Solutions to QUIZ #6, Math 253

1. Consider the tetrahedron E with vertices at $(0,0,0),(0,1,0),(1,1,0)$ and $(0,0,2)$. If $f(x,y,z)$ is a function defined on E, express the integral $\iiint_E f(x,y,z)\,dV$ as an iterated integral, with two different orders of integration by filling in the limits of integration:

   Solution:

$$\iiint_E f(x,y,z)\,dV = \int_0^2 \int_0^{1-\frac{z}{2}} \int_0^y f(x,y,z)\,dxdydz = \int_0^1 \int_0^{2-2x} \int_x^{1-\frac{z}{2}} f(x,y,z)\,dydzdx$$

   Explanation: The boundary planes of E are $z=0$, $x=0$, $x=y$, and $2y+z=2$. For the first integral, E projects onto the triangle in the yz-plane bounded by the axes and the line $2y+z=2$, and for each point of that triangle, x ranges from 0 to the plane $x=y$, giving the limits on the third integral sign.

   For the second integral, note that the projection of E onto the xy plane is also a triangle with sides along the axes and the line $x+\frac{z}{2}=1$. This gives the limits of integration on the first two integral signs. For each fixed x,z in that triangle, y ranges from the plane $y=x$ to the plane $y+\frac{z}{2}=1$, giving the limits on the last integral.

2. A spherical planet, whose radius is taken to be 1 unit, has density varying linearly with depth, so that the density equals A at the centre and B at the surface. Find the total mass of the planet, and its average density.

   Solution: In this problem, density (which we will denote by δ) is a function of the distance ρ from the origin. It has the form

$$\delta(\rho) = (1-\rho)A + \rho B = A + (B-A)\rho$$

   [Here it is unfortunate that ρ was used by our author to denote density in other contexts!]

   Mass $= \int_0^{2\pi}\int_0^{\pi}\int_0^1 \delta(\rho)\rho^2 \sin\phi\,d\rho d\phi d\theta = 2\pi \int_0^{\pi} \left[A\frac{\rho^3}{3} + (B-A)\frac{\rho^4}{4}\right]_{\rho=0}^1 \sin\phi\,d\phi =$ $2\pi(\frac{A}{12}+\frac{B}{4})\,[-\cos\phi]_0^{\pi} = \boxed{\pi(\frac{A}{3}+B)}$

   Average density $= \dfrac{mass}{volume} = \pi(B+\dfrac{A}{3})/(\dfrac{4\pi}{3}) = \boxed{\frac{1}{4}A + \frac{3}{4}B}$