots & p \end{bmatrix}$$

2. Adding the j^{th} column to the $(p-j)^{th}$ column, $1 \leq j \leq m$, yields the matrix

$$M_2 = \begin{bmatrix} 1 & 2 & \dots & m & p & p & \dots & p \\ 2 & 4 & \dots & 2m & p & p & \dots & p \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ i & R(2i) & \dots & R(mi) & p & p & \dots & p \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ m & R(2m) & \dots & R(m^2) & p & p & \dots & p \\ p & p & \dots & p & 2p & 2p & \dots & 2p \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ p & p & \dots & p & 2p & 2p & \dots & 2p \end{bmatrix}$$

3. Subtracting the $(m+1)^{st}$ row from rows $m+2, \dots, p-1$, and then subtracting the $(m+1)^{st}$ column from columns $m+2, \dots, p-1$ gives the new coefficient matrix

$$M_3 = \begin{bmatrix} 1 & 2 & \dots & m & p & 0 & \dots & 0 \\ 2 & 4 & \dots & 2m & p & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ i & R(2i) & \dots & R(mi) & p & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ m & R(2m) & \dots & R(m^2) & p & 0 & \dots & 0 \\ p & p & \dots & p & 2p & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \end{bmatrix}$$

The variables z_k for this coefficient matrix are related to the y_k by the equations

$$z_k = y_k - y_{p-k},\ 1 \leq k \leq m,\ z_{m+1} = y_{m+1} + \cdots + y_{p-1},\ z_{m+j} = y_{m+j},\ 2 \leq j \leq p-1.$$

Examination of the last $m-1$ columns of M_3 reveals that $z_{m+2}, \dots, z_{p-1}$ are completely independent; whereas, $z_1, \dots, z_{m+1}$ must satisfy the matrix equation

$$\begin{bmatrix} 1 & 2 & \dots & m & p \\ 2 & 4 & \dots & 2m & p \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ i & R(2i) & \dots & R(mi) & p \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ m & R(2m) & \dots & R(m^2) & p \\ p & p & \dots & p & 2p \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_i \\ \vdots \\ z_m \\ z_{m+1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}$$

Now we apply another sequence of row and column operations to this last coefficient matrix.

1. Subtracting i times the first row from the i^{th} row, $2 \le i \le m$, yields the coefficient matrix

$$\begin{bmatrix} 1 & 2 & \dots & j & \dots & m & p \\ 0 & 0 & \dots & 0 & \dots & 0 & -p \\ 0 & 0 & \dots & -[3j/p]p & \dots & -[3m/p]p & -2p \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & -[ij/p]p & \dots & -[im/p]p & -(i-1)p \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & -[mj/p]p & \dots & -[m^2/p]p & -(m-1)p \\ p & p & \dots & p & \dots & p & 2p \end{bmatrix}$$

2. Subtracting j times the first column from the j^{th} column, $2 \le j \le m$, yields the matrix

$$\begin{bmatrix} 1 & 0 & \dots & 0 & \dots & 0 & p \\ 0 & 0 & \dots & 0 & \dots & 0 & -p \\ 0 & 0 & \dots & -[3j/p]p & \dots & -[3m/p]p & -2p \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & -[ij/p]p & \dots & -[im/p]p & -(i-1)p \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & -[mj/p]p & \dots & -[m^2/p]p & -(m-1)p \\ p & -p & \dots & -(j-1)p & \dots & -(m-1)p & 2p \end{bmatrix}$$

The new variables w_j, after these last column operations, are related to the z_j by the equations $w_1 = z_1 + 2z_2 + \cdots + mz_m$ and $w_j = z_j$, $2 \le j \le m+1$.

It follows that $w_1 = w_{m+1} = 0$ and $w_2, \dots, w_m$ are related by the equations

$$w_2 + 2w_3 + \cdots + (m-1)w_m = 0,$$

$$\begin{bmatrix} -[9/p]p & \dots & -[3j/p]p & \dots & -[3m/p]p \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ -[3i/p]p & \dots & -[ij/p]p & \dots & -[im/p]p \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ -[3m/p]p & \dots & -[mj/p]p & \dots & -[m^2/p]p \end{bmatrix} \begin{bmatrix} w_3 \\ \vdots \\ w_j \\ \vdots \\ w_m \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

The coefficient matrix of this system can be row reduced to the matrix whose (i,j) entry, $3 \leq i,j \leq m$, is $[ij/p]p - [(i-1)j/p]p$, by first subtracting row $m-3$ from row $m-2$, then row $m-4$ from row $m-3$, etc., and then changing all signs. The resulting matrix is invertible, in fact its determinant equals $\pm p^{m-2}h_1$, where h_1 is the first factor of the class number [2]. Thus $w_j = 0,\ 1 \leq j \leq m+1$.

This proves that $\sum_{k=1}^{p-1} \frac{x_k}{\zeta^k - 1} = 0$ if, and only if, $y_k = y_{p-k}$ for $1 \leq k \leq p-1$, and

$$y_m = -y_{m+2} - \cdots - y_{p-1},$$

where $y_{m+2}, \cdots, y_{p-1}$ are completely arbitrary. Translating back to the x_k variables we have:

Corollary 6. $\sum_{k=1}^{p-1} \frac{x_k}{\zeta^k - 1} = 0$ *if, and only if,* $x_k = x_{p-k}$ *for* $1 \leq k \leq p-1$, *and*

$$x_m = -x_{m+2} - \cdots - x_{p-1},$$

where $x_{m+2}, \cdots, x_{p-1}$ *are completely arbitrary.*

We can now complete the proof of Theorem (2).

Proof: Suppose $\chi_1 = \chi_2$ are the Eichler traces of two actions, say

$$\chi_1 = 1 + \sum_{j=1}^{t} \frac{1}{\zeta^{k_j} - 1} = 1 + \sum_{k=1}^{p-1} \frac{u_k}{\zeta^k - 1},$$

$$\chi_2 = 1 + \sum_{j=1}^{u} \frac{1}{\zeta^{l_j} - 1} = 1 + \sum_{k=1}^{p-1} \frac{v_k}{\zeta^k - 1},$$

where u_k is the number of times k appears as a rotation number in χ_1, and v_k is defined similarly. We immediately get $t = u$ since $\chi_1 + \overline{\chi}_1 = 2 - t$ and $\chi_2 + \overline{\chi}_2 = 2 - u$. The equation $\chi_1 - \chi_2 = 0$ gives the linear relation $\sum_{k=1}^{p-1} \frac{x_k}{\zeta^k - 1} = 0$, where $x_k = u_k - v_k$. It follows from Corollary (6) that the vector $\vec{x} = (x_1, \cdots, x_{p-1})$ is an integral linear combination of the vectors

$$\vec{e_j} = (\cdots, 1, \cdots, -1, -1, \cdots, 1, \cdots),\ 1 \leq j \leq m-1,$$

where the $1's$ are in positions $j,\ p-j$; the $-1's$ are in positions $m,\ m+1$; and the other entries are zero.

For argument's sake suppose $\vec{x} = \vec{e_j}$ for some j. This means we can move from the vector of rotation numbers $[u_1, \cdots, u_{p-1}]$ to the vector $[v_1, \cdots, v_{p-1}]$ by replacing a cancelling pair $\{j, p-j\}$ by the cancelling pair $\{m, m+1\}$. Taking linear combinations of the $\vec{e_j}$ corresponds to a sequence of such moves.

This completes the proof of Theorem (2). ■

The remainder of this section is concerned with the proof of Theorem (1). According to Theorem (3) and the Eichler Trace Formula (1) the set of Eichler traces is given by

$$A = \left\{ \chi \in \mathbb{Z}[\zeta] \;\middle|\; \chi = 1 + \sum_{j=1}^{t} \frac{1}{\zeta^{k_j} - 1} \right\},$$

where the only restriction on the rotation numbers k_j is that $\sum_{j=1}^{t} R(k_j^{-1}) \equiv 0 \ (mod\ p)$. If we define x_k to be the number of j, $1 \le j \le t$, such that $k_j = k$, then we can characterize A by

$$A = \left\{ \chi \in \mathbb{Z}[\zeta] \;\middle|\; \chi = 1 + \sum_{k=1}^{p-1} \frac{x_k}{\zeta^k - 1},\ x_k \ge 0 \ and \ \sum_{k=1}^{p-1} R(k^{-1}) x_k \equiv 0 \ (mod\ p) \right\}. \tag{9}$$

In the next lemma we show that by passing to $\widehat{A}$ we can remove the restriction that the x_k be non-negative integers.

Lemma 2. *The set of Eichler traces modulo* $\mathbb{Z}$ *is given by*

$$\widehat{A} = \left\{ \widehat{\chi} \in \widehat{\mathbb{Z}[\zeta]} \;\middle|\; \chi = \sum_{k=1}^{p-1} \frac{x_k}{\zeta^k - 1},\ \sum_{k=1}^{p-1} R(k^{-1}) x_k \equiv 0 \ (mod\ p) \right\}.$$

Proof: First note that by choosing all $x_k = 1$ in Equation (9) we get an element $\chi \in A$. In fact a short calculation using Lemma (1) gives $\chi = 1 - (p-1)/2$, and thus this element represents 0 in $\widehat{A}$. By adding χ sufficiently many times to an element in A we can ensure that all the coefficients x_k become positive, and this does not change its value in $\widehat{A}$. ■

This description of $\widehat{A}$ contains a lot of redundancy. In fact we have the following characterization of $\widehat{A}$.

Lemma 3. *The set of Eichler traces modulo* $\mathbb{Z}$ *is given by*

$$\widehat{A} = \left\{ \widehat{\chi} \;\middle|\; \chi = \sum_{k=1}^{m} \frac{z_k}{\zeta^k - 1},\ \sum_{k=1}^{m} R(k^{-1}) z_k \equiv 0 \ (mod\ p) \right\}.$$

Proof: According to Lemma (2) a typical element $\widehat{\chi} \in \widehat{A}$ can be represented by

$$\chi = \sum_{k=1}^{p-1} \frac{x_k}{\zeta^k - 1} = \sum_{k=1}^{m} \frac{x_k}{\zeta^k - 1} + \sum_{k=1}^{m} \frac{x_{p-k}}{\zeta^{-k} - 1},$$

where the x_k are integers satisfying $\sum_{k=1}^{p-1} R(k^{-1}) x_k \equiv 0 \ (mod\ p)$. Now we use the fact that

$$\frac{1}{\zeta^k - 1} + \frac{1}{\zeta^{-k} - 1} = -1$$

to see that $\widehat{\chi} = \widehat{\psi}$, where

$$\psi = \sum_{k=1}^{m} \frac{z_k}{\zeta^k - 1},\ and\ z_k = x_k - x_{p-k}.$$

The restriction on the integers z_k becomes $\sum_{k=1}^{m} R(k^{-1})z_k \equiv 0\ (mod\ p)$, since

$$\begin{aligned}\sum_{k=1}^{p-1} R(k^{-1})x_k &= \sum_{k=1}^{m} R(k^{-1})x_k + \sum_{k=1}^{m} R((p-k)^{-1})x_{p-k} \\ &= \sum_{k=1}^{m} R(k^{-1})x_k + \sum_{k=1}^{m} (p - R(k^{-1}))x_{p-k} \\ &\equiv \sum_{k=1}^{m} R(k^{-1})z_k\ (mod\ p)\end{aligned}$$

and $\sum_{k=1}^{p-1} R(k^{-1})x_k \equiv 0\ (mod\ p)$. ■

In Definition (2) we introduced elements $\theta_1,\ \theta_2,\ \ldots,\theta_m$ and then in Corollary (3) we showed that the corresponding classes modulo $\mathbb{Z}$, that is $\widehat{\theta_1},\ \widehat{\theta_2},\ \ldots,\widehat{\theta_m}$, formed a basis of $\widehat{B}$. To determine the index of $\widehat{A}$ in $\widehat{B}$ we want to express a typical element of $\widehat{A}$ in terms of this basis. But first we need a definition.

Definition 3. For integers k, n define $C(k,n) = R(k^{-1}n) + R(k^{-1}) - p$.

The following properties of the coefficients $C(k,n)$ are easy to verify:

(i) $C(k,n) + C(p-k,n) = 0$ and $C(k,n) + C(k,p-n) = 2R(k^{-1}) - p$.
(ii) $C(1,n) = n+1-p$, $C(k,1) = 2R(k^{-1}) - p$, $C(p-1,n) = p-n-1$, and $C(k,p-1) = 0$.

Lemma 4. *The elements of $\widehat{A}$ are those elements $\widehat{\chi} \in \widehat{\mathbb{Z}[\zeta]}$ of the form*

$$\widehat{\chi} = \frac{1}{p}\sum_{n=1}^{m}\Big(\sum_{k=1}^{m} C(k,n)z_k\Big)\widehat{\theta_n},$$

where the only restriction on the integers z_k is $\sum_{k=1}^{m} R(k^{-1})z_k \equiv 0\ (mod\ p)$.

Proof: By Lemma (3) a typical Eichler trace modulo $\mathbb{Z}$ is given by $\widehat{\chi}$, where $\chi = \sum_{k=1}^{m} \frac{z_k}{\zeta^k - 1}$, and $\sum_{k=1}^{m} R(k^{-1})z_k \equiv 0\ (mod\ p)$. Using Lemma (1) we have

$$\chi = -\frac{1}{p}\sum_{jk\equiv -1} jz_k + \frac{1}{p}\sum_{n=1}^{p-2}\left(\sum_{jk\equiv n} jz_k - \sum_{jk\equiv -1} jz_k\right)\zeta^n.$$

The condition $\sum_{k=1}^{m} R(k^{-1})z_k \equiv 0\ (mod\ p)$ can be written as $\sum_{jk\equiv 1} jz_k \equiv 0\ (mod\ p)$, and so $\sum_{jk\equiv -1} jz_k = \sum_{jk\equiv 1}(p-j)z_k \equiv 0\ (mod\ p)$. Therefore, modulo $\mathbb{Z}$ we have

$$\chi \equiv \frac{1}{p}\sum_{n=1}^{p-2}\left(\sum_{jk\equiv n} jz_k - \sum_{jk\equiv -1} jz_k\right)\zeta^n \equiv \frac{1}{p}\sum_{n=1}^{p-1}\left(\sum_{jk\equiv n} jz_k - \sum_{jk\equiv -1} jz_k\right)\zeta^n.$$

Note that the term corresponding to $n = p-1$ contributes 0 to the sum. Also note that

$$\sum_{jk\equiv n} jz_k - \sum_{jk\equiv -1} jz_k = \sum_{k=1}^{m} C(k,n)z_k$$

and therefore $\chi \equiv \frac{1}{p}\sum_{n=1}^{p-1}\left(\sum_{k=1}^{m} C(k,n)z_k\right)\zeta^n$.

Next we break the sum up into two pieces, one piece for $1 \le n \le m$, the other piece for the remaining values of n, and then use properties of the coefficients $C(k,n)$.

$$\begin{aligned}
\chi &\equiv \frac{1}{p}\sum_{n=1}^{m}\left(\sum_{k=1}^{m} C(k,n)z_k\right)\zeta^n + \frac{1}{p}\sum_{n=1}^{m}\left(\sum_{k=1}^{m} C(k,p-n)z_k\right)\zeta^{p-n}\\
&\equiv \frac{1}{p}\sum_{n=1}^{m}\left(\sum_{k=1}^{m} C(k,n)z_k\right)\zeta^n + \frac{1}{p}\sum_{n=1}^{m}\left(\sum_{k=1}^{m}\left(2R(k^{-1}) - C(k,n) - p\right)z_k\right)\zeta^{-n}\\
&\equiv \frac{1}{p}\sum_{n=1}^{m}\left(\sum_{k=1}^{m} C(k,n)z_k\right)\left(\zeta^n - \zeta^{-n}\right) + \frac{1}{p}\sum_{n=1}^{m}\left(\sum_{k=1}^{m}\left(2R(k^{-1}) - p\right)z_k\right)\zeta^{-n}\\
&\equiv \frac{1}{p}\sum_{n=2}^{m}\left(\sum_{k=1}^{m} C(k,n)z_k\right)\theta_n + \left(\frac{1}{p}\sum_{k=1}^{m} C(k,1)z_k\right)\left(\zeta - \zeta^{-1}\right)\\
&+ \left(\frac{1}{p}\sum_{k=1}^{m} C(k,1)z_k\right)\left(\zeta^{m+1} + \cdots + \zeta^{p-1}\right)\\
&\equiv \frac{1}{p}\sum_{n=1}^{m}\left(\sum_{k=1}^{m} C(k,n)z_k\right)\theta_n.
\end{aligned}$$

The last equation follows from $\theta_1 = \zeta + \zeta^{m+1} + \cdots + \zeta^{p-2}$. ∎

Definition 4. Let K be the collection of $m-tuples$ $\vec{v} = [z_1, \dots, z_m]$ satisfying the condition

$$\sum_{k=1}^{m} R(k^{-1})z_k \equiv 0 \ (mod\ p).$$

Thus K is a free abelian group of rank m. We can write $z_1 = lp - \sum_{k=2}^{m} R(k^{-1})z_k$, for some integer l, and therefore a basis of K is given by the vectors

$$\vec{v_1} = [p, 0, \dots, 0],\ \vec{v_k} = [-R(k^{-1}), \dots, 1, \dots],\ 2 \le k \le m,$$

where the 1 is in the k^{th} entry, and all other entries, except the first, are zero.

Now consider the group homomorphism $L\colon K \to \widehat{A}$ defined by

$$L(\vec{v}) = \frac{1}{p}\sum_{n=1}^{m}\big(\sum_{k=1}^{m} C(k,n)z_k\big)\widehat{\theta_n}.$$

Lemma (4) implies that L is an epimorphism.

Proposition 3. *L is a group isomorphism.*

Proof: We first compute the images of the basis elements $\vec{v_1}$ and $\vec{v_k}$, $2 \le k \le m$, using properties of the coefficients $C(k,n)$:

$$\begin{aligned}
L(\vec{v_1}) &= \frac{1}{p}\sum_{n=1}^{m}\big(\sum_{k=1}^{m}C(k,n)z_k\big)\widehat{\theta_n} = \sum_{n=1}^{m}C(1,n)\widehat{\theta_n} \\
&= \sum_{n=1}^{m}(n+1-p)\widehat{\theta_n} \\
L(\vec{v_k}) &= \frac{1}{p}\sum_{n=1}^{m}\big(\sum_{k=1}^{m}C(k,n)z_k\big)\widehat{\theta_n} \\
&= \frac{1}{p}\sum_{n=1}^{m}\big(-C(1,n)R(k^{-1})+C(k,n)\big)\widehat{\theta_n} \\
&= \frac{1}{p}\sum_{n=1}^{m}\big(-(n+1-p)R(k^{-1})+R(k^{-1}n)+R(k^{-1})-p\big)\widehat{\theta_n} \\
&= \sum_{n=1}^{m}\left(-\left[\frac{jn}{p}\right]+j-1\right)\widehat{\theta_n},
\end{aligned}$$

where we have set $j = R(k^{-1})$ and used the equation $jn = [\frac{jn}{p}]p + R(jn)$.

Now consider the $m \times m$ matrix M whose (k,n) entry is given by

$$M_{(k,n)} = \begin{cases} n+1-p & \text{if } k=1, \\ -[\frac{jn}{p}]+j-1 & \text{if } k \geq 2. \end{cases}$$

To complete the proof of the proposition we need only show that $det(M) \neq 0$. In fact we will show that the determinant of this matrix is $\pm h_1$, thereby completing the proof of Theorem (1).

There are two cases to consider. The first case concerns those values of k, $2 \leq k \leq m$, for which $m+1 \leq j \leq p-1$. For each such value of k we add the first row of M to the k^{th} row, and then change signs. The resulting entries of the new k^{th} row are

$$-\left(n+1-p-\left[\frac{jn}{p}\right]+j-1\right) = -\left(n+1+\left[-\frac{jn}{p}\right]-(p-j)\right) = -\left[\frac{(p-j)n}{p}\right]+(p-j)-1.$$

Notice that the form of these entries is the same as that of the matrix M and that now $1 \leq p-j \leq m$.

In the second case, that is for those values of k such that $2 \leq k \leq m$ and $1 \leq j \leq m$, we leave the k^{th} row as it is.

Applying these elementary row operations to M results in a matrix which agrees, up to rearrangement of rows, with the matrix N whose entries are given by

$$N_{(k,n)} = \begin{cases} n+1-p & \text{if } k=1, \\ -[\frac{kn}{p}]+k-1 & \text{if } k \geq 2. \end{cases}$$

That is

$$N = \begin{bmatrix} 2-p & 3-p & 4-p & \dots & n+1-p & \dots & m+1-p \\ 1 & 1 & 1 & \dots & 1 & \dots & 1 \\ 2 & 2 & 2-[\frac{9}{p}] & \dots & 2-[\frac{3n}{p}] & \dots & 2-[\frac{3m}{p}] \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ k-1 & k-1 & k-1-[\frac{3k}{p}] & \dots & k-1-[\frac{kn}{p}] & \dots & k-1-[\frac{km}{p}] \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ m-1 & m-1 & m-1-[\frac{3m}{p}] & \dots & m-1-[\frac{mn}{p}] & \dots & m-1-[\frac{m^2}{p}] \end{bmatrix}$$

Next we apply a sequence of elementary column operations, namely we subtract column 1 from columns $2,\dots,m$. This results in the matrix

$$\begin{bmatrix} 2-p & 1 & 2 & \dots & n-1 & \dots & m-1 \\ 1 & 0 & 0 & \dots & 0 & \dots & 0 \\ 2 & 0 & -[\frac{9}{p}] & \dots & -[\frac{3n}{p}] & \dots & -[\frac{3m}{p}] \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ k-1 & 0 & -[\frac{3k}{p}] & \dots & -[\frac{kn}{p}] & \dots & -[\frac{km}{p}] \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ m-1 & 0 & -[\frac{3m}{p}] & \dots & -[\frac{mn}{p}] & \dots & -[\frac{m^2}{p}] \end{bmatrix}$$

The determinant of this matrix is just the negative of the determinant of the $(m-2)\times(m-2)$ matrix

$$\begin{bmatrix} -[\frac{9}{p}] & \dots & -[\frac{3n}{p}] & \dots & -[\frac{3m}{p}] \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ -[\frac{3k}{p}] & \dots & -[\frac{kn}{p}] & \dots & -[\frac{km}{p}] \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ -[\frac{3m}{p}] & \dots & -[\frac{mn}{p}] & \dots & -[\frac{m^2}{p}] \end{bmatrix}$$

Now subtract column $m-3$ from column $m-2$, column $m-4$ from column $m-3,\dots,$ column 1 from column 2 to get the following matrix, where $3 \le k,n \le m$.

$$\begin{bmatrix} & \vdots & \\ \dots & -[\frac{kn}{p}]+[\frac{(k-1)n}{p}] & \dots \\ & \vdots & \end{bmatrix}$$

According to [2] the determinant of this matrix is $\pm h_1$. This proves the proposition since the determinant of M has only changed by a $\pm$ sign in the course of the above elementary row and column operations. ■

The proof of Theorem (1) follows from the fact that $det(M) = \pm h_1$ since the matrix M is the coefficient matrix for expressing the basis elements of $\widehat{A}$ in the basis elements of $\widehat{B}$.

As mentioned in the introduction, Ewing proves our Theorem (1), but in a different setting. See Theorem (3.2) in [7]. To Explain how Ewing's results relate to ours we need some notation.

Let W denote the Witt group of equivalence classes $[V, \beta, \rho]$, where V is a finitely generated free abelain group, β is a skew symmetric non-degenerate bilinear form on V, and ρ is a representation of $\mathbb{Z}_p$ into the group of β-isometries of V. To an automorphism of order p, $T\colon S \to S$, we assign the Witt class $[V, \beta, \rho]$, where V is the first cohomology group, β is the cup product form, and ρ is the induced representation on cohomology. This assignment is well defined up to cobordism and so defines a group homomorphism $ab\colon \Omega \to W$, the so-called Atiyah-Bott map.

The G-signature of Atiyah and Singer defines a group homomorphism from the group of Witt classes to the complex representation ring of $\mathbb{Z}_p$, $sig\colon W \to R(\mathbb{Z}_p)$. Let $e\colon R(\mathbb{Z}_p) \to \mathbb{Z}[\zeta]$ be the homomorphism that evaluates the character of a representation at the generator $T \in \mathbb{Z}_p$. Let $s\colon \Omega \to \mathbb{Z}[\zeta]$ denote the composite $e \circ sig \circ ab\colon \Omega \to \mathbb{Z}[\zeta]$.

Ewing proves that s is a monomorphism whose image has index h_1 in the subgroup R of $\mathbb{Z}[\zeta]$ spanned by the elements $\zeta^k - \zeta^{-k}, k = 1, \dots, m$. From the Remark earlier in this section it follows that $\widehat{R}$ has index 2 in $\widehat{B}$. If $< g \mid a_1, \dots, a_t >$ denotes the cobordism class of T, see Section (5) for the notation, then

$$\sigma = s < g \mid a_1, \dots, a_t >= \sum_{j=1}^{t} \frac{\zeta^{k_j} + 1}{\zeta^{k_j} - 1} \tag{10}$$

The relationship between the G-signature σ and the Eichler trace χ is given by $\sigma = 2\chi + t - 2$, and from this it is an easy matter to translate Ewing's results into ours.

4. THE EICHLER ISOMORPHISM

We start this section with some preliminaries needed for the proofs of Theorems (4) and (5). Any sequence $[a_1, \dots, a_t]$, as in (4), determines uniquely up to topological conjugacy, a compact connected Riemann surface S and an analytical automorphism $T\colon S \to S$ having order p, orbit genus 0, and whose Eichler trace is given by the equation

$$\chi = 1 + \sum_{j=1}^{t} \frac{1}{\zeta^{k_j} - 1}, \ where\ k_j a_j \ \equiv \ 1 \ (mod\ p), \ for\ 1 \le j \le t. \tag{11}$$

Let $\chi[a_1, \dots, a_t]$ denote this Eichler trace.

According to Theorem (2), if $[a_1, \dots, a_t]$ and $[b_1, \dots, b_u]$ are two sequences corresponding to $\mathbb{Z}_p$ actions with orbit genus 0, then $\chi[a_1, \dots, a_t] = \chi[b_1, \dots, b_u]$ if, and only if, $t = u$ and the sequences agree up to rearrangement and cancelling pairs.

Define a group homomorphism η from the abelian group $\mathcal{A}$ in Definition (1) to the free abelian group $\mathbb{Z}[\zeta]/\mathbb{Z}$ by:

$$\eta\colon \mathcal{A} \to \mathbb{Z}[\zeta]/\mathbb{Z}, \ \eta\colon [a_1, \dots, a_t] \to \chi[a_1, \dots, a_t] \ (mod\ \mathbb{Z}). \tag{12}$$

Now we prove Theorem (4).

Proof: To prove that η is well defined recall that the relations used to define $\mathcal{A}$ are:

$$[a_1, \dots, a_t] + [b_1, \dots, b_u] = [a_1, \dots, a_t, b_1, \dots, b_u],$$
$$[\dots, a, \dots, p-a, \dots] = [\dots, \hat{a}, \dots, \widehat{p-a}, \dots].$$

The corresponding equations for the Eichler trace are

$$\chi[a_1,\dots,a_t]+\chi[b_1,\dots,b_u]=\chi[a_1,\dots,a_t,b_1,\dots,b_u]+1,$$

$$\chi[\dots,a,\dots,p-a,\dots]=\chi[\dots,\hat{a},\dots,\widehat{p-a},\dots]-1.$$

This follows from the Eichler Trace Formula (11). Thus the Eichler trace is not additive, but reducing modulo $\mathbb{Z}$ we see that η is a well defined group homomorphism.

By definition the image of η is $\widehat{A}$, the set of Eichler traces modulo $\mathbb{Z}$. It remains to show that η is a monomorphism. If there is an element in the kernel of η we may assume it is a generator, say $\eta[a_1,\dots,a_t]=0$. This follows from the nature of the defining relations in $\mathcal{A}$. Therefore $\chi = \chi\ [a_1,\dots,a_t] = n$ for some integer n. From the Eichler Trace Formula (11) we get $\chi+\bar{\chi}=2-t=2n$, and so $n\le 1$. A short calculation then gives $\chi=\chi[1,p-1,\dots,1,p-1]$, where there are $1-n$ cancelling pairs $\{1,p-1\}$. Now Theorem (2) implies that $[a_1,\dots,a_t]$ consists entirely of cancelling pairs, and so represents 0 in $\mathcal{A}$. This completes the proof of Theorem (4). ∎

Now we begin the proof of Theorem (5). First we show that $\mathcal{A}$ is generated by all triples. The argument used in the following lemma is analogous to an argument used by Symonds in [16].

Lemma 5. *The abelian group $\mathcal{A}$ is generated by the triples $[q,r,s]$, where $1\le q\le r\le s\le p-1$ and $q+r+s\ \equiv\ 0\ (mod\ p)$.*

Proof: We will show that any generator $[a_1,\dots,a_t]$ can be expressed as a linear combination of triples. We can assume that the sequence $[a_1,\dots,a_t]$ does not have any subsequence $[q,r,s]$ such that $q+r+s\equiv 0\ (mod\ p)$, and does not contain any cancelling pairs. Therefore $t\ge 4$. The following equation is valid because of the defining relations in $\mathcal{A}$:

$$[a_1,a_2,a_3,a_4,\dots]=[a_1,a_2,b]+[a_1+a_2,a_3,\dots],\ where\ b\equiv p-a_1-a_2\ (mod\ p).$$

Arguing by induction on the length of the sequence completes the proof. ∎

The generators in Lemma (5) are not independent as the next example shows.

Example . Let r be any integer such that $1\le r\le p-3$. Then

$$[1,r,p-r-1]+[1,r+1,p-r-2]=[1,1,r,p-r-2]=[1,1,p-2]+[2,r,p-r-2].$$

Now we complete the proof of Theorem (5), that is we show that the abelian group $\mathcal{A}$ is freely generated by the triples $[1,r,s]$, where $1\le r\le s\le p-1$ and $1+r+s\equiv 0\ (mod\ p)$.

Proof: Let $\mathcal{G}$ denote the subgroup generated by these triples. The first equation in the Example shows that all 4-tuples $[1,1,r,s]\in\mathcal{G}$, where

$$1\le r\le s\le p-1\ and\ 2+r+s\equiv 0\ (mod\ p).$$

We now set up an induction. To reduce the amount of notation we omit mentioning some of the restrictions that the following sequences must satisfy.

Assume that we have shown that for some integer $q \geq 1$ all 3-tuples of the form $[q, r, s] \in \mathcal{G}$ and all 4-tuples of the form $[1, q, r, s] \in \mathcal{G}$. The Example above establishes the initial case, $q = 1$, of the induction. Now consider the equations

$$[1, q, p-q-1] + [q+1, r, s] = [1, q, r, s], \;\; [q+1, r, s] + [1, r+q+1, s-1] = [1, q+1, r, s-1].$$

The first equation shows that all 3-tuples of the form $[q+1, r, s] \in \mathcal{G}$, and then the next equation shows that all 4-tuples of the form $[1, q+1, r, s-1] \in \mathcal{G}$. The induction ends when q is so large that there are no triples satisfying the conditions stated in Lemma (5).

This proves that $\mathcal{A}$ is generated by the triples $[1, r, s]$, where

$$1 \leq r \leq s \leq p-1 \quad and \quad 1 + r + s \equiv 0 \; (mod \; p).$$

There are $(p-1)/2$ such triples. To complete the proof we show that $\mathcal{A}$ is free abelian of rank $(p-1)/2$.

To do this recall that $\widehat{B}$ is a free abelian group of rank $(p-1)/2$, see Corollary (3). But $\widehat{A}$ is a subgroup of finite index in $\widehat{B}$, see Theorem (1), and therefore $\widehat{A}$ is also a free abelian group of rank $(p-1)/2$. Theorem (4) now implies that $\mathcal{A}$ is free abelian of rank $(p-1)/2$, and so the generators $[1, r, s]$ freely generate $\mathcal{A}$.

This completes the proof of Theorem (5). ∎

We conclude this section by answering Question (1) in the introduction. This is just a matter of determining the possible sets of rotation numbers. Thus let $\{k_1, \cdots, k_t\}$ be any set of t numbers satisfying $1 \leq k_j \leq p-1, \; 1 \leq j \leq t$, and let a_j denote that number such that $k_j a_j \equiv 1 \; (mod \; p)$ and $1 \leq a_j \leq p-1$.

Proposition 4. $1 + \sum_{j=1}^{t} \frac{1}{\zeta^{k_j} - 1} \in A, \; if, \; and \; only \; if, \; \sum_{j=1}^{t} a_j \equiv 0 \; (mod \; p).$

Proof: First suppose that $\chi = 1 + \sum_{j=1}^{t} \frac{1}{\zeta^{k_j} - 1} \in A$. Thus there is an automorphism of order p, $T \colon S \to S$, on some compact, connected Riemann surface S, such that $\chi(T) = \chi$. In fact we can assume that the genus of $S/\mathbb{Z}_p$ is zero. According to Section (2) the action of $\mathbb{Z}_p$ on S corresponds to a short exact sequence $1 \to \Pi \to \Gamma \xrightarrow{\theta} \mathbb{Z}_p \to 1$. Here Γ is abstractly isomorphic to the group presented by t generators $A_1, \dots, A_t$ and $t+1$ relations $A_1^p = \cdots = A_t^p = A_1 \cdots A_t = 1$. The epimorphism θ is determined by the equations $\theta(A_j) = T^{a_j}$, $1 \leq k_j \leq p-1$. In order that θ be well defined it is necessary that $\sum_{j=1}^{t} a_j \equiv 0 (mod \; p)$.

Next suppose that we are given a set $\{k_1, \cdots, k_t\}$ satisfying the conditions of the proposition. Then the short exact sequence above determines a Riemann surface S and an automorphism $T \colon S \to S$ realizing χ as an Eichler trace. ∎

5. EQUIVARIANT COBORDISM

In this section we prove Theorem (6). To begin with suppose $T_1 \colon S_1 \to S_1$ and $T_2 \colon S_2 \to S_2$ are automorphisms of order p on compact connected Riemann surfaces. We do not assume that the orbit genus of either S_1 or S_2 is 0. We start with a standard definition.

Definition 5. *We say that T_1 is equivariantly cobordant to T_2, written $T_1 \sim T_2$, if there exists a smooth, compact, connected 3-manifold W and a smooth $\mathbb{Z}_p$ action $T\colon W \to W$ such that*

(i) *The boundary of W is the disjoint union of S_1 and S_2, $\partial(W) = S_1 \sqcup S_2$.*
(ii) *T restricted to $\partial(W)$ agrees with $T_1 \sqcup T_2$.*

The cobordism class of an automorphism $T\colon S \to S$ depends only upon its topological conjugacy class $[g \mid a_1, \ldots, a_t]$. We denote this cobordism class by $< g \mid a_1, \ldots, a_t >$, and if the orbit genus $g = 0$, we denote it by $< a_1, \ldots, a_t >$.

The set of all cobordism classes of $\mathbb{Z}_p$ actions on compact connected Riemann surfaces is denoted by Ω. Addition of the cobordism classes of the automorphisms $T_1\colon S_1 \to S_1$, $T_2\colon S_2 \to S_2$ is defined by equivariant connected sum as follows. Find discs D_j in S_j such that $D_j,\ T_j(D_j), \ldots, T_j^{p-1}(D_j)$ are mutually disjoint for $j = 1, 2$. Then excise all discs $T^k(D_j)$, $j = 1, 2$, $k = 0, 1, \ldots, p-1$ from S_1, S_2 and take a connected sum by matching $\partial(T^k(D_1))$ to $\partial(T^k(D_2))$ for $k = 0, 1, \ldots, p-1$. The resulting surface S has p tubes joining S_1 and S_2. The automorphisms T_1, T_2 can be extended to an automorphism $T\colon S \to S$ by permuting the tubes. The cobordism class of T does not depend on the choices made.

Thus addition in Ω is given by the formula

$$< g \mid a_1, \ldots, a_t > + < h \mid b_1, \ldots, b_u > = < g + h \mid a_1, \ldots, a_t, b_1, \ldots, b_u > . \tag{13}$$

The next two lemmas show that Ω is an abelian group generated by the cobordism classes $< a_1, \ldots, a_t >$. The identity is represented by any fixed point free action, or by any cobordism class consisting entirely of cancelling pairs, and the inverse of $< g \mid a_1, \ldots, a_t >$ is represented by $< g \mid p - a_1, \ldots, p - a_t >$. The proofs are not original, but are presented here to emphasize the relationship with $\mathcal{A}$.

Lemma 6. $< g \mid a_1, \ldots, a_t > = < a_1, \ldots, a_t >$.

Proof: Let $T\colon S \to S$ represent the class $< a_1, \ldots, a_t >$. First we take the product cobordism $W_1 = S \times [0, 1]$, where T is extended over W_1 in the obvious way. Next we modify W_1 on the top end $S \times \{1\}$ as follows. Take a disc D in S such that D, $T(D), \ldots, T^{p-1}(D)$ are mutually disjoint, and then to each disc $T^k(D)$ in $S \times \{1\}$, $k = 0, 1, \ldots, p-1$, attach a copy of a handlebody H of genus g by identifying the disc $T^k(D)$ with some disc $D' \subset \partial(H)$. Let W_2 denote the resulting 3-manifold. See Figure (2). The action of $\mathbb{Z}_p$ can be extended to W_2 by permuting the handlebodies. The manifold W_2 provides the cobordism showing that $< g \mid a_1, \ldots, a_t > = < a_1, \ldots, a_t >$. ∎

Lemma 7. $< a, p - a, a_3, \ldots, a_t > = < 1 \mid a_3, \ldots, a_t > = < a_3, \ldots, a_t >$.

Proof: The proof of this lemma is similar to the proof of the last one. Start with a product cobordism W_1. Suppose P_0, P_1 are the fixed points corresponding to the cancelling pair $\{a, p - a\}$. Choose small invariant discs D_0, D_1 around P_0, P_1 respectively, and then modify the cobordism at the top end by adding a solid tube $D \times [0, 1]$ so that $D \times \{0\} = D_0$ and $D \times \{1\} = D_1$. The automorphism T can be extended over this tube, and the resulting cobordism shows that

$$< a, p - a, a_3, \ldots, a_t > = < 1 \mid a_3, \ldots, a_t > .$$

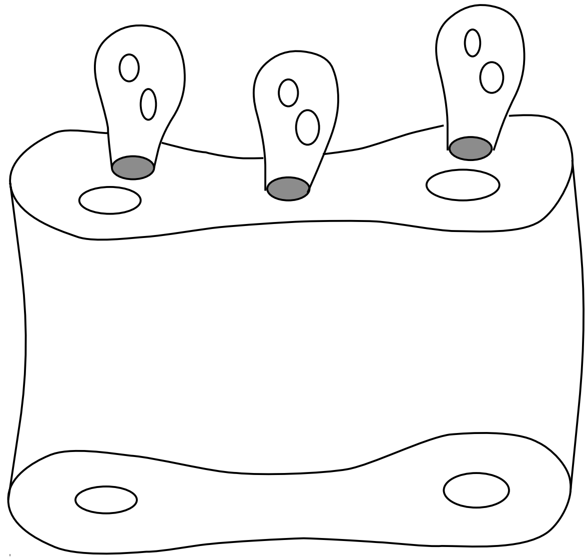

FIGURE 2.

See Figure (3). Lemma (6) completes the proof. ■

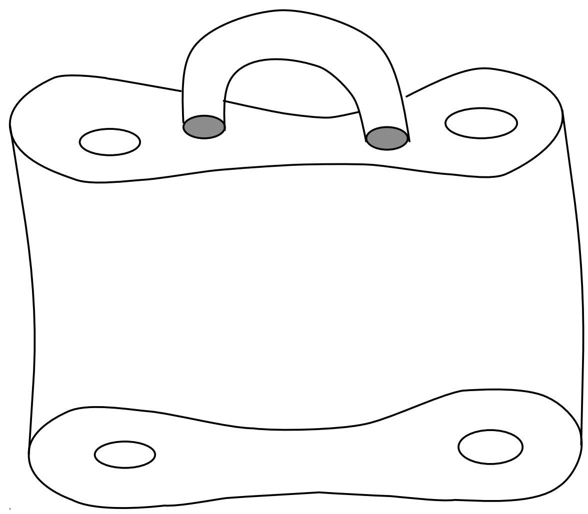

FIGURE 3.

Define the isomorphism of Theorem (6), $\phi\colon \mathcal{A} \to \Omega$, by $\phi[a_1, \ldots, a_t] =< a_1, \ldots, a_t >$. The defining relations of $\mathcal{A}$ are

(i) $[a_1, \ldots, a_t] + [b_1, \ldots, b_u] = [a_1, \ldots, \widehat{a_t, b_1}, \ldots, b_u]$.
(ii) $[\ldots, a, \ldots, p-a, \ldots] = [\ldots, \hat{a}, \ldots, \widehat{p-a}, \ldots]$.

The same relations hold for cobordism classes, see Equation (13) and Lemma (7), and therefore the mapping ϕ is a well defined group homomorphism.

Now we complete the proof of Theorem (6). The argument is analogous to one used in [5].

Proof: From the remarks above we know that $\phi\colon \mathcal{A} \to \Omega$ is a well defined group homomorphism. Lemma (6) implies that it is an epimorphism. It only remains to prove that ϕ is a monomorphism.

If there is an element in the kernel of ϕ we can assume it is a generator, say $[a_1, \dots, a_t]$. Suppose $T\colon S \to S$ represents $[a_1, \dots, a_t]$. Then there is a compact, connected, smooth 3-manifold W such that $\partial(W) = S$, and an extension of T to a smooth homeomorphism $T\colon W \to W$ of order p, also denoted by T. The fixed point set of $T\colon W \to W$ must consist of disjoint, properly embedded arcs joining fixed points in S to fixed points in S. The fixed points at the end of each arc will form a cancelling pair $\{a, p-a\}$. In this way we see that $[a_1, \dots, a_t]$ consists entirely of cancelling pairs, and hence $[a_1, \dots, a_t] = 0$ in $\mathcal{A}$. ■

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