Math 257 PDE Assignment 5

[**Periodic functions**]

1. Which of the following functions are periodic? For those that are, find the fundamental period.

$$\text{(a) } \sin x \cos x \quad \text{(b) } \sec x + \tan \sqrt{2}x \quad \text{(c) } \sin(x^2) \quad \text{(d) } \frac{\{x/3\}}{2 + \sin \pi x}.$$

The greatest integer function $[x]$ is by definition the largest integer less than or equal to x, e.g., $[3/2] = 1$ and the fractional part function is $\{x\} = x - [x]$.

2. Let $f(x)$ be the 2-periodic function defined by

$$f(x) = \begin{cases} \lambda & -1 \le x < 0 \\ \mu & 0 \le x < 1 \end{cases} \quad (\text{ where } \lambda > \mu \text{ are positive constants.})$$

(a) Graph the function $f(x)$, $-\infty < x < \infty$.

(b) Find the Fourier series of $f(x)$.

(c) Find the Fourier series of the 2-periodic function $g(x) = \begin{cases} \dfrac{\lambda - \mu}{2} & -1 \le x < 0 \\ \dfrac{\mu - \lambda}{2} & 0 \le x < 1 \end{cases}$

3. (a) Show that $\cos x = \dfrac{8}{\pi} \displaystyle\sum_{n=1}^{\infty} \frac{n}{(2n)^2 - 1} \sin(2nx),\ 0 < x < \pi.$

(b) The series in (a) converges for all x. What function does it converge to?

(c) Show that $\displaystyle\sum_{n=1,3,5,\ldots} \frac{2n}{4n^2 - 1}(-1)^{(n-1)/2} = \frac{\pi}{4\sqrt{2}}.$

[**Parseval's Identity**]

4. The Fourier series expansion of the 2π-periodic function $f(x) = x^2$, $-\pi \le x \le \pi$, is

$$x^2 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx. \quad \text{This is question 9 in section 2.2 of the text.}$$

(a) What identity do you get if you substitute in $x = \pi/2$?

(b) What identity do you get from Parseval's identity?

5. Use Parseval's identity to evaluate $\int_{-\pi}^{\pi} f(x)^2 dx$, where

$$f(x) = 1 + \sum_{n=1}^{\infty} \left(\frac{1}{2^n} \cos nx + \frac{1}{n^2} \sin nx \right).$$

Hint: consult the book on the Riemann zeta function $\zeta(s)$, in particular its value at $s = 4$. The previous question is also relevant.

6. Let $f(x) = \sum_{n=1}^{\infty} \frac{\sin nx}{n^2}$, $g(x) = \mu + \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos 2nx$, where μ is some constant. What is the value of $\int_{-\pi}^{\pi} (f(x) - g(x))^2 dx$?