SOLUTIONS TO HOMEWORK ASSIGNMENT #6

1. A culture of bacteria is found to contain 10^4 bacteria per cm^3 at the start of an experiment. After 1 day there are 10^6 bacteria. Assume that the number of bacteria increases at a rate that is proportional to the number of bacteria present.

(a) Determine the doubling time of the bacteria.

(b) How many will there be after 2 days?

Solution:

(a) If $B(t)$ is the number of bacteria at time t then $B(t) = B(0)e^{kt} = 10^4 e^{kt}$ for some constant k. We are given $B(1) = 10^6$, and thus $10^6 = 10^4 e^k \Longrightarrow k = \ln 100$. We get the doubling time τ by solving the equation $2 = e^{k\tau}$ for τ. Therefore $\tau = \dfrac{\ln 2}{k} = \dfrac{\ln 2}{\ln 100} \approx 0.15$ days ≈ 3.6 hours.

(b) The number will be $B(2) = 10^4 e^{2k} = 10^4 e^{2\ln 100} = 10^4 \times 100^2 = 10^8$.

2. In a chemical reaction it is found that a substance is broken down at a rate proportional to the amount of substance remaining. It was observed that 10 g of the substance decreased to 8 g in 1 hr.

(a) What is the differential equation satisfied by $A(t)$, the amount of substance remaining at time t?

(b) How long will it take until only 1 gm is left?

Solution:

(a) The differential equation is $\dfrac{dA}{dt} = kA$, where k is a constant.

(b) Therefore $A(t) = A(0)e^{kt}$. We are also told that $A(0) = 10$ and $A(1) = 8$, and therefore $8 = 10e^k$. Solving for k gives $k = \ln 0.8$. Thus $A(t) = 10e^{(\ln 0.8)t}$. To determine when there is only 1 gram left we must solve $1 = 10e^{(\ln 0.8)t}$ for t. We get $t = \dfrac{\ln 0.1}{\ln 0.8} \approx 10.3$ hours.

3. The time of death of a murder victim can be estimated from the temperature of the body if it is discovered early enough after the crime has occurred. Suppose that in a room whose ambient temperature is $A = 20$ degrees centigrade, the temperature of the body upon discovery is $T = 30$ degrees, and that one hour later it is $T = 25$ degrees.

(a) What is the differential equation satisfied by $T(t)$, the temperature of the body t hours after discovery?

(b) Determine the approximate time of death. You may assume that just prior to death, the temperature of the victim was 37 degrees.

Solution:

(a) The differential equation is $\dfrac{dT}{dt} = k(T - 20)$, where k is a (negative) constant.

(b) The solution of this differential equation is $T(t) = 20 + 10e^{kt}$, where the time of discovery is taken to be $t = 0$. We are given $T(1) = 25$ and therefore $k = -\ln 2$. Therefore $T(t) = 20 + 10e^{(-\ln 2)t}$, and the time of death is determined by solving $37 = 20 + 10e^{(-\ln 2)t}$ for t.

$$t = -\frac{\ln 1.7}{\ln 2} \approx -0.77, \text{ that is, approx. 0.77 hours before discovery.}$$

4. A tank that holds 1000 litres is initially full of water to which 100 kg of salt has been added. (The mixture was stirred so that the salt concentration was then 0.1 kg/litre). A concentrated solution of salt, containing 0.25 kg/litre is pumped into the tank continuously, at the rate 10 litres/min and the mixture (which is continuously stirred to keep it uniform) is pumped out at the same rate.

(a) Derive a differential equation for the amount of salt in the tank.

(b) How much salt will there be in the tank after 10 minutes?

(c) How much salt will there be in the tank after 1 hr?

Solution:

(a) Let $Q(t)$ denote the amount of salt in the tank at time t, measured in kilograms. The amount of salt in the tank is affected by the flow in and the flow out, and therefore the differential equation is $\dfrac{dQ}{dt} = 2.5 - \dfrac{Q}{100} = -\dfrac{1}{100}(Q - 250)$.

(b) The solution of the differential equation is $Q = 250 + (Q(0) - 250)e^{-\frac{t}{100}} = 250 - 150e^{-\frac{t}{100}}$. Thus the amount of salt after 10 minutes will be $Q(10) = 250 - 150e^{-1/10} \approx 114kg$.

(c) The amount of salt after 60 minutes it will be $Q(60) = 250 - 150e^{-0.6} \approx 168kg$.

5. For what values of a does the curve $y = a^x$ intersect the curve $y = x$?

Solution: Certainly $y = a^x$ intersects $y = x$ for $0 < a \leq 1$, and it is apparent that some larger values of a will also have intersection points. The largest such value of a will satisfy the equations

$$a^x = x \text{ and } a^x \ln a = 1,$$

since for this value of a the graph of $y = a^x$ will be tangent to $y = x$ at the (unique) point of intersection. Solving we get $x = 1/\ln a$ and therefore $a^{1/\ln a} = 1/\ln a$. From this last equation we see that $\ln(1/\ln a) = 1$, and therefore $a = e^{1/e} \approx 1.4$.
Therefore the values of a for which the curves $y = a^x$ and $y = x$ intersect are $0 < a \leq e^{1/e}$.