SOLUTIONS TO HOMEWORK ASSIGNMENT #9, Math 253

1. For each of the following regions E, express the triple integral $\iiint_E f(x,y,z)\,dV$ as an iterated integral in cartesian coordinates.

   (a) E is the box $[0,2]\times[-1,1]\times[3,5]$;
   Solution:

$$\iiint_E f(x,y,z)\,dV = \boxed{\int_0^2\int_{-1}^1\int_3^5 f(x,y,z)\,dz\,dy\,dx}$$

   (b) E is the pyramid with vertices $(0,0,0)$, $(1,1,1)$, $(1,1,-1)$, $(-1,1,1)$, and $(-1,1,-1)$;
   Solution:

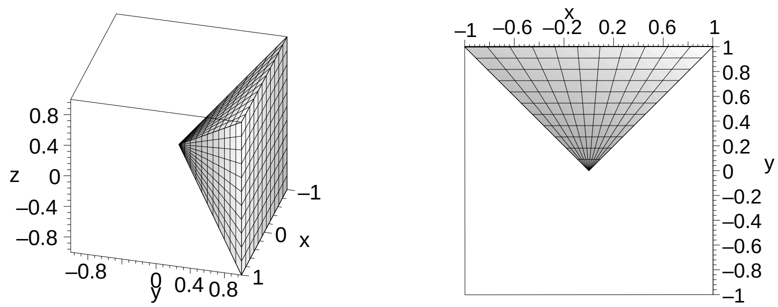

Figure 1: Q1(b): Left: The solid E; Right: The image of E on xy-plane

Top function: $z=y$ (plane passing through $(0,0,0)$, $(1,1,1)$, and $(-1,1,1)$)
Bottom function: $z=-y$ (plane passing through $(0,0,0)$, $(1,1,-1)$, and $(-1,1,-1)$)

$$\iiint_E f(x,y,z)\,dV = \iint_D\int_{-y}^{y} f(x,y,z)\,dz\,dA$$
$$= \boxed{\int_0^1\int_{-y}^{y}\int_{-y}^{y} f(x,y,z)\,dz\,dx\,dy}$$

   (c) E is the region in the first octant above the plane $y=z$ and bounded by the cylinder $x^2+z^2=1$.
   Solution:
   Left function: $y=0$
   Right function: $y=z$

$$\iiint_E f(x,y,z)\,dV = \iint_D\int_0^z f(x,y,z)\,dy\,dA$$
$$= \boxed{\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^z f(x,y,z)\,dy\,dz\,dx}$$

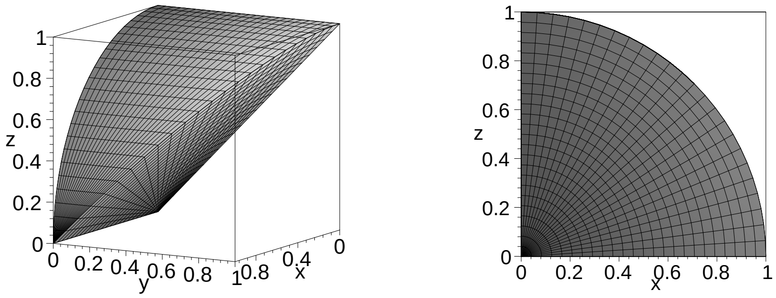

Figure 2: Q1(c): Left: The solid E; Right: The image of E on xz-plane

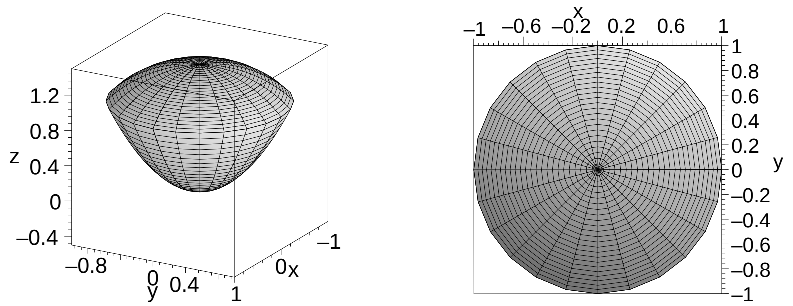

Figure 3: Q1(d): Left: The solid E; Right: The image of E on xy-plane

(d) E is the region inside the sphere $x^2+y^2+z^2=2$ and above the elliptic paraboloid $z=x^2+y^2$.

Solution:

Top function: $z=\sqrt{2-x^2-y^2}$
Bottom function: $z=x^2+y^2$
The boundary of image D on xy-plane is the intersection of top and bottom function, which is $x^2+y^2=1$

$$\iiint_E f(x,y,z)\,dV = \iint_D \int_{x^2+y^2}^{\sqrt{2-x^2-y^2}} f(x,y,z)\,dz\,dA$$
$$= \boxed{\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{x^2+y^2}^{\sqrt{2-x^2-y^2}} f(x,y,z)\,dz\,dy\,dx}$$

2. Consider the integral

$$\iiint_E f(x,y,z)\,dV = \int_{-2}^{2}\int_{x^2}^{4}\int_0^y f(x,y,z)\,dz\,dy\,dx$$

(a) Sketch the region E.

Solution:

(b) Write the other five iterated integrals which represent $\iiint_E f(x,y,z)\,dV$.

Solution:

If we project *E* onto *xy*-plane, then the top function is $z=y$, and the bottom function is $z=0$, as given in the question.

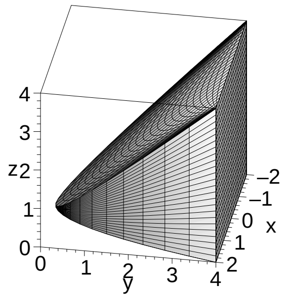

Figure 4: Q2(a): The solid E

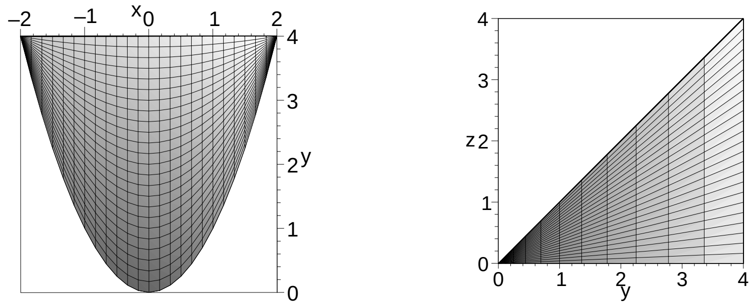

Figure 5: Q2(b): Left: The image of E on xy-plane; Right: The image of E on yz-plane

In the order of $dz\,dx\,dy$,

$$\iiint_E f(x,y,z)\,dV = \boxed{\int_0^4 \int_{-\sqrt{y}}^{\sqrt{y}} \int_0^y f(x,y,z)\,dz\,dx\,dy}$$

If we project E onto yz-plane, then the front function is $x=\sqrt{y}$, and back function is $x=-\sqrt{y}$.
In the order of $dx\,dy\,dz$,

$$\iiint_E f(x,y,z)\,dV = \boxed{\int_0^4 \int_z^4 \int_{-\sqrt{y}}^{\sqrt{y}} f(x,y,z)\,dx\,dy\,dz}$$

In the order of $dx\,dz\,dy$,

$$\iiint_E f(x,y,z)\,dV = \boxed{\int_0^4 \int_0^y \int_{-\sqrt{y}}^{\sqrt{y}} f(x,y,z)\,dx\,dy\,dz}$$

If we project E onto xz-plane, we can think of it as a parabolic cylinder $E_1 : \{(x,y,z) : x^2 < y < 4, -2 < x < 2, 0 < x < 4\}$ (whole body) with the solid $E_2 : \{(x,y,z) : x^2 < y < z, x^2 < z < 4, -2 < x < 2\}$ (the transparent part) removed. So

$$\iiint_E f(x,y,z)\,dV = \iiint_{E_1} f(x,y,z)\,dV - \iiint_{E_2} f(x,y,z)\,dV$$

In the order of $dy\,dx\,dz$,

$$\iiint_{E_1} f(x,y,z)\,dV = \int_0^4 \int_{-2}^2 \int_{x^2}^4 f(x,y,z)\,dy\,dx\,dz$$

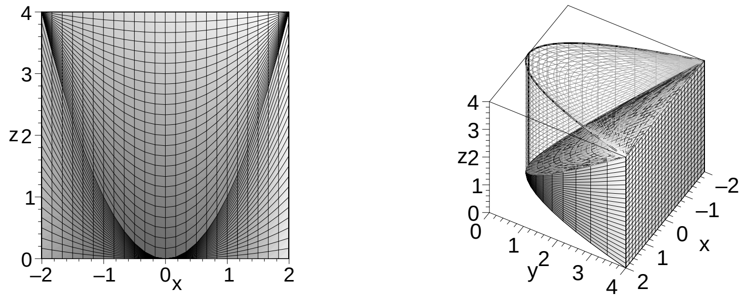

Figure 6: Q2(b): Left: The image of E on xz-plane; Right: solid: E, transparent: E_2, solid + transparent: E_1

$$\iiint_{E_2} f(x,y,z)\,dV = \int_0^4 \int_{-\sqrt{z}}^{\sqrt{z}} \int_{x^2}^4 f(x,y,z)\,dy\,dx\,dz$$

$$\Rightarrow \iiint_E f(x,y,z)\,dV = \boxed{\int_0^4 \int_{-2}^{2} \int_{x^2}^4 f(x,y,z)\,dy\,dx\,dz - \int_0^4 \int_{-\sqrt{z}}^{\sqrt{z}} \int_{x^2}^4 f(x,y,z)\,dy\,dx\,dz}$$

In the order of $dy\,dz\,dx$,

$$\iiint_{E_1} f(x,y,z)\,dV = \int_{-2}^2 \int_0^4 \int_{x^2}^4 f(x,y,z)\,dy\,dz\,dx$$

$$\iiint_{E_2} f(x,y,z)\,dV = \int_{-2}^2 \int_{x^2}^4 \int_{x^2}^4 f(x,y,z)\,dy\,dz\,dx$$

$$\Rightarrow \iiint_E f(x,y,z)\,dV = \boxed{\int_{-2}^2 \int_0^4 \int_{x^2}^4 f(x,y,z)\,dy\,dz\,dx - \int_{-2}^2 \int_{x^2}^4 \int_{x^2}^4 f(x,y,z)\,dy\,dz\,dx}$$

(c) Find the volume of E.

Solution:

The volume of E is given by

$$\iiint_E dV = \int_{-2}^2 \int_{x^2}^4 \int_0^y dz\,dy\,dx = \int_{-2}^2 \int_{x^2}^4 y\,dy\,dx = \int_{-2}^2 \left[8 - \frac{1}{2}x^4\right] dx = \boxed{\frac{128}{5}}$$

(d) Find the centre of mass of E when the density of E is constant.

Solution:

Let the constant density be $\rho(x,y,z) = c$. Then

$$m = \iiint_E c\,dV = c(\text{volume of } E) = \frac{128c}{5}$$

$$\bar{x} = \frac{1}{m} \iiint_E cx\,dV = \frac{1}{m} \int_{-2}^2 \int_{x^2}^4 \int_0^y cx\,dz\,dy\,dx = \frac{5}{128c}(0) = 0$$

$$\bar{y} = \frac{1}{m} \iiint_E cy\,dV = \frac{1}{m} \int_{-2}^2 \int_{x^2}^4 \int_0^y cy\,dz\,dy\,dx = \frac{5}{128c}\frac{512}{7} = \frac{20}{7}$$

$$\bar{z} = \frac{1}{m} \iiint_E cz\,dV = \frac{1}{m} \int_{-2}^2 \int_{x^2}^4 \int_0^y cz\,dz\,dy\,dx = \frac{5}{128c}\frac{256}{7} = \frac{10}{7}$$

The centre of mass is $\boxed{(0, 20/7, 10/7)}$.

3. Let E be the solid bounded by $z = \sqrt{x^2 + y^2}$ and $z = \sqrt{1 - x^2 - y^2}$,

   (a) Use cylindrical coordinates to find the volume of E.
   Solution:

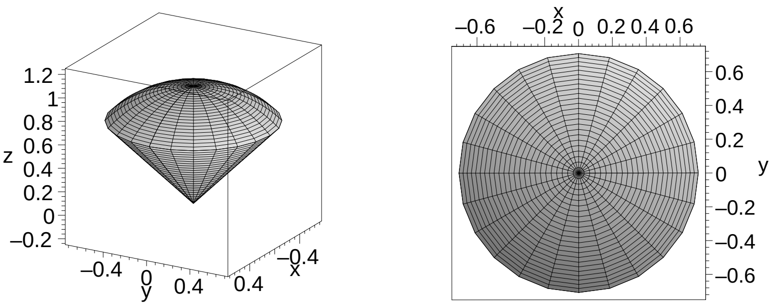

Figure 7: Q3: Left: The solid E; Right: The image of E on xy-plane

Top function: $z = \sqrt{1 - x^2 - y^2} = \sqrt{1 - r^2}$
Bottom function: $z = \sqrt{x^2 + y^2} = r$

$$V = \iiint_E dV = \iint_D \int_r^{\sqrt{1-r^2}} dz\, dA$$

D is the circular image on xy-plane. The boundary of D is given by the intersection of the top and bottom function, which is $x^2 + y^2 = 1/2$, or $r = \sqrt{1/2}$.

$$V = \int_0^{2\pi} \int_0^{\sqrt{1/2}} \int_r^{\sqrt{1-r^2}} r\, dz\, dr\, d\theta = \boxed{\frac{\pi}{3}(2 - \sqrt{2})}$$

   (b) Use spherical coordinates to find the volume of E.
   Solution:
   Write the function in spherical coordinates: $z = \sqrt{x^2 + y^2} \Rightarrow \phi = \pi/4$; $z = \sqrt{1 - x^2 - y^2} \Rightarrow \rho = 1$. So

$$V = \iiint_E dV = \int_0^{\pi/4} \int_0^{2\pi} \int_0^1 \rho^2 \sin\phi\, d\rho\, d\theta\, d\phi = \boxed{\frac{\pi}{3}(2 - \sqrt{2})}$$

4. Find the volume of the solid above the xy-plane, under the surface $z = 1 - x^2 - y^2$, and within the wedge $x \le y \le \sqrt{3}x$.

   Solution:

   Top function: $z = 1 - x^2 - y^2$
   Bottom function: $z = 0$

$$V = \iiint_E dV = \iint_D \int_0^{1-x^2-y^2} dz\, dA$$

Since on the xy-plane, $0 = 1 - x^2 - y^2 \Rightarrow r = 1$, $y = x \Rightarrow \theta = \pi/4$ and $y = \sqrt{3}x \Rightarrow \theta = \pi/3$, D is the region on the xy-plane defined by $D : \{(r, \theta) : 0 < r < 1, \pi/4 < \theta < \pi/3\}$. So

$$V = \int_{\pi/4}^{\pi/3} \int_0^1 \int_0^{1-r^2} r\, dz\, dr\, d\theta = \boxed{\frac{\pi}{48}}$$

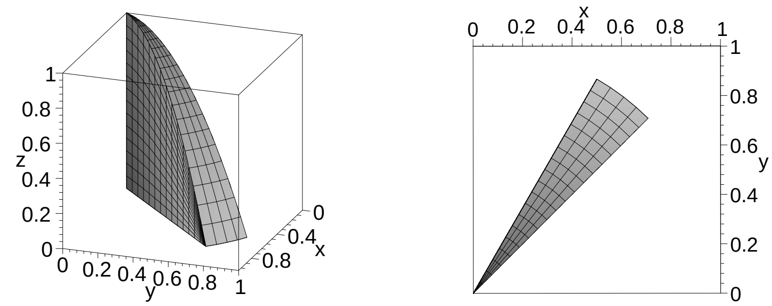

Figure 8: Q4: Left: The solid E; Right: The image of E on xy-plane

5. Find the volume remaining in a sphere of radius a after a hole of radius b is drilled through the centre. Assume $0 < b < a$.

   Solution:

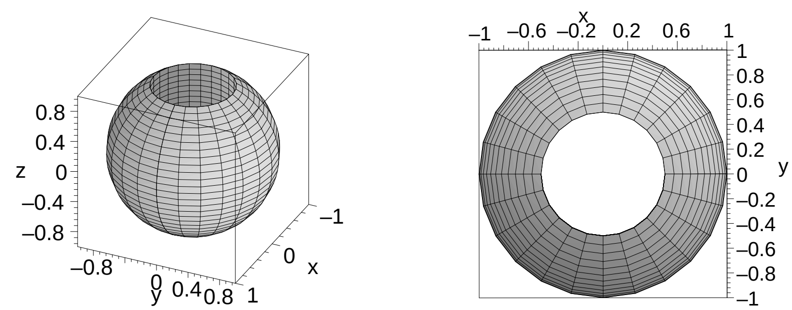

Figure 9: Q5: Left: The solid E; Right: The image of E on xy-plane

Sphere of radius a: $x^2 + y^2 + z^2 = a^2$
Top function: $z = \sqrt{a^2 - x^2 - y^2} = \sqrt{a^2 - r^2}$
Bottom function: $z = -\sqrt{a^2 - x^2 - y^2} = -\sqrt{a^2 - r^2}$
Cylindrical hole of radius b: $x^2 + y^2 = b^2 \Rightarrow r = b$

$$V = \iiint_E dV = \int_0^{2\pi} \int_b^a \int_{-\sqrt{a^2-r^2}}^{\sqrt{a^2-r^2}} r\, dz\, dr\, d\theta = \boxed{\frac{4}{3}\pi \left(a^2 - b^2\right)^{3/2}}$$

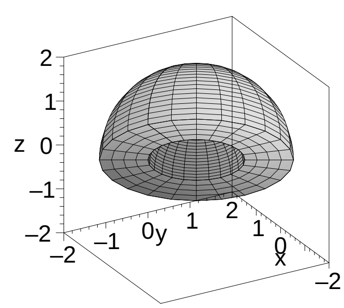

Figure 10: Q6: The solid E

6. Find the mass of the solid between the spheres $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 = 4$ above the xy-plane when the density is $\rho(x, y, z) = z$.

Solution:

Outer function: $x^2 + y^2 + z^2 = 4 \Rightarrow \rho = 2$
Inner function: $x^2 + y^2 + z^2 = 1 \Rightarrow \rho = 1$
xy-plane: $\phi = \pi/2$

$$\begin{aligned} m = \iiint_E z\,dV &= \int_0^{\pi/2} \int_0^{2\pi} \int_1^2 (\rho\cos\phi)\rho^2 \sin\phi \,d\rho\,d\theta\,d\phi \\ &= \int_0^{\pi/2} \int_0^{2\pi} \int_1^2 \rho^3 \cos\phi \sin\phi \,d\rho\,d\theta\,d\phi \\ &= \boxed{\frac{15}{4}\pi} \end{aligned}$$