SOLUTIONS TO QUIZ 2

Question 1 [8 marks]

1(a) Show that $\lim_{h\to 0} \frac{\sin h}{h} = 1$.

1(b) Show that $\lim_{h\to 0} \frac{1-\cos h}{h} = 0$. Hint: multiply $\frac{1-\cos h}{h}$ top and bottom by $1+\cos h$ and then use a trig identity and properties of limits.

Solution to Question 1:

1(a) We need only show that $\lim_{h\to 0^+} \frac{\sin h}{h} = 1$ since $\frac{\sin h}{h} = \frac{\sin(-h)}{-h}$. Thus suppose $0 < h < \pi/2$. Note that $\cos h > 0$ for this range of values of h. Now consider the diagram below of the unit circle centered at the origin.

By trig we have area($\triangle$ OAB) $= \frac{\cos h \sin h}{2}$, area(sector OCB)$= \frac{h}{2\pi}\pi = \frac{h}{2}$ and area($\triangle$ OCD) $= \frac{\tan h}{2}$. Because of the way the triangles and sector are nested within one another we have

$$\frac{\cos h \sin h}{2} < \frac{h}{2} < \frac{\tan h}{2} = \frac{\sin h}{2\cos h}.$$

This is equivalent to $\cos h < \frac{\sin h}{h} < \frac{1}{\cos h}$, for $0 < h < \pi/2$. Using the continuity of $\cos h$ at $h = 0$, the squeeze principle and the fact that $\cos 0 = 1$ we get $\lim_{h\to 0^+} \frac{\sin h}{h} = 1$.

1(b)

$$\begin{aligned}
\lim_{h\to 0} \frac{1-\cos h}{h} &= \lim_{h\to 0} \frac{(1-\cos h)(1+\cos h)}{h(1+\cos h)} \\
&= \lim_{h\to 0} \frac{\sin^2 h}{h(1+\cos h)} \\
&= \left(\lim_{h\to 0} \frac{\sin h}{h}\right)\left(\lim_{h\to 0} \sin h\right)\left(\lim_{h\to 0} \frac{1}{1+\cos h}\right) = 1 \times 0 \times \frac{1}{2} = 0.
\end{aligned}$$

Question 2 [4 marks]

Let $f(x)$ and $g(x)$ be functions satisfying

$$f(1) = 2,\ f'(1) = -2,\ g(0) = 4/\pi,\ g'(0) = 2.$$

2(a) Find $F'(1)$ for the function defined by $F(x) = \sqrt{f(x)^2 + x}$.

2(b) Find $G'(0)$ for the function defined by $G(x) = \tan\left(\frac{1}{g(x)}\right)$.

Solution to Question 2:

2(a) $F'(x) = \frac{1}{2\sqrt{f(x)^2 + x}}\left(2f(x)f'(x) + 1\right)$. Putting $x = 1$ gives

$$F'(1) = \frac{1}{2\sqrt{5}}(-7) = \frac{-7}{2\sqrt{5}}.$$

2(b) $G'(x) = \left(\sec^2\left(\frac{1}{g(x)}\right)\right)\left(\frac{-g'(x)}{g(x)^2}\right)$. Putting $x = 0$ gives

$$G'(0) = \left(\sec^2 \pi/4\right)\left(\frac{-2}{(4/\pi)^2}\right) = \frac{-\pi^2}{4}.$$

Question 3 [8 marks]

Let $f(x)$ be the function defined by $f(x) = \tan(2x),\ -\pi/4 < x < \pi/4$.

3(a) Find all points on the graph $y = f(x)$ where the slope of the tangent line is 4.

3(b) Find all points on the the graph $y = f(x)$ where the normal line is parallel to the line $y = -x/8$.

Solution to Question 3:

3(a) We have to solve the equation $f'(x) = 4$ for $x,\ -\pi/4 < x < \pi/4$:

$$f'(x) = 2\sec^2 2x = 4 \Longleftrightarrow \sec 2x = \pm\sqrt{2} \Longleftrightarrow \cos 2x = \pm\frac{1}{\sqrt{2}} \Longleftrightarrow x = \pm\pi/8.$$

For $x = \pm\pi/8$ we have $\tan 2x = \tan \pi/4 = \pm 1$. Thus the points on the graph are $\pm(\pi/8, 1)$.

3(b) Here we want to solve $f'(x) = 8$ for $x, \ -\pi/4 < x < \pi/4$:

$$f'(x) = 2\sec^2 2x = 8 \Longleftrightarrow \cos 2x = \pm 1/2 \Longleftrightarrow x = \pm\pi/6.$$

For $x = \pm\pi/6$ we have $\tan 2x = \pm\sqrt{3}$. Thus the points on the graph are $x = \pm(\pi/6, \sqrt{3})$.

Figure 1: CIRCLE OF RADIUS ONE