SOLUTIONS TO HOMEWORK ASSIGNMENT #1

1. Sketch the curve $r = 1 + \cos\theta, 0 \le \theta \le 2\pi$, and find the area it encloses.

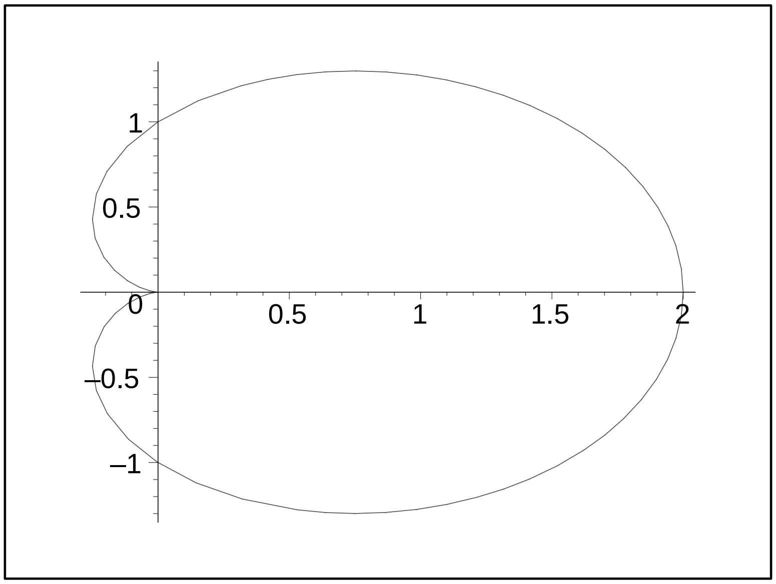

Figure 1: The curve $r = 1 + cos\theta,\ 0 \le \theta \le 2\pi$

The area is given by

$$\begin{aligned} A &= \frac{1}{2}\int_{\theta=0}^{2\pi}(1+\cos\theta)^2 d\theta = \frac{1}{2}\int_{\theta=0}^{2\pi}(1+2\cos\theta+\cos^2\theta) \\ &= \frac{1}{2}(2\pi+0+\pi) = \frac{3\pi}{2} \end{aligned}$$

2. Find the dot product $\vec{a}\cdot\vec{b}$ in the following cases:

(a) $\vec{a} =< 1, 0, -2 >, \vec{b} =< 2, 0, 1 >$. Are these vectors orthogonal?

(b) $\vec{a} =< x_2y_3 - x_3y_2, x_3y_1 - x_1y_3, x_1y_2 - x_2y_1 >, \vec{b} =< x_1, x_2, x_3 >$, where the x_i, y_i are any real numbers. Are these vectors orthogonal?

(c) $\vec{a}$ is a unit vector having the same direction as $\vec{i}+\vec{j}$ and $\vec{b}$ is a vector of magnitude 2 in the direction of $\vec{i}+\vec{j}-\vec{k}$.

Solution:

(a) $\vec{a}\cdot\vec{b} = 2 - 2 = 0$. These vectors are orthogonal.

(b) $\vec{a}\cdot\vec{b} = (x_2y_3 - x_3y_2)x_1 + (x_3y_1 - x_1y_3)x_2 + (x_1y_2 - x_2y_1)x_3 = 0$. These vectors are orthogonal.

(c) $\vec{a} = \frac{1}{\sqrt{2}}(\vec{i}+\vec{j})$ and $\vec{b} = \frac{2}{\sqrt{3}}(\vec{i}+\vec{j}-\vec{k})$ and therefore $\vec{a}\cdot\vec{b} = \frac{2\sqrt{2}}{\sqrt{3}}$.

3. Use cross products to find the following areas:

   (a) the area of the triangle through the points $P = (1, 1, 0), Q = (1, 0, 1), R = (0, 1, 1)$.

   (b) the area of the parallelogram spanned by the vectors $\vec{u} = < 1, 2, 0 >, \vec{v} = < a, b, c >$.

   (c) the areas of all 4 faces of the tetrahedron whose vertices are $(0, 0, 0), (a, 0, 0), (0, b, 0)$ and $(0, 0, c)$, where a, b, c are positive numbers.

   Solution:

   (a) Let $\vec{a} = Q - P = -\vec{j} + \vec{k}, \vec{b} = R - P = -\vec{i} + \vec{k}$. Then the area of the triangle is $\frac{1}{2}\left|\vec{a} \times \vec{b}\right| = \frac{1}{2}|(-\vec{j} + \vec{k}) \times (-\vec{i} + \vec{k})| = \frac{1}{2}| - \vec{k} - \vec{i} - \vec{j}| = \frac{\sqrt{3}}{2}$.

   (b) $\vec{u} \times \vec{v} = (\vec{i} + 2\vec{j}) \times (a\vec{i} + b\vec{j} + c\vec{k}) = 2c\vec{i} - c\vec{j} + (b - 2a)\vec{k}$. Thus the area is

$$\sqrt{5c^2 + (b - 2a)^2} = \sqrt{4a^2 - 4ab + b^2 + 5c^2}.$$

   (c) The areas of the faces in the co-ordinate planes (i.e. the x, y plane, the y, z plane and the z, x plane) are $\frac{ab}{2}, \frac{bc}{2}, \frac{ca}{2}$ respectively. To find the area of the sloping face we compute the cross product of the vectors $\vec{u} = -a\vec{i} + c\vec{k}, \vec{v} = -a\vec{i} + b\vec{j}$:

$$\vec{u} \times \vec{v} = (-a\vec{i} + c\vec{k}) \times (-a\vec{i} + b\vec{j}) = -bc\vec{i} - ca\vec{j} - ab\vec{k}.$$

   Thus the area of the sloping face is $\frac{1}{2}\sqrt{(bc)^2 + (ca)^2 + (ab)^2}$.

4. Suppose $\vec{a}$ is a vector in 3-space. Show that $\left(\frac{\vec{a} \cdot \vec{i}}{|\vec{a}|}\right)^2 + \left(\frac{\vec{a} \cdot \vec{j}}{|\vec{a}|}\right)^2 + \left(\frac{\vec{a} \cdot \vec{k}}{|\vec{a}|}\right)^2 = 1.$

   Remark: The direction cosines of the vector $\vec{a}$ are by definition

$$\cos\alpha = \frac{\vec{a} \cdot \vec{i}}{|\vec{a}|}, \cos\beta = \frac{\vec{a} \cdot \vec{j}}{|\vec{a}|}, \cos\gamma = \frac{\vec{a} \cdot \vec{k}}{|\vec{a}|}.$$

   The angles α, β, γ are the angles $\vec{a}$ makes with the positive directions of the x, y, z axes respectively.

   Solution:

   Suppose $\vec{a} = < a_1, a_2, a_3 >$. Then

$$\left(\frac{\vec{a} \cdot \vec{i}}{|\vec{a}|}\right)^2 + \left(\frac{\vec{a} \cdot \vec{j}}{|\vec{a}|}\right)^2 + \left(\frac{\vec{a} \cdot \vec{k}}{|\vec{a}|}\right)^2 = \left(\frac{a_1}{|\vec{a}|}\right)^2 + \left(\frac{a_2}{|\vec{a}|}\right)^2 + \left(\frac{a_3}{|\vec{a}|}\right)^2 = \frac{a_1^2 + a_2^2 + a_3^2}{a_1^2 + a_2^2 + a_3^2} = 1.$$

5. (a) Find all vectors of length 2 that make equal angles with the positive directions of the x, y, z axes respectively.

(b) Find all unit vectors $\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$ making respective angles of $\pi/3, \pi/4$ with the positive directions of the x, y axes.

(c) Find the angles of the triangle whose vertices are $(1,0,0), (0,2,0), (0,0,3)$.

(d) Find the angle(s) between a diagonal of a cube and one of its edges.

Solution:

(a) If $\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$ has length 2 and makes equal angles with respect to the positive directions of the 3 co-ordinate axes then

$v_1^2+v_2^2+v_3^2 = 4$ and $v_1 = v_2 = v_3 = \lambda$. Therefore $\vec{v} = \pm\dfrac{2}{\sqrt{3}} < 1,1,1 >= \pm\dfrac{2}{\sqrt{3}}(\vec{i}+\vec{j}+\vec{k})$.

(b) If $\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$ is a unit vector making angles $\pi/3$, $\pi/4$ with the positive directions of the x, y axes respectively then

$$v_1^2 + v_2^2 + v_3^2 = 1,\ v_1 = \cos(\pi/3) = 1/2 \text{ and } v_2 = \cos(\pi/4) = 1/\sqrt{2}.$$

Therefore $v_3^2 = 1 - 1/4 - 1/2 = 1/4$, that is $v_3 = \pm 1/2$. Hence $\vec{v} = (1/2, 1/\sqrt{2}, \pm 1/2)$.

(c) Let $P = (1,0,0), Q = (0,2,0), R = (0,0,3)$ and let α, β, γ be the 3 angles at P, Q, R respectively. Then

$$\cos\alpha = \frac{\overrightarrow{PQ}\cdot\overrightarrow{PR}}{|\overrightarrow{PQ}||\overrightarrow{PR}|} = \frac{1}{\sqrt{50}}, \cos\beta = \frac{\overrightarrow{QP}\cdot\overrightarrow{QR}}{|\overrightarrow{QP}||\overrightarrow{QR}|} = \frac{4}{\sqrt{65}}, \cos\gamma = \frac{\overrightarrow{RP}\cdot\overrightarrow{RQ}}{|\overrightarrow{RP}||\overrightarrow{RQ}|} = \frac{9}{\sqrt{130}}$$

Therefore $\alpha \approx 1.428899272, \beta \approx 1.051650212, \gamma \approx 0.6610431690$, all angles measured in radians. Note that $\alpha + \beta + \gamma = \pi$.

(d) One of the diagonals of a (unit) cube is $\vec{v} = \vec{i}+\vec{j}+\vec{k}$. The common angle θ between $\vec{v}$ and any of $\vec{i}, \vec{j}, \vec{k}$ satisfies $\cos\theta = \dfrac{\vec{v}\cdot\vec{i}}{\sqrt{3}} = \dfrac{1}{\sqrt{3}}$. Therefore $\theta \approx 0.9553166180$. The other possibility is the complementary angle, namely $\pi - \theta \approx 2.186276036$.

6. A straight river $400m$ wide flows due west at a constant speed of $3km/hr$. If you can row your boat at $5km/hr$ in still water, what direction should you row in if you wish to go from a point A on the south shore to the point B directly opposite on the north shore? How long will the trip take?

   Solution: We can take the velocity vector of the river to be $\vec{v} = -3\vec{i}$ and the "rowing" vector to be $\vec{u} = 5(\cos\theta\ \vec{i} + \sin\theta\ \vec{j})$, where θ is the angle of inclination with respect to the east. We want $\vec{v} + \vec{u} = (-3 + 5\cos\theta)\vec{i} + (5\sin\theta)\vec{j}$ to be a positive multiple of $\vec{j}$. Therefore $\cos\theta = 3/5$ and $\sin\theta = 4/5$. That is $\theta = \arccos(3/5) \approx 0.9272952180$. With this choice of θ our net velocity is $4\vec{j}$. Therefore it will take $\frac{1}{10}hr = 6$ minutes to get to the opposite shore.

7. Find equations of the planes satisfying the following conditions:

(a) Passing through the point $(0, 2, -3)$ and normal to the vector $4\vec{i} - \vec{j} - 2\vec{k}$.

(b) Passing through the point $(1, 2, 3)$ and parallel to the plane $3x + y - 2z = 15$.

(c) Passing through the 3 points $(\lambda, 0, 0), (0, \mu, 0), (0, 0, \nu)$, where λ, μ, ν are non-zero real numbers.

(d) Passing through the point $(-2, 0, -1)$ and containing the line which is the intersection of the 2 planes $2x + 3y - z = 0$ and $x - 4y + 2z = -5$.

Solution:

(a) The equation is $4(x - 0) - (y - 2) - 2(z + 3) = 0$, that is $4x - y - 2z = 4$.

(b) The equation is $3(x - 1) + (y - 2) - 2(z - 3) = 0$, that is $3x + y - 2z = -1$.

(c) The equation is $\frac{x}{\lambda} + \frac{y}{\mu} + \frac{z}{\nu} = 1$.

(d) The equation will have the form $\mu(2x + 3y - z) + \nu(x - 4y + 2z + 5) = 0$ for an appropriate choice of μ, ν. Putting $x = -2, y = 0, z = -1$ into this equation gives $-3\mu + \nu = 0$. Choosing $\mu = 1, \nu = 3$ gives $5x - 9y + 5z = -15$.

8. Let $v_1 = (0, -1, 0), v_2 = (0, 1, 0), v_3, v_4$ be the 4 vertices of a regular tetrahedron. Suppose $v_3 = (x, 0, 0)$ for some positive x and v_4 has a positive z component. Find v_3 and v_4.

Solution: v_1, v_2, v_3 must form an equilateral triangle in the x, y plane with each side having length 2. Therefore $v_3 = (\sqrt{3}, 0, 0)$. The vertex v_4 must lie over $\frac{1}{3}(v_1 + v_2 + v_3) = \left(\frac{1}{\sqrt{3}}, 0, 0\right)$. Therefore $v_4 = \left(\frac{1}{\sqrt{3}}, 0, z\right)$, where z is that positive number chosen such that the distance from $(0, 1, 0)$ to $\left(\frac{1}{\sqrt{3}}, 0, z\right)$ is 2. Solving we get $v_4 = \left(\frac{1}{\sqrt{3}}, 0, 2\sqrt{\frac{2}{3}}\right)$.