SOLUTIONS TO HOMEWORK ASSIGNMENT #5

1. Each of the following questions can be done with little computation. Enter your answers in the boxes and show any work in the spaces provided.
Find derivatives of the following functions and simplify as much as possible:

(a) $f(x) = \sqrt{x}\tan(\sin x)$

(b) $f(x) = e^{\sqrt{x}} + e^{-\sqrt{x}}$.

(c) $f(x) = \ln(\ln x)$.

(d) $y = x^{\sin x}$.

(e) $y = \left(1 + \dfrac{1}{x}\right)^x$.

Solution:

(a) $f'(x) = \dfrac{1}{2\sqrt{x}}\tan(\sin x) + \sqrt{x}\sec^2(\sin x)\cos x$.

(b) $f'(x) = \dfrac{1}{2\sqrt{x}}e^{\sqrt{x}} - \dfrac{1}{2\sqrt{x}}e^{-\sqrt{x}} = \dfrac{1}{2\sqrt{x}}\left(e^{\sqrt{x}} - e^{-\sqrt{x}}\right)$.

(c) $f'(x) = \dfrac{1}{x\ln x}$.

(d)

$$\begin{aligned}\ln y = \sin x \ln x \quad &\Longrightarrow \quad \frac{y'}{y} = \cos x \ln x + \frac{\sin x}{x}\\ &\Longrightarrow \quad y' = y\left(\cos x\ln x + \frac{\sin x}{x}\right) = x^{\sin x}\left(\cos x \ln x + \frac{\sin x}{x}\right)\end{aligned}$$

(e)

$$\begin{aligned}\ln y &= x\ln\left(1+\frac{1}{x}\right) \Longrightarrow \frac{y'}{y} = \ln\left(1+\frac{1}{x}\right) + \frac{x}{1+1/x}\times\frac{-1}{x^2}\\ &= \ln(1+1/x) - \frac{1}{x+1} \Longrightarrow y' = (1+1/x)^x\left(\ln(1+1/x) - \frac{1}{x+1}\right).\end{aligned}$$

2. Show that $\dfrac{d}{dx}\ln\left(x+\sqrt{1+x^2}\right) = \dfrac{1}{\sqrt{1+x^2}}$.

Solution:

$$\begin{aligned}\frac{d}{dx}\ln\left(x+\sqrt{1+x^2}\right) &= \frac{1}{x+\sqrt{1+x^2}}\times\left(1+\frac{x}{\sqrt{1+x^2}}\right)\\ &= \frac{1}{x+\sqrt{1+x^2}}\times\left(\frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}}\right) = \frac{1}{\sqrt{1+x^2}}\end{aligned}$$

3. Find the equations of the tangent lines to the graphs of $y = f(x)$ at $x = a$.

(a) $f(x) = e^{x^2-x}$, $a = 1$

(b) $f(x) = \dfrac{\ln x}{x^2}$, $a = e$.

(c) $f(x) = \dfrac{e^{x^2}}{\cos \pi x}$, $a = 1$.

Solution:

(a) $f(1) = 1$ and $f'(1) = (2x-1)e^{x^2-x} \mid_{x=1} = 1$. Therefore the tangent line has the equation $y = 1 + (x-1) = x$.

(b) $f(e) = \dfrac{1}{e^2}$ and $f'(e) = \dfrac{x - 2x\ln x}{x^4} \Big|_{x=e} = -\dfrac{1}{e^3}$. Therefore the tangent line has the equation $y = \dfrac{1}{e^2} - \dfrac{1}{e^3}(x-e) = \dfrac{2}{e^2} - \dfrac{1}{e^3}x$.

(c) $f(1) = -e$ and $f'(1) = \dfrac{(\cos \pi x)2xe^{x^2} + \pi e^{x^2}\sin \pi x}{(\cos \pi x)^2} \Big|_{x=1} = -2e$. Therefore the tangent line has the equation $y = -e - 2e(x-1) = e - 2ex$.

4. Find all values of x where the graph of $y = x - \sin 2x$ has a horizontal tangent.

Solution:

$y'(x) = 1 - 2\cos 2x = 0 \iff 2x = \pm\dfrac{\pi}{3} + 2n\pi \iff x = \pm\dfrac{\pi}{6} + n\pi,\ n = 0, \pm 1, \pm 2, \ldots$

5. Carbon extracted from an ancient skull recently unearthed contained only 1/6 as much ^{14}C as carbon extracted from present-day bone. How old is the skull?

Solution:

Let $A(t)$ denote the amount of ^{14}C in the skull t years after death. Then $A(t) = A(0)e^{rt}$ for some value of r. We know that the half life of ^{14}C is 5730 years and therefore $\dfrac{1}{2} = e^{5730r}$, and so $r = -\dfrac{\ln 2}{5730}$. To determine how old the skull is we solve the equation $e^{-\frac{\ln 2}{5730}t} = \dfrac{1}{6}$ for t. Thus the skull is $\dfrac{5730 \ln 6}{\ln 2} \approx 14812$ years old.

6. Upon the birth of their first child a couple deposited \$5000 in a savings account that pays 6% annual interest compounded continuously. How much will the account contain when the child is ready to go to college at age 18?

Solution:

The acount will contain $5000e^{(0.06)18} \approx \$14723.40$.

7. The population of a certain town was 50,000 in 1980 and 70,000 in 2000. Assuming population growth is exponential determine what it will be in 2010.

Solution: Let $P(t)$ denote the population at time t. Taking time $t = 0$ to be 1980 we have $P(t) = 50000e^{rt}$ for some value of r. Putting $t = 20$ gives $70000 = 50000e^{20r}$ and therefore $r = \dfrac{\ln(7/5)}{20}$. The population is 2010 will be $P(30) = 50000e^{3\ln(7/5)/2} \approx 82825$.

8. The half life of radioactive cobalt is 5.27 years. If a certain region has 100 times the safe level of radioactive cobalt, how long will it take for the region to again be safe?

Solution: Let $A(t)$ denote the radioactive level of cobalt. Then $A(t) = A(0)e^{rt}$ for some r. From $\dfrac{1}{2} = e^{5.27r}$ we get $r = -\dfrac{\ln 2}{5.27}$ and therefore $A(t) = A(0)e^{-\frac{\ln 2}{5.27}t}$. The region will be safe when $1 = 100e^{-\frac{\ln 2}{5.27}t}$. Solving for t gives $t = \dfrac{5.27 \ln 100}{\ln 2} \approx 35$ years.

9. The population of a certain bacteria is known to increase 10-fold over a 24 hour period. Determine the doubling time.

Solution: Let $P(t)$ denote the bacterial population at time t. Then $P(t) = P(0)e^{rt}$. From $P(24) = 10P(0) = P(0)e^{24r}$ we deduce $r = \dfrac{\ln 10}{24}$. The doubling time is determined by solving for t in the equation $2P(0) = P(0)e^{rt}$. Therefore $t = \dfrac{\ln 2}{r} = \dfrac{24 \ln 2}{\ln 10} \approx 7.22$ hours.

10. Suppose a mutation of the bacterium E. coli produces a cell line that divides into 3 daughter cells every 30 minutes. How many cells would there be after 1 day?

Solution: After 1 day a single cell will become $3^{48} \approx 8 \times 10^{22}$ cells.

11. C^{14}, a radioactive isotope of carbon, has a half life of 5730 years. (Note: this isotope is used in radiocarbon dating, a process by which the age of materials containing carbon can be estimated. W. Libby received the Nobel prize in chemistry in 1960 for developing this technique.)

(a) Determine how long it takes for a sample to fall to 0.001 of its original level of radioactivity.

(b) Each gram of C^{14} has an activity of 12 decays per minute. If a sample of material is found to have 45 decays per hour, approximately how old is it?

Solution:

(a) It will take $\dfrac{5730 \ln 100}{\ln 2} \approx 38069$ years.

(b) 45 decays per hour equals 3/4 decays per minute. The sample is t years old, where t satisfies $\dfrac{3}{4} = 12e^{(-\ln 2/5730)t}$. That is $t = \dfrac{5730 \ln 16}{\ln 2} = 4 \times 5730 = 22920$ years.

12. The human population on Earth doubles roughly every 50 years. There were 6 billion humans on earth in the year 2000.

(a) How many people will there be in the year 3010?

(b) How many people would have to inhabit each square kilometer of the planet for this population to fit on earth? (Take the circumference of the earth to be 40,000 km for the purpose of computing its surface area.)

Solution:

(a) The population at time t will be $P(t) = 6 \times 10^9 e^{rt}$, where r is determined by the equation $2 = e^{50r}$, that is $r = \dfrac{\ln 2}{50}$. In the year 3010 there will be $P(1010) = 6 \times 10^9 e^{1010 \ln 2/50} \approx 7.2 \times 10^{15}$ people on earth.

(b) The surface area of a sphere of radius r is $4\pi r^2$. Thus the number of people per square kilometer will be $\dfrac{6 \times 10^9 e^{1010 \ln 2/50}}{4\pi(20000/\pi)^2} \approx 1.4 \times 10^7$.