Solutions to Quiz #1 (version 1), Math 253

1. Given the vectors $\vec{u} = \vec{i} - 2\vec{j}$ and $\vec{v} = \vec{i} + \vec{j} - 3\vec{k}$, calculate:

   $\vec{u} \cdot \vec{v} =$ **Solution:** 1 - 2 = -1.

   $\vec{u} \times \vec{v} =$ **Solution:** $6\vec{i} + 3\vec{j} + 3\vec{k}$ by routine calculation.

2. True or false? Justify your answer: $(\vec{a} \times \vec{b}) \times \vec{c} = \vec{a} \times (\vec{b} \times \vec{c})$.

   Solution: This formula is FALSE in general. For example if $\vec{a} = \vec{i}$ and $\vec{b} = \vec{c} = \vec{j}$ we obtain different results for the two sides of the formula:

   $$(\vec{a} \times \vec{b}) \times \vec{c} = \vec{k} \times \vec{j} = -\vec{i}$$

   $$\vec{a} \times (\vec{b} \times \vec{c}) = \vec{i} \times \vec{0} = \vec{0}.$$

   Note: a counterexample was necessary to justify the answer, and obtain full credit.

3. Consider the three points $O(0,0,0)$, $A(1,0,0)$ and $B(0,1,0)$. Find the point $C(x,y,z)$ whose coordinates are all positive and such that: the angles satisfy

   $$\angle AOC = \angle BOC = \pi/3$$

   and the area of the triangle $\triangle AOC$ is equal to 1.

   Solution: The angle conditions give $1/2 = \cos(\pi/3) = \frac{\vec{OA} \cdot \vec{OC}}{|\vec{OC}||\vec{OC}|} = \frac{x}{\sqrt{x^2+y^2+z^2}}$ and similarly $\frac{y}{\sqrt{x^2+y^2+z^2}} = 1/2$. Therefore $x = y$ (one could also have deduced this by symmetry), and we write $C = (x, x, z)$.

   From the area condition we have $1 = \frac{1}{2}|\vec{OA} \times \vec{OC}| = \frac{1}{2}|\langle 0, z, x\rangle| = \frac{1}{2}\sqrt{x^2+z^2}$. This gives, after squaring,

   $$x^2 + z^2 = 4.$$

   From the earlier calculation, we have $4x^2 = x^2 + y^2 + z^2 = 2x^2 + z^2$, or simply

   $$2x^2 = z^2.$$

   Now the solution is easily seen to be

   $$x = \frac{2}{\sqrt{3}} = y, \quad z = \frac{2\sqrt{2}}{\sqrt{3}}.$$