

SOLUTIONS TO HOMEWORK ASSIGNMENT #1

1. Express the following complex numbers in the form $a + bi$.

- (a) $(1 + i)^8$
- (b) $(\sqrt{3} + i)^6$
- (c) $\frac{1+i}{1-i} - \frac{1}{1+i}$.

Solution:

- (a) $(1 + i)^8 = (2i)^4 = 16$.
- (b) $(\sqrt{3} + i)^6 = (2e^{\pi i/6})^6 = -64$.
- (c) $\frac{1+i}{1-i} - \frac{1}{1+i} = \frac{(1+i)^2 - (1-i)}{2} = \frac{-1+3i}{2} = -\frac{1}{2} + \frac{3}{2}i$.

2. Show that $\mathbb{C}$ can not be ordered (see #30 on page 7).

Solution: Suppose it is possible to order $\mathbb{C}$. Then there would be a set P of positive elements. It would have the following properties

- 1. Given any $z \in \mathbb{C}$ one, and only one of the following, would hold: $z = 0, z \in P, -z \in P$.
- 2. $z_1, z_2 \in P \implies z_1 + z_2, z_1 z_2 \in P$. That is P is closed under sums and products.

The number i would be either positive or negative (that is, either $i \in P$ or $-i \in P$). Assume $i \in P$. Then $-i = i^3 \in P$, a contradiction. If we assume $-i \in P$ we also get a contradiction since $(-i)^3 = i$. Therefore there can be no such set P , and thus $\mathbb{C}$ can not be ordered.

3. Find all solutions of the following equations. Express your answers in the form $a + bi$.

- (a) $z^2 + iz + 2 = 0$.
- (b) $z^4 - 16 = 0$.
- (c) $z^3 + 1 = 0$.

Solution:

- (a) Applying the quadratic formula we get $z = \frac{-i \pm \sqrt{-1-8}}{2} = -2i, i$.

- (b) $z^4 = 16 \iff z = \pm 2, \pm 2i$.

- (c) $z^3 + 1 = 0 \iff z = -1, e^{\pm \pi i/3} = -1, \frac{1}{2} \pm \frac{\sqrt{3}}{2}i$.

4. Show that $\frac{\pi}{4} = 4 \arctan(1/5) - \arctan(1/239)$. Hint: $(1+i)(5-i)^4$. This is question #11 on page 23.

Solution: By computation

$$\begin{aligned} (1+i)(5-i)^4 &= 956 - 4i = 4(239 - i) = 676\sqrt{2}e^{i\theta}, \quad \theta = \arctan \frac{-1}{239} = -\arctan \frac{1}{239} \\ (1+i)(5-i)^4 &= \sqrt{2}e^{\pi i/4}(\sqrt{26})^4 e^{4i\phi} = 676\sqrt{2}e^{i(\pi/4+4\phi)}, \quad \phi = \arctan \frac{-1}{5} = -\arctan \frac{1}{5}. \end{aligned}$$

Comparing arguments gives $\frac{\pi}{4} + 4\phi = \theta$, that is $\frac{\pi}{4} = 4 \arctan \frac{1}{5} - \arctan \frac{1}{239}$.