Solutions to Math 253 Midterm Exam

1. (a) Consider the line with parametric equations: $x = 2 + 2t, \quad y = 1 + 5t, \quad z = 4t$. Find the point in which this line meets the plane: $x - y + z = 3$. [6]

   Solution: Plug the parametric equations into the equation of the plane:

$$2 + 2t - (1 + 5t) + 4t = 3 \Rightarrow t = 2$$

   Hence, $x = 6$, $y = 11$, $z = 8$, and the intersection point is $\boxed{(6, 11, 8)}$.

   (b) Find the equation of the line perpendicular to the above line, parallel to the above plane, and passing through the point $(x, y, z) = (2, 3, -2)$. [7]

   Solution: The line is perpendicular to both $\langle 2, 5, 4\rangle$ and $\langle 1, -1, 1\rangle$. Choose $\vec{v} = \langle 2, 5, 4\rangle \times \langle 1, -1, 1\rangle = \langle 9, 2, -7\rangle$. So the equation of the line is

$$\boxed{\vec{r} = \langle 2, 3, -2\rangle + t\langle 9, 2, -7\rangle}.$$

   (c) Find the distance between the above plane and the origin. [7]

   Solution:

   Choose a point on the plane, e.g. $(3, 0, 0)$, and let the vector $\vec{b}$ be the vector pointing from the origin to $(3, 0, 0)$. Then $\vec{b} = \langle 3, 0, 0\rangle$. The distance from the plane to the origin is given by

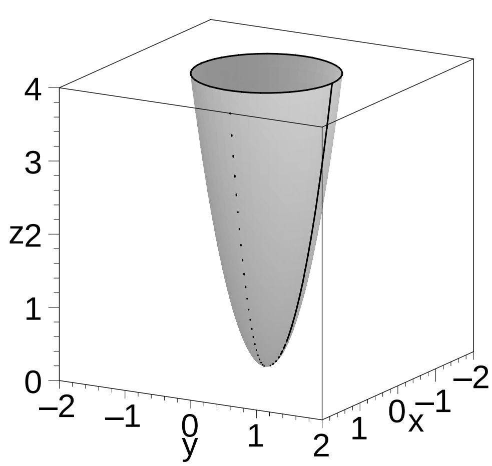
$$|\text{proj}_{\vec{n}}\vec{b}| = \frac{|\vec{n} \cdot \vec{b}|}{|\vec{n}|} = \frac{|\langle 1, -1, 1\rangle \cdot \langle 3, 0, 0\rangle|}{\sqrt{1^2 + (-1)^2 + 1^2}} = \frac{3}{\sqrt{3}} = \boxed{\sqrt{3}}$$

2. Let $h(x, y) = 4x^2 + 4y^2$.

   (a) Sketch the surface $z = h(x, y)$. [5]

   Solution:

   On xz-plane, $y = 0$ and $z = 4x^2$. This is a parabola opening in $+z$-direction.
   On yz-plane, $x = 0$ and $z = 4x^2$. This is a parabola opening in $+z$-direction.
   Level curves $z = k$ are ellipses (circles) when $k > 0$.
   This is an elliptic paraboloid opening in the +z-direction.



(b) Write the equation for this surface in cylindrical coordinates $z = z(r, \theta)$. [5]

Solution:

Since $x^2 + y^2 = r^2$, the equation $z = 4x^2 + 4y^2$ becomes $\boxed{z = 4r^2}$.

(c) Write the equation for this surface in spherical coordinates $\rho = \rho(\theta, \phi)$ [5]

Solution:

Since $x = \rho \sin\phi \cos\theta$, $y = \rho \sin\phi \sin\theta$, and $z = \rho \cos\phi$, The equation becomes

$$\rho \cos\phi = 4\rho^2 \sin^2\phi \cos^2\theta + 4\rho^2 \sin^2\phi \sin^2\theta = 4\rho^2 \sin^2\phi$$

Then the spherical representation is given by $\boxed{\rho = \frac{\cos\phi}{4\sin^2\phi}}$.

(d) Find the equation of the plane tangent to the surface $z = h(x, y)$ at the point $(x, y, z) = (1, 1, 8)$. [5]

Solution:

$h_x(x, y) = 8x$, $h_y(x, y) = 8y$. So the equation of the tangent plane is $z = 8 + h_x(1, 1)(x - 1) + h_y(1, 1)(y - 1)$, or $\boxed{z = 8 + 8(x - 1) + 8(y - 1)}$.

3. Find and classify the critical points of $f(x, y) = x^3 - y^3 - 2xy + 6$. [20]

Solution: This was a homework problem! Refer to the solutions to assignment 5, problem 2. The critical points are the solutions to $f_x = 3x^2 - 2y = 0$ and $f_y = -3y^2 - 2x = 0$. The first equation gives $y = \frac{3}{2}x^2$, and substituting into the second gives $-3(\frac{3}{2}x^2)^2 - 2x = 0$, which simplifies to $x(27x^3 + 8) = 0$, and so the critical points are $(0, 0)$ and $(-2/3, 2/3)$.

Now, to apply the second derivative test, we have

$$f_{xx} = 6x \quad f_{yy} = -6y \quad f_{xy} = -2 \quad D = -36xy - 4$$

At $(0, 0)$, $D = -4$ so it is a saddle. At $(-2/3, 2/3)$, $D = 12 > 0$ and $f_{xx} = -4 < 0$, so it is a maximum. Summarizing: $\boxed{\text{local max at } (-2/3, 2/3), \text{ saddle at } (0, 0)}$

4. (a) Calculate the gradient of the function $g(x, y) = xe^{xy}$. [6]

Solution: $\nabla g = \boxed{\langle e^{xy} + xye^{xy}, x^2e^{xy} \rangle}$

(b) At the point $(1, 0)$, in what direction will g decrease most rapidly? [7]

Solution: $\nabla g(1, 0) = \langle 1, 1 \rangle$. The direction of most rapid *decrease* is $-\nabla g(1, 0) = \boxed{\langle -1, -1 \rangle}$.

(c) What is the directional derivative in that direction? [7]

Solution: The unit vector in that direction is $\langle -1/\sqrt{2}, -1/\sqrt{2} \rangle$. The directional derivative is $D_{\vec{u}}g(1, 0) = \vec{u} \cdot \nabla g(1, 0) = -2/\sqrt{2} = \boxed{-\sqrt{2}}$

5. Find the maximum and minimum values of the function $F(x, y) = xy$ on the ellipse defined by $x^2 + 4y^2 = 4$, and the points at which the max and min occur. [20]

Solution: Using the Lagrange multiplier method to find the extremes of $F(x, y)$, subject to the constraint $g(x, y) = x^2 + 4y^2 = 4$. we need to solve the vector equation

$\nabla F = \lambda \nabla g$ together with the constraint equation. The vector equation gives the two scalar equations:

$$y = 2\lambda x, \quad x = 8\lambda y$$

which, implies (eliminating λ) that $y/2x = x/4y$ and therefore $x^2 = 4y^2$. Putting this into the constraint equation, we conclude that $x^2 + x^2 = 4$ and so $x = \pm\sqrt{2}$. It follows that $y = \pm\sqrt{2}/2$, with independent $\pm$ signs. Calculating the values of F at the four candidate points, we find that

$$\boxed{F(x,y) \text{ has a maximum value } +1 \text{ at } (\sqrt{2}, \sqrt{2}/2) \text{ and } (-\sqrt{2}, -\sqrt{2}/2)}$$

$$\boxed{F(x,y) \text{ has a minimum value -1 at } (-\sqrt{2}, \sqrt{2}/2) \text{ and } (\sqrt{2}, -\sqrt{2}/2)}$$