HOMEWORK ASSIGNMENT #4, MATH 253

1. Prove that the following differential equations are satisfied by the given functions:

   (a) $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = 0$, where $u = (x^2 + y^2 + z^2)^{-1/2}$.

   (b) $x\dfrac{\partial w}{\partial x} + y\dfrac{\partial w}{\partial y} + z\dfrac{\partial w}{\partial z} = -2w$, where $w = \left(x^2 + y^2 + z^2\right)^{-1}$.

2. Show that the function $u = t^{-1}e^{-(x^2+y^2)/4t}$ satisfies the two dimensional heat equation $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}$.

3. (a) Find an equation of the tangent plane to the surface $x^2 + y^2 + z^2 = 9$ at the point $(2, 2, 1)$.

   (b) At what points (x, y, z) on the surface in part (a) are the tangent planes parallel to $2x + 2y + z = 1$?

4. Find the points on the ellipsoid $x^2 + 2y^2 + 3z^2 = 1$ where the tangent plane is parallel to the plane $3x - y + 3z = 1$.

5. (a) Find an equation for the tangent line to the curve of intersection of the surfaces

   $$x^2 + y^2 + z^2 = 9 \text{ and } 4x^2 + 4y^2 - 5z^2 = 0 \text{ at the point } (1, 2, 2).$$

   (b) Find the radius of the sphere whose center is $(-1, -1, 0)$ and which is tangent to the plane $x + y + z = 1$.

6. Find the point(s) on the surface $z = xy$ that are nearest to the point $(0, 0, 2)$.

7. Let $f(x, y, z)$ be the function defined by $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$. Determine an equation for the normal line of the surface $f(x, y, z) = 3$ at the point $(-1, 2, 2)$.

8. Let $f(x, y, z) = \dfrac{xy}{z}$. Measurements are made and it is found that $x = 10, y = 10, z = 2$. If the maximum error made in each measurement is 1% find the approximate percentage error made in computing the value of $f(10, 10, 2)$.

9. Find all points on the surface given by

   $$(x - y)^2 + (x + y)^2 + 3z^2 = 1$$

   where the tangent plane is perpendicular to the plane $2x - 2y = 13$.

10. Find all points at which the direction of fastest change of $f(x, y) = x^2 + y^2 - 2x - 4y$ is $\vec{i} + \vec{j}$.

11. The surface $x^4 + y^4 + z^4 + xyz = 17$ passes through $(0, 1, 2)$, and near this point the surface determines x as a function, $x = F(y, z)$, of y and z.

    (a) Find F_y and F_z at $(x, y, z) = (0, 1, 2)$.

    (b) Use the tangent plane approximation (otherwise known as linear, first order or differential approximation) to find the approximate value of x (near 0) such that $(x, 1.01, 1.98)$ lies on the surface.

12. Let $f(x, y)$ be a differentiable function, and let $u = x + y$ and $v = x - y$. Find a constant α such that
$$(f_x)^2 + (f_y)^2 = \alpha((f_u)^2 + (f_v)^2).$$

13. Find the directional derivative $D_{\vec{u}}f$ at the given point in the direction indicated by the angle

    (a) $f(x, y) = \sqrt{5x - 4y}$, $(2, 1)$, $\theta = -\pi/6$.

    (b) $f(x, y) = x \sin(xy)$, $(2, 0)$, $\theta = \pi/3$.

14. Compute the directional derivatives $D_{\vec{u}}f$, where:

    (a) $f(x, y) = \ln(x^2 + y^2)$, $\vec{u}$ is the unit vector pointing from $(0, 0)$ to $(1, 2)$.

    (b) $f(x, y, z) = \dfrac{1}{\sqrt{x^2 + 2y^2 + 3z^2}}$, $\vec{u} =< 1/\sqrt{2}, 1/\sqrt{2}, 0 >$.

15. Find all points (x, y, z) such that $D_{\vec{u}}f(x, y, z) = 0$, where $\vec{u} =< a, b, c >$ is a unit vector and $f(x, y, z) = \sqrt{\alpha x^2 + \beta y^2 + \gamma z^2}$.

16. Compute the cosine of the angle between the gradient ∇f and the positive direction of the z-axis, where $f(x, y, z) = x^2 + y^2 + z^2$.

17. The temperature at a point (x, y, z) is given by $T(x, y, z) = 200e^{-x^2-3y^2-9z^2}$.

    (a) Find the rate of change of temperature at the point $P(2, -1, 2)$ in the direction towards the point $(3, -3, 3)$.

    (b) In which direction does the temperature increase the fastest at P?

    (c) Find the maximum rate of increase at P.