

SOLUTIONS TO HOMEWORK ASSIGNMENT #2

1. Find all n^{th} roots of the following complex numbers z . Express your answers in the form $a + bi$.

(a) $n = 3, z = -8i$.

(b) $n = 4, z = -2 + \sqrt{12}i$.

Solution:

(a) $-8i = 8e^{i(3\pi/2+2k\pi)} \implies$ the 3^{rd} roots are: $2e^{i(3\pi/2+2k\pi)/3}, k = 0, 1, 2$.

$$\begin{aligned} 2e^{i(\pi/2)} &= 2i \text{ (for } k = 0) \\ 2e^{i(3\pi/2+2\pi)/3} &= 2i(-1/2 + i\sqrt{3}/2) = -\sqrt{3} - i \text{ (for } k = 1) \\ 2e^{i(3\pi/2+4\pi)/3} &= 2i(-1/2 - i\sqrt{3}/2) = \sqrt{3} - i \text{ (for } k = 2) \end{aligned}$$

(b) $-2 + \sqrt{12}i = 4(-1/2 + i\sqrt{3}/2) = 4e^{i\frac{2\pi}{3}} = 4e^{i(\frac{2\pi}{3}+2k\pi)}$. Thus the 4^{th} roots are

$$\begin{aligned} \sqrt{2}e^{i(\frac{2\pi}{3}+2k\pi)/4} &= \sqrt{2}e^{i\frac{\pi}{6}}e^{ik\pi/2} = \sqrt{2}\left(\sqrt{3}/2 + i/2\right)i^k \\ &= \pm \frac{1}{\sqrt{2}}(\sqrt{3} + i), \pm \frac{i}{\sqrt{2}}(\sqrt{3} + i) \end{aligned}$$

2. Find all complex numbers z satisfying the following equations.

(a) $e^z = i$.

(b) $\cos z = 2$.

Solution:

(a) $e^z = i \iff e^x \cos y = 0$ and $e^x \sin y = 1 \iff y = \pi/2 + 2n\pi, x = 0 \iff z = i(\pi/2 + 2n\pi)$, where n is any integer.

Question: why are the numbers $z = i(\pi/2 + (2n + 1)\pi)$ not included?

(b)

$$\begin{aligned} \cos z &= \frac{e^{iz} + e^{-iz}}{2} = 2 \iff e^{2iz} - 4e^{iz} + 1 = 0 \iff e^{iz} = 2 \pm \sqrt{3} \\ &\iff e^{-y} \cos x = 2 \pm \sqrt{3} \text{ and } e^{-y} \sin x = 0 \\ &\iff x = 2n\pi \text{ and } -y = \ln(2 \pm \sqrt{3}) \\ &\iff x = 2n\pi \text{ and } y = -\ln(2 \pm \sqrt{3}) = \ln(2 \mp \sqrt{3}) \\ &\iff z = 2n\pi + \ln(2 \mp \sqrt{3})i, n = 0, \pm 1, \pm 2, \dots \end{aligned}$$

Question: Why only even multiples of π ?

3. Use De Moivre's formula to prove that $\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1 \forall \theta$.

Solution:

$$\begin{aligned} (\cos \theta + i \sin \theta)^4 &= \cos^4 \theta + 4\cos^3 \theta(i \sin \theta) + 6\cos^2 \theta(i \sin \theta)^2 + 4\cos \theta(i \sin \theta)^3 + (i \sin \theta)^4 \\ &= \cos^4 \theta - 6\cos^2 \theta(1 - \cos^2 \theta) + (1 - \cos^2 \theta)^2 + i(\text{something real}) \\ &\implies \cos 4\theta = 8\cos^4 \theta - 8\cos^2 \theta + 1 \text{ by De Moivre's Theorem} \end{aligned}$$

4. For any 2 complex numbers z_1, z_2 show that $|z_2| - |z_1| \leq |z_2 - z_1|$.

Solution: This follows from the triangle inequality:

$$|z_2| = |z_2 - z_1 + z_1| \leq |z_2 - z_1| + |z_1| \implies |z_2| - |z_1| \leq |z_2 - z_1|$$

5. Solve for z : $z^3 = \frac{2i}{1+i}$.

Solution: $\frac{2i}{1+i} = 1+i = \sqrt{2}e^{i\pi/4+2k\pi} \implies z = 2^{1/6}e^{\pi/12+2k\pi/3}$. Now $\cos(\pi/12) = \frac{1+\sqrt{3}}{2\sqrt{2}}$ and

$\sin(\pi/12) = \frac{\sqrt{3}-1}{2\sqrt{2}}$. To see this note that

$$\begin{aligned} \cos(\pi/12) + i \sin(\pi/12) &= e^{i\pi/12} = \frac{e^{i\pi/3}}{e^{i\pi/4}} = \frac{1/2 + i\sqrt{3}/2}{\frac{1}{\sqrt{2}}(1+i)} \\ &= \frac{1}{\sqrt{2}} \frac{1+i\sqrt{3}}{1+i} \times \frac{1-i}{1-i} \\ &= \frac{1}{2\sqrt{2}}(1+\sqrt{3}+i(\sqrt{3}-1)) \end{aligned}$$

After some computation we see that the solutions are

$$\frac{-1+i}{2^{1/3}}, \frac{(1-\sqrt{3}-(\sqrt{3}+1)i)}{2^{4/3}}, \frac{(1+\sqrt{3}-(-\sqrt{3}+1)i)}{2^{4/3}}$$

6. Sketch each of the following sets Ω . If Ω is open (resp. closed or connected) put the words open (resp. closed or connected) in the boxes; otherwise leave blank.

(a) $\Omega = \{z = x + iy \mid x \geq 1 \text{ or } x \leq 0\}$

(b) $\Omega = \mathbb{C} - \{z \mid 0 \leq x \leq 1, y = 0\}$

(c) $\Omega = \{z \mid 1 < |z| < 2\}$

(d) $\Omega = \{z \mid x^2 - xy + y^2 \leq 1\}$

(e) $\Omega = \{z \mid -1/2 \leq x \leq 1/2 \text{ and } |z| \geq 1\}$

(f) $\Omega = \{z = re^{i\theta} \mid r > 0 \text{ and } \pi/4 \leq \theta \leq \pi/2\}$

(g) $\Omega = \mathbb{C} - \{z \mid 0 \leq x, y = 0\}$

Solution:

(a) Closed.

(b) Open and connected.

(c) Open and connected.

(d) Closed and connected.

(e) Closed.

(f) Connected.

(g) Open and connected.