SOLUTIONS TO MIDTERM 1: MATH 100, SECTION 107

QUESTION 1: [4 marks] Below you are given the graph of $y = f'(x)$ for some function $y = f(x)$. Graph the function $y = f(x)$ assuming that $f(0) = -1$.

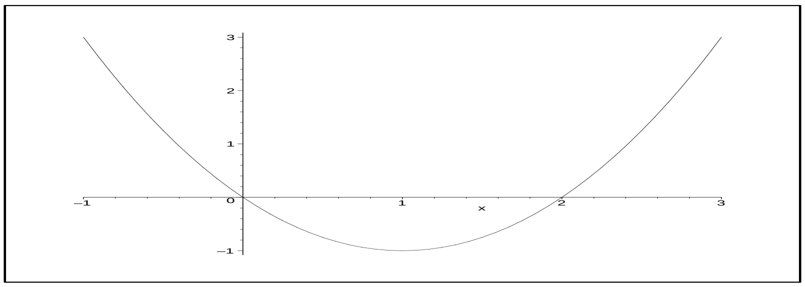

FIGURE 1. The graph of $y = f'(x)$

Solution to (1):

FIGURE 2. The graph of $y = f(x)$

QUESTION 2: [6 marks] Using the rules of differentiation find the derivatives of the following functions. DO NOT SIMPLIFY YOUR ANSWERS.

(a) $f(x) = \dfrac{x^2 - 2x}{x^3 + x + 1}$. (b) $f(x) = \sqrt{\sqrt{x} + 1}$. (c) $f(x) = \left(x^{-1} + x\right)\left(3x^2 - 2x + 10\right)$.

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Solution to (2):

(a) $f^{'}(x) = \dfrac{(x^3+x+1)(2x-2)-(x^2-2x)(3x^2+1)}{(x^3+x+1)^2}$.

(b) $f^{'}(x) = \dfrac{1}{2}\left(\sqrt{x}+1\right)^{-\frac{1}{2}}\left(\dfrac{1}{2\sqrt{x}}\right)$.

(c) $f^{'}(x) = (-x^{-2}+1)(3x^2-2x+10)+(x^{-1}+x)(6x-2)$.

QUESTION 3: [4 marks]

(a) State the tangent line approximation for a function $f(x)$ at the point x_0.

(b) A function $f(x)$ is known to satisfy $f(0) = 1.05$ and $f^{'}(0) = -0.1$. Find a reasonable approximation to $f(0.1)$.

Solution to (3):

(a) The tangent line approximation is $f(x_0+h) \approx f(x_0) + hf^{'}(x_0)$.

(b) Applying the tangent line approximation we have

$$f(0.1) \approx f(0) + (0.1)f^{'}(0) = 1.05 + (0.1) \times (-0.1) = 1.04.$$

QUESTION 4: [6 marks] Let $f(x) = x^3 - x^2,\ -\infty < x < \infty$.

(a) Determine where $f(x)$ is increasing and where it is decreasing.

(b) Determine all local maxima and local minima of $f(x)$.

(c) Does $f(x)$ have an absolute maximum or absolute minimum?

Solution to (4):

(a)

$$f^{'}(x) = 3x^2 - 2x = x(3x-2) = 0 \iff x = 0 \text{ or } \frac{2}{3}.$$

By testing at particular points we see that

$$\begin{aligned} f^{'}(x) &> 0 \text{ if either } x < 0 \text{ or } x > \frac{2}{3}. \\ f^{'}(x) &< 0 \text{ if } 0 < x < \frac{2}{3}. \end{aligned}$$

$$\begin{aligned} \text{Therefore} \qquad & f(x) \text{ is increasing if either } x < 0 \text{ or } x > \frac{2}{3} \\ \text{and} \qquad & f(x) \text{ is decreasing if } 0 < x < \frac{2}{3}. \end{aligned}$$

(b) There is a local maximum at $x = 0$ and a local minimum at $x = \frac{2}{3}$. There are no other local extrema.

(c) $f(x)$ does not have an absolute maximum since $f(x)$ is strictly increasing on the interval $\frac{2}{3} < x < \infty$. In fact $\lim_{x\to\infty} f(x) = \infty$. $f(x)$ does not have an absolute minimum since $f(x)$ is strictly increasing on the interval $-\infty < x < 0$. In fact $\lim_{x\to-\infty} f(x) = -\infty$.

QUESTION 5: [4 marks] Find the minimum value of $x + y$, where x, y are positive numbers satisfying $xy = 1$.

Solution to (5): Since $xy = 1$ we have $x + y = x + x^{-1}$. Thus we have to minimize the function $f(x) = x + x^{-1}$ for $0 < x < \infty$. Now

$$f'(x) = 1 - x^{-2} = 0 \iff x = \pm 1.$$

We discard $x = -1$ and note that

$$f'(x) < 0 \text{ if } 0 < x < 1 \text{ and } f'(x) > 0 \text{ if } x > 1.$$

Therefore the minimum ocurs at $x = 1$ and the minimum value is $f(1) = 2$.