SOLUTIONS TO HOMEWORK ASSIGNMENT #5, Math 253

1. For what values of the constant k does the function $f(x,y) = kx^3 + x^2 + 2y^2 - 4x - 4y$ have

   (a) no critical points;

   (b) exactly one critical point;

   (c) exactly two critical points?

   Hint: Consider $k = 0$ and $k \neq 0$ separately.

   Solution:

   Set $f_x = 0$ and $f_y = 0$ to find critical points:

   $$f_x = 3kx^2 + 2x - 4 = 0 \qquad (1)$$
   $$f_y = 4y - 4 = 0 \qquad (2)$$

   (2) gives $y = 1$. For (1), consider $k = 0$ and $k \neq 0$ separately.

   For $k = 0$, (1) becomes $2x - 4 = 0$, or $x = 2$. So one critical point at $(2, 1)$.

   For $k \neq 0$, use quadratic formula to solve for x.

   $$x = \frac{-2 \pm \sqrt{4 + 48k}}{6k} = \frac{-1 \pm \sqrt{1 + 12k}}{3k}$$

   So critical points are $(\frac{-1\pm\sqrt{1+12k}}{3k}, 1)$ if they exist.

   Conclusion:

   $$\boxed{\begin{array}{l} k < -1/12\text{: no critical points.} \\ k = -1/12\text{: one critical point } (4,1). \\ k > -1/12 \text{ and } k \neq 0\text{: two critical points } (\frac{-1\pm\sqrt{1+12k}}{3k}, 1). \\ k = 0\text{: one critical point } (2,1). \end{array}}$$

2. Find and classify all critical points of the following functions.

   (a) $f(x, y) = x^3 - y^3 - 2xy + 6$

   Solution:

   Step 1: find critical points

   $$f_x = 3x^2 - 2y = 0 \qquad (1)$$
   $$f_y = -3y^2 - 2x = 0 \qquad (2)$$

   (1) gives $y = \frac{3}{2}x^2$. Substituting into (2) becomes $-3\left(\frac{3}{2}x^2\right)^2 - 2x = 0$, or simplified $-x(27x^3 + 8) = 0$. Hence $x = 0$ or $-2/3$.
   If $x = 0$, then by (1) $y = 0 \Rightarrow (0, 0)$
   If $x = -2/3$, then by (1) again $y = 2/3 \Rightarrow (-2/3, 2/3)$.
   Hence, critical points at $(0, 0)$ and $(-2/3, 2/3)$.

Step 2: apply second derivative test

$$f_{xx} = 6x \quad f_{yy} = -6y \quad f_{xy} = -2$$

At $(0,0)$, $f_{xx} = 0$, $f_{yy} = 0$, $f_{xy} = -2$. So $D = f_{xx}f_{yy} - (f_{xy})^2 = -4 < 0 \Rightarrow$ saddle
At $(-2/3, 2/3)$, $f_{xx} = -4 < 0$, $f_{yy} = -4$, $f_{xy} = -2$. So $D = 12 > 0 \Rightarrow$ local max

Hence, $\boxed{\text{local max at } (-2/3, 2/3), \text{ saddle point at } (0,0)}$

(b) $f(x,y) = x^3 + y^3 + 3x^2 - 3y^2 - 8$
Solution:
Step 1: find critical points

$$f_x = 3x^2 + 6x = 0 \qquad (1)$$
$$f_y = 3y^2 - 6y = 0 \qquad (2)$$

We can solve the two equations separately. (1) gives $x = 0$ and -2. (2) gives $y = 0$ and 2. Hence, there are four critical points at $(0,0)$, $(0,2)$, $(-2,0)$, and $(-2,2)$.
Step 2: apply second derivative test

$$f_{xx} = 6x + 6 \quad f_{yy} = 6y - 6 \quad f_{xy} = 0$$

At $(0,0)$, $f_{xx} = 6$, $f_{yy} = -6$, $f_{xy} = 0$, so $D = -36 < 0 \Rightarrow$ saddle
At $(0,2)$, $f_{xx} = 6 > 0$, $f_{yy} = 6$, $f_{xy} = 0$, so $D = 36 > 0 \Rightarrow$ local min
At $(-2,0)$, $f_{xx} = -6 < 0$, $f_{yy} = -6$, $f_{xy} = 0$, so $D = 36 > 0 \Rightarrow$ local max
At $(-2,2)$, $f_{xx} = -6$, $f_{yy} = 6$, $f_{xy} = 0$, so $D = -36 < 0 \Rightarrow$ saddle
Hence, $\boxed{\text{local max at } (-2,0), \text{ local min at } (0,2), \text{ saddle at } (0,0) \text{ and } (-2,2)}$

(c) $f(x,y) = \frac{1}{x^2+y^2-1}$
Solution:
Step 1: find critical points

$$f_x = -\frac{2x}{(x^2+y^2-1)^2} = 0 \qquad (1)$$
$$f_y = -\frac{2y}{(x^2+y^2-1)^2} = 0 \qquad (2)$$

(1) gives $x = 0$ and (2) gives $y = 0$. The critical point is at $(0,0)$.
Step 2: apply second derivative test

$$f_{xx} = -\frac{2(x^2+y^2-1)^2 - 2x[2(x^2+y^2-1)(2x)]}{(x^2+y^2-1)^4}$$
$$f_{yy} = -\frac{2(x^2+y^2-1)^2 - 2y[2(x^2+y^2-1)(2y)]}{(x^2+y^2-1)^4}$$
$$f_{xy} = \frac{2x(2)(2y)}{(x^2+y^2-1)^3}$$

At $(0,0)$ $f_{xx} = -2 < 0$, $f_{yy} = -2$, $f_{xy} = 0$, So $D = 4 > 0 \Rightarrow$ local max
Hence, $\boxed{\text{local max at } (0,0)}$

(d) $f(x,y) = y \sin x$

Solution:

Step 1: find critical points

$$f_x = y\cos x = 0 \quad (1)$$
$$f_y = \sin x = 0 \quad (2)$$

(2) gives $x = n\pi$ for all $n \in \mathbb{Z}$, i.e. integers. Substituting to (1) gives $\pm y = 0$, or $y = 0$. The critical points are $(n\pi, 0)$ for all $n \in \mathbb{Z}$.

Step 2: apply second derivative test

$$f_{xx} = -y\sin x \quad f_{yy} = 0 \quad f_{xy} = \cos x$$

At all $(n\pi, 0)$, $f_{xx} = 0$, $f_{yy} = 0$, $f_{xy} = \pm 1$, so $D = -1 < 0 \Rightarrow$ saddle

Hence, $\boxed{\text{saddle points at } (n\pi, 0) \text{ for all } n \in \mathbb{Z}}$

3. Suppose $f(x,y)$ satisfies the Laplace's equation $f_{xx}(x,y) + f_{yy}(x,y) = 0$ for all x and y in $\mathbb{R}^2$. If $f_{xx}(x,y) \neq 0$ for all x and y, explain why $f(x,y)$ must not have any local minimum or maximum.

   Solution:

   Since the second derivatives exists, the first derivatives must be continuous and $f(x,y)$ must be differentiable. Also, since there is no boundary on $\mathbb{R}^2$, local max/min must occur at critical points.

   Suppose there is a critical point, then by second derivative test, $D = f_{xx}f_{yy} - f_{xy}^2$. But $f_{xx} + f_{yy} = 0 \Rightarrow f_{yy} = -f_{xx}$. It follows that $D = -f_{xx}^2 - f_{xy}^2 < 0$ when it is given that $f_{xx} \neq 0$. Therefore all critical points are saddle points.

4. Find all absolute maxima and minima of the following functions on the given domains.

   (a) $f(x,y) = 2x^2 - 4x + y^2 - 4y + 1$ on the closed triangular plate with vertices $(0,0)$, $(2,0)$, and $(2,2)$

   Solution:

   Step 1: find interior critical points

$$f_x = 4x - 4 = 0 \quad (1)$$
$$f_y = 2y - 4 = 0 \quad (2)$$

   (1) gives $x = 1$. (2) gives $y = 2$. Critical point at $(1,2)$, but not in region.

   Step 2: find boundary critical points and endpoints

   Bottom side $y = 0 \Rightarrow f(x,0) = 2x^2 - 4x + 1$.

   $\frac{df}{dy} = 4x - 4 = 0 \Rightarrow x = 1$. Critical point at $\underline{(1,0)}$

   Right side $x = 2 \Rightarrow f(2,y) = 8 - 8 + y^2 - 4y + 1 = y^2 - 4y + 1$.

   $\frac{df}{dx} = 2y - 4 = 0 \Rightarrow y = 2$. Critical point at $\underline{(2,2)}$.

   Hypotenuse $y = x \Rightarrow f(x,x) = 2x^2 - 4x + x^2 - 4x + 1 = 3x^2 - 8x + 1$

   $\frac{df}{dx} = 6x - 8 = 0 \Rightarrow x = 4/3$. So $y = 4/3$. Critical point at $\underline{(4/3, 4/3)}$.

   Together with the endpoints of all sides $\underline{(0,0)}$, $\underline{(2,0)}$, $\underline{(2,2)}$.

Step 3: compare the values of $f(x,y)$
$f(1,0) = -1$
$f(2,2) = -3$
$f(4/3, 4/3) = -13/3 \Leftarrow$ absolute min
$f(0,0) = 1 \Leftarrow$ absolute max
$f(2,0) = 1 \Leftarrow$ absolute max
Hence, $\boxed{\text{abs max at } f(2,0) = f(0,0) = 1, \text{ abs min at } f(4/3,4/3) = -13/3}$

(b) $f(x,y) = x^2 + xy + 3x + 2y + 2$ on the domain $D = \{(x,y)|x^2 \le y \le 4\}$
Solution:
Step 1: find interior critical points

$$f_x = 2x + y + 3 = 0 \quad (1)$$
$$f_y = x + 2 = 0 \quad (2)$$

(2) gives $x = -2$. Substituting to (1) gives $y = 1$. Critical point at $(-2, 1)$ but not in region.
Step 2: find boundary critical points
Top side: $y = 4 \Rightarrow f(x,4) = x^2 + 4x + 3x + 8 + 2 = x^2 + 7x + 10$
$\frac{df}{dx} = 2x + 7 = 0 \Rightarrow x = -7/2$ but not in region
Parabola: $y = x^2 \Rightarrow f(x, x^2) = x^2 + x^3 + 3x + 2x^2 + 2 = x^3 + 3x^2 + 3x + 2$
$\frac{df}{dx} = 3x^2 + 6x + 3 = 3(x+1)^2 = 0 \Rightarrow x = -1$, then $y = (-1)^2 = 1$. Critical point $\underline{(-1,1)}$.
Together with the endpoints of the two sides $\underline{(-2,4)}$, $\underline{(2,4)}$.
Step 3: Compare the values of $f(x,y)$
$f(-1,1) = 1$
$f(-2,4) = 0 \Leftarrow$ absolute min
$f(2,4) = 28 \Leftarrow$ absolute max
Hence, $\boxed{\text{absolute min at } f(-2,4) = 0, \text{ absolute max at } f(2,4) = 28}$

(c) $f(x,y) = 2x^2 + 3y^2 - 4x - 5$ on the domain $D = \{(x,y)|x^2 + y^2 \le 16\}$.
Solution:
Step 1: find interior critical points

$$f_x = 4x - 4 = 0 \quad (1)$$
$$f_y = 6y = 0 \quad (2)$$

(1) gives $x = 1$. (2) gives $y = 0$. Critical point $\underline{(1,0)}$.
Step 2: find boundary critical points
Rewrite the boundary $y^2 = 16 - x^2$ or $y = \pm\sqrt{16 - x^2}$, which the endpoints are $\underline{(4,0)}$ and $\underline{(-4,0)}$.
Then f becomes $f = 2x^2 + 3(16 - x^2) - 4x - 5 = -x^2 - 4x + 43$.
$\frac{df}{dx} = -2x - 4 = 0 \Rightarrow x = -2$, $y^2 = 16 - (-2)^2 \Rightarrow y = \pm\sqrt{12}$
Critical points at $\underline{(-2, \sqrt{12})}$ and $\underline{(-2, -\sqrt{12})}$.
Step 3: compare the values of $f(x,y)$
$f(1,0) = -7 \Leftarrow$ absolute min
$f(4,0) = 11$

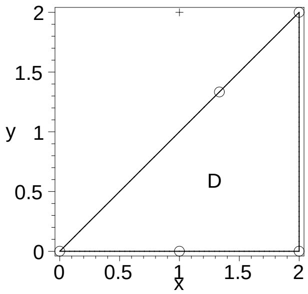

Figure 1: Q4(a)

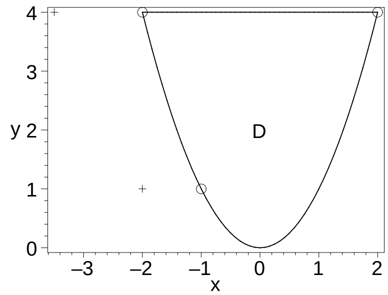

Figure 2: Q4(b)

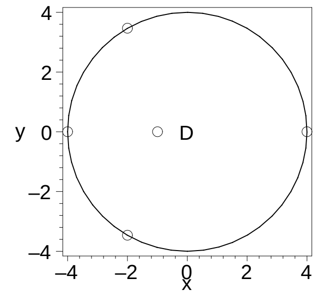

Figure 3: Q4(c)

$f(-4,0) = 43$
$f(-2,\sqrt{12}) = 47 \Leftarrow$ absolute max
$f(-2,-\sqrt{12}) = 47 \Leftarrow$ absolute max
Hence, $\boxed{\text{abs min at } f(1,0) = -7, \text{ abs max at } f(-2,\sqrt{12}) = f(-2,-\sqrt{12}) = 47}$

5. Use Lagrange multipliers to find the maximum and minimum values of the following functions subject to the given constraint(s).

   (a) $f(x,y) = xy$ subject to $x^2 + 2y^2 = 1$

   Solution:

   Step 1: Find critical points on constraint
   $f(x,y) = xy,\ f_x = y,\ f_y = x$
   $g(x,y) = x^2 + 2y^2 = 1,\ g_x = 2x,\ g_y = 4y$

$$y = 2\lambda x \qquad (1)$$
$$x = 4\lambda y \qquad (2)$$
$$x^2 + 2y^2 = 1 \qquad (3)$$

   Substituting (1) into (2) gives $x = 4\lambda(2\lambda x)$, or $x(8\lambda^2 - 1) = 0 \Rightarrow x = 0$ or $\lambda = \pm 1\sqrt{8}$.
   For $x = 0$, (2) gives $y = 0$, but contradicts with (3). No solution in this case.
   For $\lambda = 1/\sqrt{8}$, (2) gives $x = \sqrt{2}y$. Substituting into (3) gives $2y^2 + 2y^2 = 1 \Rightarrow y = \pm 1/2$. So $x = \pm 1/\sqrt{2}$. Critical points at $\underline{(1/\sqrt{2}, 1/2)}$, $\underline{(-1/\sqrt{2}, -1/2)}$.
   For $\lambda = -1/\sqrt{8}$, (2) gives $x = -\sqrt{2}y$. Substituting into (3) gives $2y^2 + 2y^2 = 1 \Rightarrow y = \pm 1/2$. So $x = \mp 1/\sqrt{2}$. Critical points at $\underline{(-1/\sqrt{2}, 1/2)}$, $\underline{(1/\sqrt{2}, -1/2)}$.
   Step 2: Compare the values of $f(x,y)$
   $f(1/\sqrt{2}, 1/2) = 1/2\sqrt{2} \Leftarrow$ absolute max
   $f(-1/\sqrt{2}, -1/2) = 1/2\sqrt{2} \Leftarrow$ absolute max
   $f(-1/\sqrt{2}, 1/2) = -1/2\sqrt{2} \Leftarrow$ absolute min
   $f(1/\sqrt{2}, -1/2) = -1/2\sqrt{2} \Leftarrow$ absolute min
   Hence, $\boxed{\text{abs max at } f(1/\sqrt{2}, 1/2) = f(-1/\sqrt{2}, -1/2) = 1/2\sqrt{2},}$
   $\boxed{\text{abs min at } f(-1/\sqrt{2}, 1/2) = f(1/\sqrt{2}, -1/2) = -1/2\sqrt{2}.}$

   (b) $f(x,y,z) = xy + z^2$ subject to $y - x = 0$ and $x^2 + y^2 + z^2 = 4$

Solution:

Step 1: Find critical points on constraints

$f(x,y) = xy + z^2,\ f_x = y,\ f_y = x,\ f_z = 2z$

$g(x,y) = y - x = 0,\ g_x = -1,\ g_y = 1,\ g_z = 0$

$h(x,y) = x^2 + y^2 + z^2 = 4,\ h_x = 2x,\ h_y = 2y,\ h_z = 2z$

$$
\begin{aligned}
y &= -\lambda + 2\mu x && (1)\\
x &= \lambda + 2\mu y && (2)\\
2z &= 2\mu z && (3)\\
y - x &= 0 && (4)\\
x^2 + y^2 + z^2 &= 4 && (5)
\end{aligned}
$$

(4) gives $y = x$. Substitute into (1) and (2)

$$
\begin{aligned}
x &= -\lambda + 2\mu x && (1a)\\
x &= \lambda + 2\mu x && (2a)
\end{aligned}
$$

$(1a) - (2a)$ gives $\lambda = 0$. (1) and (2) becomes

$$
\begin{aligned}
x &= 2\mu x && (1b)\\
y &= 2\mu y && (2b)
\end{aligned}
$$

$(1b)$ and $(2b)$ gives either $x = y = 0$ or $\mu = 1/2$.

For $\boldsymbol{x = y = 0}$, (5) gives $z = \pm 2$, and (3) gives $\mu = 1$. Critical points at $\underline{(0,0,2)}$ and $\underline{(0,0,-2)}$

For $\boldsymbol{\mu = 1/2}$, (3) gives $z = 0$. (5) becomes $x^2 + x^2 = 4 \Rightarrow x = \pm\sqrt{2}$, then $y = \pm\sqrt{2}$. Critical points at $\underline{(\sqrt{2}, \sqrt{2}, 0)}$ and $\underline{(-\sqrt{2}, -\sqrt{2}, 0)}$

Step 2: Compare the values of $f(x,y)$

$f(0,0,2) = 4 \Leftarrow$ absolute max

$f(0,0,-2) = 4 \Leftarrow$ absolute max

$f(\sqrt{2}, \sqrt{2}, 0) = 2 \Leftarrow$ absolute min

$f(-\sqrt{2}, -\sqrt{2}, 0) = 2 \Leftarrow$ absolute min

Hence, $\boxed{\text{absolute max at } f(0,0,2) = f(0,0,-2) = 4,}$

$\boxed{\text{absolute min at } f(\sqrt{2}, \sqrt{2}, 0) = f(-\sqrt{2}, -\sqrt{2}, 0) = 2.}$