SOLUTIONS TO MID TERM #2, MATH 100

1. [6 marks]

(a) Find the derivative of $f(x) = \arcsin(\sqrt{x})$. Do not simplify.

(b) Find $\dfrac{f'(x)}{f(x)}$ if $f(x) = (\ln x)^x$ and simplify.

(c) Find $f'(x)$ for $f(x) = \arctan\left(\dfrac{x-1}{x+1}\right)$ and simplify.

Solution:

(a) $f'(x) = \dfrac{1}{\sqrt{1-x}} \times \dfrac{1}{2\sqrt{x}}$.

(b) $\dfrac{f'(x)}{f(x)} = \dfrac{d}{dx}(\ln f(x)) = \dfrac{d}{dx}(x \ln(\ln x)) = \ln(\ln x) + \dfrac{1}{\ln x}$.

(c) $f'(x) = \dfrac{1}{1+\left(\frac{x-1}{x+1}\right)^2} \times \dfrac{x+1-(x-1)}{(x+1)^2} = \dfrac{2}{(x+1)^2+(x-1)^2} = \dfrac{1}{1+x^2}$.

2. [8 marks] Let $f(x)$ be the function $f(x) = x(\ln x)^2,\ x > 0$.

(a) Find all x where $f'(x) = 0$.

(b) Find all x where $f''(x) = 0$.

(c) Find all intervals where $f(x)$ is decreasing.

(d) Find all intervals where $f''(x) < 0$.

Solution:

(a) $f'(x) = (\ln x)^2 + 2\ln x = \ln x(\ln x + 2) = 0 \Longleftrightarrow x = 1, e^{-2}$.

(b) $f''(x) = 2\ln x \times \dfrac{1}{x} + \dfrac{2}{x} = \dfrac{2}{x}(\ln x + 1) = 0 \Longleftrightarrow x = e^{-1}$.

(c) $f(x)$ is decreasing for $e^{-2} < x < 1$.

(d) $f''(x) < 0 \Longleftrightarrow 0 < x < e^{-1}$.

3. [12 marks]

(a) Suppose $f(x)$ is defined for all x and satisfies $f'(x) = \dfrac{x}{1+x^2}$, $f(1) = 2$. Use a linear approximation to estimate $f(0.99)$.

(b) Locate on the graph below the approximations x_1, x_2 resulting from Newton's method, if the starting value is x_0.

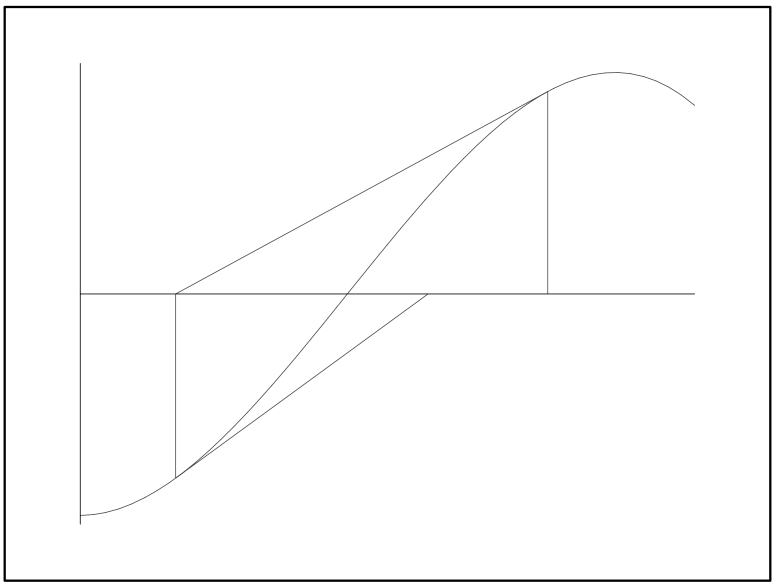
(c) State the Mean Value Theorem.

(d) Prove that $1 + \frac{x}{2} \geq \sqrt{1+x}$ for all $x \geq 0$.

Solution:

(a) $f(0.99) \approx f(1) + f'(1) \times (-0.01) = 2 + \frac{1}{2} \times (-0.01) = 1.995$.

(b)



(c) Mean Value Theorem: Suppose $f(x)$ is continuous for $a \leq x \leq b$ and differentiable for $a < x < b$. Then there exists c such that $a < c < b$ and $f'(c) = \frac{f(b) - f(a)}{b - a}$.

(d) Let $f(x) = 1 + \frac{x}{2} - \sqrt{1+x}$. Then $f(0) = 0$ and $f'(x) = \frac{1}{2} - \frac{1}{2\sqrt{1+x}} \geq 0$ for $x \geq 0$. It follows that $f(x) \geq f(0) = 0$ for $x \geq 0$, that is $1 + \frac{x}{2} \geq \sqrt{1+x}$ for all $x \geq 0$.

4. [12 marks] At time $t = 0$ a pot of boiling water is removed from a stove into a room where the ambient temperature is 20^o Celsius. 10 minutes later its temperature is 60^o Celsius. Assume the temperature of boiling water is 100^o Celsius.

(a) Determine a formula for the temperature $T(t)$ of the water at any time t.

(b) What will the temperature be after 20 minutes (that is at $t = 20$).?

Solution:

(a) The differential equation for Newton's law of cooling is $\frac{dT}{dt} = k(T - A)$, where $T(t)$ is the temperature at time t, A is the ambient temperature and k is a constant. Thus $T(t) = 20 + Ce^{kt}$, where $C = 80$. To determine k we put $t = 10$:

$$60 = 20 + 80e^{10k} \Longrightarrow k = -\frac{\ln 2}{10}; \text{ therefore } T(t) = 20 + 80e^{-(\ln 2/10)t}.$$

(b) $T(20) = 20 + 80e^{-(\ln 2/10)20} = 20 + 80e^{-2\ln 2} = 40.$

5. [12 marks] Let $y = f(x)$ be the function defined implicitly near $x = 1$ by

$$x^3 - xy + y^3 = 1, \ y = 1 \text{ when } x = 1.$$

(a) Find y' for x near 1.

(b) Is $f(x)$ increasing or decreasing near $x = 1$? You must give a cogent reason for your assertion.

(c) Determine an equation for the tangent line of $y = f(x)$ at $x = 1$.

(d) Use a linear approximation to estimate $f(1.1)$.

Solution:

(a) $3x^2 - y - xy' + 3y^2y' = 0 \Longrightarrow y' = \dfrac{y - 3x^2}{3y^2 - x}.$

(b) Putting $x = 1, y = 1$ gives $y' = -1$ and therefore $f(x)$ is decreasing near $x = 1$.

(c) The tangent line is $y - 1 = -(x - 1)$, that is $y = -x + 2$.

(d) $f(1.1) \approx f(1) + f'(1) \times 0.1 = 1 - 0.1 = 0.99.$