SOLUTIONS TO HOMEWORK ASSIGNMENT #2

1. Compute the following limits:

(a) $\lim_{x\to 0} \frac{\sin(-x)}{\sin 3x}$ (b) $\lim_{\theta\to 0} \frac{\theta^3}{(\sin\theta)^2}$ (c) $\lim_{x\to 0} \frac{\sqrt{1+2x}-\sqrt{1-2x}}{x}$

(d) $\lim_{t\to 0} \frac{t}{t+\sin t}$ (e) $\lim_{z\to\infty} \frac{z^2+1}{2z^2-1}$ (f) $\lim_{x\to -\infty} \frac{\cos x}{x^2+1}$

Solutions:
Theses questions use the limit laws (study pages 67-70 and 76) and various algebraic steps.

(a) $\lim_{x\to 0} \frac{\sin(-x)}{\sin 3x} = -\frac{1}{3}\lim_{x\to 0}\left(\frac{\sin x}{x}\frac{3x}{\sin 3x}\right) = -\frac{1}{3}\lim_{x\to 0}\left(\frac{\sin x}{x}\right) \times \lim_{x\to 0}\left(\frac{3x}{\sin 3x}\right) = -\frac{1}{3}$

(b) $\lim_{\theta\to 0} \frac{(\theta)^3}{(\sin\theta)^2} = \lim_{\theta\to 0}\theta \times \lim_{\theta\to 0}\frac{(\theta)^2}{(\sin\theta)^2} = 0 \times 1^2 = 0$

(c)

$$
\begin{aligned}
\lim_{x\to 0}\left(\frac{\sqrt{1+2x}-\sqrt{1-2x}}{x}\right) &= \lim_{x\to 0}\left(\frac{\sqrt{1+2x}-\sqrt{1-2x}}{x} \times \frac{\sqrt{1+2x}+\sqrt{1-2x}}{\sqrt{1+2x}+\sqrt{1-2x}}\right) \\
&= \lim_{x\to 0}\left(\frac{1+2x-(1-2x)}{x(\sqrt{1+2x}+\sqrt{1-2x})}\right) \\
&= \lim_{x\to 0}\left(\frac{2x}{x(\sqrt{1+x}+\sqrt{1-x})}\right) \\
&= \lim_{x\to 0}\left(\frac{4}{\sqrt{1+2x}+\sqrt{1-2x}}\right) = 2 \text{ by setting } x=0.
\end{aligned}
$$

The last step uses continuity of the function $\frac{4}{\sqrt{1+2x}+\sqrt{1-2x}}$ at $x=0$.

(d) $\lim_{t\to 0}\left(\frac{t}{t+\sin t}\right) = \lim_{t\to 0}\left(\frac{1}{1+\sin t/t}\right) = \frac{1}{\lim_{t\to 0}(1+\sin t/t)} = \frac{1}{2}$

(e) $\lim_{z\to\infty} \frac{z^2+1}{2z^2-1} = \lim_{z\to\infty}\frac{1+1/z^2}{2-1/z^2} = 2$ since $\lim_{z\to 0} 1/z^2 = 0$

(f) $\lim_{x\to -\infty} \frac{\cos x}{x^2+1} = 0$ by the "squeeze law" (see page 76). Observe that

$$\frac{-1}{x^2+1} \le \frac{\cos x}{x^2+1} \le \frac{1}{x^2+1} \text{ for } x<0 \text{ since } -1 \le \cos x \le 1$$

and therefore $0 = \lim_{x\to-\infty}\left(\frac{-1}{x^2+1}\right) \le \lim_{x\to-\infty}\left(\frac{\cos x}{x^2+1}\right) \le \lim_{x\to-\infty}\left(\frac{1}{x^2+1}\right) = 0$

2. Let $f(x)$, $-\infty < x < \infty$, be the function defined as follows:

$$f(x) = \begin{cases} x+\lambda & \text{if } x \le 2 \\ 2\lambda - x & \text{if } x > 2 \end{cases}$$

(a) Determine the constant λ so that $f(x)$ is continuous for all x.
(b) Graph the function $y = f(x)$ for the value of λ found in (a).

Solutions:
(a) The function $x + \lambda$ is continuous for $x < 2$ and the function $2\lambda - x$ is continuous for $x > 2$. Therefore, as far as the function $f(x)$ is concerned, we need only check continuity at $x = 2$. This we do by computing one-sided limits:

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (x+\lambda) = 2+\lambda, \quad \lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (2\lambda - x) = 2\lambda - 2$$

These 2 limits must be equal in order that $f(x)$ be continuous at $x = 2$. Therefore we must have $\lambda = 4$.
(b)

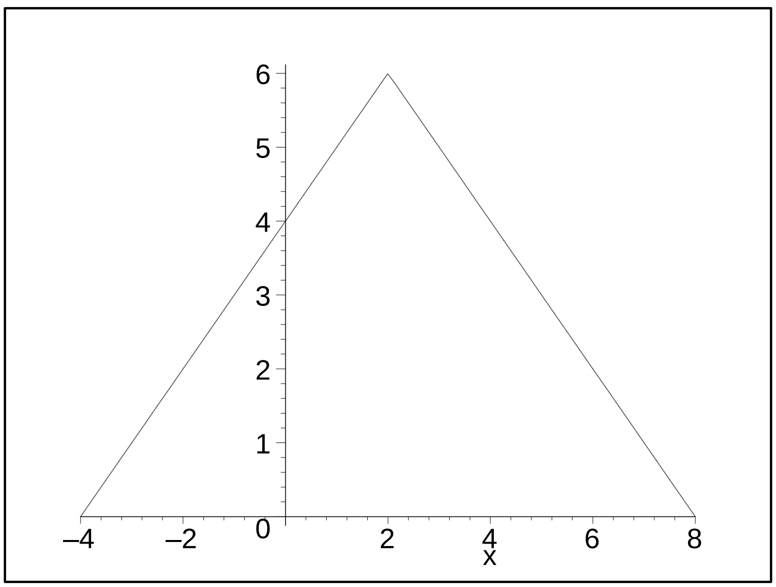

Figure 1: The graph of $y = f(x)$

3. Show that the function $f(t) = \cos t - t$ has a zero in the interval $\pi/6 < t < \pi/4$.
Solution:
The function $f(t) = \cos t - t$ is continuous, $f(\pi/6) = \dfrac{\sqrt{3}}{2} - \pi/6 = .3424266282 > 0$ and $f(\pi/4) = \dfrac{1}{\sqrt{2}} - \pi/4 = -0.0782913825 < 0$ (a sign change) and therefore there is a zero in this interval by the Intermediate Value Property (see p. 91).

4. Using only the definition of the derivative, find $f'(x)$ for the following functions:
(a) $f(x) = \dfrac{1}{\sqrt{x^2+1}}$ (b) $f(x) = \dfrac{x}{1+2x}$
Solutions:

(a)

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\frac{1}{\sqrt{(x+h)^2+1}}-\frac{1}{\sqrt{x^2+1}}}{h} \\
&= \lim_{h\to 0}\frac{\frac{1}{\sqrt{(x+h)^2+1}}-\frac{1}{\sqrt{x^2+1}}}{h}\times\frac{\frac{1}{\sqrt{(x+h)^2+1}}+\frac{1}{\sqrt{x^2+1}}}{\frac{1}{\sqrt{(x+h)^2+1}}+\frac{1}{\sqrt{x^2+1}}} \\
&= \lim_{h\to 0}\frac{\frac{1}{(x+h)^2+1}-\frac{1}{x^2+1}}{h\left(\frac{1}{\sqrt{(x+h)^2+1}}+\frac{1}{\sqrt{x^2+1}}\right)} = \lim_{h\to 0}\frac{x^2+1-((x+h)^2+1)}{h(x^2+1)((x+h)^2+1)\left(\frac{1}{\sqrt{(x+h)^2+1}}+\frac{1}{\sqrt{x^2+1}}\right)} \\
&= \lim_{h\to 0}\frac{-2xh-h^2}{h(x^2+1)((x+h)^2+1)\left(\frac{1}{\sqrt{(x+h)^2+1}}+\frac{1}{\sqrt{x^2+1}}\right)} \\
&= \lim_{h\to 0}\frac{-2x-h}{(x^2+1)((x+h)^2+1)\left(\frac{1}{\sqrt{(x+h)^2+1}}+\frac{1}{\sqrt{x^2+1}}\right)} = \frac{-x}{(x^2+1)^{3/2}}
\end{aligned}
$$

(b)

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{(x+h)/(1+2(x+h))-x/(1+2x)}{h} \\
&= \lim_{h\to 0}\frac{(x+h)(1+2x)-x(1+2x+2h)}{h(1+2x+2h)(1+2x)} \\
&= \lim_{h\to 0}\frac{h}{h(1+2x+2h)(1+2x)} = \frac{1}{(1+2x)^2}
\end{aligned}
$$