Math 253 Homework assignment 8 Solutions

1. Find the mass and centre of mass of the lamina that occupies the region bounded by $y = e^x$, $y = 0$, $x = 0$ and $x = 1$ and having density $\rho(x, y) = y$.

Solution: Mass $= \displaystyle\int_0^1 \int_0^{e^x} y\, dydx = \frac{1}{2}\int_0^1 e^{2x}\, dx = \boxed{\frac{1}{4}(e^2 - 1)}$. Centre of mass $=$ $(\bar{x}, \bar{y})$, where

$$\bar{x} = \frac{1}{m}\int_0^1 \int_0^{e^x} xy\, dydx = \frac{1}{2m}\int_0^1 xe^{2x} dx = \frac{1}{2m}\left[\frac{xe^{2x}}{2} - \frac{e^{2x}}{4}\right]_0^1 = \boxed{\frac{e^2+1}{2(e^2-1)}}$$

(integrate by parts: $u = x, dv = e^{2x}dx$)

$$\bar{y} = \frac{1}{m}\int_0^1 \int_0^{e^x} y^2\, dydx = \frac{1}{3m}\int_0^1 e^{3x}\, dx = \frac{1}{9m}\left[e^{3x}\right]_0^1 = \boxed{\frac{4(e^3-1)}{9(e^2-1)}}$$

2. Find the moments of inertia I_x, I_y and I_0 for the lamina of problem 1.

Solution: $I_x = \displaystyle\int_0^1 \int_0^{e^x} y^3\, dydx = \frac{1}{4}\int_0^1 e^{4x} dx = \boxed{\frac{e^4-1}{16}}$

$$I_y = \int_0^1 \int_0^{e^x} x^2 y\, dydx = \frac{1}{2}\int_0^1 x^2 e^{2x}\, dx = \frac{1}{2}\left[e^{2x}(\frac{x^2}{2} - \frac{x}{2} + \frac{1}{4})\right]_0^1 = \boxed{\frac{e^2-1}{8}}.$$

$I_0 = I_x + I_y = \dfrac{e^4 + 2e^2 - 3}{16}$. (Here we used integration by parts twice)

3. A square lamina of constant density ρ occupies a square with vertices $(0, 0), (a, 0), (a, a)$ and $(0, a)$. Find the moments of inertia I_x, I_y and the radii of gyration $\bar{\bar{x}}$ and $\bar{\bar{y}}$.

Solution: $I_x = \rho \displaystyle\int_0^a \int_a y^2\, dydx = \boxed{\frac{\rho a^4}{3}}$ By symmetry $I_y = \boxed{\dfrac{\rho a^4}{3}}$ Since $m = \rho a^2 m$,

$\bar{\bar{x}} = \sqrt{\dfrac{I_y}{m}} = \boxed{\dfrac{a}{\sqrt{3}}} = \bar{\bar{y}}$, again by symmetry.

4. Use integration to verify the following formula for the area of a triangle in $\mathbb{R}^3$ with vertices at $(a, 0, 0), (0, b, 0)$ and $(0, 0, c)$, with a, b, c positive numbers:

$$\text{Area} = \frac{1}{2}\sqrt{a^2b^2 + b^2c^2 + a^2c^2}.$$

Solution: The triangle lies in the plane $x/a + y/b + z/c = 1$, on which we have $\partial z/\partial x = -c/a$ and $\partial z/\partial y = -c/b$. The triangle is the part of this plane that lies above the triangle in the xy-plane bounded by the two axes and the line $x/a+y/b = 1$, or $y = b - bx/a$. Therefore

$$\text{Area} = \int_0^a \int_0^{b-bx/a} \sqrt{1 + c^2/a^2 + c^2/b^2}\, dydx = \sqrt{1 + c^2/a^2 + c^2/b^2}\int_0^a (b - bx/a)\, dx =$$

$$= \sqrt{1 + c^2/a^2 + c^2/b^2}\left[bx - bx^2/2a\right]_0^a = \frac{ab}{2}\sqrt{1 + c^2/a^2 + c^2/b^2} = \frac{1}{2}\sqrt{a^2b^2 + b^2c^2 + a^2c^2}.$$

5. Find the area of the part of the cylinder $y^2 + z^2 = 9$ that lies above the rectangle with vertices $(0,0), (4,0), (4,2)$ and $(0,2)$.

   Solution: Since $z = \sqrt{9 - y^2}$ we have $\dfrac{\partial z}{\partial x} = 0$ and $\dfrac{\partial z}{\partial y} = \dfrac{y}{\sqrt{9-y^2}}$.

   Then Area $= \displaystyle\int_0^4 dx \int_0^2 \sqrt{1 + \frac{y^2}{9-y^2}}\, dy = (4)3 \int_0^2 \frac{dy}{\sqrt{9-y^2}} = \boxed{12 \arcsin(2/3)}$.

   Note that it can also be calculated by flattening out the cylinder into a rectangle of dimensions 4 by $3 \arcsin(2/3)$.

6. Find the area of that part of the hyperbolic paraboloid $z = x^2 - y^2$ that lies inside the cylinder $x^2 + y^2 = a^2$.

   Solution: Area $= \displaystyle\int\int_D \sqrt{1 + 4x^2 + 4y^2}\, dA$, where D is the disk of radius a in the xy plane. Using polar coordinates, we have

   Area $= \displaystyle\int_0^{2\pi} d\theta \int_0^a \sqrt{1+4r^2}\, r dr$ (let $u = 1 + 4r^2$)

   $$= (2\pi)\frac{1}{8}\int_1^{1+4a^2} u^{1/2} du = \frac{\pi}{4}\left[\frac{2}{3}u^{3/2}\right]_1^{1+4a^2} = \boxed{\frac{\pi}{6}((1+4a^2)^{3/2} - 1)}.$$

7. Find the area of the part of the sphere $x^2 + y^2 + z^2 = a^2$ that lies inside the cylinder $x^2 + y^2 = ax$.

   Solutions: The sphere has $z = \sqrt{a^2 - x^2 - y^2}$ so
   $\dfrac{\partial z}{\partial x} = \dfrac{-x}{\sqrt{a^2-x^2-y^2}}$ and $\dfrac{\partial z}{\partial y} = \dfrac{-y}{\sqrt{a^2-x^2-y^2}}$

   The cylinder has base a circle of radius a in the xy-plane, centered at $(x,y) = (a/2, 0)$. It has polar equation $r = a\cos\theta$. By symmetry, the area is twice the part over the semicircle in the first quadrant, with $0 \le \theta \le \pi/2$, which we will denote by D.

   $$\text{Area} = 2\iint_D \sqrt{1 + \frac{x^2}{a^2-x^2-y^2} + \frac{y^2}{a^2-x^2-y^2}}\, dA = 2a \iint_D \frac{dA}{\sqrt{a^2-x^2-y^2}} =$$

   $$= 2a\int_0^{\pi/2}\int_0^{a\cos\theta} \frac{r dr}{\sqrt{a^2-r^2}} d\theta = 2a \int_0^{\pi/2}\left[-\sqrt{a^2-r^2}\right]_{r=0}^{a\cos\theta} d\theta = 2a^2 \int_0^{\pi/2} (1 - \sin\theta) d\theta =$$

   $$= 2a^2 \left[\theta + \cos\theta\right]_0^{\pi/2} = \boxed{a^2(\pi - 2)}$$