SOLUTIONS TO HOMEWORK ASSIGNMENT 3, Math 253

1. Calculate the following limits, or discuss why they do not exist:

   (a) $\lim_{(x,y)\to 0} \frac{y}{x^2+y^2}$

   Solution: The limit does not exist. If (x, y) approaches $(0, 0)$ along the x-axis, we have $y = 0, x \neq 0$ and the expression is identically zero, and so the limit in that direction is zero. On the other hand, approaching $(0, 0)$ along the y-axis, the expression becomes $1/y$, which increases without bound as $y \to 0$, so the limit cannot exist (even as ∞).

   (b) $\lim_{(x,y)\to 0} \frac{y^3}{x^2+y^2}$ [hint: $|y^2| \leq |x^2 + y^2|$]

   Solution: Note that $0 \leq |\frac{y^3}{x^2+y^2}| = \frac{|y^2|}{|x^2+y^2|}|y| \leq |y|$, the latter inequality following from the hint. Since $|y| \to 0$, the same must be true of $|\frac{y^3}{x^2+y^2}|$, so $\lim_{(x,y)\to 0} \frac{y^3}{x^2+y^2} = 0$.

2. For each of the following functions, give its domain and calculate $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$:

   (a) $f(x, y) = e^{3x}\cos(3y)$

   Solution: The domain is all of $\mathbb{R}^2$.
   $\frac{\partial f}{\partial x} = 3e^{3x}\cos(3y)$ and $\frac{\partial f}{\partial y} = -3e^{3x}\sin(3y)$.

   (b) $f(x, y) = \ln(1 + xy^2)$

   Solution: Since the logarithm requires a positive input, we must have $xy^2 > -1$, or equivalently $x > -\frac{1}{y^2}$. The graph of $x > -\frac{1}{y^2}$ has 2 branches, one in the third quadrant and one in the fourth. The domain of definition is everything between these graphs together the set of all points (x, y) such that $x \geq 0$, i.e.

   $$\left\{(x,y) \mid \ x < 0, -\sqrt{-1/x} < y < \sqrt{-1/x}\right\} \cup \{(x,y) \mid \ x \geq 0\}$$

   $\frac{\partial f}{\partial x} = y^2/(1 + xy^2)$ and $\frac{\partial f}{\partial y} = 2xy/(1 + xy^2)$

   (c) $f(x, y) = \frac{y}{x^2+y^2}$

   Solution: The domain is $\mathbb{R}\backslash(0, 0)$. $\frac{\partial f}{\partial x} = -2xy/(x^2+y^2)^2$ and $\frac{\partial f}{\partial y} = (x^2-y^2)/(x^2+y^2)^2$

   (d) $f(x, y) = x^y$

Solution: The standard definition is $x^y = e^{(y \ln x)}$, which requires $x > 0$, so the domain is the open right half-plane. $\dfrac{\partial f}{\partial x} = yx^{y-1}$ and $\dfrac{\partial f}{\partial y} = x^y \ln x$.

(e) $f(x, y) = \cosh(x)\cos(y)$

Solution: The domain is all of $\mathbb{R}^2$. $\dfrac{\partial f}{\partial x} = \sinh(x)\cos(y)$ and $\dfrac{\partial f}{\partial y} = -\cosh(x)\sin(y)$

(f) $f(x, y) = x^3 \arcsin y^2$

Solution: The arcsin requires the input y^2 between -1 and $+1$, which is equivalent to $-1 \leq y \leq 1$. So the range is the infinite horizontal strip between the lines $y = -1$ and $y = 1$. $\dfrac{\partial f}{\partial x} = 3x^2 \arcsin y^2$ and $\dfrac{\partial f}{\partial y} = 2x^3 y/\sqrt{1 - y^4}$.

3. Which of the above functions satisfies the Laplace equation: $f_{xx} + f_{yy} = 0$?

   Solution: (a), (c) and (e).

   For (a): $f_{xx} = 9e^x \cos 3y = -f_{yy}$.

   For (c): $f_{xx} = (6x^2 y - 2y^3)/(x^2 + y^2)^3 = -f_{yy}$.

   For (e): $f_{xx} = \cosh(x)\cos(y) = -f_{yy}$.

4. Give an equation for the tangent plane to the graph of $f(x, y) = \dfrac{y}{x^2 + y^3}$ at the point $(0, 1, 1)$. What is the normal vector at that point?

   Solution: Since $f_x(0, 1) = 0$ and $f_y(0, 1) = -2$, the tangent plane has equation $z = 1 + (-2)(y - 1)$ or $2y + z = 3$. A normal vector at $(0, 1, 1)$ is $\langle 0, 2, 1 \rangle$.

5. Find the coordinates of all points at which the surface with the following equation has a horizontal tangent plane: $z = x^4 - 4xy^3 + 6y^2 - 2$.

   Solution: There is a horizontal tangent plane when $\dfrac{\partial z}{\partial x} = \dfrac{\partial z}{\partial y} = 0$. This gives $4x^3 - 4y^3 = 0$ and $-12xy^2 + 12y = 0$. The first equation implies $x = y$. The second can be rewritten $y(1 - y^2) = 0$, and so $y = 0$ or $y = \pm 1$. Thus there are three points at which the surface has a horizontal tangent plane: $(0, 0, -2)$; $(1, 1, 1)$ and $(-1, -1, 1)$.

6. The equation $x^3 y^4 + xz^2 - yz^3 = 1$ defines a surface which passes through the point $(1, 1, 1)$. Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ at that point.

   Solution: We can find the partials by implicit differentiation. Holding y fixed and taking the derivative with respect to x (regarding z as function of x) gives

$$3x^2 y^4 + z^2 + 2xz\frac{\partial z}{\partial x} - 3yz^2\frac{\partial z}{\partial x} = 0.$$

At the point $(1,1,1)$ this becomes $3+1+2\frac{\partial z}{\partial x}-3\frac{\partial z}{\partial x}=0$, so
$\dfrac{\partial z}{\partial x}(1,1)=4$. Similarly, holding x fixed and taking the y-derivative gives

$$4x^3y^3+2xz\frac{\partial z}{\partial y}-z^3-3yz^2\frac{\partial z}{\partial y}=0,$$

which becomes $4+2\frac{\partial z}{\partial y}-1-3\frac{\partial z}{\partial y}=0$, and we find $\dfrac{\partial z}{\partial y}(1,1)=3$.

7. The radius and height of a right circular-conical tank are measured and found to be 25 ft and 21 ft respectively. Each measurement is accurate to within 0.5 in. By about how much can the calculated volume of the tank be in error?

   Solution: The volume of a right circular cone of radius r and height h is $V=\pi r^2h/3$. Then

$$dV=\frac{2\pi}{3}rhdr+\frac{\pi}{3}r^2dh=\frac{2\pi}{3}\times 25\times 21\times\frac{1}{24}+\frac{\pi}{3}(25)^2\frac{1}{24}=1675\pi/24\approx 73(ft^3).$$

8. Write an appropriate version of the chain rule for $\dfrac{\partial z}{\partial u}$, if $z=g(x,y)$, $y=f(x)$ and $x=h(u,v)$.

   Solution: $\dfrac{\partial z}{\partial u}=g_1h_1+g_2f'h_2$, or in Leibniz notation $\dfrac{\partial z}{\partial u}=\dfrac{\partial g}{\partial x}\dfrac{\partial h}{\partial u}+\dfrac{\partial g}{\partial y}\dfrac{df}{dx}\dfrac{\partial h}{\partial u}$.

9. Use two different methods to calculate $\dfrac{\partial z}{\partial x}$ given that

$$z=\arctan(u/v),\quad u=2x+y,\quad v=3x-y.$$

   Solution: Using either direct substitution or the chain rule results in the answer

$$\frac{\partial z}{\partial x}=\frac{-5y}{13x^2-2xy+2y^2}.$$

10. Calculate $\dfrac{\partial}{\partial x}f(y^2,x^2)$, assuming f has continuous partial derivatives.

    Solution: $2xf_2(y^2,x^2)$.