Math 257 PDE   Assignment 6

[**Separation of variables**]

1. Determine whether the method of separation of variables can be used to replace the following PDE's by a pair of ODE's. If so, find the equations.

   (a) $xu_{xx} + tu_t = 0$.   (b) $u_{xx} + u_{yy} = x$.   (c) $u_x + u_{xt} + u_t = 0$.

2. For each (real) constant k find all the non-zero solutions of the following boundary value problem

$$X'' = kX, \text{ for } x \in (0,1), \qquad X(0) = -X(1).$$

[**Wave Equation with periodic-BC**]

3. (a) Find the solution of the wave equation

$$\begin{aligned}
&u_{tt} = \frac{1}{4}u_{xx}, \quad (0 < x < 2,\ t > 0);\\
&u(0,t) = u(2,t),\ u_x(0,t) = u_x(2,t) \quad (t > 0);\\
&u(x,0) = 2\cos(\pi x) - \sin(2\pi x) + 4\sin 4\pi x, \quad (0 \le x \le 2),\\
&u_t(x,0) = 0, \quad (0 \le x \le 2).
\end{aligned}$$

   (b) Sound waves in a metal ring of circumference 2 could be modeled by the above equation. A sound wave is a small displacement $u(x,t)$ of a small region of the ring centred on x away from its normal position at x on the ring.

   Is there a solution with the same periodic boundary conditions but with initial conditions $u(x,0) = 0$ and $u_t(x,0) = 1$ for $0 \le x \le 2$? What would this solution mean physically in the context of a metal ring?

[**Heat equations with zero-BC**]

4. Consider the conduction of heat in a rod 40 cm in length whose ends are maintained at 0°C for all $t > 0$. Find an expression for the temperature $u(x,t)$ if the initial temperature distribution in the rod is $u(x,0) = 50$ (°C) for $0 < x < 40$. Suppose the diffusion constant ("thermal diffusivity" in the textbook) $c^2 = 1$.

5. For the same temperature distribution $u(x,t)$ in the previous problem, $u(x,t)$ converges to 0 as $t \to \infty$. Argue why this is so with a physical reasoning, and verify the solution formula directly.

[**Heat equations with nonhomogeneous BC or zero-flux BC**]

6. Let an aluminum rod of length 20 cm be initially at the uniform temperature of 25°C. Suppose that at time $t = 0$ the end $x = 0$ is cooled to 0°C while the end $x = 20$ is heated to 60°C, and both are thereafter maintained at those temperatures. Find the temperature distribution in the rod at any time t. What is the final (steady-state) temperature distribution, i.e, the limit as $t \to \infty$? The thermal diffusivity for aluminum is $c^2 = 0.86$ (cm^2/sec).

7. Find the solution of the heat conduction problem

$$
\begin{aligned}
&u_t = u_{xx}, \quad (0 < x < \pi,\ t > 0); \qquad u_x(0,t) = 0 = u_x(\pi,t), \quad (t > 0);\\
&u(x,0) = \pi - x, \quad (0 < x < \pi).
\end{aligned}
$$

Note: Final Exam is scheduled on **Friday December 9, 3:30am**. If you have difficulty making this time such as for a religious reason, or if you have three exams within a 24 hour period, with the middle one being a math exam, let me know immediately.