HOMEWORK ASSIGNMENT # 4

due in class on Friday, February 24

Student No: ________________ Name (Print): ____________________________

Note: All homework assignments are due in class one week after being assigned. They must be on standard $8\frac{1}{2} \times 11$ size paper and they must be stapled. Assignments which are not stapled will not be accepted. I will not bring a stapler to class. Please enter your student number and name (as it appears on the registrar's list) in the spaces above. SURNAME FIRST IN CAPITALS, and given name second. Please put your answers in the boxes (if provided), show any work in the spaces provided and submit these pages for your assignment.

1. Suppose $f(z)$ is defined for $|z - z_0| < \epsilon$, where ϵ is some positive number. If $f'(z_0)$ $\exists$ show that $f(z)$ is continuous at $z = z_0$.

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2. Determine the domain Σ of analyticity of the function $f(z) = \text{Log}(4 + \imath - z)$. Note: $\text{Log}z$ is analytic on the domain $\Omega = \{z \mid -\pi < Arg z < \pi.\}$

3. Show that the linear fractional transformation $L(z) = \dfrac{z - \imath}{z + \imath}$ maps the upper half plane $\mathbb{U} = \{z = x + \imath y \mid y > 0\}$ onto the interior of the unit circle. Hint: Show that the real axis is mapped to the unit circle and $z = \imath$ is mapped to 0.

4. Derive the identity $\sec^{-1} z = -\imath \log\left(\frac{1}{z} + \sqrt{\frac{1}{z^2} - 1}\right)$.

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5. Show that $\int_C e^z dz = 0$, where C is the square with vertices $0, 1, 1+\imath, \imath$, traversed once in that order.

6. Suppose $f(z)$ is analytic on the domain $D = \{z \mid |z| < 1\}$ and satisfies $|f'(z)| \leq M$ in D. Prove that $|f(z_1) - f(z_2)| \leq M|z_1 - z_2|$ for all z_1, z_2 in D. See problem #12 on page 180.

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7. Use Theorem 5 on page 170 to establish the following estimates:

   (a) $\left| \int_C \frac{dz}{z^2 + \imath} \right| \leq \frac{3\pi}{4}$, where C is the circle $|z| = 3$ traversed once.

   (b) $\left| \int_C \mathrm{Log}(z)\, dz \right| \leq \frac{\pi^2}{4}$, where C is the first quadrant portion of the circle $|z| = 1$.

8. Use the Cauchy Integral Theorem (see page 194) to prove that

$$\int_0^{2\pi} e^{\cos\theta}\cos(\sin\theta+\theta)d\theta = 0, \quad \int_0^{2\pi} e^{\cos\theta}\sin(\sin\theta+\theta)d\theta = 0$$

Hint: $\int_C e^z dz = 0$, where C is the unit circle parametrized by $z = e^{\imath\theta}, 0 \le \theta \le 2\pi$.

9. Evaluate $\displaystyle\int_C \frac{dz}{(z^2+1)^2}$, where C is the circle of radius 2 about the origin, oriented in the counterclockwise direction. Hint: $\displaystyle\frac{1}{(z^2+1)^2} = \frac{A_1}{z+\imath} + \frac{A_2}{(z+\imath)^2} + \frac{A_3}{z-\imath} + \frac{A_4}{(z-\imath)^2}$.

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10. Find a branch $f(z)$ of $\log(2z-1)$ that is analytic on $\mathbb{C} - \{x + \imath y \mid x \leq 1/2, y = 0\}$ and satisfies $f(1) = 2\pi\imath$.