SOLUTIONS TO HOMEWORK ASSIGNMENT #2, Math 253

1. Find the equation of a sphere if one of its diameters has end points $(1, 0, 5)$ and $(5, -4, 7)$.

   Solution:

   The length of the diameter is $\sqrt{(5-1)^2 + (-4-0)^2 + (7-5)^2} = \sqrt{36} = 6$, so the radius is 3. The centre is at the midpoint $(\frac{1+5}{2}, \frac{0-4}{2}, \frac{5+7}{2}) = (3, -2, 6)$. Hence, the sphere is given as $\boxed{(x-3)^2 + (y+2)^2 + (z-6)^2 = 9}$.

2. Find vector, parametric, and symmetric equations of the following lines.

   (a) the line passing through the points $(3, 1, \frac{1}{2})$ and $(4, -3, 3)$

   Solution:

   The vector between two points is $\vec{v} = \langle 4-3, -3-1, 3-\frac{1}{2}\rangle = \langle 1, -4, \frac{5}{2}\rangle$. Hence the equation of the line is

   Vector form: $\boxed{\vec{r} = \vec{r}_0 + t\vec{v} = \langle 4, -3, 3\rangle + t\langle 1, -4, \frac{5}{2}\rangle = \langle 4+t, -3-4t, 3+\frac{5}{2}t\rangle}$

   Parametric form: $\boxed{x = 4+t, \quad y = -3-4t, \quad z = 3+\frac{5}{2}t}$

   Symmetric from: Solving the parametric form for t gives $\boxed{x - 4 = \frac{y+3}{-4} = \frac{z-3}{5/2}}$

   (b) the line passing through the origin and perpendicular to the plane $2x - 4y = 9$

   Solution:

   Perpendicular to the plane $\Rightarrow$ parallel to the normal vector $\vec{n} = \langle 2, -4, 0\rangle$. Hence

   Vector form: $\boxed{\vec{r} = \langle 0, 0, 0\rangle + t\langle 2, -4, 0\rangle = \langle 2t, -4t, 0\rangle}$

   Parametric from : $\boxed{x = 2t, \quad y = -4t, \quad z = 0}$

   Symmetric form $\boxed{\frac{x}{2} = \frac{y}{-4}, \quad z = 0}$

   (c) the line lying on the planes $x + y - z = 2$ and $3x - 4y + 5z = 6$

   Solution:

   We can find the intersection (the line) of the two planes by solving z in terms of x, and in terms of y.

   $$\begin{aligned} &(1) \quad x + y - z = 2 \\ &(2) \quad 3x - 4y + 5z = 6 \end{aligned}$$

   Solve z in terms of y: $3 \times (1) - (2) \Rightarrow 7y - 8z = 0 \Rightarrow z = \frac{7}{8}y$
   Solve z in terms of x: $4 \times (1) + (2) \Rightarrow 7x + z = 14 \Rightarrow z = 14 - 7x$

   Hence, symmetric form: $\boxed{14 - 7x = \frac{7}{8}y = z}$

   Set the symmetric form $= t$, we have parametric form: $\boxed{x = \frac{14-t}{7}, \quad y = \frac{8}{7}t, \quad z = t}$

   Vector form: $\boxed{\vec{r} = \langle \frac{14-t}{7}, \frac{8}{7}t, t\rangle}$

3. Find the equation of the following planes.

   (a) the plane passing through the points $(-1, 1, -1)$, $(1, -1, 2)$, and $(4, 0, 3)$

   Solution:

   Name the points $P(-1, 1, -1)$, $Q(1, -1, 2)$, and $R(4, 0, 3)$. Set up two vectors:

$$\vec{v}_1 = \overrightarrow{PQ} = \langle 1+1, -1-1, 2+1 \rangle = \langle 2, -2, 3 \rangle \tag{1}$$
$$\vec{v}_2 = \overrightarrow{PR} = \langle 5, -1, 4 \rangle \tag{2}$$

   Choose the normal vector $\vec{n} = \vec{v}_1 \times \vec{v}_2 = \langle -5, 7, 8 \rangle$. Hence the equation of the plane is $\boxed{-5(x+1) + 7(y-1) + 4(z+1) = 0}$ using point P.

   (b) the plane passing through the point $(0, 1, 2)$ and containing the line $x = y = z$

   Solution:

   Name $Q(0, 1, 2)$. The line can be represented as $\vec{r} = \langle t, t, t \rangle$, which crosses the point $P(0, 0, 0)$ and is parallel to $\vec{v} = \langle 1, 1, 1 \rangle$. Set $\vec{b} = \overrightarrow{PQ} = \langle 0, 1, 2 \rangle$. Choose $\vec{n} = \vec{v} \times \vec{b} = \langle 1, -2, 1 \rangle$ and hence the equation of the plane is $\boxed{x - 2y + z = 0}$ using point P.

   (c) the plane containing the lines

$$L_1 : x = 1 + t, \quad y = 2 - t, \quad z = 4t$$

$$L_2 : x = 2 - s, \quad y = 1 + 2s, \quad z = 4 + s$$

   Solution:

   From L_1 and L_2, $\vec{v}_1 = \langle 1, -1, 4 \rangle$ and $\vec{v}_2 = \langle -1, 2, 1 \rangle$. Choose $\vec{n} = \vec{v}_1 \times \vec{v}_2 = \langle -9, -5, 1 \rangle$. Since L_1 crosses the point (1,2,0), the equation of the plane is $\boxed{-9(x-1) - 5(y-2) + z = 0}$.

4. Find the intersection of the line $x = t$, $y = 2t$, $z = 3t$, and the plane $x + y + z = 1$.

   Solution:

   Substitute the line into the plane: $t + 2t + 3t = 1 \Rightarrow t = \frac{1}{6}$.

   Put t back to the line: $x = \frac{1}{6}$, $y = \frac{1}{3}$, $z = \frac{1}{2}$.

   Hence the intersection point is $\boxed{(\frac{1}{6}, \frac{1}{3}, \frac{1}{2})}$.

5. Find the distance between the point $(2, 8, 5)$ and the plane $x - 2y - 2z = 1$.

   Solution:

   Name $Q(2, 8, 5)$. Choose any point on the plane, say a convenient one $(x, 0, 0)$. So $x - 2(0) - 2(0) = 1 \Rightarrow x = 1 \Rightarrow P(1, 0, 0)$. Then $\vec{b} = \overrightarrow{PQ} = \langle 1, 8, 5 \rangle$. The normal vector of the plane is $\vec{n} = \langle 1, -2, -2 \rangle$. The distance between the plane and the point is given as

$$\text{distance} = \left|\text{proj}_{\vec{n}}\,\vec{b}\right| = \frac{|\vec{n}\cdot\vec{b}|}{|\vec{n}|} = \frac{|-25|}{|3|} = \boxed{\frac{25}{3}}$$

6. Show that the lines

$$L_1 : \frac{x-4}{2} = \frac{y+5}{4} = \frac{z-1}{-3}$$
$$L_2 : \frac{x-2}{1} = \frac{y+1}{3} = \frac{z}{2}$$

are skew.

Solution:

Write the equation in parametric form.

$$L1 : x = 2t+4, \quad y = 4t-5, \quad z = -3t+1$$

$$L2 : x = s+2, \quad y = 3s-1, \quad z = 2s$$

The lines are not parallel since the vectors $\vec{v}_1 = \langle 2, 4, -3\rangle$ and $\vec{v}_2 = \langle 1, 3, 2\rangle$ are not parallel. Next we try to find intersection point by equating x, y, and z.

$$\begin{aligned} &(1) \quad 2t+4 = s+2 \\ &(2) \quad 4t-5 = 3s-1 \\ &(3) \quad -3t+1 = 2s \end{aligned}$$

(1) gives $s = 2t+2$. Substituting into (2) gives $4t-5 = 3(2t+2)-1 \Rightarrow t = -5$. Then $s = -8$. However, this contradicts with (3). So there is no solution for s and t. Since the two lines are neither parallel nor intersecting, they are skew lines.

7. Identify and sketch the following surfaces.

   (a) $4x^2 + 9y^2 + 36z^2 = 36$

   Solution:

   xy-plane: $4x^2 + 9y^2 = 36$ ellipse

   xz-plane: $4x^2 + 36z^2 = 36$ ellipse

   yz-plane: $9y^2 + 36z^2 = 36$ ellipse

   $\Rightarrow \boxed{\text{ellipsoid}}$

   (b) $4z^2 - x^2 - y^2 = 1$

   Solution:

   xy-plane: $-x^2 - y^2 = 1$ nothing, try $z =$ constants

   $z = c$: $-x^2 - y^2 = 1 - 4c^2 \Rightarrow x^2 + y^2 = 4c^2 - 1$ circles when $4c^2 - 1 > 0$

   xz-plane: $4z^2 - x^2 = 1$ hyperbola opening in z-direction

   yz-plane: $4z^2 - y^2 = 1$ hyperbola opening in z-direction

   $\Rightarrow \boxed{\text{hyperboloid of two sheets}}$

(c) $y^2 = x^2 + z^2$

Solution:

xy-plane: $y^2 = x^2$ cross

xz-plane: $0 = x^2 + z^2$ point at origin, try $y =$ constants

$y = c$: $c^2 = x^2 + z^2$ circles

yz-plane: $y^2 = z^2$ cross

$\Rightarrow$ $\boxed{\text{cone}}$

(d) $x^2 + 4z^2 - y = 0$

Solution:

xy-plane: $x^2 - y = 0 \Rightarrow y = x^2$ parabola opening in $+y$-direction

xz-plane: $x^2 + 4z^2 = 0$ point at origin, try $y =$ constants

$y = c$: $x^2 + 4z^2 - c = 0 \Rightarrow x^2 + 4z^2 = c$ ellipses when $c > 0$

yz-plane: $4z^2 - y = 0 \Rightarrow y = 4z^2$ parabola opening in $+y$-direction

$\Rightarrow$ $\boxed{\text{elliptic paraboloid}}$

(e) $y^2 + 9z^2 = 9$

Solution:

x missing: cylinder along x-direction

yz-plane: $y^2 + 9z^2 = 9$ ellipse

$\Rightarrow$ $\boxed{\text{elliptic cylinder}}$

(f) $y = z^2 - x^2$

Solution:

xy-plane: $y = z^2$ parabola opening in $+y$-direction

xz-plane: $0 = z^2 - x^2 \Rightarrow z^2 = x^2$ cross, try $y =$ constants

$y = c$: $c = z^2 - x^2$ hyperbola opening in z-direction when $c > 0$, in x-direction when $c < 0$

yz-plane: $y = -x^2$ parabola opening in $-y$-direction

$\Rightarrow$ $\boxed{\text{hyperbolic paraboloid}}$

8. Find the polar equation for the curve represented by the following Cartesian equation.

(a) $x = 4$

Solution:

$x = 4 \Rightarrow r\cos\theta = 4 \Rightarrow \boxed{r = 4\sec\theta}$

(b) $x^2 + y^2 = -2x$

Solution:

$x^2 + y^2 = -2x \Rightarrow r^2 = -2r\cos\theta \Rightarrow \boxed{r = -2\cos\theta}$

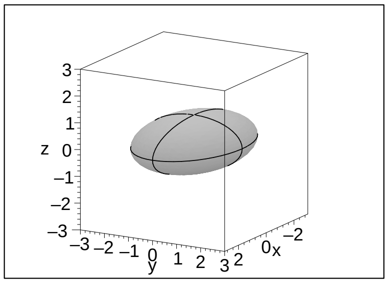

Figure 1: Q7(a)

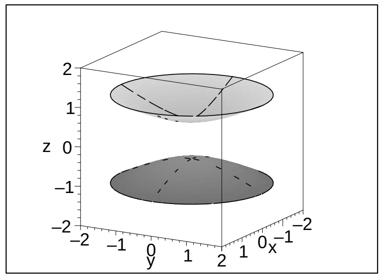

Figure 2: Q7(b)

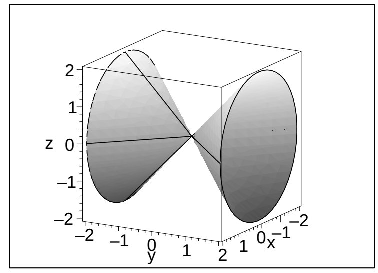

Figure 3: Q7(c)

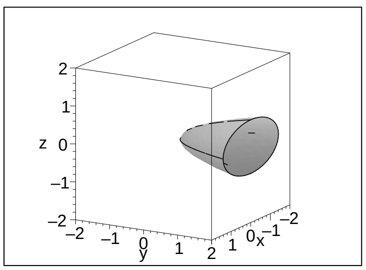

Figure 4: Q7(d)

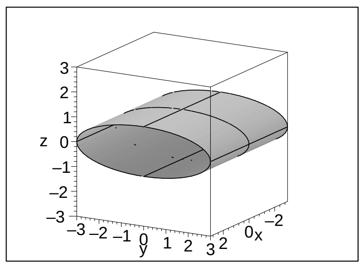

Figure 5: Q7(e)

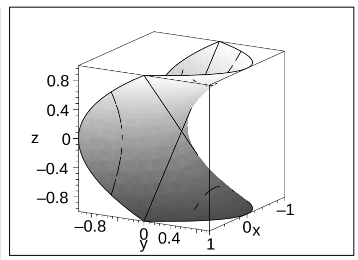

Figure 6: Q7(f)

(c) $x^2 - y^2 = 1$

Solution:

$x^2 - y^2 = 1 \Rightarrow r^2\cos^2\theta - r^2\sin^2\theta = 1 \Rightarrow r^2(\cos^2\theta - \sin^2\theta) = 1 \Rightarrow r^2\cos 2\theta = 1$
$\Rightarrow r^2 = \sec 2\theta \Rightarrow \boxed{r = \pm\sqrt{\sec 2\theta}}$

9. Sketch the curve of the following polar equations.

   (a) $r = 5$

   (b) $\theta = \frac{3\pi}{4}$

   (c) $r = 2\sin\theta$

   (d) $r = 3(1 - \cos\theta)$

10. (a) Change $(3, \frac{\pi}{3}, 1)$ from cylindrical to rectangular coordinates

    Solution:

    $x = r\cos\theta = 3\cos\frac{\pi}{3} = \frac{3}{2}$, $y = r\sin\theta = 3\sin\frac{\pi}{3} = \frac{3\sqrt{3}}{2}$, $z = 1$. Hence $(x, y, z) = \boxed{(\frac{3}{2}, \frac{3\sqrt{3}}{2}, 1)}$

    (b) Change $(\sqrt{3}, 1, 4)$ from rectangular to cylindrical coordinates

    Solution:

    $r = \sqrt{x^2 + y^2} = \sqrt{3+1} = 2$, $\tan\theta = \frac{y}{x} = \frac{1}{\sqrt{3}} \Rightarrow \theta = \frac{\pi}{6}$ in first quadrant, $z = 4$. Hence $(r, \theta, z) = \boxed{(2, \frac{\pi}{6}, 4)}$

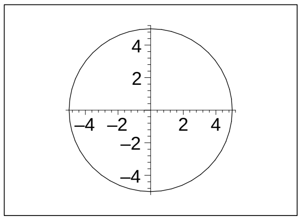

Figure 7: Q9(a)

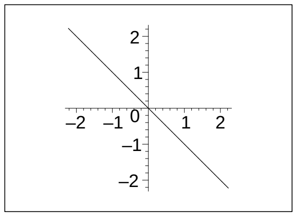

Figure 8: Q9(b)

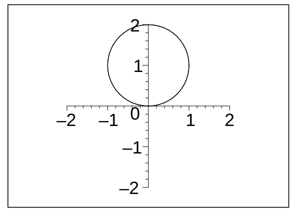

Figure 9: Q9(c)

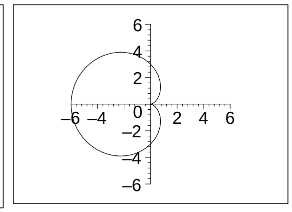

Figure 10: Q9(d)

(c) Change $(\sqrt{3}, 1, 2\sqrt{3})$ from rectangular to spherical coordinates
Solution:
$\rho = \sqrt{x^2+y^2+z^2} = \sqrt{3+1+12} = 4$, $\tan\theta = \frac{y}{x} = \frac{1}{\sqrt{3}} \Rightarrow \theta = \frac{\pi}{6}$ in first quadrant, $\phi = \cos^{-1}\frac{z}{\rho} = \cos^{-1}\frac{2\sqrt{3}}{4} = \cos^{-1}\frac{\sqrt{3}}{2} = \frac{\pi}{6}$. Hence $(\rho, \theta, \phi) = \boxed{(4, \frac{\pi}{6}, \frac{\pi}{6})}$

(d) Change $(4, \frac{\pi}{4}, \frac{\pi}{3})$ from spherical to cylindrical coordinates
Solution:
$r = \rho\sin\phi = 4\sin\frac{\pi}{3} = 2\sqrt{3}$, $\theta = \frac{\pi}{4}$, $z = \rho\cos\phi = 4\cos\frac{\pi}{3} = 2$. Hence $(r, \theta, z) = \boxed{(2\sqrt{3}, \frac{\pi}{4}, 2)}$