SOLUTIONS TO QUIZ 1

Question 1 [6marks]

Let $f(x)$ be the function defined by $f(x) = \sqrt{1 + 1/x}$, $x > 0$.

1(a) Find the derivative $f'(x)$ using only first principles.

1(b) Find an equation of the tangent line to the graph of $y = f(x)$ at $x = 1/3$.

Solution to Question 1:

1(a)

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{1 + 1/(x+h)} - \sqrt{1 + 1/x}}{h}$$

$$= \lim_{h\to 0} \frac{1 + 1/(x+h) - (1 + 1/x)}{h(\sqrt{1 + 1/(x+h)} + \sqrt{1 + 1/x})}$$

$$= \lim_{h\to 0} \frac{-1}{x(x+h)(\sqrt{1 + 1/(x+h)} + \sqrt{1 + 1/x})}$$

$$= \frac{-1}{2x^2\sqrt{1 + 1/x}}.$$

1(b) By a simple computation $f(1/3) = 2$ and $f'(1/3) = -9/4$, and therefore the tangent line is given by $y - 2 = -9/4(x - 1/3)$.

Question 2 [6 marks]

Compute the following limits.

2(a) $\lim_{x\to\infty} (\sqrt{x^2 + ax} - \sqrt{x^2 - ax})$, where a is a constant.

2(b) $\lim_{x\to 1} (x^2 - 1)/(\sqrt{x + 8} - 3)$.

2(c) $\lim_{x\to 0} (2x + 3x^2)/(3x - 2x^2)$.

Solution to Question 2:

2(a)

$$\lim_{x\to\infty} (\sqrt{x^2 + ax} - \sqrt{x^2 - ax}) = \lim_{x\to\infty} \frac{x^2 + ax - (x^2 - ax)}{(\sqrt{x^2 + ax} + \sqrt{x^2 - ax})}$$

$$= \lim_{x\to\infty} \frac{2a}{\sqrt{1+a/x}+\sqrt{1-a/x}} = a$$

2(b)

$$\lim_{x\to 1}(x^2-1)/(\sqrt{x+8}-3) = \lim_{x\to 1}\frac{(x^2-1)(\sqrt{x+8}+3)}{x+8-9}$$
$$= \lim_{x\to 1}(x+1)(\sqrt{x+8}+3) = 12$$

2(c)

$$\lim_{x\to 0}(2x+3x^2)/(3x-2x^2) = \lim_{x\to 0}(2+3x)/(3-2x) = 2/3$$

Question 3 [4 marks]

Let $f(x)$ be any function such that $\lim_{x\to 1}\frac{f(x)-3}{x-1} = 2$ and $f(1)=3$.

3(a) Find an equation of the tangent line to the graph of $y=f(x)$ at $x=1$.

3(b) Show that $f(x)$ is continuous at $x=1$.

Solution to Question 3:

3(a) It is immediate that $f'(1)=2$ and so the equation is $y-3=2(x-1)$.

3(b) $\lim_{x\to 1} f(x) = \lim_{x\to 1}(f(1)+(\frac{f(x)-f(1)}{x-1})(x-1)) = f(1)+f'(1)\times 0 = f(1)$. This proves that $f(x)$ is continuous at $x=1$.

Question 4 [4 marks]

Let $f(x)$ and $g(x)$ be two functions satisfying

$$f(1)=1,\ g(1)=2,\ f'(1)=-1,\ g'(1)=0.$$

Compute the derivatives of the following functions at $x=1$.

4(a) $F(x)=(\sqrt{x}+f(x))(x^2+g(x))$.

4(b) $G(x)=(1+1/x+f(x))/(1-1/x+g(x))$.

Solution to Question 4:

4(a) $F'(x) = (\frac{1}{2\sqrt{x}} + f'(x))(x^2 + g(x)) + (\sqrt{x} + f(x))(2x + g'(x))$ Evaluating at $x = 1$ gives $F'(1) = \frac{5}{2}$.

4(b) $G'(x) = \dfrac{(1 - 1/x + g(x))(\frac{-1}{x^2} + f'(x)) - (1 + 1/x + f(x))(\frac{1}{x^2} + g'(x))}{(1 - 1/x + g(x))^2}.$

Evaluating at $x = 1$ gives $G'(1) = -\frac{7}{4}$.