Math 253 Homework assignment 6

1. Consider the integral $\int_R xy^2 dA$, where A is the rectangle $[0,1] \times [0,1]$.

   (a) Calculate the Riemann sum corresponding to this integral, with the subdivision corresponding to $\Delta x = \Delta y = 0.2$ and using the centre of each small rectangle as the sample point (x_{ij}^*, y_{ij}^*).

   Solution: The Riemann sum has 25 terms: $((.1)(.1)^2 + (.1)(.3)^2 + (.1)(.5)^2 + (.1)(.7)^2 + (.1)(.9)^2 + (.3)(.1)^2 + (.3)(.3)^2 + \ldots)(.2)(.2) = \boxed{0.16500}$.

   (b) Using an iterated integral, calculate the value exactly.

   Solution: $\displaystyle\int_0^1 \int_0^1 xy^2 dy dx = \int_0^1 \left[xy^3/3\right]_{y=0}^1 dx = \frac{1}{3}\int_0^1 x dx =$
   $\displaystyle= \frac{1}{3}\left[x^2/2\right]_0^1 = \boxed{\frac{1}{6} = 0.16666\ldots}$

2. Find the volume of the solid bounded by the planes $x = 1$, $x = 2$, $y = 0$, $y = \pi/2$, $z = 0$ and the surface $z = x\cos y$.

   Solution: Vol $\displaystyle= \int_0^{\pi/2}\int_1^2 x\cos y dx dy = \int_0^{\pi/2}\left[\frac{x^2\cos y}{2}\right]_{x=1}^2 dy = \frac{3}{2}\int_0^{\pi/2}\cos y dy =$
   $\displaystyle\frac{3}{2}\left[\sin y\right]_0^{\pi/2} = \boxed{\frac{3}{2}}$

3. Calculate $\displaystyle\iint_R \sqrt{x+y}dA$, where $R = [0,1]\times[0,3]$.

   Solution: $\displaystyle\iint_R \sqrt{x+y}dA = \int_0^3\int_0^1 \sqrt{x+y}dx dy = \int_0^3\left[\frac{2}{3}(x+y)^{3/2}\right]_{x=0}^1 dy =$
   $\displaystyle= \frac{2}{3}\int_0^3((1+y)^{3/2} - y^{3/2})dy = \frac{4}{15}\left[(1+y)^{5/2} - y^{5/2}\right]_0^3 = \frac{4}{15}[(4^{5/2} - 3^{5/2}) - 1] =$
   $\displaystyle\boxed{\frac{4}{15}[31 - 9\sqrt{3}]}$

4. Find $\displaystyle\iint_R (x^2+y^2)dA$, where R is the rectangle $0 \le x \le a, \quad 0 \le y \le b$.

   Solution: $\displaystyle\iint_R (x^2+y^2)dA = \int_0^a\int_0^b (x^2+y^2)dy dx = \int_0^a\left[x^2y + \frac{y^3}{3}\right]_{y=0}^b dx =$
   $\displaystyle= \int_0^a (bx^2 + \frac{b^3}{3})dx = \left[\frac{bx^3}{3} + \frac{b^3x}{3}\right]_0^a = \boxed{\frac{a^3b + ab^3}{3}}$

5. Calculate the iterated integral $\displaystyle\int_0^\pi\int_{-x}^x \cos y dy dx$.

   Solution: $\displaystyle\int_0^\pi\int_{-x}^x \cos y dy dx = \int_0^\pi \left[\sin y\right]_{y=-x}^x = 2\int_0^\pi \sin x dx = -2\left[\cos x\right]_0^\pi = \boxed{4}$.

6. Find the volume under the surface $z = \dfrac{1}{x+y}$ and above the region in the xy-plane bounded by $x = 1$, $x = 2, y = 0$ and $y = x$.

Solution: Vol $= \displaystyle\int_1^2 \int_0^x \frac{1}{x+y} dydx = \int_1^2 [\ln(x+y)]_{y=0}^{y=x} = \int_1^2 (\ln(2x) - \ln(x))dx = \int_1^2 \ln(2)dx = \boxed{\ln 2}$

7. Using a double integral, calculate the volume of the tetrahedron in the first quadrant bounded by the coordinate planes and the plane which intersects the x- y- and z-axes at a, b and c, respectively, where a, b, c are positive numbers.

   Solution: The plane has equation $\dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c} = 1$, and it intersects the xy-plane in the triangle bounded by the axes and the line $\dfrac{x}{a} + \dfrac{y}{b} = 1$, or $y = b(1 - \frac{b}{a}x)$. So we may compute the volume as

   $$\int_0^a \int_0^{b(1-\frac{x}{a})} c\left(1 - \frac{x}{a} - \frac{y}{b}\right) dydx = c\int_0^a \left[y - \frac{xy}{a} - \frac{y^2}{2b}\right]_{y=0}^{b(1-\frac{x}{a})} dx =$$
   $$bc\int_0^a \left(1 - \frac{x}{a} - \frac{x}{a}(1-\frac{x}{a}) - \frac{1}{2}(1-\frac{x}{a})^2\right) dx = bc\int_0^a \left(\frac{1}{2} - \frac{x}{a} + \frac{x^2}{2a^2}\right) dx =$$
   $$bc\left[\frac{x}{2} - \frac{x^2}{2a} + \frac{x^3}{6a^2}\right]_0^a = \boxed{\frac{abc}{6}}$$

8. Calculate the integral $I = \displaystyle\int_0^1 \int_{\sqrt{x}}^1 e^{y^3} dydx$.

   Solution: There is no nice expression for the antiderivative of e^{y^3}, which by convention means $e^{(y^3)}$, so we solve this problem by reversing the order of integration. Notice that the region of integration is the set in $\mathbb{R}^2$ defined by the inequalities $0 \le x \le 1$ and $\sqrt{x} \le y \le 1$, or in other words, the region bounded by the y-axis, the line $y = 1$ and the curve $x = y^2$. Thus we can calculate the double integral with the order of integration reversed:

   $$I = \int_0^1 \int_0^{y^2} e^{y^3} dxdy = \int_0^1 y^2 e^{y^3} dy = \left[\frac{e^{y^3}}{3}\right]_0^1 = \boxed{\frac{e-1}{3}}$$