SOLUTIONS TO HOMEWORK ASSIGNMENT #3

1. Each of the following questions can be done with little computation. Suppose $f(x)$, $g(x)$ are functions satisfying $f(a) = \alpha,\ g(a) = \beta,\ f'(a) = \gamma$ and $g'(a) = \delta$ for some $a, \alpha, \beta, \gamma, \delta$.

(a) Compute the derivative of $f(x) + g(x)$ at $x = a$.

(b) Compute the derivative of $f(x)g(x)$ at $x = a$.

(c) Compute the derivative of $f(x)/g(x)$ at $x = a$.

(d) Compute the derivative of $x^2 f(x) - x^3 g(x)$ at $x = a$.

Solutions:

(a) $\frac{d}{dx}(f(x) + g(x))\,\big|_{x=a} = (f'(x) + g'(x))\,\big|_{x=a} = f'(a) + g'(a) = \gamma + \delta.$

(b) $\frac{d}{dx}(f(x)g(x))\,\big|_{x=a} = (f'(x)g(x) + f(x)g'(x))\,\big|_{x=a} = f'(a)g(a) + f(a)g'(a) = \gamma\beta + \alpha\delta$

(c) $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right)\Big|_{x=a} = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}\Big|_{x=a} = \frac{\beta\gamma - \alpha\delta}{\beta^2}$

(d)

$$\begin{aligned} \frac{d}{dx}\left(x^2 f(x) - x^3 g(x)\right)\Big|_{x=a} &= \left(2xf(x) + x^2 f'(x) - 3x^2 g(x) - x^3 g'(x)\right)\Big|_{x=a} \\ &= 2a\alpha + a^2\gamma - 3a^2\beta - a^3\delta \end{aligned}$$

2. Each of the following questions can be done with little computation. Suppose $f(x)$, $g(x)$ are functions satisfying $f(2) = 4,\ g(2) = 4,\ f'(2) = -1$ and $g'(2) = 1$.

(a) Compute the derivative of $f(xg(x) - 6)$ at $x = 2$.

(b) Compute the derivative of $f(x)/\sqrt{g(x)}$ at $x = 2$.

Solutions:

(a)

$$\begin{aligned} \frac{d}{dx}(f(xg(x) - 6))\,\big|_{x=2} &= f'\left((xg(x) - 6)\,\big|_{x=2}\right) \times \left((g(x) + xg'(x))\,\big|_{x=2}\right) \\ &= f'(2) \times 6 = -6 \end{aligned}$$

(b)

$$\begin{aligned} \frac{d}{dx}\left(\frac{f(x)}{\sqrt{g(x)}}\right)\Big|_{x=2} &= \frac{\sqrt{g(x)}f'(x) - f(x) \times \frac{1}{2}(g(x))^{-1/2} \times g'(x)}{g(x)}\Big|_{x=2} \\ &= -\frac{3}{4} \end{aligned}$$

3. Each edge of an equilateral triangle is increasing at the rate of 4 cm/sec. At what rate is the area of the triangle changing when each edge is 20 cm?

Solution:

We are given $\frac{dx}{dt} = 4$, where $x(t)$ is the length of each edge at time t. See the diagram. The height is $h = \frac{\sqrt{3}}{2}x$, and therefore the area of the triangle is given by $A = \frac{\sqrt{3}}{4}x^2$. Hence $\frac{dA}{dt} = \frac{\sqrt{3}}{4} \times 2x \times \frac{dx}{dt} = 40\sqrt{3}\ cm^2/sec$ when $x = 20\ cm$.

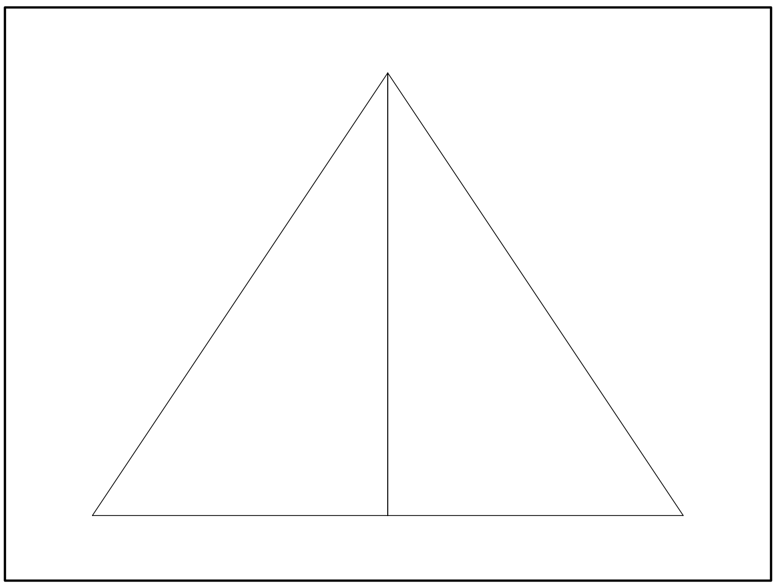

Figure 1: An equilateral triangle with sides $x(t)$

4. The area of a circle is decreasing at the rate of $3\pi\ cm^2/sec$. At what rate is the radius of the circle changing when its area is 100 cm^2?

Solution:

We are given that $\frac{dA}{dt} = -3\pi\ cm^2/sec$, where $A(t)$ is the area of the circle at time t. Since $A = \pi r^2$ we have $r = \sqrt{\frac{A}{\pi}}$, and therefore

$$\frac{dr}{dt} = \frac{1}{\sqrt{\pi}}\frac{1}{2\sqrt{A}}\frac{dA}{dt} = \frac{1}{\sqrt{\pi}}\frac{1}{2\sqrt{100}} \times (-3\pi) = -\frac{3}{20}\sqrt{\pi}\ cm/sec$$

when $A = 100$. Therefore the radius is decreasing at the rate of $\frac{3}{20}\sqrt{\pi}\ cm/sec$.

5. Find the derivatives of the following functions. DO NOT SIMPLIFY YOUR ANSWERS.

(a) $y = x/\sqrt{x^2+1}$

(b) $y = (\sin^4(x) + \cos^4(x))^2$

(c) $y = (x - 1/x)^4$

(d) $f(x) = \sqrt{x + \sqrt{x}}$

(e) $g(x) = \sqrt{\dfrac{x-1}{x+1}}$

(f) $f(t) = \left(\dfrac{1}{t} + \dfrac{1}{t^2} + \dfrac{1}{t^3}\right)^{-1}$

Solution:

(a) If $y = x/\sqrt{x^2+1}$ then $\dfrac{dy}{dx} = \dfrac{\sqrt{x^2+1} - x \times (1/2)(x^2+1)^{-1/2}(2x)}{x^2+1}$

(b) If $y = (\sin^4(x)+\cos^4(x))^2$ then $\dfrac{dy}{dx} = 2(\sin^4(x)+\cos^4(x))(4\sin^3(x)\cos x - 4\cos^3(x)\sin x)$

(c) If $y = (x - 1/x)^4$ then $\dfrac{dy}{dx} = 4(x-1/x)^3(1+1/x^2)$

(d) If $f(x) = \sqrt{x+\sqrt{x}}$ then $f'(x) = \dfrac{1}{2\sqrt{x+\sqrt{x}}}\left(1 + \dfrac{1}{2\sqrt{x}}\right)$

(e) If $g(x) = \sqrt{\dfrac{x-1}{x+1}}$ then $g'(x) = \dfrac{1}{2}\left(\dfrac{x-1}{x+1}\right)^{-1/2} \times \left(\dfrac{x+1-(x-1)}{(x+1)^2}\right)$

(f) If $f(t) = \left(\dfrac{1}{t} + \dfrac{1}{t^2} + \dfrac{1}{t^3}\right)^{-1}$ then $f'(t) = -\left(\dfrac{1}{t} + \dfrac{1}{t^2} + \dfrac{1}{t^3}\right)^{-2} \times \left(-\dfrac{1}{t^2} - \dfrac{2}{t^3} - \dfrac{3}{t^4}\right)$

6. Let $f(x)$ be the function defined by $f(x) = \begin{cases} x^2\sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

(a) Compute $f'(x)$ for $x \neq 0$.

(b) Prove that $f'(0) = 0$.

(c) Prove that $f'(x)$ is not continuous at $x = 0$.

Solution:

(a) For $x \neq 0$ we can use the rules: $f'(x) = 2x\sin(1/x) - \cos(1/x)$.

(b) To compute $f'(0)$ we must go back to the definition: $f'(0) = \lim\limits_{h\to 0} \dfrac{f(h)-f(0)}{h} = \lim\limits_{h\to 0} h\sin(1/h) = 0$.

(c) $\lim\limits_{x\to 0} f'(x) = \lim\limits_{x\to 0}(2x\sin(1/x) - \cos(1/x)) = \lim\limits_{x\to 0} 2x\sin(1/x) - \lim\limits_{x\to 0}\cos(1/x) = -\lim\limits_{x\to 0}\cos(1/x) \neq f'(0)$ since $\lim_{x\to 0}\cos(1/x))$ does not exist.