SOLUTIONS TO HOMEWORK ASSIGNMENT #8

1. Graph the following functions showing all work:

(a) $f(x) = \dfrac{x^2}{x-1}$.

(b) $f(x) = e^{-x^2}$, $-\infty < x < \infty$.

(c) $f(x) = xe^{-x}$, $-\infty < x < \infty$.

(d) $f(x) = x^2 e^{-|x|}$.

Solution:

(a) First notice that the function is not defined at $x = 1$. In fact $\lim_{x \to 1^+} f(x) = +\infty$ and $\lim_{x \to 1^-} f(x) = -\infty$. Thus $x = 1$ is a vertical asymptote. Also note that

$$f(0) = 0,\ f(x) > 0 \text{ for } x > 1 \text{ and } f(x) < 0 \text{ for } x < 1.$$

Next notice that $y = x+1$ is a slant asymptote since long division gives $\dfrac{x^2}{x-1} = x+1+\dfrac{1}{x-1}$. In fact this tells us that $f(x)$ approaches the line $y = x+1$ from above as $x \to \infty$ and from below as $x \to -\infty$.
Now we do the calculus:

1. $f'(x) = 1 - (x-1)^{-2} = 0 \iff x = 0, 2$.
2. $f'(x) > 0 \iff -\infty < x < 0$ or $2 < x < \infty$ and $f'(x) < 0 \iff 0 < x < 1$ or $1 < x < 2$.
3. $f''(x) = 2(x-1)^{-3}$ is never 0. $f''(x) > 0 \iff x > 1$ and $f''(x) < 0 \iff x < 1$.

It follows that:

1. $f(x)$ is increasing for $x < 0$ or $x > 2$ and decreasing for $0 < x < 1$ or $1 < x < 2$. Therefore there is a local maximum at $x = 0$ and a local minimum at $x = 2$. The local maximum and minimum values are $f(0) = 0$ and $f(2) = 4$ resp.
2. The graph is concave up for $1 < x < \infty$ and concave down for $-\infty < x < 1$. There are no inflection points.

Now assemble all this information into the graph. See the diagram at the end.

(b) $f(x) = e^{-x^2}$ is defined for all x (therefore no vertical asymptotes). Also notice the following: $f(0) = 1$, $f(x) > 0$ for all x, $f(-x) = f(x)$ (i.e. the function is even) and so the graph is symmetric about the y-axis, and $\lim_{x \to \pm\infty} f(x) = 0$.
Now we do the calculus:

1. $f'(x) = -2xe^{-x^2} = 0 \iff x = 0$, $f'(x) > 0 \iff x < 0$ and $f'(x) < 0 \iff x > 0$.

2. $f''(x) = -2e^{-x^2} + 4x^2 e^{-x^2} = (4x^2 - 2)e^{-x^2} = 0 \iff x = \pm\dfrac{1}{\sqrt{2}}$.

3. $f''(x) > 0 \iff\ x > \dfrac{1}{\sqrt{2}}$ or $x < -\dfrac{1}{\sqrt{2}}$.

4. $f''(x) < 0 \iff\ -\dfrac{1}{\sqrt{2}} < x < \dfrac{1}{\sqrt{2}}$.

It follows that:

1. The graph is increasing for $-\infty < x < 0$ and decreasing for $0 < x < \infty$. There is an absolute maximum at $x = 0$ and the absolute maximum is $f(0) = 1$. There is no absolute maximum and no absolute minimum.

2. The graph is concave up for $-\infty < x < -\dfrac{1}{\sqrt{2}}$ or $\dfrac{1}{\sqrt{2}} < x < \infty$ and concave down for $-\dfrac{1}{\sqrt{2}} < x < \dfrac{1}{\sqrt{2}}$. The inflection points are $\left(\pm 1/\sqrt{2}, 1/\sqrt{e}\right)$.

Now assemble all this information into the graph. See the diagram at the end.

(c) The function $f(x) = xe^{-x}$ is defined for all x so there are no vertical asymptotes. Moreover, note the following: $f(0) = 0$, $f(x) > 0$ if $x > 0$, $f(x) < 0$ if $x < 0$, $\lim_{x\to\infty} f(x) = 0$ and $\lim\limits_{x\to-\infty} f(x) = -\infty$.
Now the calculus:

1. $f'(x) = (1 - x)e^{-x} = 0 \iff x = 1, f'(x) > 0 \iff x < 1$ and $f'(x) < 0 \iff x > 1$.

2. $f''(x) = (x - 2)e^{-x} = 0 \iff x = 2, f''(x) > 0 \iff x > 2$ and $f''(x) < 0 \iff x < 2$.

It follows that:

1. The graph is increasing for $-\infty < x < 1$ and decreasing for $1 < x < \infty$. There is an absolute maximum at $x = 1$ and the absolute maximum value is $f(1) = 1/e$.

2. The graph is concave down for $-\infty < x < 2$ and concave up for $2 < x < \infty$. There is an inflection point at $(2, 2/e^2)$.

Now assemble all this information into the graph. See the diagram at the end.

(d) Observe that $f(x) = x^2 e^{-|x|}$ is defined for all x, $f(0) = 0$, $f(x) > 0$ for $x \neq 0$, $f(x)$ is an even function (i.e. $f(-x) = f(x)$) and $\lim\limits_{x\to\pm\infty} f(x) = 0$.
Now assume $x \geq 0$. Then $f(x) = x^2 e^{-x}$ and

1. $f'(x) = (2x - x^2)e^{-x} = 0 \iff x = 0, 2$.

2. $f'(x) > 0 \iff 0 < x < 2\ \ f'(x) < 0 \iff x > 2$ (recall we are assuming $x \geq 0$).

3. $f''(x) = (x^2 - 4x + 2)e^{-x} = 0 \iff x = 2 \pm \sqrt{2}$.

4. $f''(x) > 0 \iff 0 < x < 2-\sqrt{2}$ or $x > 2+\sqrt{2}$, $f''(x) < 0 \iff 2-\sqrt{2} < x < 2+\sqrt{2}$.

Now use the symmetry about the y-axis to complete the analysis.

1. The graph is increasing for $-\infty < x < -2$ and $0 < x < 2$.
2. The graph is decreasing for $-2 < x < 0$ and $2 < x < \infty$.
3. The graph is concave up for $-\infty < x < -2-\sqrt{2}$, $-(2-\sqrt{2}) < x < 2-\sqrt{2}$ and $2+\sqrt{2} < x < \infty$.
4. The graph is concave down for $-2-\sqrt{2} < x < -2+\sqrt{2}$ and $2-\sqrt{2} < x < 2+\sqrt{2}$.
5. Inflection points occur at $x = -2-\sqrt{2}, -2+\sqrt{2}, 2-\sqrt{2}, 2+\sqrt{2}$.

Now assemble all this information into the graph. See the diagram at the end.

2. Compute the following limits:

(a) $\lim\limits_{x\to 0} \dfrac{1-\cos(x^2)}{x^2\sin(x^2)}$

(b) $\lim\limits_{x\to 0} \dfrac{\sin(x)\sin(2x)}{x^2+x^4}$

(c) $\lim\limits_{x\to 0} \dfrac{\ln(1+x)}{x}$

Solution:

(a) $\lim\limits_{x\to 0} \dfrac{1-\cos(x^2)}{x^2\sin(x^2)} = \lim\limits_{x\to 0} \dfrac{1-(1-x^4/2)+\cdots}{x^2(x^2-\cdots)} = \lim\limits_{x\to 0} \dfrac{1/2+\cdots}{1+\cdots} = 1/2.$

(b) $\lim\limits_{x\to 0} \dfrac{\sin(x)\sin(2x)}{x^2+x^4} = \lim\limits_{x\to 0} \dfrac{(x+\cdots)(2x+\cdots)}{x^2+\cdots} = 2.$

(c) $\lim\limits_{x\to 0} \dfrac{\ln(1+x)}{x} = \lim\limits_{x\to 0} \dfrac{x-\cdots}{x} = 1.$

3. Find the Maclaurin series (Taylor series at $x=0$) for the following functions:

(a) $f(x) = \ln(1+x^2)$

(b) $f(x) = \dfrac{1-e^{-x}}{x}$

(c) $f(x) = \tan x$ out to and including terms of order 5.

(d) $f(x) = e^{\sin x}$ out to and including terms of order 3.

Solution:

(a) $\ln(1+x^2) = x^2 - \frac{1}{2}x^4 + \frac{1}{3}x^6 - \frac{1}{4}x^8 + \cdots$

(b)

$$\begin{aligned}\frac{1-e^{-x}}{x} &= \frac{1-(1-x+x^2/2!-x^3/3!+x^4/4!-+\cdots)}{x} \\ &= 1-\frac{1}{2!}x+\frac{1}{3!}x^2-\frac{1}{4!}x^3+\cdots\end{aligned}$$

(c)

$$\begin{aligned}\tan x &= \frac{\sin x}{\cos x} = \frac{x-\frac{1}{3!}x^3+\frac{1}{5!}x^5+\cdots}{1-(\frac{1}{2!}x^2-\frac{1}{4!}x^4+\cdots)} \\ &= \left(x-\frac{1}{3!}x^3+\frac{1}{5!}x^5+\cdots\right)\left(1+\left(\frac{1}{2!}x^2-\frac{1}{4!}x^4+\cdots\right)+\left(\frac{1}{2!}x^2-\frac{1}{4!}x^4+\cdots\right)^2+\cdots\right) \\ &= x+\frac{1}{3}x^3+\frac{2}{15}x^5+\cdots \text{ (after a lot of algebra.)}\end{aligned}$$

(d)

$$\begin{aligned}e^{\sin x} &= 1+\sin x+\frac{1}{2!}(\sin x)^2+\frac{1}{3!}(\sin x)^3+\cdots \\ &= 1+x-\frac{1}{3!}x^3+\frac{1}{2!}\left(x-\frac{1}{3!}x^3\right)^2+\frac{1}{3!}\left(x-\frac{1}{3!}x^3\right)^3+\cdots \\ &= 1+x+\frac{1}{2}x^2+\cdots \text{ (the third order terms cancel).}\end{aligned}$$

4. Suppose $f(x)$ is a function satisfying $f(0)=10$ and $f'(x)=\dfrac{1}{1+x^4}$ for all x. Compute the linear approximation L to $f(0.1)$ and show that $L-2\times10^{-5}<f(0.1)<L$.

Solution: The linearization to $f(x)$ at $x=0$ is $L(x)=f(0)+f'(0)x=10+x$. Thus $f(0.1)\approx 10.1$. To see how accurate this is we need Taylor's theorem with remainder, namely

$$f(x)=f(a)+f'(a)(x-a)+\frac{f''(\bar{x})}{2!}(x-a)^2,$$

where $\bar{x}$ is some point between a and x. We apply this formula to our function with $a=0$, using $f''(x)=-\dfrac{4x^3}{(1+x^4)^2}$:

$$f(0.1)=f(0)+f'(0)\times 0.1+\frac{f''(\bar{x})}{2!}(0.1)^2=10.1-\frac{2\bar{x}^3}{(1+\bar{x}^4)^2}\times(0.1)^2,$$

where $0<\bar{x}<0.1$. This certainly implies $f(0.1)<10.1$. Since

$$\frac{2\bar{x}^3}{(1+\bar{x}^4)^2}\times(0.1)^2<2\times(0.1)^3\times(0.1)^2=2\times10^{-5}$$

we see that $10.1 - 2 \times 10^{-5} < f(0.1) < 10.1$.

5. Suppose $f(x)$ is a function which is twice differentiable for $-\infty < x < \infty$ and satisfies $f(0) = f(1) = f(2) = 0$. Show that there exists x such that $0 < x < 2$ and $f''(x) = 0$.

Solution: Applying the Mean Value Theorem we see that there are points x_1, x_2 such that $0 < x_1 < 1 < x_2 < 2$, $f'(x_1) = 0$ and $f'(x_2) = 0$. Applying the Mean Value Theorem to the function $f'(x)$ on the interval $x_1 \leq x \leq x_2$ we see that there is a point x such that $x_1 < x < x_2$ and $f''(x) = 0$.

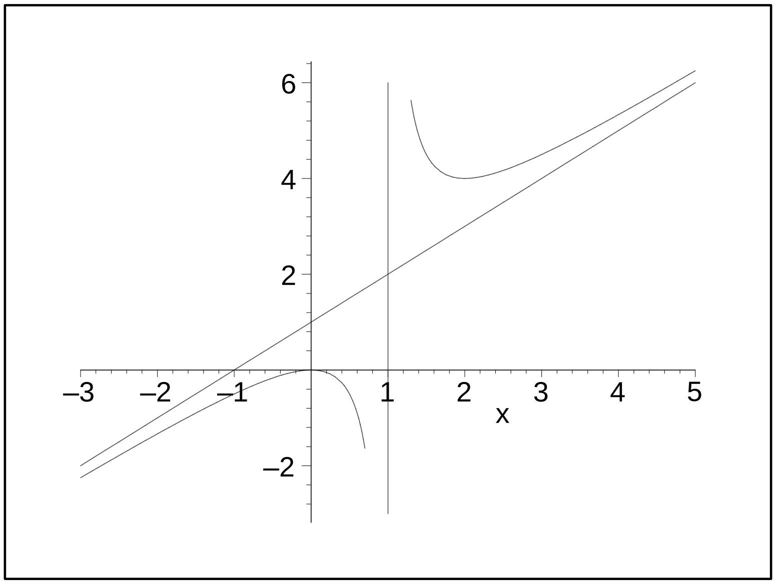

Figure 1: $y = \dfrac{x^2}{x-1}$

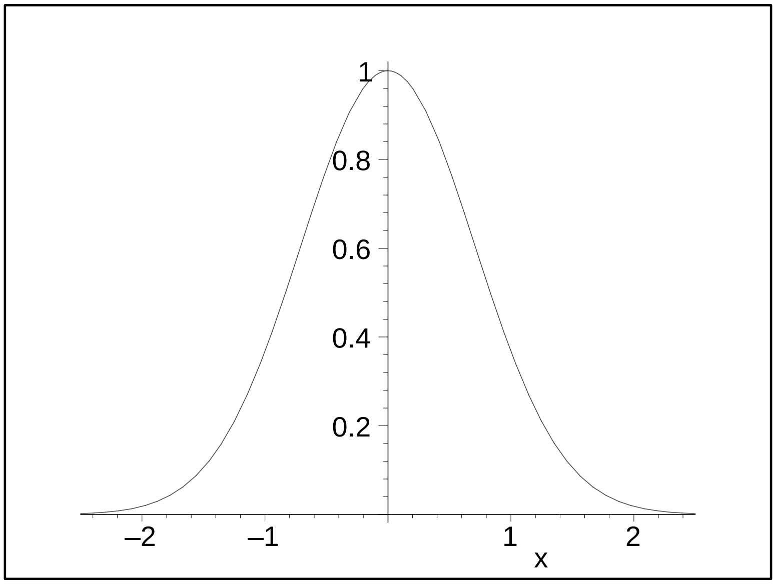

Figure 2: $y = e^{-x^2}$

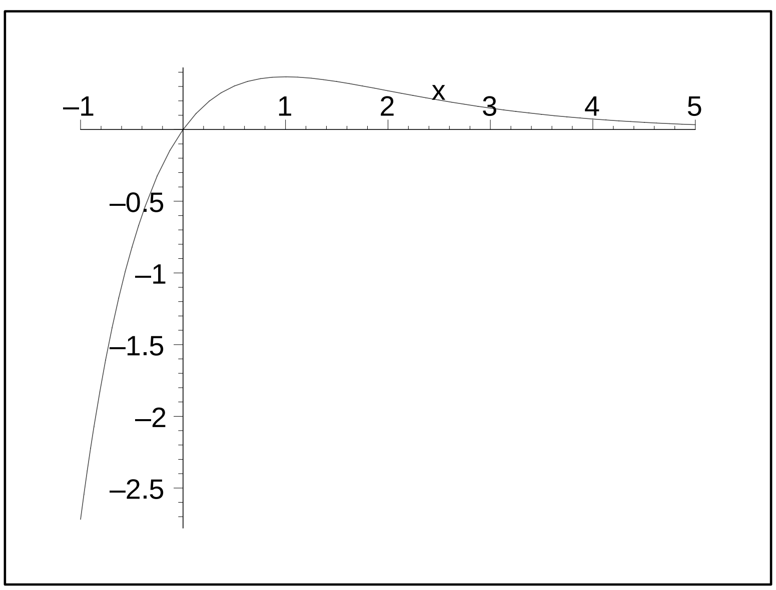

Figure 3: $y = xe^{-x}$

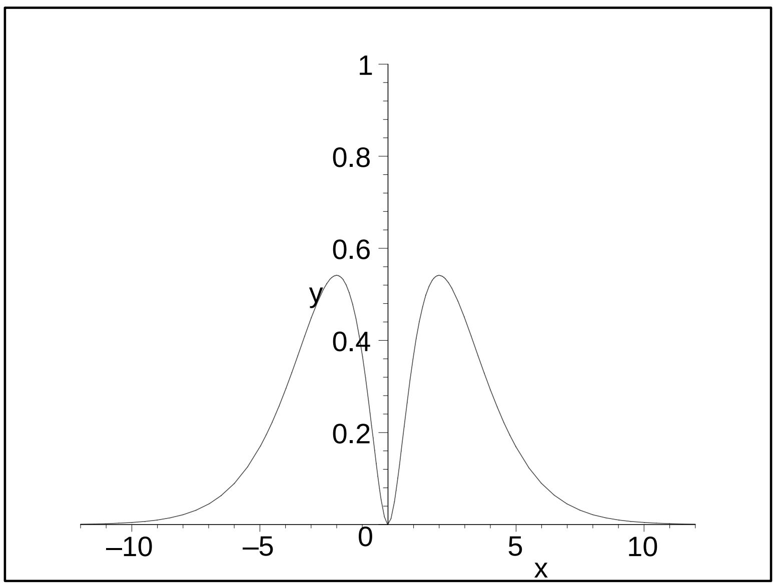

Figure 4: $y = x^2 e^{-|x|}$