SOLUTIONS TO ASSIGNMENT #7

1. The iterated integral $I = \int_{x=0}^{x=1} \left(\int_{y=0}^{y=\sqrt{x}} \sin\left(\frac{\pi(y^3-3y)}{2}\right) dy \right) dx$ is equal to the double integral $\int\int_R \sin\left(\frac{\pi(y^3-3y)}{2}\right) dA$ for a region$R$ in the x, y plane.

   (a) Sketch R.

   (b) Write the integral with the order of integration reversed.

   (c) Compute I.

   Solution:

   (a) See diagram at the end.

   (b) $I = \int_{y=0}^{y=1} \left(\int_{x=y^2}^{x=1} \sin\left(\frac{\pi(y^3-3y)}{2}\right) dx \right) dy.$

   (c)

$$I \;=\; \int_{y=0}^{y=1} \sin\left(\frac{\pi(y^3-3y)}{2}\right)(1-y^2)dy = \frac{2}{3\pi}\cos\left(\frac{\pi(y^3-3y)}{2}\right)\Big|_0^1 = -\frac{4}{3\pi}$$

2. Let D be the region bounded by $y = x$ and $y = 6 - x^2$.

   (a) Sketch D.

   (b) Find $\int\int_D x^2 dA$.

   Solution:

   (a) See the diagram at the end. Note that $6 - x^2 = x \Longleftrightarrow x = -3, 2$.

   (b)

$$\begin{aligned}\int\int_D x^2 dA &= \int_{x=-3}^{x=2} dx \int_{y=x}^{y=6-x^2} x^2 dy = \int_{-3}^{2} x^2(6-x^2-x)dx \\ &= -\frac{x^5}{5}\Big|_{-3}^{2} - \frac{x^4}{4}\Big|_{-3}^{2} + 2x^3\Big|_{-3}^{2} = \frac{125}{4}\end{aligned}$$

3. Let D be the region, described in polar coordinates by, $0 \le \theta \le \pi, 0 \le r \le 1 + \cos\theta$.

   (a) Sketch D.

   (b) Compute the area of D.

   (c) Find the average value of distances of points in D from the origin.

Solution:

(a) See the diagram at the end.

(b) The area of D is

$$\begin{aligned} A &= \int_{\theta=0}^{\theta=\pi}\int_{r=0}^{r=1+\cos\theta} r\,dr\,d\theta = \int_0^\pi \frac{(1+\cos\theta)^2}{2}d\theta \\ &= \frac{1}{2}\int_0^\pi (1+2\cos\theta+\cos^2\theta)d\theta = \frac{1}{2}(\pi+\pi/2) = \frac{3\pi}{4} \end{aligned}$$

(c) By definition, the average value of a function $f(x,y)$ over a domain D is

$$\text{average value} = \frac{1}{\text{area}(D)}\int\int_D f(x,y)dxdy.$$

In this case we have

$$\begin{aligned} \text{average value} &= \frac{4}{3\pi}\int_{\theta=0}^{\theta=\pi}\int_{r=0}^{r=1+\cos\theta} r^2dr = \frac{4}{3\pi}\int_{\theta=0}^{\theta=\pi}\frac{(1+\cos\theta)^3}{3}d\theta \\ &= \frac{4}{9\pi}\int_{\theta=0}^{\theta=\pi}(1+3\cos\theta+3\cos^2\theta+\cos^3\theta)d\theta \\ &= \frac{4}{9\pi}(\pi+3\pi/2) = \frac{10}{9} \end{aligned}$$

4. Determine the following integrals:

(a) $\int\int_D (|x|+|y|)dA$, where D is the region $x^2+y^2\le a^2$ and a is a positive constant.

(b) $\int\int_T \sqrt{a^2-x^2}dA$, where T is the triangle with vertices $(0,0),(a,0),(a,a)$.

(c) $\int\int_D \frac{1}{x^2+y^2}dA$, where D is the region in the first quadrant bounded by

$$y=0, y=x, x^2+y^2=1/4, x^2+y^2=1.$$

(d) $\int\int_R \left(\sin xy + x^2 - y^2 + 3\right)dxdy$, where R is the region inside the circle $x^2+y^2 = a^2$ and outside the circle $x^2+y^2=b^2$, and a,b are constants satisfying $0<b<a$.

Solution:

(a)

$$\begin{aligned} \int\int_D(|x|+|y|)dA &= 4\int_{\theta=0}^{\theta=\pi/2}d\theta\int_{r=0}^{r=a}(r\cos\theta+r\sin\theta)r\,dr \\ &= \frac{4a^3}{3}\int_{\theta=0}^{\theta=\pi/2}(\cos\theta+\sin\theta)d\theta = \frac{8a^3}{3} \end{aligned}$$

(b)

$$\int\int_T \sqrt{a^2-x^2}dA = \int_{x=0}^{x=a} dx \int_{y=0}^{y=x} \sqrt{a^2-x^2}dy = \int_{x=0}^{x=a} x\sqrt{a^2-x^2}dx$$
$$= -\frac{1}{3}(a^2-x^2)^{3/2} \Big|_0^a = \frac{a^3}{3}.$$

(c) $\displaystyle\int\int_D \frac{1}{x^2+y^2}dA = \int_{\theta=0}^{\theta=\pi/4}\int_{r=1/2}^{r=1} \frac{1}{r}drd\theta = \frac{\pi \ln 2}{4}.$

(d) By symmetry $\displaystyle\int\int_R \sin xydxdy = 0$ and $\displaystyle\int\int_R x^2dxdy = \int\int_R y^2dxdy$, and therefore $\displaystyle\int\int_R \big(\sin xy + x^2 - y^2 + 3\big)\,dxdy = 3\text{area } R = 3\pi(a^2-b^2).$

5. Find the volume above the x, y plane, below the surface $z = e^{-(x^2+y^2)}$ and inside the cylinder $x^2+y^2=4$.

Solution: The volume is $\displaystyle V = \int_{\theta=0}^{\theta=2\pi} d\theta \int_{r=0}^{r=2} e^{-r^2}rdr = 2\pi\frac{e^{-r^2}}{-2}\Big|_{r=0}^{r=2} = \pi(1-e^{-4}).$

6. Find the volume above the x, y plane and below the surface $z = e^{-(x^2+y^2)}$.

Solution: The volume is $\displaystyle V = \int_{\theta=0}^{\theta=2\pi} d\theta \int_{r=0}^{r=\infty} e^{-r^2}rdr = 2\pi\frac{e^{-r^2}}{-2}\Big|_{r=0}^{r=\infty} = \pi.$

7. The iterated integral $\displaystyle\int_{x=0}^{x=4}\left(\int_{y=\sqrt{x}}^{y=2} e^{y^3}dy\right)dx$ can be written in the form $\displaystyle\int\int_D e^{y^3}dA$ for a region D.

(a) Sketch D.

(b) Evaluate $\displaystyle\int_{x=0}^{x=4}\left(\int_{y=\sqrt{x}}^{y=2} e^{y^3}dy\right)dx.$

Solution:

(a) See the diagram at the end.

(b) $\displaystyle\int_{x=0}^{x=4}\left(\int_{y=\sqrt{x}}^{y=2} e^{y^3}dy\right)dx = \int_{y=0}^{y=2} dy \int_{x=0}^{x=y^2} e^{y^3}dx = \int_{y=0}^{y=2} y^2e^{y^3}dy = \frac{e^{y^3}}{3}\Big|_0^2 = \frac{e^8-1}{3}$

8. Compute the double integral $\int\int_D (x+y)\,dA$, where D is the domain that lies to the right of the y-axis and between the circles $x^2+y^2=1,\ x^2+y^2=4$.

Solution:

$$\int\int_D (x+y)\,dA = \int_{\theta=-\pi/2}^{\theta=\pi/2}\int_{r=1}^{r=2} (r\cos\theta + r\sin\theta)rdrd\theta$$
$$= \frac{7}{3}\int_{\theta=-\pi/2}^{\theta=\pi/2} (\cos\theta+\sin\theta)d\theta = \frac{14}{3}$$

9. Find the area that is common to the polar curves $r = \cos\theta,\ r = \sin\theta$.

   Solution: The area is

$$A = 2\int_{\theta=0}^{\theta=\pi/4}\int_{r=0}^{r=\sin\theta} r\,dr\,d\theta = \int_{\theta=0}^{\theta=\pi/4} \sin^2\theta\,d\theta = \pi/8$$

10. Find the area that is inside the polar curve $r = 4\sin\theta$ and outside the circle $r = 2$.

    Solution:

$$\begin{aligned} A &= \int_{\theta=\pi/6}^{\theta=5\pi/6}\int_{r=2}^{r=4\sin\theta} r\,dr\,d\theta = \int_{\theta=\pi/6}^{\theta=5\pi/6}(8\sin^2\theta - 2)d\theta \\ &= \int_{\theta=\pi/6}^{\theta=5\pi/6}(4(1-\cos 2\theta) - 2)d\theta = \int_{\theta=\pi/6}^{\theta=5\pi/6}(2 - 4\cos 2\theta)d\theta \\ &= \frac{4\pi}{3} - 2\sin 2\theta \Big|_{\theta=\pi/6}^{\theta=5\pi/6} = \frac{4\pi}{3} - 2\left(\sin\frac{5\pi}{3} - \sin\frac{2\pi}{3}\right) = \frac{4\pi}{3} + 2\sqrt{3} \end{aligned}$$

11. Find the volume that is above the cone $z = \sqrt{x^2+y^2}$ and below the sphere $x^2+y^2+z^2 = 1$.

    Solution:

$$\begin{aligned} V &= \int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=1/\sqrt{2}}(\sqrt{1-r^2} - r)r\,dr\,d\theta = 2\pi\int_{r=0}^{r=1/\sqrt{2}}(\sqrt{1-r^2} - r)r\,dr \\ &= 2\pi\left(-\frac{1}{3}(1-r^2)^{3/2}\Big|_{r=0}^{r=1/\sqrt{2}} - \frac{r^3}{3}\Big|_{r=0}^{r=1/\sqrt{2}}\right) = \frac{2\pi}{3}(1 - 1/\sqrt{2}) \end{aligned}$$

12. A cylindrical hole of radius a is drilled through a sphere of radius b ($a < b$). Find the volume of the solid that remains.

    Solution:

    The volume of the drilled out piece is

$$V = 2\int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=a}\sqrt{b^2-r^2}\,r\,dr\,d\theta = -\frac{4\pi}{3}(b^2-r^2)^{3/2}\Big|_{r=0}^{r=a} = \frac{4\pi}{3}\left(b^3 - (b^2-a^2)^{3/2}\right)$$

    Therefore the volume of the remaining piece is $\frac{4\pi}{3}\left(b^2-a^2\right)^{3/2}$.

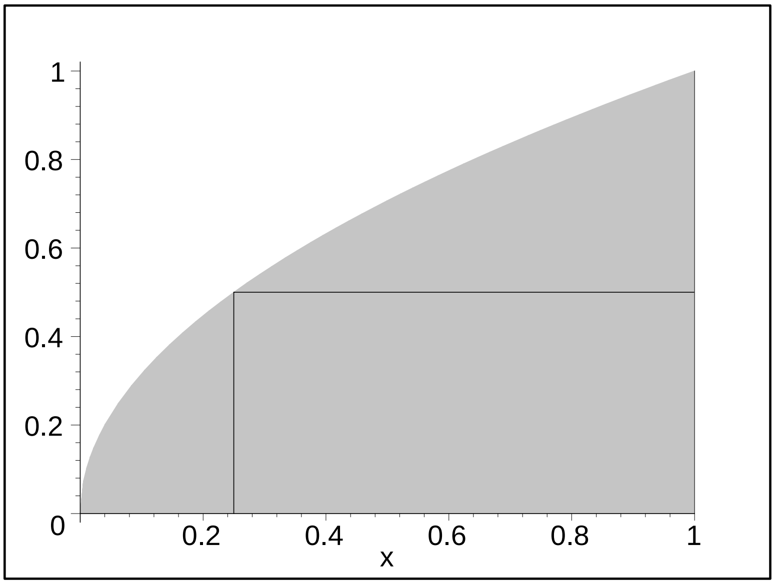

Figure 1: Question 1(a), $0 \leq x \leq 1, 0 \leq y \leq \sqrt{x}$

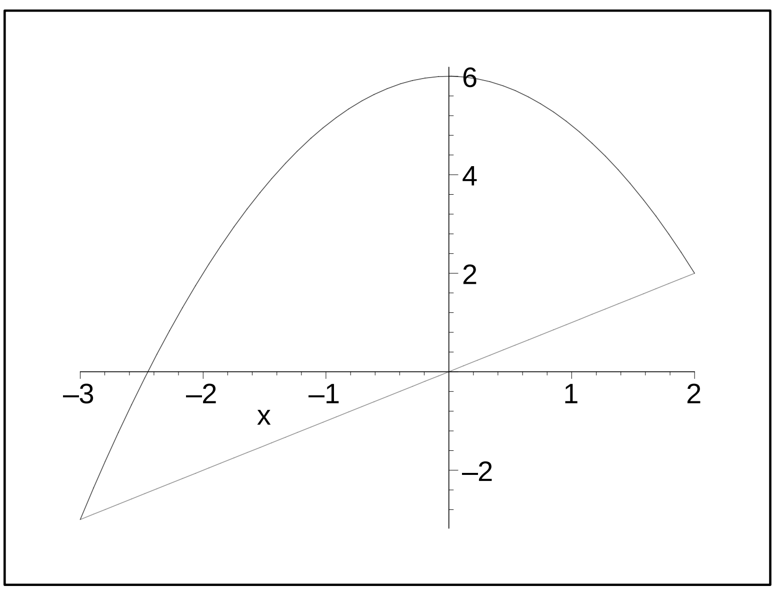

Figure 2: Question 2(a), the region bounded by $y = x, y = 6 - x^2$

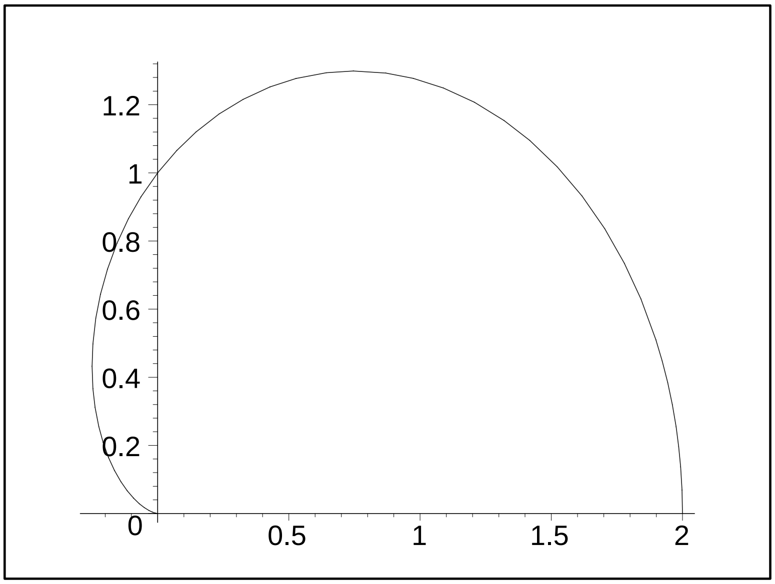

Figure 3: Question 3(a), $0 \le \theta \le \pi, 0 \le r \le 1 + \cos\theta$

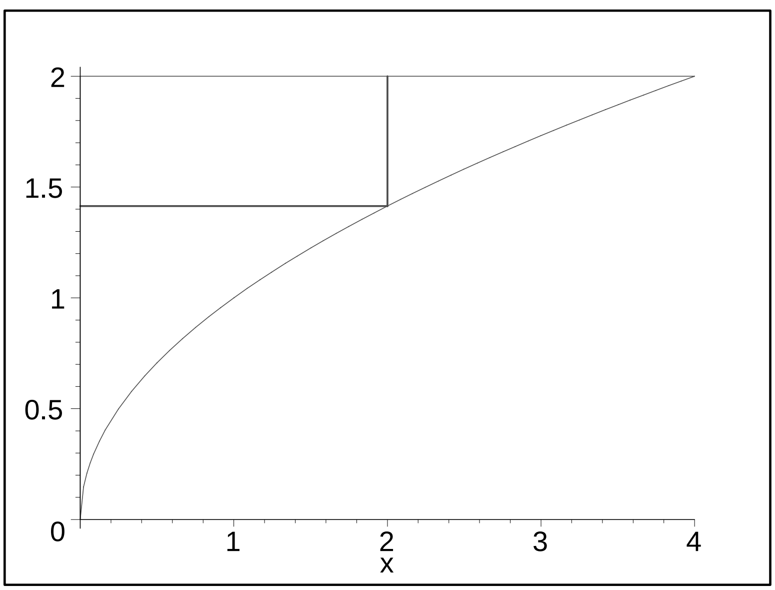

Figure 4: Question 7(a), $0 \le x \le 4, \sqrt{x} \le y \le 2$