MATH 300 REVIEW

1. The Arithmetic of Complex Numbers

1.1. The Algebra of Complex Numbers.

(1) $1, 2, 3,$ many
(2) $1, 2, 3, \ldots$
(3) Operations on positive integers; addition and multiplication: $a + b$ and ab
(4) The Greeks thought of numbers in terms of lengths or areas; the number line; interpreting numbers in terms of Euclidean geometry.
(5) Solving the linear equation $x + b = a$: solution $x = b - a$.
(6) The number 0 and negative numbers.
(7) Solving the linear equation $bx = a$: solution $x = a/b$.
(8) The rational numbers $\mathbb{Q}$.
(9) Arithmetic properties of $\mathbb{Q}$: commutativity of addition and multiplication; existence of 0 and 1; inverses for addition and multiplication; associativity of addition and multiplication; the distributive law.
(10) Are there more numbers?
(11) The diagonal of the unit square; irrationality of $\sqrt{2}$; the number field $\mathbb{Q}(\sqrt{2})$.
(12) The real numbers $\mathbb{R}$; interpret in terms of the number line.
(13) Are there more numbers? Consider quadratic equations $ax^2 + bx + c = 0$, where $a \neq 0$. The solution is $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. If $b^2 - 4ac < 0$ this makes no sense, unless of course we introduce some new numbers. In particular $\imath$ will stand for a particular square root of -1.
(14) Adjoin $\imath$ to $\mathbb{R}$ to get the complex number field $\mathbb{C}$; $a + b\imath$, where a and b are real numbers.
(15) Arithmetic properties of the complex number field.
(16) Algebraic numbers, transcendental numbers.

1.2. Point Representation of Complex Numbers.

(1) The Argand diagram.
(2) Real and imaginary axes; real and imaginary parts of a complex number $z = x + \imath y$ are $Re(z) = x, Im(z) = y$.
(3) The modulus of $z = x + \imath y$ is $|z| := \sqrt{x^2 + y^2}$; the conjugate of a complex number, $\overline{z} = \overline{x + \imath y} = x - \imath y$; algebraic properties of conjugation; $z = \overline{z} \iff z$ is real.
(4) $|z - z_0| = r$ represents a circle.

1.3. Vectors and Polar Forms.

(1) The complex number $z = x + \imath y$ can be identified with the vector (x, y).
(2) The triangle inequality $|z_1 + z_2| \leq |z_1| + |z_2|$.

(3) Exercise $|z_2| - |z_1| \leq |z_2 - z_1|$.
(4) The arguments of a complex number; the principal argument $(-\pi < \theta \leq \pi.)$
(5) $z = x + \imath y = r(\cos\theta + \imath \sin\theta)$, where $r = |z|$.
(6) $cis\theta = \cos\theta + \imath\sin\theta$;
$$z_1 z_2 = r_1 cis\theta_1 r_2 cis\theta_2 = r_1 r_2 cis(\theta_1 + \theta_2).$$
(7) Examples $1 + \sqrt{3} = 2cis\pi/3$; $\dfrac{1+\imath}{\sqrt{3}-\imath} = \dfrac{1}{\sqrt{2}}cis5\pi/12$.

1.4. **The Complex Exponential.**

(1) $exp(x) = e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots$ converges for all x.
(2) How do we define $e^z = e^{x+\imath y}$ for a complex number z? Want the usual laws of exponents to hold for the complex exponential function, for example $e^{z_1}e^{z_2} = e^{z_1+z_2}$. The correct definition is
$$exp(z) = e^x = 1 + z + \frac{1}{2!}z^2 + \frac{1}{3!}z^3 + \cdots$$
This converges for all z.
(3) $e^{x+\imath y} = e^x(\cos y + \imath\sin y)$; $|e^z| = e^x$.
(4) Polar form of a complex number $e^z = re^{\imath\theta}$.
(5) $z_1 z_2 = r_1 r_2 e^{\imath(\theta_1+\theta_2)}$.
(6) De Moivre's formula $(\cos\theta + \imath\sin\theta)^n = \cos n\theta + \imath\sin n\theta$.

1.5. **Powers and Roots.**

(1) De Moivre's formula allows us to find simple formulas for the n^{th} roots of a complex number $z = re^{\imath\theta}$. There are n roots, they are
$$\omega_k = r^{1/n}e^{(\theta+2k\pi)/n}, \ k = 0, 1, \dots, n-1.$$
(2) The n^{th} roots of unity on the unit circle.
(3) Examples: the $3^{th}, 4^{th}$ and 6^{th} roots of unity; the 8^{th} roots of unity.

We end this section by explaining how to solve a cubic equation $a_0 z^3 + a_1 z^2 + a_2 z + a_3 = 0$. By simple substitutions we can assume the cubic has the form $z^3 + az + b = 0$. Then we make the substitution
$$z = u + v, \ \text{where we choose } u, v \text{ to satisfy } 3uv + a = 0.$$
Thus
$$\begin{aligned} z^3 + az + b &= (u+v)^3 + a(u+v) + b \\ &= u^3 + 3u^2v + 3uv^2 + v^3 + a(u+v) + b \\ &= u^3 + v^3 + (3uv + a)(u+v)) + b \\ &= u^3 - \frac{a^3}{27u^3} + b \end{aligned}$$

Therefore

$$\begin{aligned} z^3+az+b &= 0 \Longleftrightarrow u^3-\frac{a^3}{27u^3}+b=0 \Longleftrightarrow u^6+bu^3-\frac{a^3}{27}=0 \Longleftrightarrow \\ & u^3=\frac{-b\pm\sqrt{b^2+4a^3/27}}{2} \Longleftrightarrow u=\left(\frac{-b\pm\sqrt{b^2+4a^3/27}}{2}\right)^{1/3} \end{aligned}$$

Theorem 1. *The solutions of the cubic* $z^3+az+b=0$ *are*

$$\begin{aligned} z_1 &= \left(\frac{-b+\sqrt{b^2+4a^3/27}}{2}\right)^{1/3}+\left(\frac{-b-\sqrt{b^2+4a^3/27}}{2}\right)^{1/3} \\ z_2 &= \omega\left(\frac{-b+\sqrt{b^2+4a^3/27}}{2}\right)^{1/3}+\omega^2\left(\frac{-b-\sqrt{b^2+4a^3/27}}{2}\right)^{1/3} \\ z_3 &= \omega^2\left(\frac{-b+\sqrt{b^2+4a^3/27}}{2}\right)^{1/3}+\omega\left(\frac{-b-\sqrt{b^2+4a^3/27}}{2}\right)^{1/3} \end{aligned}$$

where $\omega=e^{2\pi\imath/3}$ *and the cube roots are chosen so that*

$$\left(\frac{-b+\sqrt{b^2+4a^3/27}}{2}\right)^{1/3}\times\left(\frac{-b-\sqrt{b^2+4a^3/27}}{2}\right)^{1/3}=-\frac{a}{3}$$

Example: If we apply this theorem to the cubic $z^3-15z-4=0$ we find that the solutions are

$$\begin{aligned} z_1 &= (2+11\imath)^{1/3}+(2-11\imath)^{1/3} \\ z_2 &= \omega(2+11\imath)^{1/3}+\omega^2(2-11\imath)^{1/3} \\ z_3 &= \omega^2(2+11\imath)^{1/3}+\omega(2-11\imath)^{1/3}, \end{aligned}$$

where the cube roots are chosen so that $(2+11\imath)^{1/3}\times(2-11\imath)^{1/3}=5$.

We end this section with an important theorem.

Theorem 2. The Fundamental Theorem of Algebra *The solutions of any polynomial equation* $a_0z^n+a_1z^{n-1}+\cdots+a_{n-1}z+a_n=0$ *all lie in* $\mathbb{C}$.

1.6. **Planar Sets.**

(1) Functions of a real variable are defined on intervals.
(2) $|z-z_0|>r$ is an open disc.
(3) Interior points of a set S in $\mathbb{C}$; open subsets of $\mathbb{C}$;
(4) Examples $|z|>R$; $|z|\geq R$; $\left\{z=x+\imath y \;\middle|\; y\neq 0\right\}$
(5) An open set is connected if any 2 points can be joined by a polygonal path
(6) A domain is an open connected set.
(7) The boundary or frontier of a subset; closed subsets.

1.7. **The Riemann Sphere and Stereographic projection.**

(1) The unit sphere in $\mathbb{R}^3$ is given by $x_1^2 + x_2^2 + x_3^2 = 1$; a sketch of the unit sphere and its equatorial plane; the line through the north pole $(0,0,1)$ and a point $z = x + \imath y$ in the complex plane $\mathbb{C}$.
(2) A one-to-one correspondence between points $Z = (x_1, x_2, x_3)$ on the unit sphere, other than the north pole, and points $z = x + \imath y$ in the complex plane $\mathbb{C}$
(3)
$$x_1 = \frac{2x}{x^2+y^2+1}, \; x_2 = \frac{2y}{x^2+y^2+1}, \; x_3 = \frac{x^2+y^2-1}{x^2+y^2+1}$$
(4) The inverse transformation is
$$x = \frac{x_1}{1-x_3}, \; y = \frac{x_2}{1-x_3}$$
(5) Circles on the Riemann sphere correspond to circles and lines in the complex plane $\mathbb{C}$.
(6) The north pole $(0,0,1)$ corresponds to a point at ∞ in the complex plane; $\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$.
(7) There is only one point at infinity in $\mathbb{C}$.
(8) Some topology of the extended complex plane.

2. Analytic Functions

2.1. **Analytic Functions.** In this section we study complex valued functions of a complex variable, that is $w = f(z)$, where $z \in \mathbb{C}$ and $w \in \mathbb{C}$.

$$\text{If } z = x + \imath y \text{ we write } w = f(z) = u(x,y) + \imath v(x,y).$$

In particular we want to differentiate and integrate such functions.

Example 1. Describe the image of the unit disc under the map $f(z) = z^n$.

Example 2. The mapping $f(z) = 1/z$ corresponds to rotation by π about the x_1 axis on the Riemann sphere.

2.2. **Limits and Continuity.**

Definition 1. *A sequence of complex numbers $\{z_n\}_{n\geq 1}$ converges to z_0 if*

$$\text{given any } \epsilon > 0 \; \exists N \text{ such that } |z_n - z_0| < \epsilon \text{ if } n \geq N$$

Definition 2. *Suppose $f(z)$ is a function defined in some open neighbourhood of z_0, except possibly at z_0 itself. We say that the limit of $f(z)$ as z approaches z_0 is w_0, and write $\lim_{z\to z_0} f(z) = w_0$ (equivalently; $f(z) \to w_0$ as $z \to z_0$) if, given any $\epsilon > 0 \; \exists \; \delta > 0$ such that*

$$|z - z_0| < \delta \Longrightarrow |f(z) - f(z_0)| < \epsilon.$$

Definition 3. *Suppose $f(z)$ is defined in a neighbourhood of z_0. Then $f(z)$ is continuous at z_0 if $\lim_{z\to z_0} = f(z_0)$.*

Some obvious arithmetic properties of limits and continuity. So for example all polynomials are continuous on $\mathbb{C}$.

2.3. **Analyticity.** The basic notion here is the concept of the derivative of a complex valued function $w = f(z)$:

Definition 4.

$$\frac{df}{dz} = f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

In this definition we are allowed to let Δz approach zero in any way possible.

Example 3. *$f(z) = \bar{z}$ is nowhere differentiable. On the other hand the power functions $z \to z^n$, for n an integer, are differentiable everywhere.*

Rules of differentiation. All polynomials are differentiable everywhere. Differentiable functions are continuous.

Definition 5. *A function $f(z)$ is analytic on an open set Ω if $f'(z) \exists \forall z \in \Omega$.*

2.4. **The Cauchy-Riemann Equations.**

Theorem 3. *A necessary condition for a function $w = f(z) = u(x, y) + \imath v(x, y)$ to be analytic at a point $z = x + \imath y$ is that the Cauchy-Riemann equations*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \ \textit{(Cauchy-Riemann equations)}$$

are satisfied at that point.

To derive the Cauchy-Riemann equations let $\Delta z \to 0$ through real values ($\Delta z = \Delta x$) or complex values ($\Delta z = \imath \Delta y$). This gives

$$f'(z) = \frac{\partial u}{\partial x} + \imath \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - \frac{\partial v}{\partial x}$$

Theorem 4. *Suppose $f(z) = u(x, y) + \imath v(x, y)$ is defined in an open set Ω containing z_0. If all first partial derivatives of u and v exist in Ω, are continuous at z_0 and the Cauchy-Riemann equations hold at z_0 then $f(z)$ is differentiable at z_0.*

2.5. **Harmonic Functions.** Solutions of Laplace's equation $\nabla^2\phi := \dfrac{\partial^2\phi}{\partial x^2}+\dfrac{\partial^2\phi}{\partial y^2} = 0$ are called harmonic functions.

Example: The 2-dimensional heat equation on a region Ω is

$$\frac{\partial u}{\partial t} = c^2\left(\frac{\partial u^2}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right),\ (x,y) \in \Omega$$

Steady state heat flow is governed by Laplace's equation. We often impose boundary conditions of the form

$$u(x,y) = \text{some given function on the boundary of } \Omega$$

Definition 6. *A real valued function $\phi(x,y)$ is called harmonic in a domain Ω if all of its second order partials are continuous on Ω and Lalace's equation holds in Ω.*

Theorem 5. *If $f(z) = u(x,y) + v(x,y)\imath$ is analytic in Ω then $u(x,y)$ and $v(x,y)$ are harmonic in Ω.*

Proof. Assuming analyticity implies continuity of all second partials this follows from the Cauchy-Riemann equations. □

Examples:

(1) z^3 and e^z.
(2) Show that $u = x^3 - 3xy^2 + y$ is harmonic and find a harmonic conjugate $v(x,y)$.
(3) Find a function $\phi(x,y)$ that is harmonic in the region $\Omega = \{z \mid x > 0, 2 \le x^2 - y^2 \le 4\}$ and which satisfies

$$u(x,y) = \begin{cases} \alpha & \text{if } x > 0 \text{ and } x^2 - y^2 = 2 \\ \beta & \text{if } x > 0 \text{ and } x^2 - y^2 = 4 \end{cases}$$

(4) Find a function $\phi(x,y)$ that is harmonic in the region $\Omega = \{z \mid y > 0, 2 \le xy \le 4\}$ and which satisfies

$$u(x,y) = \begin{cases} \alpha & \text{if } y > 0 \text{ and } xy = 2 \\ \beta & \text{if } y > 0 \text{ and } xy = 4 \end{cases}$$

Theorem 6. *If $f(z) = y(x,y) + v(x,y)\imath$ is analytic then the family curves $u(x,y) = const.$ is mutually perpendicular to the family of curves $v(x,y) = const$ at each point of intersection (x,y) if $f'(z) \neq 0$ at the point of intersection.*

Level curves of a harmonic function and one of its conjugates intersect at right angles at all points where the gradients are $\neq 0$.

Examples:

(1) $f(z) = z^2 = x^2 - y^2 + 2xy\imath$.

2.6. **Steady State Temperature as a Harmonic Function.** Omit this section.

2.7. **Iterated Maps: Julia and Mandelbrot Sets.** Omit this section.

3. Elementary Functions

3.1. **Polynomials and Rational Functions.** The general polynomial function is

$$P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n, \text{ where the } a_i \in \mathbb{C} \text{ and } a_n \neq 0$$

The degree of $P(z)$ is $deg(P) = n$.

Definition 7. *A rational function $R(z)$ is a function of the form*

$$R(z) = \frac{P(z)}{Q(z)} = \frac{a_0 + a_1 z + a_2 z^2 + \cdots + a_m z^m}{b_0 + b_1 z + b_2 z^2 + \cdots + b_n z^n}, \text{ where } a_m \neq 0, b_n \neq 0$$

We assume that $P(z)$ and $Q(z)$ have no common factor. By the fundamental theorem of algebra we have

$$R(z) = \frac{P(z)}{Q(z)} = \frac{a_m}{b_n} \frac{(z-z_1)^{d_1}(z-z_2)^{d_2}\cdots(z-z_r)^{d_r}}{(z-\zeta_1)^{e_1}(z-\zeta_2)^{e_2}\cdots(z-\zeta_s)^{e_s}}$$

where $\sum_{i=1}^{r} d_i = m$ and $\sum_{i=1}^{s} e_i = n$. There is no intersection amongst the sets $\{z_1, \ldots, z_r\}$ and $\{\zeta_1, \ldots, \zeta_s\}$.

Definition 8. *We say that ζ_i is a pole of $R(z)$ of order e_i.*

Example: $\lim_{z \to \zeta} \frac{1}{(z-\zeta)^e} = \infty$.

Lemma 1. $\lim_{z \to \zeta_i} R(z) = \infty$.

Example: The equation $e^z = w$, where w is any non-zero complex number, has infinitely many solutions in any neighbourhood of ∞. Let $w = re^{\imath\theta}$ and $z = x + \imath y$. Then

$$e^{x+\imath y} = e^x e^{\imath y} = re^{\imath\theta} \iff e^x = r \text{ and } e^{\imath y} = e^{\imath\theta} \iff x = \ln r \text{ and } y = \theta + 2k\pi.$$

Exercise: Show that given any complex number $w = re^{\imath\theta} \neq 0$ $\exists$ a sequence $\{z_n\}$ such that

$$\text{all } z_n \neq 0, \; z_n \to 0 \text{ as } n \to \infty, \text{ and } e^{1/z} = w.$$

Partial Fraction Decomposition:

$$\begin{aligned} R(z) &= \frac{A_{1,e_1-1}}{z-\zeta_1} + \frac{A_{1,e_1-2}}{(z-\zeta_1)^2} + \cdots + \frac{A_{1,0}}{(z-\zeta_1)^{e_1}} + \\ &\quad \frac{A_{2,e_2-1}}{z-\zeta_2} + \frac{A_{2,e_2-2}}{(z-\zeta_2)^2} + \cdots + \frac{A_{2,0}}{(z-\zeta_2)^{e_2}} + \\ &\quad \cdots + \frac{A_{s,e_s-1}}{z-\zeta_s} + \frac{A_{s,e_s-2}}{(z-\zeta_s)^2} + \cdots + \frac{A_{s,0}}{(z-\zeta_s)^{e_s}} \end{aligned}$$

The coefficients are computed by the formula

$$A_{k,j} = \frac{1}{j!}\frac{d^j}{dz^j}R(z)(z-\zeta_k)^{e_k}\Big|_{z=\zeta_k},\ 1 \le k \le s, 0 \le j \le e_k - 1$$

Example: The partial fraction decomposition of the functions $\dfrac{z^2-1}{(z^2+1)^2}$ and $\dfrac{1}{z^n-1}$.

3.2. **The Exponential, Trigonometric and Hyperbolic Functions.**

$$\begin{aligned} e^z &= e^{x+\imath y} := e^x e^{\imath y} = e^x \cos y + (e^x \sin y)\imath \\ &= 1 + z + \frac{1}{2!}z^2 + \frac{1}{3!}z^3 + \cdots + \frac{1}{n!}z^n + \cdots \end{aligned}$$

Lemma 2. $e^{z_1} = e^{z_2} \iff z_1 = z_1 + 2k\pi\imath$ *for some integer* k.

Thus the function e^z is periodic with period $2\pi\imath$. A fundamental region for e^z is any one of the following sets:

$$S_n = \{x + \imath y \mid -\infty < x < \infty,\ (2n-1)\pi < \ y \le (2n+1)\pi\}$$

Definition 9.

$$\begin{aligned} \cos z &:= \frac{e^{\imath z}+e^{-\imath z}}{2},\ \sin z := \frac{e^{\imath z}-e^{-\imath z}}{2\imath} \\ \cosh z &:= \frac{e^{z}+e^{-z}}{2},\ \sinh z := \frac{e^{z}-e^{-z}}{2} \end{aligned}$$

From these definitions all the usual formulas for single variable calculus follow (formulas for derivatives, various identities, etc.)

3.3. **The Logarithmic Function.** Single valued functions vs multi-valued functions. Taking the inverse of a single valued function often leads to a multi-valued function.

Example:

$$z = e^w \iff \log(z) = w.$$

The function e^w is a single valued function of w, but $\log z$ is multi-valued. If $w = u + \imath v$ then $\log(z) = u + \imath(v + 2k\pi)$. The periodicity of the exponential function becomes multi-valuedness for the inverse function.

Definition 10. *For any complex number* $z \ne 0$ *we define the multi-valued function* $\log(z)$ *by* $\log(z) = \ln|z| + \imath arg z$, *where* $arg(z)$ *is any argument for* z.

Example: $ln(-1) = (2n+1)\imath$ for any integer n. The usual laws of logarithms hold for the complex valued logarithm so long as we remember the fact that $\log z$ is multi-valued. For example $\log(z_1 z_2) = \log z_1 = \log z_2$ means that some value of $\log(z_1 z_2)$ equals some value of $\log(z_1) + \log(z_2)$, and vice-versa.

Example: $\log(1) = \log(-1)(-1) = \log(-1) + \log(-1)$. The values of $\log 1$ are $2k\pi\imath$, $k = 0, \pm 1, \pm 2, \ldots$, whereas the values of $\log(-1) + \log(-1)$ are $(2m+1)\imath + (2n+1)\imath$.

Definition 11. *The principal branch of the logarithm is defined on the cut domain*

$$\Omega = \mathbb{C} - \{x + \imath y \mid x \le 0, y = 0\} \quad \text{by the formula } Log(z) = Log|z| + \imath Arg(z).$$

Recall that the possible arguments θ for the principal argument satisfy $-\pi < \theta \le \pi$.

Theorem 7. *$Log(z)$ is analytic in Ω, and $\frac{d}{dz} Log z = \frac{1}{z}$.*

Corollary 1. *$Log|z|$ and $Arg(z)$ are harmonic in Ω.*

3.4. **Complex Powers and Inverse Trigonometric Functions.** We define the various inverse trig functions in the standard way. For example

$$\begin{aligned} \arcsin z &= w \Longleftrightarrow z = \sin w \\ \arccos z &= w \Longleftrightarrow z = \cos w \\ \arctan z &= w \Longleftrightarrow z = \tan w \end{aligned}$$

Periodicity of the trig functions $sin w, \cos w, \tan w$ means that the corresponding inverse trig fuynctions are multi-valued.

Example: The multi-valued function $f(z) = \sqrt{z-a}$ has a single valued branch on any cut domain that prevents going once around the point $z = a$.

Example: The function $f(z) = \sqrt{z^2 - 1}$ has a single valued branch of the cut domain

$$\Omega = \{z = x + \imath y \mid \text{if } -1 \le x \le 1 \text{ then } y \ne 0\}.$$

Exercise: Prove that the function in the last example is an odd function.

Theorem 8. *The inverse trig functions are given by*

$$\begin{aligned} \arcsin z &= -\imath \log\left(\imath z + \sqrt{1 - z^2}\right) \\ \arccos z &= -\imath \log\left(z + \sqrt{z^2 - 1}\right) \\ \arctan z &= \frac{1}{2\imath} \log \frac{1 + \imath z}{1 - \imath z} \end{aligned}$$

Definition 12. *The power function is defined by $z^\alpha = e^{\alpha \log z}$.*

Thus the power function will in general have multiple values.

Exercise: Prove the following statements:

(1) If $z = re^{\imath\theta}$ then $z^\alpha = r^\alpha e^{\imath\alpha(\theta + 2k\pi)}$, where k is any integer.
(2) z^α has a single value if α is an integer.

(3) z^α has finitely many values if α is a rational number.
(4) z^α has finitely infinitely many values in all other cases.

Example 4. *The values of* $\imath^{\pm\imath}$.

Example 5. *The solutions of* $\cos z = 2\imath$.

$$\begin{aligned}\cos z &= 2\imath \iff \frac{e^{\imath z}+e^{-\imath z}}{2} = 2\imath \iff e^{2\imath z} - 4\imath e^{\imath z} + 1 = 0 \\ &\iff e^{\imath z} = \frac{4\imath \pm \sqrt{-16-4}}{2} = (2 \pm \sqrt{5})\imath \\ &\iff z = -\imath \log((2\pm\sqrt{5})\imath)\end{aligned}$$

The values of $\log((2+\sqrt{5})\imath)$ *are* $\ln(2+\sqrt{5}) + (\pi/2 + 2k\pi)\,\imath$ *and the values of* $\log((2-\sqrt{5})\imath)$ *are* $\ln(\sqrt{5}-2) + (3\pi/2+2k\pi)\,\imath$, *where* k *is any integer. Therefore*

$$\cos z = 2\imath \iff z = -\imath\ln(2+\sqrt{5}) - \pi/2 - 2k\pi,\ z = -\imath\ln(\sqrt{5}-2) - 3\pi/2 - 2k\pi$$

4. Complex Integration

4.1. Contours.

(1) A smooth arc is given by a continuously differentiable function $z(t), a \le t \le b$, such that: $z'(t) \ne 0$ for $a \le t \le b$ and $z(t)$ is one-to-one for $a < t < b$. A smooth closed curve is the range of a smooth arc satisfying: $z(a) = z(b)$ and the right hand derivative of $z(t)$ at $t = a$ equals the left hand derivative of $z(t)$ at $t = b$.
(2) Any smooth arc or smooth closed curve has 2 natural orientations.
(3) A contour is a sequence of oriented smooth arcs $\gamma_1, \gamma_2, \ldots, \gamma_k$ such that the end point of γ_i is the beginning point of $\gamma_{i+1}, i = 1, \ldots, k-1$.
(4) Parametrizations of contours.
(5) Jordan Curve Theorem: Any simple closed contour in $\mathbb{C}$ separates $\mathbb{C}$ into 2 domains, the interior of the contour and the exterior.

4.2. **Contour Integrals.** Let $f(z)$ be a continous function in some domain D and let C be a contour in D. Then the integral $\int_C f(z)dz$ is defined to be a limit of Riemann sums in the obvious way. This integal exists even if $f(z)$ is only piecewise continuous. An immediate consequence of this definition is the so-called ML inequality.

Theorem 9. *Suppose* $f(z)$ *is continuous on a contour* C *of lenth* L *and* $\exists$ *a constant* M *such that* $|f(z)| \le M\ \forall\ z \in C$. *Then* $\left|\int_C f(z)dz\right| \le ML$.

Theorem 10. (The Fundamental Theorem of Calculus)

Suppose C is a smooth curve parametrized by $z(t), a \leq t \leq b$, and suppose $F(z)$ is an antiderivative of $f(z)$, that is $F'(z) = f(z)$. Then

$$\int_C f(z)dz = \int_{t=a}^{t=b} f(z(t))z'(t)dt = F(z(b)) - F(z(a))$$

In particular $\int_C f(z)dz$ is independent of the parametrization.

4.3. **Independence of Path.** The fundamental theorem of calculus can be expressed in the form

Theorem 11. *Suppose $f(z)$ is continuous in a domain D and $F(z)$ is an antiderivative. Then $\int_C f(z)dz = F(Q) - F(P)$, where C is any contour in D with beginning point P and ending point Q.*

This is saying that integration is independent of path.

Corollary 2. *Suppose $f(z)$ is continuous in a domain D and has an antiderivative. Then $\int_C f(z)dz = 0$ for all closed contours C in D.*

The following corollary is easy to prove.

Corollary 3. *Suppose $f(z)$ is continuous on a domain D. Then integration is independent of path $\iff \int_C f(z)dz = 0$ for all closed contours C in D.*

Theorem 12. *Suppose $f(z)$ is continuous on a domain D. Then the following statements are equivalent:*

1. $\exists$ *a function $F(z)$ defined on D and satisfying $F'(z) = f(z)$ $\forall z \in D$.*
2. $\int_C f(z)dz = 0$ $\forall$ *closed contours C in D.*
3. *Integration is independent of path, that is $\int_{C_1} f(z)dz = \int_{C_2} f(z)dz$ for all piecewise smooth curves C_1, C_2 in D having the same beginning and ending points.*

Proof. The above theorem and corollaries establish the implications (a) $\Longrightarrow$ (b) $\Longrightarrow$ (c), so we need only prove that (c) $\Longrightarrow$ (a). Define a function $F(z)$, for $z \in D$, by $F(z) = \int_C f(\zeta)d\zeta$, where C is any smooth path from a fixed reference point $z_0 \in D$ to z. Then $F'(z) = f(z)$. □

4.4. **Cauchy's Integral Theorem.** Deforming one contour C_0 in a domain D into another C_1. Lots of examples of deformations. Simply connected domains.

Theorem 13. (The Cauchy Integral Theorem) *If $f(z)$ is analytic in a simply connected domain D then $\int_C f(z)dz = 0$ for all closed contours C in D.*

Proof. Let $z = x + \imath y$ and $f(z) = u(x, y) + \imath v(x, y)$. Then $dz = dx + \imath dy$ and

$$\int_C f(z)dz = \int_C (u + \imath v)(dx + \imath dy) = \int_C (udx - vdy) + \imath \int_C (vdx + udy)$$

Since $f(z)$ is analytic u, v satisfy the Cauchy-Riemann equations: $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$. Let Ω be the interior of C. Then applying Green's Theorem from Vector Calculus (Math 317) we have

$$\begin{aligned} \int_C f(z)dz &= \int_C (u + \imath v)(dx + \imath dy) = \int_C (udx - vdy) + \imath \int_C (vdx + udy) \\ &= \int\int_\Omega \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dA + \imath \int\int_\Omega \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dA = 0 \end{aligned}$$

□

Corollary 4. *In a simply connected domain an analytic function has an antiderivative.*

Theorem 14. *Suppose $f(z)$ is an analytic function on the domain D. If C_0 and C_1 are closed contours in D that can be deformed into each other in D then $\int_{C_0} f(z)dz = \int_{C_1} f(z)dz$.*

4.5. **Cauchy's Integral Formula and Consequences.** An important corollary of Cauchy's Integral Theorem is the Cauchy Integral Formula:

Theorem 15. *Suppose $f(z)$ is analytic in some simply connected domain D, C is a positively oriented simple closed contour in D, and z_0 is a point interior to C. Then*

$$f(z_0) = \frac{1}{2\pi\imath} \int_C \frac{f(z)}{z - z_0} dz.$$

This theorem is rather amazing since it is saying that the values of an analytic function inside a simple closed contour are determined from the values of $f(z)$ for $z \in C$ in a very simple way.

Proof. First note that $\int_C \frac{f(z)}{z - z_0} dz = \int_{Cr} \frac{f(z)}{z - z_0} dz$, where C_r is the positively oriented circle $|z - z_0| = r$ and r is chosen sufficiently small so that the disc $\{|z - z_0| \leq r\}$ is interior to C.

This follows from the fact that in a simply connected domain we can always deform one simple closed curve into any other one. Then

$$\begin{aligned}\int_C \frac{f(z)}{z-z_0}dz &= \int_{Cr} \frac{f(z)}{z-z_0}dz = \int_{Cr} \frac{f(z_0)}{z-z_0}dz + \int_{Cr} \frac{f(z)-f(z_0)}{z-z_0} \\ &= 2\pi\imath f(z_0) + \int_{Cr} \frac{f(z)-f(z_0)}{z-z_0}dz\end{aligned}$$

Thus $\displaystyle\int_{Cr} \frac{f(z)-f(z_0)}{z-z_0}dz$ must be independent of r since it equals $\displaystyle\int_C \frac{f(z)}{z-z_0}dz - 2\pi\imath f(z_0)$, which is obviously independent of r. But then

$$\int_{Cr} \frac{f(z)-f(z_0)}{z-z_0}dz = \lim_{r\to 0^+} \int_{Cr} \frac{f(z)-f(z_0)}{z-z_0}dz = 0 \text{ (by the } ML \text{ inequality)}$$

Note: since $f(z)$ is analytic in D We can find a bound for $\dfrac{f(z)-f(z_0)}{z-z_0}$ for r small. Moreover the length of the contour C_r is $2\pi r$, which tends to 0 as $r \to 0^+$. □

Theorem 16. *Let $g(\zeta)$ be a function which is continuous on some contour C. Then the function defined by $G(z) = \displaystyle\int_C \frac{g(\zeta)}{\zeta - z}d\zeta$ is analytic $\forall z \notin C$. Moreover $G'(z) = \displaystyle\int_C \frac{g(\zeta)}{(\zeta-z)^2}d\zeta$.*

This theorem is saying that $\displaystyle\frac{d}{dz}\int_C \frac{g(\zeta)}{\zeta - z}d\zeta = \int_C \frac{d}{dz}\left(\frac{g(\zeta)}{\zeta - z}\right)d\zeta$, that is we can differentiate inside the integral.

Corollary 5. *If $f(z)$ is analytic in some domain D then $f(z)$ has derivatives of all orders.*

This follows from the last theorem and the Cauchy Integral Formula. In fact the derivatives are given by:

Theorem 17. *Suppose $f(z)$ is analytic inside and on the positively oriented simple closed contour C and z is any point interior to C. In fact the n^{th} derivative is given by*

$$f^{(n)}(z) = \frac{n!}{2\pi\imath}\int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}}d\zeta$$

4.6. **Bounds for Analytic Functions.** A simple corollary of the last theorem is

Corollary 6. *Suppose $f(z)$ is analytic inside and on the cirlce $C_R = \{|z-z_0| < R\}$ and $|f(z)| \le M$ $\forall z \in C_R$. Then*

$$|f^{(n)}(z_0)| \le \frac{n!M}{R^n},\ n = 1, 2, \ldots$$

Proof. Parametrize C_R by $z = z_0 + Re^{\imath\theta}$ where θ goes from $\theta = 0$ to $\theta = 2\pi$. Then

$$\begin{aligned} |f^{(n)}(z_0)| &= \left|\frac{n!}{2\pi\imath}\int_{C_R}\frac{f(z)}{(z-z_0)^{n+1}}dz\right| = \frac{n!}{2\pi}\left|\int_{\theta=0}^{\theta=2\pi}\frac{f(z_0+Re^{\imath\theta})}{R^{(n+1)}e^{(n+1)\imath\theta}}R\imath e^{\imath\theta}d\theta\right| \\ &= \frac{n!}{2\pi R^n}\left|\int_{\theta=0}^{\theta=2\pi}\frac{f(z_0+Re^{\imath\theta})}{e^{n\imath\theta}}d\theta\right| \le \frac{n!}{2\pi R^n}M\times 2\pi = \frac{n!M}{R^n} \text{ } (ML \text{ inequality}) \end{aligned}$$

□

Corollary 7. (Liouville's Theorem) *A bounded entire function is a constant.*

Corollary 8. (Fundamental Theorem of Algebra) *Any polynomial of degree ≥ 1 has a root.*

Theorem 18. (Maximum Modulus Principle) *Suppose $f(z)$ is analytic inside and on the simple closed contour C. Then the maximum value of $|f(z)|$ occurs on C.*

Theorem 19. *Suppose $f(z)$ is a non-constant function analytic inside and on a simple closed contour C. Then $max_{\zeta\in C}|f(\zeta)| > |f(z)|$ $\forall z$ interior to C.*

5. Series Representations for Analytic Functions

5.1. **Sequences and Series.**

(1) Sequences and their limits.
(2) Convergence of series
(3) Absolute convergence of series.
(4) The geometric series.
(5) The comparison Test.
(6) The Ratio Test.
(7) Pointwise convergence of a sequence of functions $\{f_n(z)\}_{n=1}^{n=\infty}$ on a domain D.
(8) Uniform convergence of a sequence of functions $\{f_n(z)\}_{n=1}^{n=\infty}$ on a domain D.

Definition 13.

Suppose $f_n(z)$ is a sequence of functions on a domain Ω. We say that the sequence converges pointwise to a function $f(z)$ if given any $\epsilon > 0$ and any $z \in \Omega$, $\exists$ a positive integer N such that

$$n \geq N \Longrightarrow |f_n(z) - f(z)| < \epsilon$$

Definition 14. *Suppose $f_n(z)$ is a sequence of functions on a domain Ω. We say that the sequence converges uniformly to a function $f(z)$ if given any $\epsilon > 0$, $\exists$ a positive integer N such that*

$$n \geq N \Longrightarrow |f_n(z) - f(z)| < \epsilon \textbf{ for all } z \in \Omega$$

This is definitely stronger than pointwise convergence. In the case of pointwise convergence the N we choose depends on both ϵ and the point $z \in \Omega$, but in the case of uniform convergence N depends only on ϵ. The choice of N is uniform for all $z \in \Omega$.

5.2. **Taylor Series.** Let $f(z)$ be a function which is infinitely differentiable at some point z_0. Then the Taylor n^{th} Taylor polynomial of $f(z)$ at z_0 is $P_n(z) = \sum_{j=0}^{N} \frac{f^{(j)}(z_0)}{j!}(z-z_0)^j$. $P_n(z)$ is the unique polynomial of degree $\leq n$ such that $f^{(j)}(z_0) = P^{(j)}(z_0)$ for $j \leq n$.

The Taylor series of $f(z)$ at $z = z_0$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0) = \lim_{n\to\infty} P_n(z)$$

Theorem 20. *If $f(z)$ is analytic in the disc $D = \{|z-z_0| < R\}$ then the Taylor series converges to $f(z)$ $\forall z \in D$. Moreover this convergence is uniform on the domain $\{|z-z_0| \leq R'\}$ for any $R' < R$.*

5.3. **Power Series.**

Definition 15. *A power series is a series of the form* $\sum_{j=0}^{\infty} a_j(z-z_0)^j := \lim_{N\to\infty} \sum_{j=0}^{N} a_j(z-z_0)^j$.

We set $f(z) = \sum_{j=0}^{\infty} a_j(z-z_0)^j$ if this series converges for z.

Questions:

(1) For what values of z does the series converge?
(2) Is $f(z)$ analytic in its domain of convergence?
(3) What is the relationship between Taylor series and power series?

Lemma 3. *If a power series $\sum_{j=0}^{\infty} a_j(z-z_0)^j$ converges for some z_1 with $|z_1-z_0| = r$ then it converges for all z with $|z-z_0| < r$.*

An easy corollary of this lemma is:

Corollary 9. *Given any power series $\sum_{j=0}^{\infty} a_j(z-z_0)^j$ $\exists$ a number R such that*

(1) $\sum_{j=0}^{\infty} a_j(z-z_0)^j$ *converges for all z with $|z-z_0| < R$.*
(2) $\sum_{j=0}^{\infty} a_j(z-z_0)^j$ *diverges for all z with $|z-z_0| > R$.*

No conclusion is possible if $|z-z_0| = R$. *Moreover the convergence is uniform on any subdomain* $\{z \mid |z-z_0| \le R'\}$, *where* R' *is any number satisfying* $R' < R$.

R is called the radius of convergence of the power series. It can be computed using the ratio test.

Theorem 21. *Suppose* $\lim \frac{a_{n+1}}{a_n} = L$. *Then the radius of convergence of the power series* $\sum_{j=0}^{\infty} z^j$ *is* $R = 1/L$.

Example 6. *The geometric series* $\sum_{j=0}^{\infty} z^j$ *converges for* $|z| < 1$, *and in fact* $\sum_{j=0}^{\infty} z^j = \frac{1}{1-z}$ $\forall\ |z| < 1$. *It diverges for* $|z| \ge 1$. **Notice** *that* $\frac{1}{1-z}$ *makes sense for all* $z \ne 1$.

Example 7. *The radius of convergence of the power series* $\sum_{j=0}^{\infty} \frac{z^j}{j^2}$ *is* $R = 1$. *In this case the power series converges for all* $|z| \le 1$. *To see this use make a comparison with the series* $\sum_{j=1}^{\infty} \frac{1}{j^2}$.

Example 8. *Consider the power series* $f(z) = \sum_{n=0}^{\infty} \frac{3^{2n}}{(n+1)^2} z^{4n}$. *To compute the radius of convergence we consider ratios of consecutive non-zero terms. That is we compute*

$$\lim_{n\to\infty} \left| \frac{3^{2n+2} z^{4n+4}/(n+2)^2}{3^{2n} z^{4n}/(n+1)^2} \right| = 9|z^4| < 1 \Longleftrightarrow |z| < \frac{1}{\sqrt{3}}.$$

Thus the radius of convergence is $r = 1/\sqrt{3}$.

Theorem 22. *Suppose* $f_n(z)$ *is a sequence of continuous functions on some domain* Ω, *converging uniformly to a function* $f(z)$ *on* Ω. *Then* $f(z)$ *is continuous on* Ω.

Proof. Let z_0 be any point in Ω. To prove continuity of $f(z)$ at $z = z_0$ we must show that given any $\epsilon > 0$, there exists $\delta > 0$ such that $|f(z) - f(z_0)| < \epsilon$ if $|z - z_0| < \delta$.

Since the sequence $f_n(z)$ converges uniformly to $f(z)$ on Ω we can find a positive integer N such that $|f_n(z) - f(z)| < \epsilon/3$ for all $n \ge N$ and **all z in Ω**. Now the function $f_N(z)$ is continuous at $z = z_0$ and therefore $\exists\ \delta > 0$ such that $|f_N(z) - f_N(z_0)| < \epsilon/3$ if $|z - z_0| < \delta$.

Then

$$\begin{aligned} |f(z) - f(z_0)| &= |f(z) - f_N(z) + f_N(z) - f_N(z_0) + f_N(z_0) - f(z_0)| \\ &\le |f(z) - f_N(z)| + |f_N(z) - f_N(z_0)| + |f_N(z_0) - f(z_0)| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \end{aligned}$$

□

Corollary 10. *Suppose $f_n(z)$ is a sequence of continuous functions converging uniformly to a function $f(z)$ on a domain Ω. If C is a contour in Ω then $\int_C f_n(z)dz$ converges to $\int_C f(z)dz$.*

Proof. Let ϵ be any positive number. Then $\exists\ N$ such that

$$|f_n(z) - f(z)| < \frac{\epsilon}{L}\ \forall\ n \geq N \text{ and } \forall\ z \in \Omega.\ \text{ where } L \text{ is the length of } C.$$

Now we apply the ML inequality: $\left|\int_C (f_n(z) - f(z))dz\right| \leq \frac{\epsilon}{L}L = \epsilon.$ □

Remark: This is saying that $\lim_{n\to\infty} \int_C f_n(z)dz = \int_C \lim_{n\to\infty} f_n(z)dz.$

Corollary 11. *Suppose $f_n(z)$ is a sequence of analytic functions converging uniformly to a function $f(z)$ on a domain Ω. Then $f(z)$ is analytic.*

Proof. By the Cauchy integral theorem $\int_C f_n(z)dz = 0$ for any closed contour C in Ω. Then the last corollary gives $\int_C f(z)dz = \lim_{n\to\infty} \int_C f_n(z)dz = 0$. By Morera's theorem (see page 210) it follows that $f(z)$ is analytic. □

From this it follows that a power series $\sum_{j=0}^{\infty} a_j(z - z_0)^j$ with radius of convergence R converges to a function $f(z)$ which is analytic for $|z - z_0| < R$. Moreover we must have $a_n = \frac{f^{(n)}(z_0)}{n!}$ for $n = 0, 1, 2, \ldots$

5.4. **Mathematical Theory of Convergence.** Omit this section.

5.5. **Laurent Series.**

Theorem 23. *Suppose $f(z)$ is analytic in the annulus $\Omega = \{z \mid r < |z - z_0| < R\}$. Then*

$$f(z) = \sum_{j=-\infty}^{\infty} a_j(z - z_0)^j,\ \textit{where } a_j = \frac{1}{2\pi\imath}\int_C \frac{f(\zeta)}{(\zeta - z_0)^{j+1}}d\zeta,\ -\infty < j < \infty,$$

and C is any positively oriented simple closed contour going once around the annulus. Moreover the series converges uniformly in any proper annulus.

Such a series is called a Laurent series. It splits into a singular part (the part with negative powers of $z - z_0$) and an analytic part (the part with non-negative powers of $z - z_0$):

$$f(z) = \sum_{j=-\infty}^{-1} a_j(z - z_0)^j + \sum_{j=0}^{\infty} a_j(z - z_0)^j.$$

Definition 16. *A function $f(z)$ has an isolated singularity at $z = z_0$ if $\exists$ some $R > 0$ such that $f(z)$ is analytic in the annulus $\{z \mid 0 < |z - z_0| < R\}$. The function may or may not be analytic at $z = z_0$.*

By the above theorem a function $f(z)$ with an isolated singularity at z_0 has a Laurent series representation at z_0 since we simply choose $r = 0$.

5.6. Zeros and Singularities.

Definition 17. *Let $f(z)$ be analytic in a neighbourhood of z_0. Then $f(z)$ has a zero of order m at z_0 if its Taylor series at z_0 has the form*

$$f(z) = a_m(z - z_0)^m + a_{m+1}(z - z_0)^{m+1} + a_{m+2}(z - z_0)^{m+2} + \cdots, \text{ where } a_m \neq 0.$$

This is equivalent to saying that $f^{(n)}(z_0) = 0$ for $n = 0, 1, \ldots, m - 1$, but $f^{(m)}(z_0) \neq 0$. It is also equivalent to saying that in a neighbourhood of z_0 we have $f(z) = (z - z_0)^m g(z)$, where $g(z)$ is analytic in a neighbourhood of z_0 and $g(z_0) \neq 0$.

Definition 18. *Suppose $f(z)$ has an isolated singularity at z_0 and $f(z) = \sum_{j=-\infty}^{\infty} a_j(z - z_0)^j$ is its Laurent series. Then*

(1) *$f(z)$ has a removable singularity at z_0 if $a_j = 0$ for all $j < 0$. In this case there are no negative powers in the Laurent series and $f(z)$ is analytic at z_0.*
(2) *$f(z)$ has a pole at z_0 if $\exists$ a* **negative integer** *N such that $a_j = 0$ for all $j < N$ and $a_N \neq 0$. In this case there are some negative powers in the Laurent series, but only finitely many, say*
$$f(z) = a_{-n}(z - z_0)^{-n} + a_{-n+1}(z - z_0)^{-n+1} + \cdots + a_{-1}(z - z_0)^{-1} + a_0 + \cdots$$
where $a_{-n} \neq 0$. We say that the order of the pole at z_0 is n.
(3) *$f(z)$ has an essential singularity at z_0 if $a_j \neq 0$ for infinitely many negative j. Thus there are infinitely many negative powers in the Laurent series.*

Example 9.

The function $f(z) = \dfrac{P(z)}{Q(z)}$, where $P(z), Q(z)$ are polynomials with no common factors, has poles at the zeros of the denominator $Q(z)$. If z_0 is a zero of order n of $Q(z)$ then z_0 is a pole of order n of $f(z)$. On the other hand the function $f(z) = e^{1/z}$ has an essential singularity at $z = 0$.

Theorem 24. *If $f(z)$ has a pole at z_0 then* $\lim_{z \to z_0} |f(z)| = +\infty$, *or equivalently* $\lim_{z \to z_0} f(z) = \infty$ *in the extended complex plane.*

Contrast this with the following theorem, which we will not prove.

Theorem 25. (Picard) *Suppose $f(z)$ has an essential singularity at $z = z_0$. Let ϵ be any positive number. Then for all w in the complex plane $\mathbb{C}$, except possibly for one value of w, the equation $f(z) = w$ has a solution satisfying* $0 < |z - z_0| < \epsilon$.

In other words, in every neighbourhood Ω of z_0, no matter how small, the set $\{f(z) \mid z \in \Omega\}$ is either the entire complex plane $\mathbb{C}$ or the complex plane $\mathbb{C}$ minus a single point! This is an amazing result.

Theorem 26. *If $f(z)$ has a pole of order n at z_0 then $\exists$ an analytic function $g(z)$ such that*

$$f(z) = \frac{g(z)}{(z - z_0)^n} \text{ and } g(z_0) \neq 0$$

Poles aren't so bad.