SOLUTIONS TO MID TERM #1, MATH 100

1. [6 marks] Using only the definition of the derivative, and not the rules, find $f'(x)$ for the function $f(x) = \sqrt{x^2+1}$.

Solution:

$$\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\sqrt{(x+h)^2+1}-\sqrt{x^2+1}}{h} \\
&= \lim_{h\to 0}\left(\frac{\sqrt{(x+h)^2+1}-\sqrt{x^2+1}}{h}\times\frac{\sqrt{(x+h)^2+1}+\sqrt{x^2+1}}{\sqrt{(x+h)^2+1}+\sqrt{x^2+1}}\right) \\
&= \lim_{h\to 0}\frac{(x+h)^2+1-(x^2+1)}{h(\sqrt{(x+h)^2+1}+\sqrt{x^2+1})} = \lim_{h\to 0}\frac{2hx+h^2}{h(\sqrt{(x+h)^2+1}+\sqrt{x^2+1})} \\
&= \lim_{h\to 0}\frac{2x+h}{\sqrt{(x+h)^2+1}+\sqrt{x^2+1}} = \frac{x}{\sqrt{x^2+1}}
\end{aligned}$$

2. [12 marks] Find the derivatives of the following functions.

(a) $f(x) = \left(\sin^3 x + \cos^3 x\right)^2$.

(b) $f(x) = \sqrt{1+\sqrt{x+x^2}}$.

(c) $f(x) = \dfrac{x^2+1}{x^2-1}$.

(d) $f(x) = (x^2+x+1)(x^3+1)$.

Solution:

(a) $f'(x) = 2\left(\sin^3 x + \cos^3 x\right)(3\sin^2 x \times \cos x + 3\cos^2 x \times (-\sin x))$.

(b) $f'(x) = \dfrac{1}{2}\left(1+\sqrt{x+x^2}\right)^{-1/2} \times \left(\dfrac{1}{2}(x+x^2)^{-1/2}(1+2x)\right)$.

(c) $f'(x) = \dfrac{(x^2-1)2x-(x^2+1)2x}{(x^2-1)^2}$.

(d) $f'(x) = (2x+1)(x^3+1) + (x^2+x+1)3x^2$.

3. [8 marks]

(a) Determine $\displaystyle\lim_{\theta\to 0}\frac{\tan 2\theta}{\theta}$.

(b) Find the absolute maximum and minimum of the function $f(x) = x^2 + \dfrac{1}{x^2}$ on the interval $\dfrac{1}{2} \le x \le 3$.

(c) Find all x where the derivative of $y = \sin x + \cos x$ is 0.

(d) Determine $\lim_{x\to 1} \frac{1}{x-1}\left(\frac{1}{\sqrt{x}} - 1\right)$.

Solution:

(a) $\lim_{\theta\to 0} \frac{\tan 2\theta}{\theta} = \lim_{\theta\to 0} \frac{\sin 2\theta}{\theta \cos 2\theta} = \lim_{\theta\to 0} \left(\frac{\sin 2\theta}{2\theta} \frac{2}{\cos 2\theta}\right) = \lim_{\theta\to 0} \frac{\sin 2\theta}{2\theta} \times \lim_{\theta\to 0} \frac{2}{\cos 2\theta} = 2.$

(b) $f'(x) = 2x - 2x^{-3} = 0 \iff x = 1$ (recall that $\frac{1}{2} \le x \le 3$). By computation

$$f(1/2) = \frac{17}{4}, f(1) = 2, f(3) = \frac{82}{9}.$$

Therefore the absolute maximum is $\frac{82}{9}$ and the absolute minimum is 2.

(c) $y' = \cos x - \sin x = 0 \iff \tan x = 1 \iff x = \frac{\pi}{4} + n\pi,\ \ n = 0, \pm 1, \pm 2, \ldots$

(d)

$$\begin{aligned}
\lim_{x\to 1} \frac{1}{x-1}\left(\frac{1}{\sqrt{x}} - 1\right) &= \lim_{x\to 1} \frac{1-\sqrt{x}}{\sqrt{x}(x-1)} = \lim_{x\to 1}\left(\frac{1-\sqrt{x}}{\sqrt{x}(x-1)} \times \frac{1+\sqrt{x}}{1+\sqrt{x}}\right) \\
&= \lim_{x\to 1} \frac{1-x}{\sqrt{x}(x-1)(1+\sqrt{x})} = -\lim_{x\to 1} \frac{1}{\sqrt{x}(1+\sqrt{x})} = -\frac{1}{2}
\end{aligned}$$

The last step follows by putting $x = 1$ and using the continuity of $\frac{1}{\sqrt{x}(1+\sqrt{x})}$.

4. [8 marks] Find the dimensions of the largest rectangle which can be inscribed in a right triangle with sides 6, 8 and 10. Assume 2 sides of the rectangle lie on the legs of the right triangle.

Solution:

We must maximize the function $A = xy$, where (x, y) is an arbitrary point on the hypotenuse. See the diagram. The equation of the hypotenuse is $\frac{x}{6} + \frac{y}{8} = 1$, and therefore $A(x) = 8x\left(1 - \frac{x}{6}\right) = 8x - \frac{4x^2}{3}, 0 \le x \le 6$. Then $A'(x) = 8 - \frac{8x}{3} = 0 \iff x = 3$. This clearly gives the maximum as $A(0) = A(6) = 0$. If $x = 3$ then $y = 4$.

5. [6 marks] A cannon ball is shot vertically upwards from the ground with initial velocity $v_o = 98m/s$. It is determined that the height of the ball, $y(t)$ (in meters), as a function of the time, t (in sec), is given by $y = v_o t - 4.9t^2$.

(a) When does the cannon ball reach its highest point?

(b) At what time does the cannon ball hit the ground?

(c) What is the velocity of the cannon ball when it hits the ground?

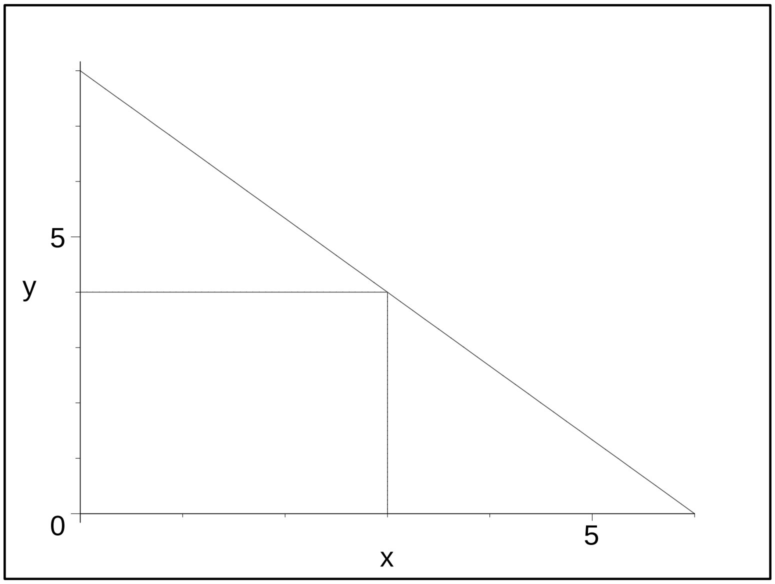

FIGURE 1. Diagram for question 4

Solution:

(a) The velocity of the cannon ball is $v(t) = \dfrac{dy}{dt} = v_0 - 9.8t = 98 - 9.8t$. It reaches its highest point when $v(t) = 0$, that is when $t = 10s$.

(b) Setting $y(t) = 0$ and solving for t gives $t = 0, 20$. Thus the cannon ball hits the ground when $t = 20s$ (it is fired at $t = 0$).

(c) The velocity upon impact is $v(20) = v_0 - 9.8 \times 20 = -98m/s$.

6. [10 marks] Let $f(x)$ be the function defined by $f(x) = \begin{cases} x^2\cos(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

(a) Compute $f'(x)$ for $x \neq 0$.

(b) Prove that $f'(0) = 0$.

(c) Prove that $f'(x)$ is not continuous at $x = 0$.

Solution:

(a) To calculate $f'(x)$ for $x \neq 0$ we can use various rules: $f'(x) = 2x\cos(1/x) + \sin(1/x)$.

(b) By definition $f'(0) = \lim\limits_{h\to 0} \dfrac{f(h) - f(0)}{h} = \lim\limits_{h\to 0} h\cos(1/h) = 0$.

(c) Using the calculation of $f'(x)$ in (a) we see that

$$\lim_{x\to 0} f'(x) = \lim_{x\to 0}(2x\cos(1/x) + \sin(1/x)) = \lim_{x\to 0} 2x\cos(1/x) + \lim_{x\to 0}\sin(1/x) = \lim_{x\to 0}\sin(1/x).$$

But this last limit does not exist and therefore, by definition, $f'(x)$ is not continuous at $x = 0$.