MIDTERM 1: MATH 100, SECTION 109, WEDNESDAY OCT. 7

QUESTION 1: [4 marks] Below you are given the graph of $y = f'(x)$ for some function $y = f(x)$. Graph the function $y = f(x)$ assuming that $f(0) = 1$.

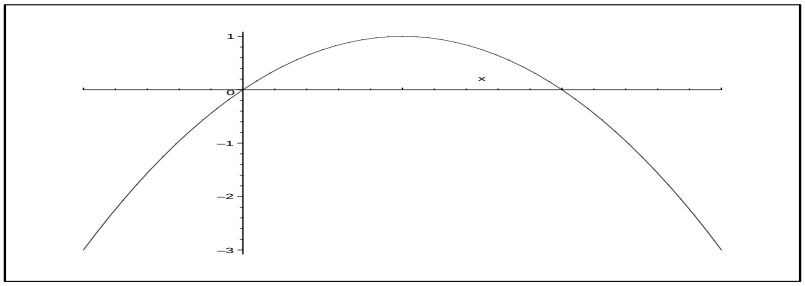

FIGURE 1. The graph of $y = f'(x)$

Solution to (1):

FIGURE 2. The graph of $y = f(x)$

QUESTION 2: [6 marks] Using the rules of differentiation find the derivatives of the following functions. DO NOT SIMPLIFY YOUR ANSWERS.

(a) $f(x) = \dfrac{x^2 + 2x}{x^3 - x - 1}$. (b) $f(x) = \dfrac{1}{\sqrt{x} + 1}$. (c) $f(x) = \left(x^{-2} + x^2\right)\left(x - x^{-1}\right)$.

Date: October 2, 2002.

Solution to (2):

(a) $f'(x) = \dfrac{(x^3 - x - 1)(2x + 2) - (x^2 + 2x)(3x^2 - 1)}{(x^3 - x - 1)^2}$.

(b) $f'(x) = -(\sqrt{x} + 1)^{-2} \left(\dfrac{1}{2\sqrt{x}} \right)$.

(c) $f'(x) = (-2x^{-3} + 2x)(x - x^{-1}) + (x^{-2} + x^2)(1 + x^{-2})$.

QUESTION 3: [4 marks] Let $g(x) = (f(x))^3$. You are given that $f(0) = 2$ and $f'(0) = -1$. Find $g'(0)$.

Solution to (3): By the chain rule $g'(x) = 3(f(x))^2 f'(x)$. Putting $x = 0$, and using $f(0) = 2$ and $f'(0) = -1$, gives $g'(0) = -12$.

QUESTION 4: [6 marks] Let $f(x) = x^4 - 2x^2$, $-\infty < x < \infty$.

(a) Determine where $f(x)$ is increasing and where it is decreasing.

(b) Determine all local maxima and local minima of $f(x)$.

(c) Does $f(x)$ have an absolute maximum or absolute minimum?

Solution to (4):

(a) $f'(x) = 4x^3 - 4x = 4x(x^2 - 1) = 0 \iff x = 0, \pm 1$. By testing values of the derivative we see that

$$\begin{aligned} f'(x) > 0 \quad & \text{if } -1 < x < 0 \text{ or } x > 1. \\ f'(x) < 0 \quad & \text{if } -\infty < x < -1 \text{ or } 0 < x < 1. \\ \text{Therefore} \quad & f(x) \text{ is increasing if } -1 < x < 0 \text{ or } x > 1 \\ \text{and} \quad & f(x) \text{ is decreasing if } -\infty < x < -1 \text{ or } 0 < x < 1. \end{aligned}$$

(b) There is a local maximum of $f(x)$ at $x = 0$ and there are local minima at $x = \pm 1$. There are no other local extrema.

(c) There is no absolute maximum since $f(x)$ is strictly increasing for $x > 1$. It is also strictly decreasing for $x < -1$. In fact $\lim_{x \to \pm\infty} f(x) = \infty$. There is an absolute minimum, and it occurs at either $x = 1$ or $x = -1$.

QUESTION 5: [4 marks] A ball is at rest 490 meters above ground when it is dropped. After t seconds it is $s(t) = 490 - 4.9t^2$ meters above ground.

(a) At what time does the ball hit the ground?

(b) What is the velocity of the ball when it hits the ground?

Solution to (5):

(a) The ball hits the ground when $s(t) = 0$, that is when $490 - 4.9t^2 = 0$. Solving for the positive value of t gives $t = 10$.

(b) The velocity of the ball is $v(t) = \dfrac{ds}{dt} = -9.8t$ for $0 \leq t \leq 10$. Thus the velocity when the ball hits the ground is -98 m/sec.