SOLUTIONS TO ASSIGNMENT #10, Math 253

1. Compute the total mass of the solid which is inside the sphere $x^2+y^2+z^2=a^2$ and outside the sphere $x^2+y^2+z^2=b^2$ if the density is given by $\rho(x,y,z)=\dfrac{c}{\sqrt{x^2+y^2+z^2}}$.

   Here a,b,c are positive constants and $0<b<a$.

   Solution: The total mass is
$$m=\int_{\theta=0}^{\theta=2\pi}\int_{\phi=0}^{\phi=\pi}\int_{\rho=b}^{\rho=a}\frac{c}{\rho}\rho^2\sin\phi\, d\rho d\phi d\theta = 2\pi c\frac{a^2-b^2}{2}(-\cos\phi)\Big|_{\phi=0}^{\phi=\pi}=2\pi c(a^2-b^2)$$

2. Compute the volume of the solid region which is inside the sphere $x^2+y^2+z^2=2$ and above the paraboloid $z=x^2+y^2$.

   Solution:

   The intersection of the sphere and the paraboloid is the circle $x^2+y^2=1, z=1$. Therefore
$$\begin{aligned} V &= \int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=1}\int_{z=r^2}^{z=\sqrt{2-r^2}} dz r dr d\theta = 2\pi\int_{r=0}^{r=1} r(\sqrt{2-r^2}-r^2)dr \\ &= -\frac{2\pi}{3}(2-r^2)^{3/2}\Big|_{r=0}^{r=1} -\frac{\pi}{2} = -\frac{2\pi}{3}+\frac{2\pi}{3}2\sqrt{2}-\frac{\pi}{2} \\ &= \left(\frac{4\sqrt{2}}{3}-\frac{7}{6}\right)\pi \end{aligned}$$

3. Find the volume inside the sphere $\rho=a$ that lies between the cones $\phi=\dfrac{\pi}{6}$ and $\phi=\dfrac{\pi}{3}$.

   Solution:
$$\begin{aligned} V &= \int_{\theta=0}^{\theta=2\pi}\int_{\phi=\pi/6}^{\phi=\pi/3}\int_{\rho=0}^{\rho=a}\rho^2\sin\phi\, d\rho d\phi d\theta = 2\pi\frac{a^3}{3}(-\cos\pi/3+\cos\pi/6) \\ &= 2\pi\frac{a^3}{3}\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}\right) = \frac{\pi(\sqrt{3}-1)}{3}a^3 \end{aligned}$$

4. Find the surface area of that part of the sphere $z=\sqrt{a^2-x^2-y^2}$ which lies within the cylinder $x^2+y^2=ay$. Here a is a positive constant.

   Solution: The surface area of the graph of $z=f(x,y)$ over a domain D in the x,y plane is $S=\displaystyle\int\int_D\sqrt{1+(\partial z/\partial x)^2+(\partial z/\partial x)^2}dxdy$. In this case D is the interior of the circle $x^2+y^2=ay$, which in polar coordinates is $0\le\theta\le\pi, 0\le r\le a\sin\theta$. Therefore
$$\begin{aligned} S &= \int\int_{x^2+y^2\le ay}\sqrt{1+\frac{x^2+y^2}{a^2-x^2-y^2}}dxdy = \int_{\theta=0}^{\theta=\pi}d\theta\int_{r=0}^{r=a\sin\theta}\frac{a}{\sqrt{a^2-r^2}}rdr \\ &= a\int_{\theta=0}^{\theta=\pi}-\sqrt{a^2-r^2}\Big|_{r=0}^{r=a\sin\theta}d\theta = a\int_{\theta=0}^{\theta=\pi}(-a|\cos\theta|+a)d\theta = a^2(\pi-2) \end{aligned}$$

5. Find the surface area of that part of the hemisphere of radius $\sqrt{2}$ centered at the origin that lies above the square $-1 \le x \le 1, -1 \le y \le 1$.

   Solution: The surface area is what you get by subtracting 4 half caps from the area of the hemisphere $z = \sqrt{2 - x^2 - y^2}$. By question 2 on quiz 5 each half cap has area $\pi\sqrt{2}(\sqrt{2} - 1)$, and therefore the surface area of the canopy is

$$2\pi(\sqrt{2})^2 - 4\pi\sqrt{2}(\sqrt{2} - 1) = 4\pi(\sqrt{2} - 1)$$

6. Find the centroid of the region inside the cube $0 \le x, y, z \le a$ and below the plane $x + y + z = 2a$.

   Solution: The centroid of R is given by

$$\bar{x} = \frac{\int\int\int_R xdV}{V},\ \bar{y} = \frac{\int\int\int_R ydV}{V},\ \bar{z} = \frac{\int\int\int_R zdV}{V} \text{ where } V \text{ is the volume of } R$$

   In this case the plane $x+y+z = 2a$ cuts off $1/6$ of the cube so $V = 5a^3/6$. We evaluate $\displaystyle\int\int\int_R xdV$ by splitting the region R into 2 separate pieces, R_1 and R_2.

$$\begin{aligned}
\int\int\int_{R_1} xdV &= \int_{x=0}^{x=a} dx \int_{y=0}^{y=a-x} dy \int_{z=0}^{z=a} xdz = \int_{x=0}^{x=a} ax(a-x)dx = \frac{a^4}{6} \\
\int\int\int_{R_2} xdV &= \int_{x=0}^{x=a} dx \int_{y=a-x}^{y=a} dy \int_{z=0}^{z=2a-x-y} xdz = \int_{x=0}^{x=a} \int_{y=a-x}^{y=a} x(2a - x - y)dy \\
&= \int_{x=0}^{x=a} \left(x(2a-x)x - \frac{x}{2}(a^2 - (a-x)^2) \right) dx \\
&= \int_{x=0}^{x=a} \left(ax^2 - \frac{x^3}{2} \right) dx = \frac{a^4}{3} - \frac{a^4}{8} = \frac{5a^4}{24}
\end{aligned}$$

   Therefore $\bar{x} = \dfrac{a^4/6 + 5a^4/24}{5a^3/6} = \dfrac{9a}{20}$, and by symmetry we know that $\bar{x} = \bar{y} = \bar{z} = \dfrac{9a}{20}$.

7. Find the total mass of the region which is above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = 1$, if the density function is $\delta(x, y, z) = 1 + \kappa\sqrt{x^2 + y^2 + z^2}$, for a positive constant κ.

   Solution: The cone and sphere intersect where $\phi = \pi/4$ and therefore the total mass is

$$\begin{aligned}
M &= \int_{\theta=0}^{\theta=2\pi} d\theta \int_{\phi=0}^{\phi=\pi/4} d\phi \int_{\rho=0}^{\rho=1} \rho^2 \sin\phi(1 + \kappa\rho)d\rho \\
&= \int_{\theta=0}^{\theta=2\pi} d\theta \int_{\phi=0}^{\phi=\pi/4} d\phi \int_{\rho=0}^{\rho=1} \rho^2 \sin\phi d\rho + \kappa \int_{\theta=0}^{\theta=2\pi} d\theta \int_{\phi=0}^{\phi=\pi/4} d\phi \int_{\rho=0}^{\rho=1} \rho^3 \sin\phi d\rho \\
&= \frac{2\pi}{3} \int_0^{\pi/4} \sin\phi d\phi + \frac{\kappa\pi}{2} \int_0^{\pi/4} \sin\phi d\phi = \pi(1 - 1/\sqrt{2})(\kappa/2 + 2/3)
\end{aligned}$$

8. Find the total mass of a planet of radius a whose density at distance R from the center is $\delta(x, y, z) = A/(B + R^2)$.

Solution: The total mass is

$$\begin{aligned}
M &= \int_{\theta=0}^{\theta=2\pi} d\theta \int_{\phi=0}^{\phi=\pi} d\phi \int_{\rho=0}^{\rho=a} \rho^2 \sin\phi \frac{A}{B+\rho^2} d\rho = 2\pi A \int_{\phi=0}^{\phi=\pi} \sin\phi d\phi \int_{\rho=0}^{\rho=a} \frac{\rho^2}{B+\rho^2} d\rho \\
&= 4\pi A \int_{\rho=0}^{\rho=a} \left(1 - \frac{B}{B+\rho^2}\right) d\rho = 4\pi A \left(a - \sqrt{B}\arctan(\rho/\sqrt{B}) \Big|_0^a\right) \\
&= 4\pi A \left(a - \sqrt{B}\arctan(a/\sqrt{B})\right)
\end{aligned}$$

9. Let R be the region in the first quadrant of the x, y plane bounded by the curves

$$y = x, y = 3x, xy = 2, xy = 3.$$

Let a change of variables be given by $u = xy, v = y/x$.

(a) What is R in u, v coordinates?

(b) Compute the Jacobian determinant $\dfrac{\partial(x, y)}{\partial(u, v)}$.

(c) Compute $\displaystyle\int\int_R y^2 dxdy$ by using this change of variables.

Solution:

(a) This is just the rectangle $2 \le u \le 3, 1 \le v \le 3$.

(b) We can solve for x, y in terms of u, v, namely $x = \sqrt{u/v}, y = \sqrt{uv}$, and then compute the Jacobian determinant: $\dfrac{\partial(x, y)}{\partial(u, v)} = \dfrac{1}{2v}$. Another method is to compute the Jacobian determinant $\dfrac{\partial(u, v)}{\partial(x, y)}$ and then use the fact that $\dfrac{\partial(x, y)}{\partial(u, v)} = \left(\dfrac{\partial(u, v)}{\partial(x, y)}\right)^{-1}$:

$$\frac{\partial(u, v)}{\partial(x, y)} = det \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix} = \begin{bmatrix} y & x \\ \frac{-y}{x^2} & \frac{1}{x} \end{bmatrix} = \frac{2y}{x} = 2v$$

Therefore $\dfrac{\partial(x, y)}{\partial(u, v)} = \dfrac{1}{2v}$.

(c) The general change of variables formula, for the substitution $x = x(u, v), y = y(u, v)$, is

$$\int\int_R f(x, y) dxdy = \int\int_S g(u, v) \left|\frac{\partial(x, y)}{\partial(u, v)}\right| dudv,$$

where $g(u,v) = f(x(u,v), y(u,v))$ and S is the description of R in u, v coordinates. Thus

$$\int\int_R y^2 dxdy = \int_{u=2}^{u=3} du \int_{v=1}^{v=3} uv \frac{1}{2v} dv = \frac{5}{2}.$$

10. Evaluate $\displaystyle\int\int_R (x^2 - xy + y^2) dA$, where R is the region bounded by the ellipse $x^2 - xy + y^2 = 2$. Hint: make the substitution $x = \sqrt{2}u - \sqrt{2/3}v, y = \sqrt{2}u + \sqrt{2/3}v$.

    Solution: $x^2 - xy + y^2 = (\sqrt{2}u - \sqrt{2/3}v)^2 - (\sqrt{2}u - \sqrt{2/3}v)(\sqrt{2}u + \sqrt{2/3}v) + (\sqrt{2}u + \sqrt{2/3}v)^2 = 2(u^2 + v^2)$, and therefore the region R in u, v coordinates becomes the circle $u^2 + v^2 = 1$. By computation the Jacobian determinant is $4/\sqrt{3}$ and therefore $\displaystyle\int\int_R (x^2 - xy + y^2) dA = \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=1} 2r^2 r dr d\theta = \pi.$