SOLUTIONS TO MIDTERM #1: MATH 102, SECTIONS 102 & 105

QUESTION 1: [6 marks]

(a) Give the definition of the derivative $f'(x)$ of a function $f(x)$.

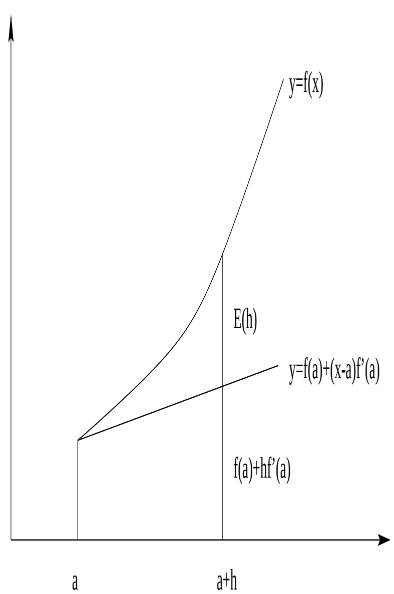
(b) In a few brief sentences give several interpretations of the derivative of a function $f(x)$.

(c) State the formula for the tangent line approximation.

(a) The derivative is defined to be $f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$.

(b) The derivative $f'(x_0)$ can be interpreted as the slope of the graph $y = f(x)$ when $x = x_0$. It is also the instantaneous rate of change of $y = f(x)$ with respect to x. If $y = f(t)$ represents distance travelled than $y'(t) = v(t)$ represents velocity and $v'(t) = y''(t) = a(t)$ represents acceleration.

(c) The tangent line approximation is $f(a+h) \approx f(a) + hf'(a)$. Equivalently the tangent line approximation is $f(a+h) = f(a) + hf'(a) + E(h)$, where $E(h)$ represents the error made in the approximation $f(a+h) \approx f(a) + hf'(a)$.



QUESTION 2: [6 marks]

(a) Using only the definition of the derivative, and not the rules of differentiation, find the derivative of $f(x) = 1/\sqrt{x}$.

(b) Find the equation of the tangent line of $y = 1/\sqrt{x}$ when $x = 25$.

(c) Using the tangent line approximation estimate the value of $1/\sqrt{26}$.

(a)

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{1/\sqrt{x+h} - 1/\sqrt{x}}{h} \\ &= \lim_{h\to 0} \frac{\sqrt{x}-\sqrt{x+h}}{h\sqrt{x+h}\sqrt{x}} = \lim_{h\to 0} \frac{\sqrt{x}-\sqrt{x+h}}{h\sqrt{x+h}\sqrt{x}} \times \frac{\sqrt{x}+\sqrt{x+h}}{\sqrt{x}+\sqrt{x+h}} \\ &= \lim_{h\to 0} \frac{x-(x+h)}{h\sqrt{x+h}\sqrt{x}(\sqrt{x}+\sqrt{x+h})} \\ &= \lim_{h\to 0} \frac{-h}{h\sqrt{x+h}\sqrt{x}(\sqrt{x}+\sqrt{x+h})} \\ &= \lim_{h\to 0} \frac{-1}{\sqrt{x+h}\sqrt{x}(\sqrt{x}+\sqrt{x+h})} = \frac{-1}{\sqrt{x}\sqrt{x}(\sqrt{x}+\sqrt{x})} = \frac{-1}{2x\sqrt{x}}. \end{aligned}$$

(b) The slope of the tangent line when $x = 25$ is $f'(25) = -\dfrac{1}{50\sqrt{25}} = -\dfrac{1}{250} = -0.004$. Thus the equation is $\dfrac{y-1/5}{x-25} = -0.004$. We can rewrite this as $y = -(0.004)x + 0.3$.

(c) We apply the tangent line approximation $f(x_0 + h) \approx f(x_0) + hf'(x_0)$ to the situation where $f(x) = 1/\sqrt{x}$, $x_0 = 25$ and $h = 1$. This gives

$$\frac{1}{\sqrt{26}} \approx \frac{1}{\sqrt{25}} + 1 \times \frac{-1}{250} = \frac{1}{5} - \frac{1}{250} = \frac{49}{250} = 0.196.$$

QUESTION 3: [4 marks] Below you are given the graph of $y = f'(x)$ for some function $y = f(x)$. Sketch the graph of $y = f(x)$ assuming that $f(0) = -1$.

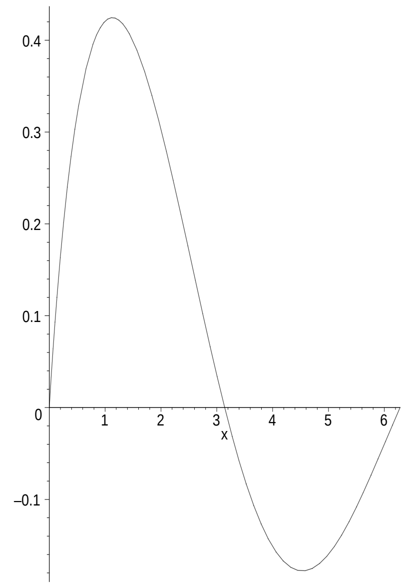


FIGURE 1. The graph of $y = f'(x)$

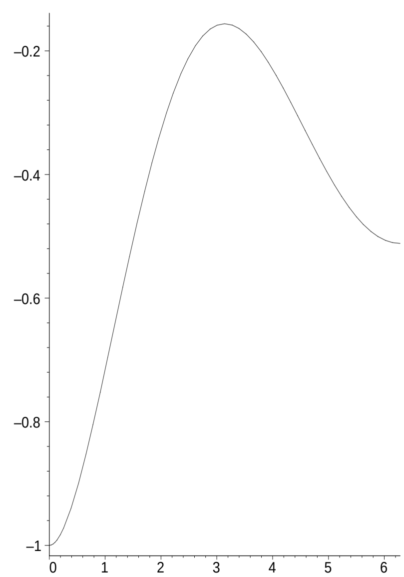


FIGURE 2. The graph of $y = f(x)$

QUESTION 4: [6 marks] Using the rules of differentiation find the derivatives of the following functions. DO NOT SIMPLIFY YOUR ANSWERS.

(a) $f(x) = \dfrac{x^2 - 2x}{x^3 + x + 1}$. (b) $f(x) = \sqrt{\sqrt{x} + 1}$. (c) $f(x) = \left(x^{-1} + x\right)\left(3x^2 - 2x + 10\right)$.

(a) $f'(x) = \dfrac{(x^3 + x + 1)(2x - 2) - (x^2 - 2x)(3x^2 + 1)}{(x^3 + x + 1)^2}$.

(b) $f'(x) = \dfrac{1}{2}\left(\sqrt{x} + 1\right)^{-1/2}\dfrac{1}{2}x^{-1/2}$.

(c) $f'(x) = (-x^{-2} + 1)(3x^2 - 2x + 10) + (x^{-1} + x)(6x - 2)$.

QUESTION 5: [6 marks] Let $f(x) = \dfrac{x^2 - 1}{x^2 + 1}$, $-\infty < x < \infty$.

(a) Compute $f'(x)$.

(b) Find all local maxima and minima of $f(x)$.

(c) Determine all intervals in which $f(x)$ is increasing or decreasing.

(a) $f'(x) = \dfrac{(x^2 + 1)2x - (x^2 - 1)2x}{(x^2 + 1)^2} = \dfrac{4x}{(x^2 + 1)^2}$.

(b) $f'(x) = 0 \iff x = 0$. Checking we see that $f'(x) < 0$ if $x < 0$ and $f'(x) > 0$ if $x > 0$. Thus $x = 0$ gives a local minimum value of $f(0) = -1$. There are no local maxima.

(c) It follows from part (b) that $f(x)$ is increasing if $x > 0$ and decreasing if $x < 0$.

QUESTION 6: [6 marks] A bullet train leaves from its station at time $t = 0$ and travels to the end of its route and back. Its distance from its starting point at time t is given by

$$y(t) = t^3(36 - t),\ 0 \le t \le 36.$$

(a) Find its velocity $v(t)$ and acceleration $a(t)$.

(b) When does it get to the furthest point away from its starting point, and how far away is this furthest point?

(c) When is the tram moving at its fastest rate, and what is this fastest rate?

(a) Since $y(t) = t^3(36 - t) = 36t^3 - t^4$ we see that
$v(t) = y'(t) = 108t^2 - 4t^3 = 4t^2(27 - t)$ and $a(t) = v'(t) = y''(t) = 216t - 12t^2 = 12t^2(18 - t)$.

(b) To determine the farthest away point se solve $y'(t) = 4t^2(27 - t) = 0$ for t. Clearly $t = 0$ or $t = 27$. Since $y(0) = y(36) = 0$ we see that the farthest away point occurs when $t = 27$, and the farthest away distance is $y(27) = 177147$. **NOTE: just because you have found a critical point does not mean you have necessarily found the maximum. You may have found the minimum or it may be neither.**

(c) We are asked to maximize $v(t)$. The first step is

$$v'(t) = 12t^2(18 - t) = 0 \Longleftrightarrow t = 0, 18.$$

Since $v'(t) > 0$ for $0 < t < 18$ and $v'(t) < 0$ for $18 < t < 36$ it follows that $t = 18$ gives the maximum. The maximum value is $v(18) = 11664$. **NOTE: just because you have found a critical point does not mean you have necessarily found the maximum. You may have found the minimum or it may be neither.**

QUESTION 7: [6 marks] Determine the dimensions (radius and height) of a tin can of maximum volume that can be made out of 100 in^2 of tin. Assume that the tin can has a bottom and a top.

We must have $2\pi r^2 + 2\pi r h = 100$, and therefore $h = \dfrac{100 - 2\pi r^2}{2\pi r} = \dfrac{50}{\pi r} - r$. The range of values of r is $0 < r \leq \sqrt{\dfrac{50}{\pi}}$. Thus the volume is $V = \pi r^2 h = \pi r^2 \left(\dfrac{50}{\pi r} - r\right) = 50r - \pi r^3$. To maximize we set the derivative equal to 0 and then test to see if this gives a maximum, a minimum or neither.

$$V'(r) = 50 - 3\pi r^2 = 0 \Longleftrightarrow r = \sqrt{\frac{50}{3\pi}}.$$

Since $V'(r) > 0$ for $r < \sqrt{\dfrac{50}{3\pi}}$ and $V'(r) < 0$ for $r > \sqrt{\dfrac{50}{3\pi}}$ the maximum volume occurs at $r = \sqrt{\dfrac{50}{3\pi}} \approx 2.303$ The corresponding value of h is $h = 2\sqrt{\dfrac{50}{3\pi}} \approx 4.606$.