Math 316/257 PDE Assignment 9

1. Find all eigenvalues and corresponding eigenfunctions for the following problem

$$-y'' = \lambda y \quad (0 < x < 1), \quad y(0) = 0, \quad y'(1) = 0.$$

[**Sturm-Liouville eigenvalue problems**]

2. Consider the eigenvalue problem of Euler type:

$$x^2 y'' + xy' + \lambda y = 0 \quad (1 < x < e), \quad y(1) = 0 = y(e). \tag{1}$$

(a) Complete this sentence:
To say that the number λ is an eigenvalue in (1) means, precisely, that ...

(b) Present a careful case-by-case analysis to find all real eigenvalues in (1), and all the corresponding eigenfunctions.

(c) Rewrite Eq. (1) in the Sturm-Liouville form.

3. Consider the differential operator $L = -\frac{d^2}{dx^2}$ on the vector space E of smooth functions $u : [0, \pi] \to \mathbf{R}$ satisfying $u'(0) = 0 = u(\pi)$.

(a) Verify that the following functions lie in E:

$$\phi_n(x) = \cos\left(\frac{2n-1}{2}\,x\right) \qquad n = 1, 2, 3, \ldots$$

Show that each ϕ_n is an eigenfunction for L and find its corresponding eigenvalue.

(b) Let $f(x) = x$. Show how to use orthogonality and inner products to find constants $a_1, a_2, \ldots$ for which

$$f(x) = a_1\phi_1(x) + a_2\phi_2(x) + \cdots = \sum_{n=1}^{\infty} a_n\phi_n(x), \quad (0 < x < \pi).$$

(c) With $f(x)$ as in (b), find constants $b_1, b_2, b_3, \ldots$ so that $u(x) = \sum_{n=1}^{\infty} b_n\phi_n(x)$ solves $Lu = f$ in E, i.e.,

$$-u''(x) = f(x), \quad (0 < x < \pi), \qquad u'(0) = 0 = u(\pi). \tag{2}$$

(This is similar to Poisson equation.)

(d) Show that the problem in (2) also has a polynomial solution, by finding it directly.

(e) Solve the non-homogeneous heat equation for $u(x,t)$:

$$u_t = u_{xx} + f, \quad (0 < x < \pi,\, t > 0),$$

$$u(0,x) = \phi_1(x), \quad (0 < x < \pi),$$
$$u'(0,t) = 0 = u(\pi,t) \quad (t > 0).$$

(Hint: the solution is the sum of the solution of the homogeneous heat equation and that in (c).)

4. Consider the Sturm-Liouville problem

$$\begin{aligned}&\big([1+\sin(\pi x))]y'\big)' + \lambda(1+x^2)y = 0, \;\; 0 < x < 1\\ &y(0) = -y(1), y'(0) = -y'(1).\end{aligned}$$

Use the equation

$$(\lambda_k - \lambda_j)\int_a^b y_j(x)y_k(x)r(x)\,dx = \Big[p(x)\big(y_k y_j' - y_j y_k'\big)\Big]_a^b \tag{3}$$

which is derived on page 337 of the book to answer the following question: if y_j, y_k are eigenfunctions corresponding to eigenvalues $\lambda_j \neq \lambda_k$, then are these eigenfunctions orthogonal and what orthogonality relation do they satisfy?

Final Exam time
Friday, December 9, 3:30pm.