SOLUTIONS TO QUIZ #5, Math 253

1. Find the mass and center of mass of the lamina that occupies the region bounded by the lines $y = x, y = -x, x = 1$ if the density is $\rho(x, y) = x$.

   Solution: The total mass is

$$\begin{aligned} m &= \int\int_D \rho(x,y)dA \text{ (where } D \text{ is given by } 0 \le x \le 1, -x \le y \le x) \\ &= \int_{x=0}^{x=1} \int_{y=-x}^{y=x} x dy dx = \int_{x=0}^{x=1} 2x^2 = \frac{2}{3}. \end{aligned}$$

   The moment about the y-axis is

$$M_y = \int\int_D x\rho(x,y)dA = \int_{x=0}^{x=1} \int_{y=-x}^{y=x} x^2 dy dx = \int_{x=0}^{x=1} 2x^3 dx = \frac{1}{2}.$$

   Therefore the x coordinate for the center of mass is $\bar{x} = \dfrac{M_y}{m} = \dfrac{3}{4}$. By symmetry the y-coordinate is $\bar{y} = 0$.

2. Find the surface area of that part of the sphere $x^2 + y^2 + z^2 = a^2$ that lies above the plane $z = b$, where $0 < b < a$.

   Solution:

   The formula for surface area of a surface $z = f(x, y)$ over a domain D in the plane is $S = \displaystyle\int\int_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial x}\right)^2} dA$. In this case $z = \sqrt{a^2 - x^2 - y^2}$ and D is determined by the inequality: $x^2 + y^2 \le a^2 - b^2$. Now $\dfrac{\partial z}{\partial x} = \dfrac{-x}{\sqrt{a^2 - x^2 - y^2}}, \dfrac{\partial z}{\partial y} = \dfrac{-y}{\sqrt{a^2 - x^2 - y^2}}$ and therefore

$$\begin{aligned} S &= \int\int_D \sqrt{1 + \frac{x^2}{a^2 - x^2 - y^2} + \frac{y^2}{a^2 - x^2 - y^2}} dA \\ &= \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=\sqrt{a^2-b^2}} \sqrt{\frac{a^2}{a^2 - r^2}} r dr d\theta \\ &= 2\pi a \int_{r=0}^{r=\sqrt{a^2-b^2}} \frac{r}{\sqrt{a^2 - r^2}} dr \\ &= -2\pi a (a^2 - r^2)^{1/2} \Big|_{r=0}^{r=\sqrt{a^2-b^2}} = 2\pi a (a - b) \end{aligned}$$