SOLUTIONS TO MIDTERM #1

QUESTION 1: [10 marks]

(a) Give the definition of the derivative $f'(x)$ of a function $f(x)$.

(b) What is the equation of the tangent line to the graph of $y = f(x)$ at $x = a$?

(c) **Using only the definition of the derivative** find $f'(x)$ for the function $f(x) = \dfrac{1}{\sqrt{x}}$.

Solutions:

(a) $f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h) - f(x)}{h}$.

(b) $y - f(a) = f'(a)(x - a)$.

(c) The derivative of $f(x) = \sqrt{x}$ is calculated as follows:

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{1/\sqrt{x+h} - 1/\sqrt{x}}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{x} - \sqrt{x+h}}{h\sqrt{x}\sqrt{x+h}} = \lim_{h\to 0} \left(\frac{\sqrt{x} - \sqrt{x+h}}{h\sqrt{x}\sqrt{x+h}} \times \frac{\sqrt{x} + \sqrt{x+h}}{\sqrt{x} + \sqrt{x+h}} \right) \\
&= \lim_{h\to 0} \frac{x - (x+h)}{h\sqrt{x}\sqrt{x+h}(\sqrt{x} + \sqrt{x+h})} = \lim_{h\to 0} \frac{-1}{\sqrt{x}\sqrt{x+h}(\sqrt{x} + \sqrt{x+h})} \\
&= -\frac{1}{2x\sqrt{x}} \text{ by putting } h = 0 \text{ and using continuity.}
\end{aligned}$$

QUESTION 2: [12 marks]

(a) Find the derivative of $f(x) = \sqrt{g(x)}$ at $x = -1$ if $g(-1) = 4$ and $g'(-1) = -2$.

(b) Find the derivative of $f(x) = \dfrac{x-1}{g(x)+1}$ at $x = 0$ if $g(0) = 2$ and $g'(0) = 2$.

(c) Find the derivative of $f(x) = \sin(\pi g(x))$ at $x = a$ if you are given that $g(a) = \dfrac{2}{3}$ and $g'(a) = b$.

(d) Let $f(x)$ be a function which is differentiable at $x = a$ (that is $f'(a)$ exists). Determine the x-coordinate where the tangent line to the graph of $y = f(x)$ at $x = a$ crosses the x-axis. Your answer should involve $a, f(a), f'(a)$.

Solutions:

(a) $f'(-1) = \dfrac{g'(x)}{2\sqrt{g(x)}} \Big|_{x=-1} = -\dfrac{1}{2}$.

(b) $f'(0) = \dfrac{(g(x)+1)\times 1 - (x-1)g'(x)}{(g(x)+1)^2}\Big|_{x=0} = \dfrac{5}{9}$.

(c) $f'(a) = \cos(\pi g(x)) \times \pi g'(x)\Big|_{x=a} = \cos\dfrac{2\pi}{3} \times \pi b = -\dfrac{\pi b}{2}$.

(d) The equation of the tangent line is $y - f(a) = f'(a)(x-a)$. To determine the x-coordinate where this line crosses the x-axis put $y = 0$ and solve for x. The answer is $x = a - \dfrac{f(a)}{f'(a)}$. See Figure 1.

QUESTION 3: [12 marks] Using the rules of differentiation find the derivatives of the following functions.

(a) $f(x) = \dfrac{e^{x^2}}{x}$.

(b) $f(x) = \cos(\sin(\pi x))$

(c) $f(x) = \left(x + \dfrac{1}{x}\right)^{-1/2}$

(d) $f(x) = e^2 + x^e$

Solutions:

(a) $f'(x) = \dfrac{x(2xe^{x^2}) - e^{x^2}}{x^2}$.

(b) $f'(x) = -\sin(\sin(\pi x)) \times \pi\cos(\pi x)$.

(c) $f'(x) = -\dfrac{1}{2}\left(x + \dfrac{1}{x}\right)^{-3/2}\left(1 - \dfrac{1}{x^2}\right)$.

(d) $f'(x) = ex^{e-1}$. Here e is a constant, so e^2 differentiates to 0.

QUESTION 4: [12 marks] Compute the following limits.

(a) $\displaystyle\lim_{x\to 0} \frac{x}{\sqrt{x+4} - 2}$

(b) $\displaystyle\lim_{x\to -2} \frac{(2+x)\cos(\pi x)}{4 - x^2}$

(c) $\displaystyle\lim_{x\to 0} \frac{\sin(x)\sin(2x)}{x^2}$

(d) $\displaystyle\lim_{x\to \infty} \frac{x^3 + x^2 + 10^{100}}{2x^3 - 1}$

Solutions:

(a)

$$\begin{aligned}
\lim_{x\to 0}\frac{x}{\sqrt{x+4}-2} &= \lim_{x\to 0}\left(\frac{x}{\sqrt{x+4}-2}\times\frac{\sqrt{x+4}+2}{\sqrt{x+4}+2}\right)\\
&= \lim_{x\to 0}\frac{x(\sqrt{x+4}+2)}{x} = \lim_{x\to 0}(\sqrt{x+4}+2)\\
&= 4
\end{aligned}$$

(b) $\displaystyle\lim_{x\to -2}\frac{(2+x)\cos(\pi x)}{4-x^2} = \lim_{x\to -2}\frac{(2+x)\cos(\pi x)}{(2+x)(2-x)} = \lim_{x\to -2}\frac{\cos\pi x}{2-x} = \frac{1}{4}$

(c) $\displaystyle\lim_{x\to 0}\frac{\sin(x)\sin(2x)}{x^2} = 2\lim_{x\to 0}\frac{\sin(x)}{x}\times\lim_{x\to 0}\frac{\sin(2x)}{2x} = 2$

(d) $\displaystyle\lim_{x\to\infty}\frac{x^3+x^2+10^{100}}{2x^3-1} = \lim_{x\to\infty}\frac{1+1/x+10^{100}/x^3}{2-1/x^3} = \frac{1}{2}$

QUESTION 5: [4 marks] Show that the function $f(x)$ defined by

$$f(x) = \begin{cases} x\cos(1/x) & \text{if } x\neq 0\\ 0 & \text{if } x = 0\end{cases}$$

does not have a derivative at $x=0$.

Solution:

If this function had a derivative at $x=0$ it would be given by $\displaystyle\lim_{h\to 0}\frac{h\cos(1/h)}{h} = \lim_{h\to 0}\cos(1/h)$. But this limit does not exist as $\cos(1/h)$ assumes the values ± 1 infinitely often in any interval of $h=0$. In fact

$$\begin{aligned}
\cos(1/h) &= 1 \Longleftrightarrow h = \frac{1}{2n\pi},\ \text{where } n = \pm 1, \pm 2, \cdots \text{ and}\\
\cos(1/h) &= -1 \Longleftrightarrow h = \frac{1}{(2n-1)\pi},\ \text{where } n = 0, \pm 1, \pm 2, \cdots
\end{aligned}$$

See Figure 2.

Figure 1. The graph for question 2(d)

Figure 2. The graph for question 5