Unpublished note: Detailed calculation of μ with $O(s^6)$ error bounds, Calculation of amplitudes A and D, Detailed calculation of p_c with $O(s^4)$ error bounds

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Abstract

In this unpublished note, we consider in detail (1) the $O(s^6)$ calculation of μ for the self-avoiding-walk, (2) the calculation of amplitude A and diffusion constant D for the self-avoiding-walk, and (3) detailed $O(s^3)$ calculation of the critical point p_c for bond percolation. Each of these is treated in separate sections, preceded by a brief supplement to Appendix A.

B Supplement to Appendix A

B.1 More on estimating Fourier Integrals

In this section, we supplement Lemma 4.1 so that it can be applied to more general situation.

Lemma B.1 *Let f be a real function on $\mathbf{Z}^d$ which respects lattice symmetry, and for which there exists a (small) positive ϵ such that*

$$\sum_{x \neq 0} |f(x)| \leq \epsilon. \tag{B.18}$$

Let $\Delta(k)$ be a real function on $[-\pi, \pi]^d$ which satisfies

$$|\Delta(k)| \leq \frac{1}{2}[1 - \hat{D}(k)], \quad |\Delta(k)| \leq 2\epsilon, \quad \textit{for} \quad k \in [-\pi, \pi]^d. \tag{B.19}$$

Then for any integer $n \geq 1$ and for $d \geq 2n + 1$, $\alpha \geq 1$,

$$\int \frac{d^d k}{(2\pi)^d} \frac{[\hat{f}(0) - \hat{f}(k)] e^{ik\cdot x}}{\{\alpha[1 - \hat{D}(k)]\}^{n-1} \{\alpha[1 - \hat{D}(k)] + \Delta(k)\}} = O\left(s\epsilon + \epsilon^2\right) + \frac{\delta_{0,x}}{\alpha^n} \sum_{y \neq 0} f(y). \tag{B.20}$$

Here $O(A)$ denotes a term bounded in absolute value by KA, where K is a positive constant independent of ϵ, d, α (but dependent on n), and as usual $s = \frac{1}{2d}$.

Proof of Lemma B.1 The proof goes in exactly the same way as that of Lemma 4.1, if we now write $\hat{f}(0) - \hat{f}(k)$ in the denominator of Lemma 4.1 as $\Delta(k)$.

□

As a corollary, we obtain the next lemma, which will be used repeatedly in our analysis of percolation.

Lemma B.2 *Let f be a real function on $\mathbf{Z}^d$ which respects lattice symmetry, and for which there exists a (small) positive ϵ such that*

$$\sum_{x \neq 0} |f(x)| \leq \epsilon. \tag{B.21}$$

Also let $\hat{h}(k)$ be a real function on $[-\pi, \pi]^d$ which satisfies (for $\delta \leq \epsilon$)

$$|\hat{h}(k)| \leq \delta, \quad \textit{for} \quad k \in [-\pi, \pi]^d, \tag{B.22}$$

and defining $\Delta(k) \equiv \hat{f}(0) - \hat{f}(k) + \hat{h}(k)$,

$$|\Delta(k)| \leq \frac{1}{2}[1 - \hat{D}(k)], \quad \textit{for} \quad k \in [-\pi, \pi]^d. \tag{B.23}$$

Then for any integer $n \geq 1$ and for $d \geq 2n+1$, $\alpha \geq 1$,

$$\int \frac{d^dk}{(2\pi)^d} \frac{[\Delta(k)]e^{ik\cdot x}}{\{\alpha[1-\hat{D}(k)]\}^{n-1}\{\alpha[1-\hat{D}(k)]+\Delta(k)\}} = O\left(\delta + s\epsilon + \epsilon^2\right) + \frac{\delta_{0,x}}{\alpha^n} \sum_{y \neq 0} f(y). \tag{B.24}$$

Here $O(A)$ denotes a term bounded in absolute value by KA, where K is a positive constant independent of ϵ, d, α (but dependent on n), and as usual $s = \frac{1}{2d}$.

Proof of Lemma B.2 We note an identity

$$\begin{aligned} &\int \frac{d^dk}{(2\pi)^d} \frac{[\Delta(k)]e^{ik\cdot x}}{\{\alpha[1-\hat{D}(k)]\}^{n-1}\{\alpha[1-\hat{D}(k)]+\Delta(k)\}} \\ &\quad = \int \frac{d^dk}{(2\pi)^d} \frac{[\hat{f}(0)-\hat{f}(k)]e^{ik\cdot x}}{\{\alpha[1-\hat{D}(k)]\}^{n-1}\{\alpha[1-\hat{D}(k)]+\Delta(k)\}} \\ &\quad + \int \frac{d^dk}{(2\pi)^d} \frac{[\hat{h}(k)]e^{ik\cdot x}}{\{\alpha[1-\hat{D}(k)]\}^{n-1}\{\alpha[1-\hat{D}(k)]+\Delta(k)\}}. \end{aligned} \tag{B.25}$$

The first term is estimated by Lemma B.1. The second term is bounded by taking its absolute value.

□

B.2 More on $1/d$–expansions for Gaussian quantities

To deal with percolation, we will need (for $x \in \mathbf{Z}^d$)

$$I_{3,0}(x) = \delta_{x,0} + O(s). \tag{B.26}$$

C Detailed Explanation of $O(s^6)$ Calculation

Here we explain in detail the calculations needed to obtain (4.1)–(4.6), thus completing the proof of Theorem 1.2.

C.1 Third Iteration: B

Part B of the Third Iteration is rather easy. That is, we have only to summarize what we have (or what we can easily get) for the two point function $G(0,x)$ at this stage.

First, by our current estimate on β_c, we have

$$2d\beta_c = 1 + s + 2s^2 + 6s^3 + 27s^4 + O(s^5) \tag{C.1}$$

and for α in (4.56),

$$\alpha \equiv 2d\beta_c + s^2 + 6s^3 = 1 + s + 3s^2 + 12s^3 + 27s^4 + O(s^5). \tag{C.2}$$

So, (4.57) immediately gives:

$$G(0,x) = \begin{cases} s + 2s^2 + 7s^3 + 35s^4 + O(s^5) & (x = e_1) \\ s^2 + 5s^3 + 29s^4 + O(s^5) & (x = 2e_1) \\ 2s^2 + 10s^3 + 50s^4 + O(s^5) & (x = e_1 + e_2) \\ O(s^3) & (|x| > 2). \end{cases} \tag{C.3}$$

We also note that in a similar way we have

$$B(x) = \begin{cases} s + 6s^2 + 41s^3 + 323s^4 + O(s^5) & (x = 0) \\ 4s^2 + 34s^3 + 288s^4 + O(s^5) & (x = e_1) \\ s^2 + 14s^3 + 148s^4 + O(s^5) & (x = 2e_1) \\ 2s^2 + 28s^3 + 272s^4 + O(s^5) & (x = e_1 + e_2) \\ O(s^3) & (|x| > 2). \end{cases} \tag{C.4}$$

For later use, we also note that

$$\tilde{B}(0) \equiv \sum_{|x|>1} G(0,x)^2 = B(0) - 2dG(0,e_1)^2 = 2s^2 + O(s^3). \tag{C.5}$$

C.2 Fourth Iteration: A

We now proceed to diagrammatic estimates. Because diagrams of six-loops and more contribute only $O(s^6)$, we have only to consider diagrams of less than six-loops.

Comment on terminology:

We first introduce notation (congruence of points): $x \cong y$ for $x, y \in \mathbf{Z}^d$ means x and y can be related by lattice rotation and reflection.

We say the contribution from a diagram is *irrelevant*, if we get $O(s^6)$ result after summing those diagram contributions over other diagram contributions related by symmetry. To give an example, if $f(e_1) = O(s^6)$, this is relevant for the sum $\sum_x f(x)$, because $\sum_x f(x)$ contains $2d$ points congruent to e_1, thus

$$\sum_x f(x) = 2df(e_1) + ... = O(s^5)...$$

In the following always keep in mind that we have

$$\left.\begin{array}{l} 1 \\ 2d \\ 2d \\ 2d(d-1) \end{array}\right\} \text{points congruent to} \left\{\begin{array}{l} 0 \\ e_1 \\ 2e_1 \\ e_1+e_2 \end{array}\right. . \tag{C.6}$$

This in particular means that a contribution to $\Pi(x)$ is irrelevant if it is

$$\begin{array}{ll} O(s^6) & (x=0) \\ O(s^7) & (x \cong e_1) \\ O(s^7) & (x \cong 2e_1) \\ O(s^8) & (x \cong e_1+e_2) \end{array} \tag{C.7}$$

We also make heavy use of diagrams. In the diagrams, a solid line connecting x and y denotes a connection (with possibly some constraints) while a thick (or wiggly) line denotes a two-point function from x to y. Also a short thick line represents a bond.

C.2.1 Five-loop diagram

The estimate of the five-loop diagram is the simplest. For an upper bound, we just do as we did for the four-loop diagram,

$$\sum_x \Pi^{(5)}(x) \le \left[\sup_{x\neq 0} G(0,x)\right]\left[\sup_{x\neq 0} B'(x)\right]\left[\sup_{x\neq 0} B'(x)\right]\left[\sup_{x\neq 0} B'(x)\right] B(0) = s^5 + O(s^6). \tag{C.8}$$

For a lower bound, we consider the simplest one, where all the slashed lines are zero, and the other vertex is $|y|=1$ [see Fig. C.5]:

$$\Pi^{(5)}(0) \ge 2d{\beta_c}^6 = s^5 + O(s^6). \tag{C.9}$$

Combining these, we have

$$\Pi^{(5)}(0) = s^5 + O(s^6), \quad \sum_{x\neq 0} \Pi^{(5)}(x) = O(s^6). \tag{C.10}$$

We have thus proved (4.5).

C.2.2 Four-loop diagram

The four-loop diagram is given by Fig. C.4, (0). We consider three cases separately, according to the number of slashed lines.

(1) Both of the slashed are nonzero: In this case we have [Fig. C.4, (1)]

$$\begin{aligned} \sum_x \left[\Pi^{(4)}(x),\ \text{case (1)}\ \right] &\le \left[\sup_{x\neq 0} G(0,x)\right]\left[\sup_{x\neq 0} B(x)\right]\left[\sup_{x\neq 0} B(x)\right] B(0) \\ &= O(s)O(s^2)^2 O(s) = O(s^6), \end{aligned} \tag{C.11}$$

and thus this contribution is irrelevant.

(2) One of the slashed is zero. This is the most complicated among the four-loops. We have two cases: $0 = y$ or $z = x$ in (4.32). These two give the same contribution, so I will consider $z = x$ case. We classify into two: (2-a) $|x| = 1$ and (2-b) $|x| > 1$.

Case (2-b). When $|x| > 1$ [Fig. C.4, (2-b)],

$$\begin{aligned} \sum_{|x|>1} \Big[\Pi^{(4)}(x),\ \text{case (2)}\ \Big] &\leq \tilde{B}(0) \left[\sup_{|x|>1} B(x)\right] \left[\sup_{x\neq 0} G(0,x)\right]^2 \\ &= O(s^2)O(s^2)O(s)^2 = O(s^6), \end{aligned} \tag{C.12}$$

and thus is irrelevant.

Case (2-a). $|x| = 1$. This is the most difficult. First, as a lower bound, we just collect the simplest contribution where $y = e_1 + e_j$ $(j = \pm 2, \pm 3, \ldots, \pm d)$ [Fig. C.4, (2-a)],

$$\Big[\Pi^{(4)}(e_1),\ \text{case (2)}\ \Big] \geq (2d-2){\beta_c}^7 = s^6 + O(s^7). \tag{C.13}$$

For an upper bound, we further classify by the value of y [Fig. C.4, (2-a)]. First, when $|y - e_1| > 1$,

$$\begin{aligned} \Big[\Pi^{(4)}(e_1),\ \text{case (2)},\ |y-e_1| > 1\Big] &\leq G(0,e_1)^2 B(e_1) \left[\sup_{|y-e_1|>1} G(y,e_1)\right]^2 \\ &= O(s)^2 O(s^2) O(s^2)^2 = O(s^8), \end{aligned} \tag{C.14}$$

and is irrelevant.

Second, if $y = 2e_1$,

$$\begin{aligned} \Big[\Pi^{(4)}(e_1),\ \text{case (2)},\ y = 2e_1\Big] &\leq G(0,e_1)^2 G(e_1,2e_1)^3 \{G(0,2e_1) - {\beta_c}^2\} \\ &= O(s)^2 O(s)^3 O(s^3) = O(s^8), \end{aligned} \tag{C.15}$$

and is irrelevant. [We had $-{\beta_c}^2$, because the line connecting 0 and y should avoid $x = e_1$.]

Third, if $y = e_1 + e_j$ $(j = \pm 2, \pm 3, \ldots, \pm d)$,

$$\begin{aligned} \Big[\Pi^{(4)}(e_1),\ \text{case (2)},\ y = e_1 + e_2\Big] &\leq G(0,e_1)^2 G(e_1, e_1+e_2)^3 \{G(0, e_1+e_2) - {\beta_c}^2\} \\ &= s^7 + O(s^8), \end{aligned} \tag{C.16}$$

where we had $-{\beta_c}^2$ as before.

Combining these, we have [because we have $(2d-2)$ contributions of the third case]

$$\Big[\Pi^{(4)}(e_1),\ \text{case (2)}\ \Big] \leq s^6 + O(s^7), \tag{C.17}$$

and thus

$$\Big[\Pi^{(4)}(e_1),\ \text{case (2)}\ \Big] = s^6 + O(s^7). \tag{C.18}$$

(3) Both lines are zero. As a diagram, this is simple. We have just five lines connecting 0 and x. We classify according to $|x| = 1$ or $|x| > 1$.

Contribution from $|x| > 1$ is irrelevant:

$$\sum_{|x|>1} \Big[\Pi^{(4)}(x),\ \text{case (3)}\ \Big] \leq \left[\sup_{|x|>1} G(0,x)\right]^3 \tilde{B}(0) = O(s^2)^3 O(s^2) = O(s^8). \tag{C.19}$$

On the other hand,

$$\Big[\Pi^{(4)}(e_1),\ \text{case (3)}\ \Big] \le G(0,e_1)^5 = (s+2s^2+\dots)^5 = s^5+10s^6+O(s^7), \tag{C.20}$$

and by Fig. C.4, (3),

$$\Big[\Pi^{(4)}(e_1),\ \text{case (3)}\ \Big] \ge {\beta_c}^5 + 5(2d-2){\beta_c}^7 = s^5+10s^6+O(s^7). \tag{C.21}$$

So we have

$$\Big[\Pi^{(4)}(e_1),\ \text{case (3)}\ \Big] = s^5+10s^6+O(s^7). \tag{C.22}$$

(4) Summary: Summarizing (C.11), (C.12), (C.18), (C.19), (C.22) [don't forget that we have twice of (C.12), (C.18)], the four loop diagram contributes as

$$\Pi^{(4)}(e_1) = s^5+12s^6+O(s^7), \qquad \sum_{|x|\neq 1} \Pi^{(4)}(x) = O(s^6). \tag{C.23}$$

We have in particular proved (4.4).

C.2.3 Three-loop Diagram

We consider 5 cases separately, according to the value of $|x|$.

(1) $|x|>2$. This is simple [Fig. C.3, (1)].

$$\sum_{|x|>2} \Pi^{(3)}(x) \le \Big[\sup_{|x|>2} G(0,x)\Big] \Big[B(0)\tilde{B}(0)+\tilde{B}(0)B(0)\Big] = O(s^3)O(s^3) = O(s^6), \tag{C.24}$$

and thus is irrelevant. In the above second factor, we used the fact that $|x|>2$ means either $|y|>1$ or $|x-y|>1$ in Fig. C.3, (1).

(2) $|x|=2$, $x \cong 2e_1$. This will turn out to be irrelevant. This is seen as follows. We classify by the values of $|y|$. [See Fig. C.3, (2)] Then

$$\begin{aligned}[y=e_1 \text{ contribution}] &\le G(0,e_1)^4\{G(0,2e_1)-{\beta_c}^2\} \\ &= O(s)^4 O(s^3) = O(s^7)\end{aligned} \tag{C.25}$$

$$\begin{aligned}[y=e_2 \text{ contribution}] &\le G(0,e_2)^2 G(e_2,2e_1)^2 G(0,2e_1) \\ &= O(s)^2 O(s^3)^2 O(s^2) = O(s^{10})\end{aligned} \tag{C.26}$$

$$[y=e_2+2e_1 \text{ contribution}] = [\ y=e_2 \text{ contribution}] = O(s^{10}) \tag{C.27}$$

$$\begin{aligned}[y=-e_1 \text{ contribution}] &\le G(0,e_1)^2 G(-e_1,2e_1)^2 G(0,2e_1) \\ &= O(s)^2 O(s^3)^2 O(s^2) = O(s^{10})\end{aligned} \tag{C.28}$$

$$[y=3e_1 \text{ contribution}] = [y=-e_1 \text{ contribution}] = O(s^{10}) \tag{C.29}$$

$$\begin{aligned}[|y|,|y-2e_1|>1 \text{ contribution}] &\le B(2e_1)\Big[\sup_{|y|>1} G(0,y)\Big]^2 G(0,2e_1) \\ &= O(s^2)O(s^2)^2 O(s^2) = O(s^8)\end{aligned} \tag{C.30}$$

So in all we have

$$\Pi^{(3)}(2e_1) \le O(s^7), \tag{C.31}$$

and thus is irrelevant.

(3) $|x| = 2$, and $x \cong e_1 + e_2$. This is analyzed in a similar way. It will turn out that in Fig. C.3, (3a), only the first two terms (i.e. $y = e_1$ or $y = e_2$) are relevant.

$$\begin{aligned} [y = e_3 \text{ contribution}] &\le G(0,e_3)^2 G(e_3, e_1+e_2)^2 G(0, e_1+e_2) \\ &= O(s)^2 O(s^3)^2 O(s^2) = O(s^{10}), \end{aligned} \tag{C.32}$$

$$\begin{aligned} [y = -e_1 \text{ contribution}] &\le G(0,-e_1)^2 G(-e_1, e_1+e_2)^2 G(0, e_1+e_2) \\ &= O(s)^2 O(s^3)^2 O(s^2) = O(s^{10}), \end{aligned} \tag{C.33}$$

$$[y = e_1 + 2e_2 \text{ contribution}] = [y = -e_1 \text{ contribution}] = O(s^{10}), \tag{C.34}$$

$$\begin{aligned} [|y|, |y-(e_1+e_2)| > 1 \text{ contribution}] &\le B(e_1+e_2) \left[\sup_{|y|>1} G(0,y) \right]^2 G(0, e_1+e_2) \\ &= O(s^2) O(s^2)^2 O(s^2) = O(s^8). \end{aligned} \tag{C.35}$$

All these sum up to $O(s^8)$. Now the problem is $y = e_1, e_2$ contribution. This is a bit tricky.

We start from a lower bound: see Fig. C.3 (3b). We add those configurations

$$[y = e_1 \text{ contribution}] \ge {\beta_c}^6 + 6(2d-3){\beta_c}^8 + 2(2d-4){\beta_c}^8 = s^6 + 14s^7 + O(s^8) \tag{C.36}$$

On the other hand, as an upper bound [Fig. C.3, (3c)],

$$[y = e_1 \text{ contribution}] \le G(0,e_1)^4 \times \text{ (the line connecting 0 and } e_1+e_2) \,. \tag{C.37}$$

Now, this line connecting 0 and $e_1 + e_2$ has a rather complicated avoiding constraint. In particular, it should not have contributions marked “excluded” in Fig. C.3. (3c), because otherwise this line does not obey the constraint “loops should be saw”. So

$$\begin{aligned} [y = e_1 \text{ contribution}] &\le G(0,e_1)^4 \times [G(0, e_1+e_2) - \{{\beta_c}^2 + (2d-3){\beta_c}^4 + (2d-3){\beta_c}^4\}] \\ &= s^6 + 14s^7 + O(s^8). \end{aligned} \tag{C.38}$$

As a result, we have (because we have $y = e_2$ also)

$$\Pi^{(3)}(e_1+e_2) = 2[s^6 + 14s^7 + O(s^8)]. \tag{C.39}$$

(4) $|x| = 1$. This is the most complicated of 3-loops. It will turn out that this contribution is irrelevant.

Again, we classify according to $|y|, |y - e_1|$. Just as in the above, it can be easily shown that [See Fig. C.3 (4a)]

$$\Pi^{(3)}(e_1) = (2d-2)\,[\; y = e_2,\; y = e_1 + e_2 \text{ contribution }] + O(s^7). \tag{C.40}$$

But we now claim that the first two terms are also irrelevant, due to avoidance constraint. See Fig. C.3. (4b). There we have five walks A–E which make the diagram for $y = e_2$.

We now claim that at least one of walks D or E cannot go through neither 0 nor e_1+e_2. If this is true, the contribution from that walk is bounded by $G(e_2, e_1) - 2{\beta_c}^2 = O(s^3)$, and the diagram is bounded by

$$[y = e_2,\ y = e_1 + e_2 \text{ contribution }] \leq G(0, e_1)G(0, e_2)^2 G(e_1, e_2)O(s^3) = O(s^8), \quad \text{(C.41)}$$

and thus contributes $O(s^7)$ to (C.40). ($y = e_1 + e_2$ is similar, although the details are different.)

The claim is proven by the following steps. (1) Because the loop (B,C,D) should be SAW, the walk D cannot go through 0. (2) If the walk D does not go through $e_1 + e_2$ either, there is nothing to prove. So I assume it goes through $e_1 + e_2$. (3) But then, the loop (D,E) has to be a SAW also. This means the walk E cannot go through $e_1 + e_2$. (4) However, the walk E cannot go thorough 0 either. Why? e_1 should be the *first* intersection between the loop (B,C,D) and the walk E. If the walk E goes through 0, this 0 is the first intersection, not e_1! QED.

So as a result,

$$\Pi^{(3)}(e_1) = O(s^7). \quad \text{(C.42)}$$

(5) $x = 0$. This is also complicated. Again we classify by $|y|$. It is easily shown that

$$[\ |y| > 1 \text{ contribution }] \leq \left[\sup_{|y|>1} G(0, y)\right]^2 \tilde{B}(0) = O(s^6) \quad \text{(C.43)}$$

and thus is irrelevant.

Now $|y| = 1$ contribution is by definition, because we have $2d$ such y's,

$$[\ |y| = 1 \text{ contribution }] = 2dA_1 \quad \text{(C.44)}$$

$$A_1 \equiv \sum_{\omega_j : 0 \to e_1} \rho_1\rho_2\rho_3\rho_4 I_{1,2} I_{2,3} I_{3,1} I_{3,4} \quad \text{(C.45)}$$

where $\rho_j \equiv {\beta_c}^{|\omega_j|}$, and $I_{i,j}$ is the indicator function of the event that $\omega_i \cap \omega_j = \{0, e_1\}$.

Now we use inclusion-exclusion for the last indicator function

$$I_{3,4} = 1 - \mathrm{I}[\omega_3 \cap \omega_4 \neq \{0, e_1\}].$$

Substitution into (C.45) gives

$$A_1 = G(0, e_1)\Pi^{(2)}(e_1) - A_2 \quad \text{(C.46)}$$

with

$$A_2 = \sum_{\omega_j : 0 \to e_1} \rho_1\rho_2\rho_3\rho_4 I_{1,2} I_{2,3} I_{3,1} \mathrm{I}[\omega_3 \cap \omega_4 \neq \{0, e_1\}]. \quad \text{(C.47)}$$

Now, neglecting mutual avoidance, and using

$$\mathrm{I}[\omega_3 \cap \omega_4 \neq \{0, e_1\}] \leq \sum_{z \neq 0, e_1} \mathrm{I}[\omega_3 \cap \omega_4 \ni z],$$

we have an upper bound on A_2, which shows its irrelevancy:

$$A_2 \le G(0,e_1)^2 \sum_{z\neq 0,e_1} G(0,z)^2 G(z,e_1)^2 \le O(s)^2 O(s^2)^2 O(s) = O(s^7). \tag{C.48}$$

In the last step, we used the fact that at least one of $|z|$ and $|z-e_1|$ is greater than one, and bounded the longer lines by $\sup_{|z|>1} G(0,z) = O(s^2)$.

The first term on the other hand is shown by (C.91) and (C.3) to be equal to $s^4 + 8s^5 + 52s^6 + O(s^7)$.

So summarizing above, we have

$$A_1 = s^4 + 8s^5 + 52s^6 + O(s^7) \tag{C.49}$$

and thus

$$\Pi^{(3)}(0) = s^3 + 8s^4 + 52s^5 + O(s^6). \tag{C.50}$$

(6) Summary: we have shown by the above (1)–(5) that

$$\Pi^{(3)}(x) = \begin{cases} s^3 + 8s^4 + 52s^5 + O(s^6) & (x=0) \\ O(s^7) & (x = e_1, 2e_1) \\ 2s^6 + 28s^7 + O(s^8) & (x = e_1 + e_2) \end{cases} . \tag{C.51}$$

And recalling the number of equivalent points, (C.6), we have

$$\sum_x \Pi^{(3)}(x) = \begin{cases} s^3 + 8s^4 + 52s^5 + O(s^6) & (x=0) \\ O(s^6) & (x \cong e_1, 2e_1) \\ s^4 + 12s^5 + O(s^6) & (x \cong e_1 + e_2) \\ O(s^6) & (|x| > 2) \end{cases} . \tag{C.52}$$

We have in particular proved (4.3).

C.2.4 Two-loop diagram

We estimate the diagram using inclusion-exclusion. We classify according to the value of x.

(0) First, in general, we have by inclusion-exclusion

$$\begin{aligned} \Pi^{(2)}(x) &\equiv \sum_{\omega_j:0\to x} \rho_1\rho_2\rho_3 \mathrm{I}[\omega_i \cap \omega_j = \{0,x\}] \\ &= \sum_{\omega_j:0\to x} \rho_1\rho_2\rho_3 \left\{1 - I_{1,2} - I_{2,3} - I_{3,1} + I_{1,2}I_{2,3} + I_{2,3}I_{3,1} + I_{3,1}I_{1,2} - I_{1,2}I_{2,3}I_{3,1}\right\} \\ &\equiv G(0,x)^3 - 3\Pi_a(x) + 3\Pi_b(x) - \Pi_c(x) \end{aligned} \tag{C.53}$$

where

$$\rho_j \equiv \beta_c{}^{|\omega_j|}, \quad I_{i,j} \equiv \mathrm{I}[\omega_i \cap \omega_j \neq \{0,x\}] \tag{C.54}$$

and we defined Π_* by the last equality, using symmetry between ω_i and ω_j. In the following, we consider these terms one by one. Note here that because Π_a has less constraint than Π_b, etc. we have

$$G(0,x)^3 \ge \Pi_a(x) \ge \Pi_b(x) \ge \Pi_c(x). \tag{C.55}$$

(1) First, contributions from $|x| > 2$ is shown to be irrelevant:

$$\sum_{|x|>2} \Pi^{(2)}(x) \le \sum_{|x|>2} G(0,x)^3 \le \left[\sup_{|x|>2} G(0,x)\right]\left[\sum_{|x|>2} G(0,x)^2\right] = O(s^3)O(s^3) = O(s^6). \tag{C.56}$$

So we have only to consider the cases congruent to $x = e_1$, $x = 2e_1$, $x = e_1 + e_2$ in the following.

(2) By simple calculation, we have

$$G(0,x)^3 = \begin{cases} s^3 + 6s^4 + 33s^5 + 197s^6 + O(s^7) & (x = e_1) \\ s^6 + 15s^7 + O(s^8) & (x = 2e_1) \\ 8s^6 + 120s^7 + O(s^8) & (x = e_1 + e_2) \end{cases}. \tag{C.57}$$

This takes care of the first term of (C.53).

(3) $\Pi_a(x)$, $|x| = 2$, upper bounds.

By overcounting

$$I_{1,2} \le \sum_{y \neq 0,x} \mathrm{I}[y \in \omega_1]\mathrm{I}[y \in \omega_2], \tag{C.58}$$

and thus [See Fig. C.2.(3a)]

$$\Pi_a(x) \le G(0,x) \sum_{y\neq 0,x} \sum_{\omega_1,\omega_2: 0\to y\to x} \rho_1\rho_2 \le G(0,x) \sum_{y\neq 0,x} G(0,y)^2 G(y,x)^2. \tag{C.59}$$

Now we estimate the second factor of the RHS for each x. First note that

$$\begin{aligned} \sum_{y\neq 0,x} G(0,y)^2 G(y,x)^2 &= \sum_{y\neq 0,x} G(0,y)^2 G(y,x)^2 \mathrm{I}[|y| = 1 \text{ or } |y-x| = 1] \\ &\quad + \sum_{y\neq 0,x} G(0,y)^2 G(y,x)^2 \mathrm{I}[|y| > 1 \text{ and } |y-x| > 1]. \end{aligned} \tag{C.60}$$

(3a) The contribution from the second term is irrelevant for $|x| \le 2$ as follows.

$$\begin{aligned} &\sum_{y} G(0,y)^2 G(y,x)^2 \mathrm{I}[|y| > 1 \text{ and } |y-x| > 1] \\ &\qquad \le \left[\sup_{|y|>1} G(0,y)^2\right]\left[\sum_{|y|>1} G(0,y)^2\right] = O(s^2)^2 O(s^2) = O(s^6). \end{aligned} \tag{C.61}$$

This, substituted into (C.59), gives $O(s^2)O(s^6) = O(s^8)$ contribution for $|x| = 2$, and $O(s)O(s^6) = O(s^7)$ contribution for $|x| = 1$. Even after multiplied by the number of equivalent points given by (C.6), the contribution is $O(s^6)$, which is irrelevant.

(3b) Now, for the first term, we count explicitly all y configurations. For $x = 2e_1$, we just use the simple bound as above [Fig. C.3, (b3)]:

$$\begin{aligned} &\sum_{y\neq 0,x} G(0,y)^2 G(y,x)^2 \mathrm{I}[|y| = 1 \text{ or } |y-x| = 1] \\ &= \left[(2d-2)G(0,e_2)^2 G(e_2,2e_1)^2 + G(0,-e_1)^2 G(-e_1,2e_1)^2\right] \times 2 + G(0,e_1)^2 G(e_1,2e_1)^2 \\ &= O(2d)O(s)^2 O(s^3)^2 + G(0,e_1)^2 G(e_1,2e_1)^2 = s^4 + O(s^5). \end{aligned} \tag{C.62}$$

For $x = e_1 + e_2$, we have similarly,

$$\begin{aligned} &= \Big[(2d-4)G(0,e_3)^2G(e_3,e_1+e_2)^2 + 2G(0,-e_1)^2G(-e_1,e_1+e_2)^2\Big] \times 2 \\ &\quad +2G(0,e_1)^2G(e_1,e_1+e_2)^2 \\ &= O(2d)O(s)^2O(s^3)^2 + 2G(0,e_1)^2G(e_1,e_1+e_2)^2 = 2s^4 + 16s^5 + O(s^6). \quad \text{(C.63)} \end{aligned}$$

(3c) Upper bounds summary. We now have

$$\Pi_a(2e_1) \le G(0,2e_1)\left[s^4 + O(s^5)\right] = s^6 + O(s^7), \tag{C.64}$$

$$\Pi_a(e_1+e_2) \le G(0,e_1+e_2)\left[2s^4 + 16s^5 + O(s^6)\right] = 4s^6 + 52s^7 + O(s^8). \tag{C.65}$$

(4) $\Pi_a(x)$, $|x| = 2$, lower bounds. This is not so complicated. We have only to find out the main contributions. By Fig. C.2. (4),

$$\begin{aligned} \Pi_a(2e_1) &\ge G(0,2e_1){\beta_c}^4 = s^6 + O(s^7), & \text{(C.66)} \\ \Pi_a(e_1+e_2) &\ge G(0,e_1+e_2)[2{\beta_c}^4 + 8(2d-3){\beta_c}^6] = 4s^6 + 52s^7 + O(s^8). & \text{(C.67)} \end{aligned}$$

By (3) and (4), we have

$$\begin{aligned} \Pi_a(2e_1) &= s^6 + O(s^7), & \text{(C.68)} \\ \Pi_a(e_1+e_2) &= 4s^6 + 52s^7 + O(s^8). & \text{(C.69)} \end{aligned}$$

(5) $\Pi_a(x)$, $|x| = 1$. This is more subtle. We restart from the definition of $\Pi_a(e_1)$, and rewrite it as

$$\begin{aligned} \Pi_a(e_1) &= G(0,e_1) \sum_{\omega_1,\omega_2:0\to e_1} \rho_1\rho_2 \mathrm{I}[\omega_1 \cap \omega_2 \ne \{0,e_1\}] \\ &= G(0,e_1) \sum_{\omega_1,\omega_2:0\to e_1} \rho_1\rho_2 \mathrm{I}[\omega_1 \cap \omega_2 \ni y; |y| = 1] \\ &+ G(0,e_1) \sum_{\omega_1,\omega_2:0\to e_1} \rho_1\rho_2 \mathrm{I}[\omega_1 \cap \omega_2 \ni y; |y| > 1] \\ &\quad \times \mathrm{I}[\omega_1 \cap \omega_2 \not\ni z; |z| = 1] \qquad \text{(C.70)} \end{aligned}$$

(5a) Now the first term consists of $(2d-2)$ y's which are equivalent to e_2 and one $y = -e_1$. But here, the sum over ω_1 (for $y = e_2$) is bounded by $G(0,e_2)G(e_2,e_1) - {\beta_c}^3$ (and so is the sum over ω_2). The reason is that because ω_1 as a whole should be SAW, it cannot backtrack, and therefore there is no contribution where ω_1 consists of the following three steps: $(0,e_2) \circ (e_2,0) \circ (0,e_1) = {\beta_c}^3$. Considering similarly for $y = -e_1$, we have for the first term

$$\begin{aligned} (\text{first}) &\le G(0,e_1)\left[(2d-2)\{G(0,e_2)G(e_2,e_1) - {\beta_c}^3\}^2 + \{G(0,-e_1)G(-e_1,e_1) - {\beta_c}^3\}^2\right] \\ &= s^6 + O(s^7). \qquad \text{(C.71)} \end{aligned}$$

(5b) The second term is shown to be $O(s^7)$, and thus is irrelevant. To see this, first note that

$$(\text{second}) \le G(0,e_1) \sum_{|y|>1} \sum_{\omega_1,\omega_2:0\to y\to e_1} \rho_1\rho_2 \mathrm{I}[\omega_1 \cap \omega_2 \not\ni z; |z| = 1] \tag{C.72}$$

The sum over y where $|y-e_1|>1$ is bounded as

$$\leq G(0,e_1)\left[\sup_{|y|>1} G(0,y)\right]^2 \left[\sum_{|y-e_1|>1} G(0,y)^2\right] = O(s)O(s^2)^2O(s^2) = O(s^7), \tag{C.73}$$

and is irrelevant.

The remaining contribution is $|y-e_1|=1$, and this consists of $(2d-2)$ $y=e_1+e_2$ and one $y=2e_1$. We consider $y=e_1+e_2$ first. If we do not take avoidance into account, we simply have $\leq G(0,e_1)G(0,e_1+e_2)^2G(e_1+e_2,e_1)^2 = O(s^7)$. However, we now claim we in fact have

$$\leq G(0,e_1)\left[\{G(0,e_1+e_2)-{\beta_c}^2\}^2-{\beta_c}^4\right]G(e_1+e_2,e_1)^2 = O(s)O(s^5)O(s)^2 = O(s^8).$$

The reason is as follows: (1) $\omega_j : 0 \to (e_1+e_2) \to e_1$ should be SAW, so its $0 \to (e_1+e_2)$ part cannot go through e_1. Thus we do not have, at least ${\beta_c}^2$ contribution, coming from two steps: $(0,e_1)\circ(e_1,e_1+e_2)$. This explains $-{\beta_c}^2$ in the braces. (2) However, at least one of ω_j should avoid e_2 also, because otherwise we would have $\omega_1\cap\omega_2 \ni z, |z|=1$. So we have to subtract ${\beta_c}^4$ [where both ω_j consist of two steps: $(0,e_2)\circ(e_2,e_1+e_2)$].

Similarly, for $y=2e_1$, we have an upper bound $G(0,e_1)\{G(0,2e_1)-{\beta_c}^2\}^2G(e_1,2e_1)^2 = O(s^9)$.

As a result, this remaining contribution is bounded by

$$(2d-2)O(s^8)+O(s^8) = O(s^7), \tag{C.74}$$

and for (C.72), we have

$$\text{(second)} \leq O(s^7) \tag{C.75}$$

and thus is irrelevant.

(5c) Upper bound summary. We have shown by (C.71), (C.75),

$$\Pi_a(e_1) \leq s^6 + O(s^7). \tag{C.76}$$

(5d) $\Pi_a(x)$, $|x|=1$, Lower bound. We just count the main contribution. See Fig. C.2 (5d).

$$\Pi_a(e_1) \geq (2d-2)G(0,e_1){\beta_c}^6 = s^6 + O(s^7). \tag{C.77}$$

(5e) $\Pi_a(x)$, $|x|=1$, summary.

$$\Pi_a(e_1) = s^6 + O(s^7). \tag{C.78}$$

(6) $\Pi_b(x)$.

Now we proceed to $\Pi_b(x)$. We can use simple-minded upper bounds. That is, again using (C.58) for $I_{1,2}$ and $I_{2,3}$, we have an upper bound given by 4-loop diagram (but with different slashes), see Fig. C.2. (6):

$$\begin{aligned}\Pi_b(x) &\leq \sum_{y\neq 0,x} G(0,y)^3G(y,x)^3 + 2\sum_{y\neq 0,x,z;z\neq 0,x} G(0,z)^2G(0,y)G(y,z)G(z,x)G(y,x)^2 \\ &\equiv S_1(x)+S_2(x). \end{aligned}\tag{C.79}$$

In the above, we separated the contribution where $y=z$ and defined this to be $S_1(x)$.

(6a) For $x = 2e_1$,

$$\begin{aligned} S_1(2e_1) &\le G(0,e_1)^6 + \sum_{y \neq 0, e_1, 2e_1} G(0,y)^3 G(y,2e_1)^3 \\ &\le G(0,e_1)^6 + 2\left[\sup_{|y|>1} G(0,y)^3\right]\left[\sup_{|y-2e_1|\neq 0} G(y,2e_1)\right]\left[\sum_{|y-2e_1|\neq 0} G(y,2e_1)^2\right] \\ &= G(0,e_1)^6 + O(s^6)O(s)O(s) = s^6 + O(s^7). \end{aligned} \tag{C.80}$$

where we have used the fact that when $y \neq e_1, 0, 2e_1$ then either $|y| > 1$ or $|y - 2e_1| > 1$.

Similarly, for $x = e_1 + e_2$,

$$\begin{aligned} S_1(e_1+e_2) &\le 2G(0,e_1)^6 + \sum_{y \neq 0, e_1, e_2, e_1+e_2} G(0,y)^3 G(y,e_1+e_2)^3 \\ &\le 2G(0,e_1)^6 \\ &\quad +2\left[\sup_{|y|>1} G(0,y)^3\right]\left[\sup_{|y-(e_1+e_2)|\neq 0} G(y,e_1+e_2)\right]\left[\sum_{|y-(e_1+e_2)|\neq 0} G(y,e_1+e_2)^2\right] \\ &= 2G(0,e_1)^6 + O(s^6)O(s)O(s) = 2s^6 + 24s^7 + O(s^8). \end{aligned} \tag{C.81}$$

and for $x = e_1$,

$$\begin{aligned} S_1(e_1) &\le \sum_{y \neq 0, e_1} G(0,y)^3 G(y,e_1)^3 \\ &\le 2\left[\sup_{|y|>1} G(0,y)^3\right]\left[\sup_{|y-e_1|\neq 0} G(y,e_1)\right]\left[\sum_{|y-e_1|\neq 0} G(y,e_1)^2\right] \\ &= O(s^6)O(s)O(s) = O(s^8). \end{aligned} \tag{C.82}$$

(6b) Now for S_2, we can show this is irrelevant. We classify the sum defining S_2 according to whether $|z| > 1$ or $|y - x| > 1$, or not. When $|z| > 1$ or $|y - x| > 1$, we have [Fig. C.2, (6b-1)]

$$\begin{aligned} &[|z| > 1 \text{ or } |x-y| > 1 \text{ contribution to } S_2(x)] \\ &\le 2\left[\sup_{|z|>1} G(0,y)\right]^2 \left[\sup_{y\neq 0} G(0,y)\right]\left[\sup_{y \neq x} B(y-x)\right] B(0) \\ &= O(s^2)^2 O(s) O(s^2) O(s) = O(s^8). \end{aligned} \tag{C.83}$$

On the other hand, the remaining contribution is just $|z| = 1$ and $|y - x| = 1$. Consider first $x = 2e_1, e_1 + e_2$. For such contributions the geometry is constrained so that we must have $|y - z| > 1$. Thus by decomposing the diagram a bit differently [Fig. C.2, (6b-2)],

$$\begin{aligned} &[|z| = 1 \text{ and } |x-y| = 1 \text{ contribution to } S_2(x);\ |x| = 2] \\ &\le 2B(x)^2 \left[\sup_{|y-z|>1} G(z,y)\right]\left[\sup_{y\neq 0} G(0,y)\right]^2 \\ &= O(s^2)^2 O(s^2) O(s)^2 = O(s^8). \end{aligned} \tag{C.84}$$

Finally, for $|x| = 1$ and $|z| = 1$, $|y - x| = 1$, we must have $|x - z| = 2$, and thus we have the bound

$$\begin{aligned}
&[|z| = 1 \text{ and } |x-y| = 1 \text{ contribution to } S_2(x);\ |x| = 1] \\
\leq\ & B(0)\left[\sup_{|x-z|=2} B(z,x)\right]\left[\sup_{|x-z|=2} G(z,x)\right]\left[\sup_{y\neq 0} G(0,y)\right]^2 \\
=\ & O(s)O(s^2)O(s^2)O(s)^2 = O(s^7). \qquad (C.85)
\end{aligned}$$

In summary, we have proved that

$$S_2(x) = O(s^7). \qquad (C.86)$$

(6c) To summarize, we now have

$$\Pi_b(x) \leq S_1(x) + S_2(x) = \begin{cases} O(s^7) & (|x| = 1) \\ s^6 + O(s^7) & (x = 2e_1) \\ 2s^6 + 24s^7 + O(s^8) & (x = e_1 + e_2) \end{cases}. \qquad (C.87)$$

(7) $\Pi_b(x)$, Lower bounds, and summary. As a lower bound, we collect terms of Fig. C.2. (7).

$$\begin{aligned}
\Pi_b(2e_1) &\geq {\beta_c}^6 = s^6 + O(s^7), && (C.88) \\
\Pi_b(e_1 + e_2) &\geq 2{\beta_c}^6 + 4 \cdot 3 \cdot (2d-3){\beta_c}^8 = 2s^6 + 24s^7 + O(s^8). && (C.89)
\end{aligned}$$

So we have

$$\Pi_b(x) = \begin{cases} O(s^7) & (x = e_1) \\ s^6 + O(s^7) & (x = 2e_1) \\ 2s^6 + 24s^7 + O(s^8) & (x = e_1 + e_2) \end{cases}. \qquad (C.90)$$

(8) $\Pi_c(x)$. This is easy. First, because $\Pi_c(x) \leq \Pi_b(x)$, the upper bound (C.87) also holds for $\Pi_c(x)$. Second, the configurations which gave lower bounds on $\Pi_b(x)$ happen to contribute to $\Pi_c(x)$ also. So we have the same lower bound as $\Pi_b(x)$. To summarize, we have the *same* result as $\Pi_b(x)$.

(9) Summary. To summarize the above all, we have from (C.57), (C.68), (C.69), (C.78), (C.90),

$$\Pi^{(2)}(x) = \begin{cases} 0 & (x = 0) \\ s^3 + 6s^4 + 33s^5 + 194s^6 + O(s^7) & (x = e_1) \\ O(s^7) & (x = 2e_1) \\ 12s^7 + O(s^8) & (x = e_1 + e_2) \end{cases} \qquad (C.91)$$

and thus recalling the number of equivalent points, (C.6),

$$\sum_x \Pi^{(2)}(x) = \begin{cases} 0 & (x = 0) \\ s^2 + 6s^3 + 33s^4 + 194s^5 + O(s^6) & (x \cong e_1) \\ O(s^6) & (x \cong 2e_1) \\ 6s^5 + O(s^6) & (x \cong e_1 + e_2) \\ O(s^6) & (|x| > 2) \end{cases}. \qquad (C.92)$$

This in particular implies (4.2).

C.2.5 Diagrammatic estimate, summary

In the above, we have proven (4.2)–(4.5). Also we now have the following, which will be used in the next subsection: By (C.91), (C.51), (C.23), (C.10),

$$\Pi^{\geq 2}(x) \equiv \sum_{n=2}^{\infty}(-1)^n \Pi^{(n)}(x) = \begin{cases} -s^3 - 8s^4 - 53s^5 + O(s^6) & (x=0) \\ s^3 + 6s^4 + 34s^5 + 206s^6 + O(s^7) & (x = e_1) \\ O(s^7) & (x = 2e_1) \\ -2s^6 - 16s^7 + O(s^8) & (x = e_1 + e_2) \end{cases}, \tag{C.93}$$

$$\sum_{|x|>2} \Pi^{\geq 2}(x) = O(s^6), \tag{C.94}$$

and thus

$$\Pi^{\geq 2}(x) = \mathrm{I}[x=0]\Pi^{\geq 2}(0) + \mathrm{I}[|x|=1]\Pi(e_1) + \mathrm{I}[x \cong e_1 + e_2]\Pi(e_1+e_2) + h(x), \tag{C.95}$$

where

$$\sum_x |h(x)| = O(s^6). \tag{C.96}$$

C.3 Fourth Iteration, B

Out remaining task is to prove (4.1). For this, we have only to derive the estimate on $2d\beta_c G(0, e_1)$. This is done as follows.

From the previous subsection, we now have

$$\hat{\Pi}(0) - \hat{\Pi}(k) = 2d\Pi(e_1)[1 - \hat{D}(k)] + \Pi(e_1+e_2) \sum_{x \cong e_1+e_2} [1 - e^{ik\cdot x}] + \hat{h}(0) - \hat{h}(k). \tag{C.97}$$

Using the identity

$$(2d)^2[1-\hat{D}]^2 = \sum_{|e|=|f|=1} (1 - e^{ik\cdot e}e^{ik\cdot f}) = 2\sum_{x\cong e_1+e_2}[1 - e^{ik\cdot x}] + \sum_{x \cong 2e_1}(1 - e^{ik\cdot x}), \tag{C.98}$$

this gives

$$\hat{\Pi}(0) - \hat{\Pi}(k) = 2d\Pi(e_1)[1-\hat{D}(k)] + 2d^2\Pi(e_1+e_2)[1 - \hat{D}(k)^2] + \hat{g}(0) - \hat{g}(k) \tag{C.99}$$

with

$$\hat{g}(k) \equiv \hat{h}(k) - \frac{1}{2}\Pi(e_1+e_2)\sum_{|f|=1} e^{ik\cdot 2f} \tag{C.100}$$

or

$$g(x) \equiv h(x) - \frac{1}{2}\Pi(e_1+e_2)\mathrm{I}[x \cong 2e_1]. \tag{C.101}$$

Thus we have

$$\sum_{x\neq 0}|g(x)| \leq \sum_{x \neq 0}|h(x)| + \frac{1}{2}(2d)\Pi(e_1+e_2) = O(s^5). \tag{C.102}$$

We define locally

$$\alpha \equiv 2d\beta_c + 2d\Pi(e_1), \quad \gamma \equiv 2d^2\Pi(e_1+e_2) = O(s^4), \quad \Delta(k) \equiv \hat{g}(0) - \hat{g}(k), \tag{C.103}$$

$$A \equiv \alpha[1-\hat{D}(k)], \quad E \equiv \gamma[1-\hat{D}(k)^2]. \tag{C.104}$$

By (C.1) and (C.93),

$$\alpha = 1+s+3s^2+12s^3+61s^4+O(s^5), \quad \gamma = -s^4-8s^5+O(s^6). \tag{C.105}$$

Using the above definitions we can write

$$G(0,e_1) = \int \frac{d^dk}{(2\pi)^d} \frac{\hat{D}(k)}{\alpha[1-\hat{D}(k)]+\gamma[1-\hat{D}(k)^2]+\Delta(k)} = \int \frac{d^dk}{(2\pi)^d} \frac{\hat{D}}{A+E+\Delta}. \tag{C.106}$$

Now applying

$$\frac{1}{A+E+\Delta} = \frac{1}{A} - \frac{E+\Delta}{A\{A+E+\Delta\}} \tag{C.107}$$

twice, we have

$$G(0,e_1) = \int \frac{d^dk}{(2\pi)^d} \left[\frac{\hat{D}}{A} - \frac{E\hat{D}}{A^2} - \frac{\Delta\hat{D}}{A^2} + \frac{\{E+\Delta\}^2\hat{D}}{A^2\{A+E+\Delta\}} \right]. \tag{C.108}$$

Now, the first term is $I_{1,0}(e_1)/\alpha$. The second term is computed as

$$\begin{aligned}
(\text{second}) &= \int \frac{d^dk}{(2\pi)^d} \frac{\gamma[1-\hat{D}(k)^2]\hat{D}(k)}{\{\alpha[1-\hat{D}(k)]\}^2} = \frac{\gamma}{\alpha^2} \int \frac{d^dk}{(2\pi)^d} \frac{\hat{D}(k)[1+\hat{D}(k)]}{1-\hat{D}(k)} \\
&= \frac{\gamma}{\alpha^2}\left[I_{1,1}(0)+I_{1,2}(0)\right] = \frac{2\gamma}{\alpha^2} I_{1,0}(e_1).
\end{aligned} \tag{C.109}$$

The third term is bounded using Lemma B.1, by $O(\Delta s + \Delta^2) = O(s^6)$. The fourth term is simply bounded by its absolute value as $O(\{E+\Delta\}^2) = O(s^8)$.

Summarizing we have

$$\begin{aligned}
G(0,e_1) &= \frac{I_{1,0}(e_1)}{\alpha}\left[1 - \frac{2\gamma}{\alpha}\right] + O(s^6) \\
&= s+3s^2+11s^3+52s^4+304s^5+O(s^6).
\end{aligned} \tag{C.110}$$

This completes the proof.

D Calculation of A and D.

In this Appendix, we explain briefly how to derive results for the amplitude A and the connective constant D. Due to complexity involved with increasing orders, we restrict ourselves here to estimates which leads to error bounds of $O(s^3)$.

D.1 Expressions of A and D in terms of diagrams

By Eq.(3.3) and Eq.(3.5) of [16], A and D are given by

$$A = \frac{1}{\beta_c\left[2d + \partial_\beta \hat{\Pi}_{\beta_c}(0)\right]}, \quad D = A\left[2d\beta_c - \nabla_k^2 \hat{\Pi}_{\beta_c}(0)\right]. \tag{D.1}$$

So to get an expansion for these quantities with error bounds of $O(s^3)$, we have only to get such expansions for $\beta_c\partial_\beta\hat{\Pi}_{\beta_c}(0)$ and $\nabla_k^2\hat{\Pi}_{\beta_c}(0)$.

In addition to estimates derived so far, we make use of estimates derived in [26] (actually we have to work slightly harder than [26], to take extra care with slashed lines):

$$\sup_x \|x\|_2^2 G(0,x) = O(s), \tag{D.2}$$

$$\left|\nabla_k^2\hat{\Pi}_{\beta_c}^{(n)}(0)\right| \le \sum_x \|x\|_2^2 \Pi_{\beta_c}^{(n)}(x) = O(s^n). \tag{D.3}$$

In the following, we drop the label β_c. All the quantities are at β_c, except otherwise stated.

D.2 Estimates on $\nabla_k^2\hat{\Pi}(0)$.

By simple inspection,

$$-\nabla_k^2\hat{\Pi}(0) = \sum_{n=2}^{\infty}(-1)^n \sum_x \|x\|_2^2 \Pi^{(n)}(x). \tag{D.4}$$

But (D.3) guarantees that $n \ge 3$ terms in (D.4) only contributes $O(s^3)$. So in effect, we have only to consider $n = 2$.

However, this is also simple. As a lower bound, we just count the contribution where $|x| = 1$ [Fig. D.2]:

$$\sum_x \|x\|_2^2 \Pi^{(2)}(x) \ge 2d\Pi^{(2)}(e_1) = 2d \times \beta_c^3 = s^2 + O(s^3). \tag{D.5}$$

On the other hand, as an upper bound, we simply use

$$\begin{aligned}
\sum_x \|x\|_2^2 \Pi^{(2)}(x) &\le \sum_x \|x\|_2^2 G(0,x)^3 = 2dG(0,e_1)^3 + \sum_{|x|>1} \|x\|_2^2 G(0,x)^3 \\
&\le 2dG(0,e_1)^3 + \left[\sup_{|x|>1} \|x\|_2^2 G(0,x)\right]\left[\sum_{|x|>1} G(0,x)^2\right] \\
&\le 2dG(0,e_1)^3 + O(s)O(s^2) = s^2 + O(s^3).
\end{aligned} \tag{D.6}$$

as a result, we have from (D.4)

$$-\nabla_k^2\hat{\Pi}(0) = s^2 + O(s^3). \tag{D.7}$$

D.3 Estimates on $\beta\partial_\beta\hat{\Pi}_\beta(0)$

We will apply the elementary formula

$$\beta\partial_\beta \beta^{|\omega|} = |\omega|\beta^{|\omega|}, \tag{D.8}$$

where ω denotes a walk which is not necessarily self-avoiding, and $|\omega|$ its number of steps. Noting that $|\omega| = \sum_z \mathrm{I}[z \in \omega] - 1$, where the sum over z is taken with multiplicity along

ω (e.g. when ω visit a point x twice, this x appears twice in the sum), we can express each term of $\beta\partial_\beta\hat{\Pi}_\beta(0)$ as [Fig. D.3, (1)]

$$\beta\partial_\beta\hat{\Pi}^{(1)}(0) = \hat{\Pi}^{(1)}(0) + \sum_{z\neq 0}\sum_{\omega:0\to z\to 0}\beta^{|\omega|} \equiv \hat{\Pi}^{(1)}(0) + R(0) \tag{D.9}$$

and [Fig. D.3, (2)]

$$\begin{aligned}\beta\partial_\beta\hat{\Pi}^{(2)}(0) &= 3\hat{\Pi}^{(2)}(0) + 3\sum_{x\neq 0}\sum_{z\neq 0,x}\sum_{\omega_1,\omega_2,\omega_3:0\to x}\rho_1\rho_2\rho_3 \mathrm{I}[\omega_1\ni z]I_{1,2}I_{2,3}I_{3,1} \\ &\equiv 3\hat{\Pi}^{(2)}(0) + 3Q_2.\end{aligned} \tag{D.10}$$

Higher order terms are similarly given by diagrams with an extra vertex on one of its edges.

Now it is now quite elementary to show that even with such an extra vertex, the n-loop diagram is $O(s^n)$. So for our current purpose, we have only to consider the one-loop and two-loop diagrams, which are given by (D.9) and (D.10).

Neglecting mutual avoidance of three lines making up Q_2, we have [Fig. D.3, (3)]

$$Q_2 \leq \sum_{x\neq 0}\sum_{z\neq 0,x} G(0,x)^2 G(0,z)G(z,x) \leq B(0)\left[\sup_{x\neq 0} B(x)\right] = O(s)O(s^2) = O(s^3). \tag{D.11}$$

and thus using (4.2)

$$\beta\partial_\beta\hat{\Pi}^{(2)}(0) = 3\hat{\Pi}^{(2)}(0) + O(s^3) = 3s^2 + O(s^3). \tag{D.12}$$

On the other hand,

$$\begin{aligned}R(0) &= \sum_{x\neq 0}\sum_{\omega_1,\omega_2:0\to x}\rho_1\rho_2 I_{1,2} = \sum_{x\neq 0} G(0,x)^2 - \sum_{x\neq 0}\sum_{\omega_1,\omega_2:0\to x}\rho_1\rho_2\mathrm{I}[\omega_1\cap\omega_2\neq\{0,x\}] \\ &\equiv B(0) - Q_1\end{aligned} \tag{D.13}$$

Now, by neglecting mutual avoidance [Fig. D.3, (4)],

$$\begin{aligned}Q_1 &\leq \sum_{x\neq 0}\sum_{z\neq 0,x}\sum_{\omega_1,\omega_2:0\to x}\rho_1\rho_2\mathrm{I}[\omega_1\ni z,\omega_2\ni z] \leq \sum_{x\neq 0}\sum_{z\neq 0,x} G(0,z)^2 G(z,x)^2 \\ &= B(0)^2 = s^2 + O(s^3).\end{aligned} \tag{D.14}$$

And collecting the configuration which is congruent to $\omega_j = (0, e_1, e_1 + e_2)$ [Fig. D.3, (5)], which in fact is the lowest order contributions to $x \cong e_1 + e_2$, we have

$$Q_1 \geq 2d(d-1)\times 2{\beta_c}^4 = s^2 + O(s^3) \tag{D.15}$$

and thus

$$R(0) = B(0) - \left[s^2 + O(s^3)\right] = s + 5s^2 + O(s^3), \tag{D.16}$$

$$\beta\partial_\beta\hat{\Pi}^{(1)}(0) = 2s + 8s^2 + O(s^3). \tag{D.17}$$

And thus we have

$$\beta\partial_\beta\hat{\Pi}(0) = -2s - 5s^2 + O(s^3). \tag{D.18}$$

D.4 Summary

So we have

$$\frac{1}{A} = 2d\beta_c + \beta\partial_\beta\hat{\Pi}(0) = 1 - s - 3s^2 + O(s^3), \tag{D.19}$$

or

$$A = 1 + s + 4s^2 + O(s^3). \tag{D.20}$$

Similarly,

$$\frac{D}{A} = 2d\beta_c - \nabla_k^2\hat{\Pi}(0) = 1 + s + 3s^2 + O(s^3), \tag{D.21}$$

and so

$$D = 1 + 2s + 8s^2 + O(s^3). \tag{D.22}$$

E $1/d$-expansion for bond-percolation p_c

In this section we prove Theorem 1.4, i.e. that

$$p_c(\text{bond}) = s + s^2 + \frac{7}{2}s^3 + O(s^4) \tag{E.1}$$

where as before $s \equiv \frac{1}{2d}$. Comparing with our result for $1/d$-expansions for $\beta_c \equiv 1/\mu$ for self-avoiding walk, we can see that the difference begins at the $O(s^3)$-term.

E.1 The lace expansion and p_c

The proof parallels that for μ, using the lace expansion for bond percolation, derived in [14]. The lace expansion for nearest-neighbour bond percolation with $p < p_c$ reads [14]:

$$\begin{aligned}\tau(0,x) &= \delta_{0,x} + \sum_{n=0}^{N}(-1)^n g^{(n)}(0,x) \\ &\quad + p\sum_{|u|=1}\tau(u,x) + p\sum_{n=0}^{N}(-1)^n\sum_{|y-y'|=1} g^{(n)}(0,y)\tau(y',x) + (-1)^{N+1}R^{(N)}(0,x)\end{aligned} \tag{E.2}$$

or in Fourier transformed form:

$$\hat{\tau}(k) = \frac{1 + \sum_{n=0}^{N}(-1)^n\hat{g}^{(n)}(k) + (-1)^{N+1}\hat{R}^{(N)}(k)}{1 - 2dp\hat{D}(k)\{1 + \sum_{n=0}^{N}(-1)^n\hat{g}^{(n)}(k)\}}. \tag{E.3}$$

In [14], it was shown that for $d \geq d_0$, the above identity holds for all $p < p_c$, where $R_N \to 0$ as $N \to \infty$, and the sum over n absolutely converges.

In the following, we are interested in the limit $N \to \infty$ of the above identity, which reads

$$\tau(0,x) = \delta_{0,x} + g(0,x) + p\sum_{|u|=1}\tau(u,x) + p\sum_{|y-y'|=1} g(0,y)\tau(y',x) \tag{E.4}$$

or

$$\hat{\tau}(k) = \frac{1 + \hat{g}(k)}{1 - 2dp\hat{D}(k)\{1 + \hat{g}(k)\}} \tag{E.5}$$

with

$$g(0,x) = \sum_{n=0}^{\infty} (-1)^n g^{(n)}(0,x) \tag{E.6}$$

As $p \nearrow p_c$, the susceptibility $\hat{\tau}(0)$ diverges, while $\hat{g}(k)$ remains uniformly bounded. So letting $k = 0$, $p \nearrow p_c$ in (E.5), we have

$$2dp_c = \frac{1}{1+\hat{g}(0)}, \tag{E.7}$$

where on the right hand side, $\hat{g}(0)$ in fact denotes $\lim_{p \nearrow p_c} \hat{g}(0)$. So to get a $1/d$-expansion for p_c, it is sufficient to get a $1/d$-expansion for $\hat{g}(0)$ at the critical point, $p = p_c$. *In the following, we always consider quantities at* $p = p_c - 0$.

To derive $1/d$-expansions for $\hat{g}(0)$, our strategy is basically the same as that in the previous sections. That is, we start from simple bounds derived in [14], and by an iterative procedure derive better bounds on x-space quantities and k-space quantities.

We use notation which is slightly different from that of [14], but which is more consistent with that of previous sections. We define

$$B(x) = \sum_{y \neq 0,x} \tau(0,y)\tau(y,x), \quad B''(x) \equiv \sum_{w} \tau(0,w)\tau(w,x)$$

$$T'''(x) = \sum_{y,z} \tau(0,y)\tau(y,z)\tau(z,x)$$

Note that in [14] we wrote $T+1$ instead of T''' here.

E.2 First iteration: A

In [14], it was proven that (for $d \geq d_0$; currently we can take $d_0 = 19$)

$$0 \leq \sum_x g^{(n)}(0,x) \leq \begin{cases} K/d & (n=0,1) \\ (K/d)^n & (n \geq 2) \end{cases} \tag{E.8}$$

and

$$0 \leq \sum_x |x|^2 g^{(n)}(0,x) \leq \begin{cases} K/d & (n=0,1) \\ n^2 (K/d)^{n-1} & (n \geq 2) \end{cases}, \tag{E.9}$$

with a calculable constant K which is independent of d.

As an immediate consequence of (E.8), we have

$$2dp_c = \frac{1}{1+\hat{g}(0)} = \frac{1}{1+O(s)} = 1 + O(s). \tag{E.10}$$

Now we estimate $\hat{\tau}(k)$. We will find the following identity extremely useful:

$$\hat{\tau}(k) = \frac{1}{2dp_c[1-\hat{D}(k)] + \Delta(k)}, \tag{E.11}$$

where

$$\Delta(k) \equiv \frac{\hat{g}(0) - \hat{g}(k)}{\{1+\hat{g}(0)\}\{1+\hat{g}(k)\}}. \tag{E.12}$$

To derive (E.11), first note that

$$\hat{\tau}(k) = \frac{1+\hat{g}(k)}{1-2dp_c\hat{D}(k)\{1+\hat{g}(k)\}} = \left[\frac{1}{1+\hat{g}(k)} - 2dp_c\,\hat{D}(k)\right]^{-1}.$$

But because of (E.7), we have

$$\frac{1}{1+\hat{g}(k)} - 2dp_c\hat{D}(k) = \frac{\hat{g}(0)-\hat{g}(k)}{\{1+\hat{g}(0)\}\{1+\hat{g}(k)\}} + 2dp_c\,\{1-\hat{D}(k)\}.$$

and (E.11) follows immediately. From (E.11), $\hat{\tau}(k)$ looks quite similar to $\hat{G}(k)$ for SAW, (4.8).

E.3 First iteration: B

Here we prove the following estimates:

$$\tau(0,x) = \begin{cases} s+O(s^2) & (|x|=1) \\ O(s^2) & (|x|>1) \end{cases} \tag{E.13}$$

$$B(x) = \begin{cases} s+O(s^2) & (x=0) \\ O(s^2) & (x\neq 0) \end{cases} \tag{E.14}$$

$$\tilde{B}(0) \equiv \sum_{x:|x|>1} \tau(0,x)^2 = O(s^2) \tag{E.15}$$

$$T'''(x) = \delta_{x,0} + O(s). \tag{E.16}$$

We prove these estimates in a way similar to SAW, this time by applying Lemma B.2 with $\Delta(k)$ defined by (E.12), and

$$\alpha \equiv 2dp_c > 1, \quad f(x) \equiv \frac{g(x)}{[1+\hat{g}(0)]^2}, \tag{E.17}$$

$$\hat{h}(k) \equiv \Delta(k) - [\hat{f}(0)-\hat{f}(k)] = \frac{[\hat{g}(0)-\hat{g}(k)]^2}{[1+\hat{g}(0)]^2[1+\hat{g}(k)]}. \tag{E.18}$$

Note that (E.8) and (E.10) imply that we can take

$$\epsilon = O(s), \quad \delta = O(s^2). \tag{E.19}$$

We begin with $\tau(0,x)$. As for SAW, we first note by (E.11),

$$\begin{aligned} \tau(0,x) &= \int \frac{d^dk}{(2\pi)^d} \frac{e^{ik\cdot x}}{2dp_c[1-\hat{D}(k)]+\Delta(k)} \\ &= \int \frac{d^dk}{(2\pi)^d} \frac{e^{ik\cdot x}}{2dp_c[1-\hat{D}(k)]} \\ &- \int \frac{d^dk}{(2\pi)^d} \frac{\Delta(k)e^{ik\cdot x}}{\left\{2dp_c[1-\hat{D}(k)]\right\}\left\{2dp_c[1-\hat{D}(k)]+\Delta(k)\right\}}. \end{aligned} \tag{E.20}$$

The first term is just $I_{1,0}(x)/(2dp_c)$. We can apply Lemma B.2 to the above second term and conclude (for $x \neq 0$) that it is bounded by $O(\delta + \epsilon s + \epsilon^2) = O(s^2)$. So we have

$$\tau(0,x) = \frac{1}{2dp_c} I_{1,0}(x) + O(s^2) - \frac{\delta_{0,x}}{(2dp_c)^2} \sum_{y \neq 0} f(y). \tag{E.21}$$

Then (E.13) follows from the bounds on $I_{1,0}(x)$ of (A.8)–(A.11), (A.16), and on p_c of (E.10).

Proceeding now to $B(x)$, as we did for the self-avoiding walk, we begin with the related quantity $B''(x) \equiv \sum_y \tau(0,y)\tau(y,x)$. Writing $\alpha = 2dp_c$, and using the fact that the Fourier transform of a convolution is the product of Fourier transforms, we have

$$\begin{aligned} B''(x) &= \int \frac{d^d k}{(2\pi)^d} e^{ik\cdot x} \left(\frac{1}{\alpha[1-\hat{D}(k)] + \Delta(k)} \right)^2 \\ &= \int \frac{d^d k}{(2\pi)^d} e^{ik\cdot x} \left(\frac{1}{\alpha[1-\hat{D}(k)]} - \frac{\Delta(k)}{\{\alpha[1-\hat{D}(k)]\}\{\alpha[1-\hat{D}(k)] + \Delta(k)\}} \right)^2 . \end{aligned} \tag{E.22}$$

After expanding the square, the first term gives $\alpha^{-2} I_{2,0}(x)$. The cross term is bounded using Lemma B.2 as was done for $\tau(0,x)$, and the final term simply by taking absolute values. The result is

$$B''(x) = \frac{1}{(2dp_c)^2} I_{2,0}(x) + O(s^2) - \delta_{x,0} \frac{2}{(2dp_c)^3} \sum_{y \neq 0} f(y). \tag{E.23}$$

With (E.21), the identity

$$B''(x) = B(x) + 2\tau(0,x) - \delta_{x,0} \tag{E.24}$$

leads to

$$\begin{aligned} B(x) &= \frac{1}{(2dp_c)^2} \left[I_{2,0}(x) - 2(2dp_c) I_{1,0}(x) + (2dp_c)^2 \delta_{x,0} + O(s^2) \right] \\ &= \frac{1}{(2dp_c)^2} \left[I_{2,0}(x) - \delta_{x,0} - 2(2dp_c)\left\{ I_{1,0}(x) - \delta_{x,0} \right\} + (2dp_c - 1)^2 \delta_{x,0} + O(s^2) \right] . \end{aligned} \tag{E.25}$$

[Here $\sum_{y \neq 0} f(y)$ in (E.21) and (E.23) have almost canceled, with difference $O(s^2)$]. Appealing to (E.10) and (A.8)–(A.16), we obtain the desired estimates (E.14). Then (E.15) follows from (E.14) and (E.13) as for SAW, by the analogue of (4.26).

Arguing in a way similar to that for $B''(x)$, we can easily conclude by Lemma B.2,

$$T'''(x) = \delta_{x,0} + O(s). \tag{E.26}$$

E.4 Second iteration: A

We follow the notation of Section 5.5.2 of [22], apart from writing $\tilde{C}^{\{u,v\}}(x)$ instead of $C_{\{u,v\}}(x)$. Superscript (n) on an event indicates which of the nested expectations is relevant for the event.

Here we prove

$$\sum_{x\neq 0} |g(0,x)| = O(s^2), \tag{E.27}$$

$$\hat{g}(0) \geq -s + O(s^2) \tag{E.28}$$

and

$$2dp_c = 1 + s + O(s^2) \tag{E.29}$$

We now introduce a shorthand notation. For $x, y \in \mathbf{Z}^d$, we denote the event that x and y are connected by $x \leftrightarrow y$, and the event that x and y are doubly connected by $x \Leftrightarrow y$. And for nearest-neighbour x and y, we denote the event that the bond (x, y) is occupied by $x \bullet y$, and the event that the bond (x, y) is not occupied by $x \not\bullet y$.

E.4.1 Bound on $g^{(0)}$

We begin with an estimate on $g^{(0)}(0, u)$, for $u \neq 0$. This is simple [Fig. H-0]. By definition, we have the lower bound

$$g^{(0)}(0, u) = \text{Prob}[0 \Leftrightarrow u] \geq 0. \tag{E.30}$$

For an upper bound, we divide the sum into two. First, for $|u| > 1$, we just use the BK inequality and our bound (E.15) on $\tilde{B}(0)$ to get

$$\sum_{u:|u|>1} g^{(0)}(0, u) \leq \sum_{u:|u|>1} \tau(0, u)^2 = \tilde{B}(0) = O(s^2). \tag{E.31}$$

Second, for $|u| = 1$, we first classify the contributions according to whether $(0, e_1)$ is occupied or not. We can write

$$\begin{aligned} g^{(0)}(0, e_1) &= \text{Prob}[0 \Leftrightarrow e_1] = \text{Prob}[(0 \bullet e_1) \cap (0 \Leftrightarrow e_1)] + \text{Prob}[(0 \not\bullet e_1) \cap (0 \Leftrightarrow e_1)] \\ &= \text{Prob}[(0 \bullet e_1) \cap (e_1 \in \tilde{C}^{\{0,e_1\}}(0))] \\ &+ \text{Prob}[(0 \not\bullet e_1) \cap \{(e_1 \in \tilde{C}^{\{0,e_1\}}(0)) \circ (e_1 \in \tilde{C}^{\{0,e_1\}}(0))\}], \end{aligned} \tag{E.32}$$

where here and in the following, $E_1 \circ E_2$ denotes the event that E_1 and E_2 occur disjointly.

To bound the above, we will use the following lemma, which with its improved version, Lemma E.2, will be used repeatedly later.

Lemma E.1 *We have*

$$\text{Prob}[e_1 \in \tilde{C}^{\{0,e_1\}}(0)] \leq O(s^2). \tag{E.33}$$

Proof. For the event in question to occur, there should be at least one occupied path from 0 to e_1, in addition to (possibly occupied) $(0, e_1)$. In other words, we can always find two occupied bonds $(0, e)$ and $(e_1, e_1 + f)$, and an occupied path (this occupation is done disjointly from two previous occupied bonds) connecting e and $(e_1 + f)$ [here $e \neq e_1$ and $e_1 + f \neq 0$]. Overcounting, then applying BK, this gives

$$\begin{aligned} \text{Prob}[e_1 \in \tilde{C}^{\{0,e_1\}}(0)] \leq & \sum_{\substack{|e|=1:e\neq e_1 \\ |f|=1:e_1+f\neq 0}} \text{Prob}[(0 \bullet e) \circ (e \leftrightarrow (e_1 + f)) \circ ((e_1 + f) \bullet e_1)] \\ \leq & \; {p_c}^2 \sum_{\substack{|e|=1:e\neq e_1 \\ |f|=1:e_1+f\neq 0}} \tau(e, e_1 + f) \\ = & \; {p_c}^2 [(2d-2)\tau(0, e_1) + (2d-2)(2d-4)\tau(0, e_1 + e_2 + e_3) \\ + & \; 3(2d-2)\tau(0, 2e_1 + e_2) + \tau(0, 3e_1)] = O(s^2) \end{aligned} \tag{E.34}$$

□

Returning to (E.32), we employ BK and Lemma E.1 to conclude

$$\begin{aligned}\text{Prob}[0 \Leftrightarrow e_1] &\leq p_c \text{Prob}[e_1 \in \tilde{C}^{\{0,e_1\}}(0)] + \text{Prob}[e_1 \in \tilde{C}^{\{0,e_1\}}(0)]^2 \\ &\leq O(s^3) \end{aligned} \tag{E.35}$$

As a result, we have by (E.30), (E.31) and (E.35),

$$\sum_u g^{(0)}(0,u) = O(s^2). \tag{E.36}$$

E.4.2 Bound on $g^{(1)}$

Now we proceed to $g^{(1)}$. By definition, (see Fig. H-1)

$$g^{(1)}(0,v) \equiv p_c \sum_{|u-u'|=1} \left\langle \left\langle \text{I}[0 \Leftrightarrow u]\, E_2(u',v;\tilde{C}_0^{\{u,u'\}}(0)) \right\rangle^{(1)} \right\rangle^{(0)} \tag{E.37}$$

where (for details, see [14, 22]) $\tilde{C}_0^{\{u,u'\}}(0)$ is the connected cluster of 0 after setting (u,u') vacant, and $E_2(u',v;A)$ is the event that u' is connected to v through A and there is no pivotal bond for the connection from u' to v whose first endpoint is connected to u' through A. Graphically, this is represented as Fig. H-1, where either or both of the two triangles can be points.

We first note that if at least one of these triangles is not shrunk to a point, we have in essence a two-loop or three-loop diagram, which are at most $O(s^2)$. More precisely, we have

$$\left[\text{more than one-loop contribution to } \sum_v g^{(1)}(0,v)\right] = O(s^2).$$

So we have only to consider the case where both triangles are shrunk to a point, to get $O(s)$ results.

To bound this one-loop term, we first introduce

$$g_1^{(1)}(0,v) \equiv \left\langle \left\langle E_2(e_1,v;\tilde{C}_0^{\{0,e_1\}}(0)) \right\rangle^{(1)} \right\rangle^{(0)} \tag{E.38}$$

and note again by symmetry, we can write [Fig. H-1-1] (notice: now $u=0, u'=e_1$)

$$\left[\text{one-loop contributions to} \sum_v g^{(1)}(0,v)\right] = 2dp_c \sum_v g_1^{(1)}(0,v).$$

We classify the above sum by the values of v. First, the sum for $v \neq 0, e_1$ is simply bounded by BK as

$$\sum_{v:v\neq 0,e_1} g_1^{(1)}(0,v) \leq \sum_{v:v\neq 0,e_1} \text{Prob}^{(0)}[0 \leftrightarrow v]\text{Prob}^{(1)}[e_1 \leftrightarrow v] \leq B(e_1) = O(s^2).$$

Second, $v=0$ is bounded as

$$g_1^{(1)}(0,0) \leq \text{Prob}^{(1)}[e_1 \leftrightarrow 0] = \tau(0,e_1) = s + O(s^2)$$

[The reason why the above is NOT an equality is that we have to have a condition that $v = 0$ is the "first" intersection of the connection $e_1 \leftrightarrow v$ with $\tilde{C}(0)$. This condition is rather hard to treat exactly, though it will be done later, and here we will be satisfied with an upper bound only.] Third, when $v = e_1$,

$$g_1^{(1)}(0, e_1) \leq \text{Prob}^{(0)}[e_1 \in \tilde{C}_0^{\{0,e_1\}}(0)] = O(s^2)$$

by Lemma E.1.

Summing up all these three, we have

$$\sum_{v \neq 0} g_1^{(1)}(0, v) \leq O(s^2) + O(s^2) = O(s^2), \quad g_1^{(1)}(0, 0) \leq s + O(s^2).$$

E.4.3 Higher order contributions, and a new bound on p_c

Contributions from n-loop diagrams ($n \geq 2$) are at most $O(s^n)$, by (E.8). Summing over n we get (E.27) and (E.28).

Now a new bound on p_c. To get an upper bound, we use our bounds just derived for $\hat{g}^{(n)}$, combined with (E.7):

$$2dp_c = \frac{1}{1 + \hat{g}(0)} \leq \frac{1}{1 - s + O(s^2)} = 1 + s + O(s^2).$$

For a lower bound, we just use (β_c is the critical point of the self-avoiding walk)

$$p_c \geq \beta_c = s + s^2 + O(s^3).$$

(It is possible to get the above bound solely in the percolation context, but this would require calculations like those of later sections.)

E.5 Second iteration: B

Now we prove

$$\tau(0, x) = \begin{cases} s + 2s^2 + O(s^3) & (|x| = 1) \\ s^2 + O(s^3) & (x = 2e_1) \\ 2s^2 + O(s^3) & (x = e_1 + e_2) \\ O(s^3) & (|x| > 2) \end{cases} \tag{E.39}$$

$$B(x) = \begin{cases} s + 6s^2 + O(s^3) & (x = 0) \\ 4s^2 + O(s^3) & (|x| = 1) \\ s^2 + O(s^3) & (x = 2e_1) \\ 2s^2 + O(s^3) & (x = e_1 + e_2) \\ O(s^3) & (|x| > 2) \end{cases} \tag{E.40}$$

$$B''(x) = \begin{cases} 2s + 4s^2 + O(s^3) & (|x| = 1) \\ 3s^2 + O(s^3) & (x = 2e_1) \\ 6s^2 + O(s^3) & (x = e_1 + e_2) \\ O(s^3) & (|x| > 2) \end{cases} \tag{E.41}$$

$$\tilde{B}(0) = 2s^2 + O(s^3), \qquad \sum_{x:|x|>2} \tau(0, x)^2 = O(s^3) \tag{E.42}$$

Again, we use Lemma B.2, where f, $\hat{h}$, Δ, α are defined as in the proof of (E.14), i.e. by (E.12), (E.17), (E.18). But thanks to our effort in the previous section, in particular due to (E.27), we have instead of (E.19) the bounds:

$$\epsilon = O(s^2), \quad \delta = O(s^4). \tag{E.43}$$

We start from $\tau(0,x)$. This time Lemma B.2 applied to (E.20) gives

$$\tau(0,x) = \frac{1}{2dp_c} I_{1,0}(x) + O(s^3) - \frac{\delta_{0,x}}{(2dp_c)^2} \sum_{y\neq 0} f(y), \tag{E.44}$$

and if we estimate the right side using the expansions of Section A.2, we get our goal, (E.39).

Now, we proceed to $B(x)$. Lemma B.2 applied to (E.22) now gives

$$B''(x) = \frac{1}{(2dp_c)^2} I_{2,0}(x) + O(s^3) - \delta_{x,0} \frac{2}{(2dp_c)^3} \sum_{y\neq 0} f(y). \tag{E.45}$$

and thus

$$B(x) = \frac{1}{(2dp_c)^2} \left[I_{2,0}(x) - \delta_{x,0} - 2(2dp_c)\left\{ I_{1,0}(x) - \delta_{x,0} \right\} + (2dp_c - 1)^2 \delta_{x,0} + O(s^3) \right]. \tag{E.46}$$

Using (A.13)–(A.16), we get (E.40). And (E.42) now follows in exactly the same way as for SAW, using the analogue of (4.26).

E.6 Third iteration: A

We prove

$$\begin{aligned} \hat{g}^{(0)}(0) &= \frac{3}{2}s^2 + O(s^3) \\ \hat{g}^{(1)}(0) &= s + 6s^2 + O(s^3) \\ \hat{g}^{(2)}(0) &= 2s^2 + O(s^3), \end{aligned} \tag{E.47}$$

and thus get

$$2dp_c = \frac{1}{1+\hat{g}(0)} = 1 + s + \frac{7}{2}s^2 + O(s^3), \tag{E.48}$$

completing the proof.

This part is the most complicated. First, let us recall again that an n-loop diagram is bounded by s^n, by (E.8). Because we want to get estimates which are accurate to $O(s^2)$, we have only to consider one- and two-loop diagrams.

E.6.1 Bound on $g^{(0)}$

We start from $g^{(0)}$. By definition, (see Fig. H-0)

$$\begin{aligned} \hat{g}^{(0)}(0) &= \sum_u g^{(0)}(0,u) = \sum_{u\neq 0} \mathrm{Prob}[0 \Leftrightarrow u] \\ &= 2d\, \mathrm{Prob}[0 \Leftrightarrow e_1] + 2d(d-1)\, \mathrm{Prob}[0 \Leftrightarrow (e_1+e_2)] + 2d\, \mathrm{Prob}[0 \Leftrightarrow (2e_1)] \\ &+ \sum_{|u|>2} \mathrm{Prob}[0 \Leftrightarrow u]. \end{aligned} \tag{E.49}$$

We bound each term one by one. First,

$$0 \leq \sum_{|u|>2} \text{Prob}[0 \Leftrightarrow u] \leq \sum_{|u|>2} \tau(0,u)^2 = O(s^3) \tag{E.50}$$

by (E.42). And

$$0 \leq 2d \, \text{Prob}[0 \Leftrightarrow (2e_1)] \leq 2d \, \tau(0, 2e_1)^2 = O(s^3) \tag{E.51}$$

by (E.39).

To deal with $\text{Prob}[0 \Leftrightarrow e_1]$ we use the following refinement of Lemma E.1.

Lemma E.2 *We have*

$$\text{Prob}[e_1 \in \tilde{C}^{\{0,e_1\}}(0)] = s^2 + O(s^3). \tag{E.52}$$

Proof. The upper bound is now easy. We just employ (E.34), now with our improved estimates (E.39) on $\tau(0,x)$.

For a lower bound, we reason in a way similar to the proof of Lemma E.1, but now concentrate on configurations where $e = f$ there. Unfortunately, we have to be quite careful not to overcount in getting lower bounds. For this, we introduce a notation: Let F_j ($j = \pm 2, \pm 3, \ldots, \pm d$) be the event

$$F_j \equiv (0 \bullet e_j) \cap (e_j \bullet (e_j + e_1)) \cap ((e_j + e_1) \bullet e_1). \tag{E.53}$$

Then

$$\text{Prob}[e_1 \in \tilde{C}^{\{0,e_1\}}(0)] \geq \text{Prob}[\cup_{j=\pm 2}^{\pm d} F_j]$$

and now we use inclusion-exclusion to get

$$\begin{aligned} &\geq \sum_{j=\pm 2}^{\pm d} \text{Prob}[F_j] - \sum_{i \neq j: i,j=\pm 2}^{\pm d} \text{Prob}[F_i \cap F_j] \\ &\geq (2d-2){p_c}^3 - (2d)^2 {p_c}^6 = s^2 + O(s^3). \end{aligned}$$

□

Now by (E.32) and the above lemma,

$$\text{Prob}[0 \Leftrightarrow e_1] \geq \text{Prob}[(0 \bullet e_1) \cap (e_1 \in \tilde{C}_0^{\{0,e_1\}}(0))] = p_c \, \text{Prob}[e_1 \in \tilde{C}_0^{\{0,e_1\}}(0)] = s^3 + O(s^4).$$

Combining this with (E.35), we have

$$\text{Prob}[0 \Leftrightarrow e_1] = s^3 + O(s^4). \tag{E.54}$$

Next we move on to $\text{Prob}[0 \Leftrightarrow (e_1 + e_2)]$. For a lower bound,

$$\begin{aligned} \text{Prob}[0 \Leftrightarrow (e_1 + e_2)] &\geq \text{Prob}[(0 \bullet e_1) \cap (e_1 \bullet (e_1 + e_2)) \cap ((e_1 + e_2) \bullet e_2) \cap (e_2 \bullet 0)] \\ &= {p_c}^4 = s^4 + O(s^5). \end{aligned} \tag{E.55}$$

For an upper bound, we note that there should be two distinct occupied bonds, emanating from the origin, both connected to $(e_1 + e_2)$:

$$\text{Prob}[0 \Leftrightarrow (e_1 + e_2)] \leq \frac{1}{2} \sum_{|e|=|f|=1: e \neq f} \text{Prob}[(0 \bullet e) \circ (0 \bullet f) \circ (e \leftrightarrow (e_1 + e_2)) \circ (f \leftrightarrow (e_1 + e_2))]$$

$$\leq \frac{1}{2} \sum_{|e|=|f|=1: e\neq f} {p_c}^2 \tau(e, e_1+e_2)\tau(f, e_1+e_2).$$

We classify terms according to whether (i) $\{e, f\} = \{e_1, e_2\}$; (ii) $|\{e, f\} \cap \{e_1, e_2\}| = 1$; (iii) $\{e, f\} \cap \{e_1, e_2\} = \phi$. The contribution to the sum from each class of terms is (i) ${p_c}^2\tau(0, e)^2 = s^4 + O(s^5)$; (ii) $O(d)O(s^2)O(s)O(s^3)$; (iii) $O(d^2)O(s^2)O(s^3)^2$. Therefore

$$\text{Prob}[0 \Leftrightarrow (e_1 + e_2)] = s^4 + O(s^5). \tag{E.56}$$

Summarizing, by (E.49)–(E.56), we now get $\hat{g}^{(0)}(0) = \frac{3}{2}s^2 + O(s^3)$.

E.6.2 Bound on $g^{(1)}$

Unfortunately, $g^{(1)}$ is more complicated. This is given by Diagram H-1, and by (E.37).

This diagram usually has three loops and gives $O(s^3)$, except when one or both of the triangles are shrunk to a point. We consider separately the cases where both $u = 0$ and $z' = v$ (i.e. both triangles shrunk to a point), and where one of the triangles is not shrunk to a point. Here we are using the notation[1] of Figure 4 of [14].

Case 1. $u = 0, z' = v$. This case is schematically depicted in Fig. H-1-1. As was noted in Section E.4.2, we have

$$\left[u = 0, z' = v \text{ contribution to } \hat{g}^{(1)}(0)\right] = 2dp_c \sum_v g_1^{(1)}(0, v),$$

where $g_1^{(1)}(0, v)$ was defined in (E.38). Note that we can write

$$g_1^{(1)}(0, v) \equiv \left\langle \mathrm{I}[v \in \tilde{C}_0^{\{0,e_1\}}(0)] \left\langle \mathrm{I}[E_2(e_1, v; \tilde{C}_0^{\{0,e_1\}}(0)]\right\rangle^{(1)}\right\rangle^{(0)} \tag{E.57}$$

in this special case. The inserted indicator comes from the definition of E_2, and is redundant, but will be useful later. We consider five cases (A)–(E) separately, depending on the values of v.

(A) First, $v = e_1$ [Fig. H-1-2]. We have by Lemma E.2,

$$g_1^{(1)}(0, e_1) = \text{Prob}^{(0)}[e_1 \in \tilde{C}_0^{\{0,e_1\}}(0)] = s^2 + O(s^3).$$

(B) Second, $v = 0$ [Fig. H-1-3]. An upper bound is easy:

$$g_1^{(1)}(0, 0) \leq \tau(0, e_1) = s + 2s^2 + (s^3). \tag{E.58}$$

The lower bound is complicated by the fact that the inner expectation has the constraint that $v = 0$ is the first site which is "through" $\tilde{C}_0(0)$. To express this fact, right now we can only proceed as follows: We classify inner configurations according to whether $(0, e_1)$ is occupied or not,

$$\begin{aligned} g_1^{(1)}(0, 0) &= \left\langle\left\langle \mathrm{I}[E_2(e_1, 0; \tilde{C}_0^{\{0,e_1\}}(0)]\right\rangle^{(1)}\right\rangle^{(0)} \\ &= \left\langle\left\langle \mathrm{I}^{(1)}[(e_1 \bullet 0) \cap E_2(e_1, 0; \tilde{C}_0^{\{0,e_1\}}(0))]\right\rangle^{(1)}\right\rangle^{(0)} \\ &+ \left\langle\left\langle \mathrm{I}^{(1)}[(0 \not\bullet e_1) \cap E_2(e_1, 0; \tilde{C}_0^{\{0,e_1\}}(0))]\right\rangle^{(1)}\right\rangle^{(0)}. \end{aligned} \tag{E.59}$$

[1]There is an error in (2.9) of [14]: in the case where $z' = x$ in Fig. 4 of [14], we cannot say that z' is connected to v in $\mathbf{Z}^d \backslash A$. Instead, we should have said z, in place of z'. This error is avoided in [22].

In the above, the first term can be rewritten as

$$\Big\langle \mathrm{I}[e_1 \notin \tilde{C}_0^{\{0,e_1\}}(0)] \left\langle \mathrm{I}^{(1)}[e_1 \bullet 0] \right\rangle^{(1)} \Big\rangle^{(0)} = \mathrm{Prob}^{(0)}[e_1 \notin \tilde{C}_0^{\{0,e_1\}}(0)]\, \mathrm{Prob}^{(1)}[e_1 \bullet 0]$$
$$= \Big\{ 1 - \mathrm{Prob}^{(0)}[e_1 \in \tilde{C}_0^{\{0,e_1\}}(0)] \Big\}\, p_c. \tag{E.60}$$

The reason why we could insert $\mathrm{I}[e_1 \notin \tilde{C}_0^{\{0,e_1\}}(0)]$ was that if $e_1 \in \tilde{C}_0^{\{0,e_1\}}(0)$, we could have cut at e_1, and thus the contribution is $v = e_1$, not $v = 0$. Using Lemma E.2 to bound $\mathrm{Prob}^{(0)}[e_1 \in \tilde{C}_0^{\{0,e_1\}}(0)] = O(s^2)$, we now have

$$[\text{the first term of (E.59)}] = s + s^2 + O(s^3).$$

To deal with the second term, we use the notation F_j defined in (E.53). Extracting the shortest path contribution to the second term, we write

$$\begin{aligned}
&[\text{second term of (E.59) }] \\
&\geq \Big\langle \Big\langle \mathrm{I}[(0 \not\bullet e_1) \cap \{\cup_{j=\pm 2}^{j=\pm d} F_j^{(1)}\} \cap E_2^{(1)}(e_1, 0; \tilde{C}_0^{\{0,e_1\}}(0))] \Big\rangle^{(1)} \Big\rangle^{(0)} \\
&\geq \sum_{j=\pm 2}^{\pm d} \Big\langle \Big\langle \mathrm{I}[(0 \not\bullet e_1) \cap F_j^{(1)} \cap E_2^{(1)}(e_1, 0; \tilde{C}_0^{\{0,e_1\}}(0))] \Big\rangle^{(1)} \Big\rangle^{(0)} \\
&- \sum_{i \neq j : i,j=\pm 2}^{\pm d} \Big\langle \Big\langle \mathrm{I}[F_i^{(1)} \cap F_j^{(1)} \cap E_2^{(1)}(e_1, 0; \tilde{C}_0^{\{0,e_1\}}(0))] \Big\rangle^{(1)} \Big\rangle^{(0)},
\end{aligned} \tag{E.61}$$

where we removed the constraint $(0 \not\bullet e_1)$ in the last term via an upper bound. If $F_j^{(1)}$ occurs and in addition $e_1, e_2, e_1 + e_2 \notin \tilde{C}_0^{\{0,e_1\}}(0)$, then the event $F_j^{(1)} \cap E_2^{(1)}(e_1, 0; \tilde{C}_0^{\{0,e_1\}}(0))$ occurs. Therefore the first sum of (E.61) is bounded below by

$$[\text{the first sum of (E.61) }] \geq (1 - p_c)(2d - 2){p_c}^3\, \mathrm{Prob}^{(0)}[e_1, e_2, e_1 + e_2 \notin \tilde{C}_0^{\{0,e_1\}}(0)].$$

Now we apply inclusion-exclusion to rewrite the above as

$$\begin{aligned}
&[\text{the first sum of (E.61) }] \\
&\geq (1 - p_c)(2d - 2){p_c}^3 \Big[1 - \mathrm{Prob}^{(0)}[e_1,\ \text{or } e_2,\ \text{or } e_1 + e_2 \in \tilde{C}_0^{\{0,e_1\}}(0)] \Big] \\
&\geq (1 - p_c)(2d - 2){p_c}^3 \Big[1 - \mathrm{Prob}^{(0)}[e_1 \in \tilde{C}_0^{\{0,e_1\}}(0)] - \mathrm{Prob}^{(0)}[e_2 \in \tilde{C}_0^{\{0,e_1\}}(0)] \\
&\quad - \mathrm{Prob}^{(0)}[e_1 + e_2 \in \tilde{C}_0^{\{0,e_1\}}(0)] \Big] \\
&\geq (1 - p_c)(2d - 2){p_c}^3 \Big[1 - O(s^2) - \tau(e_2, 0) - \tau(e_1 + e_2, 0) \Big] = s^2 + O(s^3).
\end{aligned} \tag{E.62}$$

On the other hand, the second term of (E.61), without the minus sign, is bounded as

$$\Big\langle \Big\langle \mathrm{I}[F_i^{(1)} \cap F_j^{(1)} \cap E_2^{(1)}(e_1, 0; \tilde{C}_0^{\{0,e_1\}}(0))] \Big\rangle^{(1)} \Big\rangle^{(0)} \leq \mathrm{Prob}^{(1)}[F_i \cap F_j] \leq {p_c}^6 = O(s^6).$$

Adding these two, we have

$$[\text{the second term of } (E.59)] \geq \ \text{(E.61)}\ \geq s^2 + O(s^3), \tag{E.63}$$

and thus by (E.58), (E.60) and (E.63)

$$g_1^{(1)}(0,0) = s + 2s^2 + O(s^3).$$

(C) Third, $|v| = 1, v \neq e_1$ [Fig. H-1-4]. For an upper bound we argue

$$\sum_{|v|=1, v\neq e_1} g_1^{(1)}(0,v) \leq \sum_{|v|=1, v\neq e_1} \left\langle \mathrm{I}[v \in C_0(0)] \left\langle \mathrm{I}[v \in \tilde{C}_1^{\{0,e_1\}}(e_1)] \right\rangle^{(1)} \right\rangle^{(0)}$$

where in the innermost expectation, we have $v \in \tilde{C}_1^{\{0,e_1\}}(e_1)$ rather than simple $v \in C_1(e_1)$, because if $v \in C_1(e_1)$ but $v \notin \tilde{C}_1^{\{0,e_1\}}(e_1)$ we could have "cut" at 0, rather than at v, to define g_1. (The above inequality appears to be strict, because there are cases where the RHS contains an "earlier cut" case.) So arguing in a way similar to the proof of Lemmas E.1, E.2,

$$\begin{aligned} &\leq \sum_{|v|=1, v\neq e_1} \left\langle \mathrm{I}[v \in C_0(0)] \left\langle \sum_{|f|=1: f+e_1\neq 0} \mathrm{I}[(e_1 \bullet (e_1+f)) \circ ((e_1+f) \leftrightarrow v)] \right\rangle^{(1)} \right\rangle^{(0)} \\ &\leq \sum_{|v|=1, v\neq e_1} \sum_{|f|=1: f+e_1\neq 0} \tau(0,v)\,\tau(0,f)\,\tau(v, e_1+f) = s^2 + O(s^3). \end{aligned} \tag{E.64}$$

[In the above, most of the terms, except for $f = v$, contribute $O(s^3)$ in all.]

For a lower bound,

$$\begin{aligned} \sum_{|v|=1, v\neq e_1} g_1^{(1)}(0,v) &\geq (2d-2) g_1^{(1)}(0,e_2) \\ &\geq (2d-2)\mathrm{Prob}^{(0)}[0\bullet e_2]\,\mathrm{Prob}^{(1)}[e_2 \bullet (e_1+e_2)]\,\mathrm{Prob}^{(1)}[(e_1+e_2)\bullet e_1] \\ &= s^2 + O(s^3). \end{aligned} \tag{E.65}$$

(D) Fourth, $|v - e_1| = 1, v \neq 0$ [Fig. H-1-5]. We can argue similarly. Concretely,

$$\begin{aligned} \sum_{|v-e_1|=1, v\neq 0} g_1^{(1)}(0,v) &\leq \sum_{|v-e_1|=1, v\neq 0} \left\langle \mathrm{I}[v \in \tilde{C}_0^{\{0,e_1\}}(0)] \left\langle \mathrm{I}[v \in C_1(e_1)] \right\rangle^{(1)} \right\rangle^{(0)} \\ &\leq \sum_{|v-e_1|=1, v\neq 0} \left\langle \sum_{|f|=1, f\neq e_1} \mathrm{I}[(0\bullet f) \circ (f \leftrightarrow v)] \right\rangle^{(0)} \tau(v, e_1) \\ &\leq \sum_{|v-e_1|=1, v\neq 0} \sum_{|f|=1, f\neq e_1} p_c \tau(f,v) \tau(v, e_1) \\ &= s^2 + O(s^3). \end{aligned} \tag{E.66}$$

For a lower bound, similarly,

$$\begin{aligned} \sum_{|v-e_1|=1, v\neq 0} g_1^{(1)}(0,v) &\geq (2d-2) g_1^{(1)}(0, e_1+e_2) \\ &\geq (2d-2)\mathrm{Prob}^{(0)}[0\bullet e_2]\mathrm{Prob}^{(0)}[e_2 \bullet (e_1+e_2)]\mathrm{Prob}^{(1)}[(e_1+e_2)\bullet e_1] \\ &= s^2 + O(s^3) \end{aligned} \tag{E.67}$$

(E) Finally, when $|v|, |v-e_1| > 1$, we can simply bound as

$$\begin{aligned}\sum_{v:|v|,|v-e_1|>1} g_1^{(1)}(0,v) &\le \sum_{v:|v|,|v-e_1|>1} \tau(0,v)\tau(e_1,v) \\ &\le B(e_1) - 2(2d-2)\tau(0,e_1+e_2)\tau(e_1,e_1+e_2) \\ &= B(e_1) - \left[4s^2 + O(s^3)\right] = O(s^3). \end{aligned} \tag{E.68}$$

Combining all these (A)–(E), we have

$$\sum_v g_1^{(1)}(0,v) = s + 5s^2 + O(s^3).$$

Case 2: either $u \neq 0$ or $z' \neq v$. These are easier, and are $O(s^3)$.

We have to consider two cases, depicted in Fig. H-1-8, H-1-9. We denote them respectively by $g_2^{(1)}(0,v;u',z)$ and $g_3^{(1)}(0,v;u,u')$. We just use the BK inequality.

Remark. The reason why y is not included as an argument is that there can be several such y's, and it cannot appear in the summation (if we want to avoid overcounting). Also, we use the convention that $g_2^{(1)}(0,v;u',z)$ does not carry the factor of p_c of the pivotal bond. Thus we want to show that these quantities are $O(s^2)$.

Case 2-1. We begin with $g_2^{(1)}(0,v;u',z)$. As shown in the Figure, z and v should be nontrivially doubly connected. And in this case there should be a y as in the figure. We first classify the following three cases. (A) Only such y is $y=v$, (B) Only such y is $y=z$, (C) Other cases, i.e. we can choose $y \neq v, z$.

(C) First, when $y \neq v, z$. By BK, we have (by symmetry we have changed u' into e_1 and multiplied by $2d$, and similarly below)

$$\begin{aligned}&\left[\text{Case (C) contribution to } \sum_{v,u',z} g_2^{(1)}(0,v;u',z)\right] \\ &\le 2d \sum_{v,z,y:v\neq z\neq y\neq v} \tau(0,y)\tau(y,v)\tau(v,z)\tau(z,y)\tau(z,e_1) \\ &= 2d \sum_{z,y:z\neq y} \tau(0,y)\tau(z,y)\tau(z,e_1)B(y-z) \\ &\le 2dT'''(e_1)\sup_{x\neq 0} B(x) = 2dO(s)O(s^2) = O(s^2). \end{aligned} \tag{E.69}$$

(A) Second, when $y=v$. We use BK only to connections between $0, v=y$ and e_1, z

$$\begin{aligned}&\left[\text{Case (A) contribution to } \sum_{v,u',z} g_2^{(1)}(0,v;u',z)\right] \le 2d \sum_{v,z:v\neq z} \tau(0,v)\tau(z,e_1)\text{Prob}^{(1)}[v\Leftrightarrow z] \\ &= 2d \sum_{v,z:v\neq 0} \text{Prob}^{(1)}[0\Leftrightarrow v]\tau(v,z)\tau(z,e_1) = 2d\sum_{v\neq 0} \text{Prob}^{(1)}[0\Leftrightarrow v]B''(v-e_1) \end{aligned} \tag{E.70}$$

where we interchanged the order of convolution. We further classify whether $v=e_1$ or not:

$$= 2d\left\{\text{Prob}^{(1)}[0\Leftrightarrow e_1]B''(0) + \sum_{v:v\neq 0,e_1} \text{Prob}^{(1)}[0\Leftrightarrow v]B''(v-e_1)\right\}$$

$$\leq \quad 2d\left\{\text{Prob}^{(1)}[0\Leftrightarrow e_1]B''(0)+\hat{g}^{(0)}(0)\sup_{v:v\neq e_1}B''(v-e_1)\right\}$$
$$= \quad (2d)\left[O(s^3)O(1)+O(s^2)O(s)\right]=O(s^2). \qquad \text{(E.71)}$$

Thus we finally get

$$[\text{Case (A) contribution}]=O(s^2).$$

(B) Third, when $z=y$. In a similar way,

$$\left[\text{Case (B) contribution to } \sum_{v,u',z} g_2^{(1)}(0,v;u',z)\right] \leq 2d\sum_{v,z:v\neq z}\tau(0,z)\tau(z,e_1)\text{Prob}^{(1)}[v\Leftrightarrow z]$$
$$= \quad 2d\left\{B''(e_1)\sum_{x\neq 0}\text{Prob}^{(1)}[0\Leftrightarrow x]\right\}=2dO(s)O(s^2)=O(s^2). \qquad \text{(E.72)}$$

Combining (A)–(C), we finally get

$$\sum_{v,u',z} g_2^{(1)}(0,v;u',z)=O(s^2).$$

Case 2-2. Next, $g_3^{(1)}(0,v;u,u')$; this is almost the same as Case 2-1. We use BK; see Fig. H-1-9. We classify according to the values of y,v,u,u', but first, we use translation invariance to rewrite Fig. H-1-9 into H-1-9' (the graph is with those for section C).

Now in the figure, 0 and z are nontrivially doubly connected (in particular $z\neq 0$). So there should be a y as in the figure. Then we classify into three cases: (A) only such y is $y=0$. (B) only such y is $y=z$., (C) Other cases, where we can take $y\neq 0,z$.

(C) When we can take $y\neq 0,z$, we use BK to obtain

$$\left[\text{Case (C) contribution to } \sum_{v,u',z} g_2^{(1)}(0,v;u,u')\right]$$
$$\leq \quad 2d\sum_{y,v,z:z\neq 0;y\neq 0,z}\tau(0,z)\tau(z,y)\tau(y,0)\tau(y,v)\tau(v,e_1). \qquad \text{(E.73)}$$

We classify according to the values of $|y-e_1|$. First, when $y=e_1$,

$$2d\sum_{y,v,z:z\neq 0;y\neq 0,z}\tau(0,z)\tau(z,y)\tau(y,0)\tau(y,v)\tau(v,e_1)\text{I}[y=e_1]$$
$$= \quad 2d\sum_{v,z:z\neq 0,e_1}\tau(0,z)\tau(z,e_1)\tau(e_1,0)\tau(e_1,v)\tau(v,e_1)$$
$$= \quad 2dB(e_1)\tau(e_1,0)B''(0)=2dO(s^2)O(s)O(1)=O(s^2). \qquad \text{(E.74)}$$

Second, when $|y-e_1|=1$,

$$2d\sum_{y,v,z:z\neq 0;y\neq 0,z}\tau(0,z)\tau(z,y)\tau(y,0)\tau(y,v)\tau(v,e_1)\text{I}[|y-e_1|=1]$$
$$= \quad 2d\sum_{z\neq 0;y\neq 0,z}\tau(0,z)\tau(z,y)\tau(y,0)B''(y-e_1)\text{I}[|y-e_1|=1]$$
$$\leq \quad (2d)B''(e_1)\sum_{0\neq z\neq y\neq 0}\tau(0,z)\tau(z,y)\tau(y,0)\text{I}[|y-e_1|=1]$$
$$= \quad (2d)B''(e_1)\left[(2d-2)B(e_1+e_2)\tau(e_1+e_2,0)+B(2e_1)\tau(2e_1,0)\right]$$
$$= \quad (2d)O(s)\left[(2d-2)O(s^2)O(s^2)+O(s^2)O(s^2)\right]=O(s^3). \qquad \text{(E.75)}$$

Third, when $|y - e_1| > 1$,

$$
\begin{aligned}
&2d \sum_{y,v,z:z\neq 0;y\neq 0,z} \tau(0,z)\tau(z,y)\tau(y,0)\tau(y,v)\tau(v,e_1)\mathrm{I}[|y-e_1|>1] \\
&\leq (2d) \sum_{y,v,z:z\neq 0;y\neq 0,z} \tau(0,z)\tau(z,y)\tau(y,0)\left[\sup_{|y-e_1|>1} \tau(y,v)\tau(v,e_1)\right] \\
&\leq (2d)\left[T'''(0)-1\right]\left[\sup_{|y-e_1|>1} B''(y-e_1)\right] = (2d)O(s)O(s^2) = O(s^2). \qquad \text{(E.76)}
\end{aligned}
$$

(A) $y = 0$. In this case, we should have $0 \leftrightarrow v$ and a connection $0 \Leftrightarrow z$, disjointly. So we can use BK

$$
\begin{aligned}
&\left[\text{Case (A) contribution to } \sum_{v,u',z} g_2^{(1)}(0,v;u,u')\right] \leq 2d \sum_{z\neq 0} \mathrm{Prob}^{(0)}[0\Leftrightarrow z] \sum_v \tau(0,v)\tau(v,e_1) \\
&\leq (2d)\hat{g}^{(0)}(0)B''(e_1) = (2d)O(s^2)O(s) = O(s^2). \qquad \text{(E.77)}
\end{aligned}
$$

(B) $y = z$. Similar.

$$
\begin{aligned}
&\left[\text{Case (B) contribution to } \sum_{v,u',z} g_2^{(1)}(0,v;u,u')\right] \leq 2d \sum_{z\neq 0} \mathrm{Prob}^{(0)}[0\Leftrightarrow z] \sum_v \tau(z,v)\tau(v,e_1) \\
&\leq 2d \left\{ \mathrm{Prob}^{(0)}[0\Leftrightarrow e_1]B''(0) + \left[\sum_{z\neq 0,e_1} \mathrm{Prob}^{(0)}[0\Leftrightarrow z]\right]\left[\sup_{z\neq e_1} B''(z-e_1)\right]\right\} \\
&\leq 2d\left[O(s^3)O(1) + O(s^2)O(s)\right] = O(s^2), \qquad \text{(E.78)}
\end{aligned}
$$

where we used (E.54) to bound $\mathrm{Prob}^{(0)}[0\Leftrightarrow e_1]$.

So from (A)–(C), we have

$$
\sum_{v,u',z} g_2^{(1)}(0,v;u,u') = O(s^2). \qquad \text{(E.79)}
$$

E.6.3 Bound on $g^{(2)}$

To deal with $g^{(2)}$, we again emphasize that we only have to consider the effectively 2-loop diagrams. There is only one such diagram, i.e. the diagram in which all shaded triangles are shrunk to a point. This diagram is schematically presented in Fig. H-2-1. Again we do not include "p_c" of a pivotal bond in the figure.

By symmetry,

$$
\begin{aligned}
&\sum_w \left[\text{"truly two-loop" contribution to } g^{(2)}(0,w)\right] \\
&= 2d{p_c}^2 \sum_{w,v,v':|v-v'|=1} [\text{Fig. H-2-1}] \equiv 2d{p_c}^2 \sum_{w,v,v':|v-v'|=1} g_1^{(2)}(0,w;v,v') \qquad \text{(E.80)}
\end{aligned}
$$

In the following, we consider several cases, depending on the values of v, w (and t in the figure).

(1) First, $v = 0, w = e_1$ case.

For an upper bound, just use BK as

$$\sum_{v':|v'|=1} g_1^{(2)}(0,e_1;0,v') \le \sum_{v':|v'|=1} \text{Prob}^{(1)}[e_1 \leftrightarrow 0]\text{Prob}^{(2)}[e_1 \leftrightarrow v'] = s + O(s^2).$$

For a lower bound, we just pick up the $v=0, w=v'=e_1$ term:

$$\begin{aligned}
&\sum_{v':|v'|=1} g_1^{(2)}(0,e_1;0,v') \ge g_1^{(2)}(0,e_1;0,e_1) \\
\ge\ & \left\langle \left\langle E_2^{(1)}(e_1,0;\tilde{C}_0^{\{0,e_1\}}(0)) \right\rangle^{(1)} \right\rangle^{(0)} \ge \text{Prob}^{(0)}[e_1 \notin \tilde{C}_0^{\{0,e_1\}}(0)]\text{Prob}^{(1)}[e_1 \bullet 0] \\
\ge\ & \left\{ 1 - \text{Prob}^{(0)}[e_1 \in \tilde{C}_0^{\{0,e_1\}}(0)] \right\} p_c = s + O(s^2), \qquad \text{(E.81)}
\end{aligned}$$

where we used Lemma E.2 in the last step.

(2) Second, $v=0, w=0$ case.

For an upper bound, again just use BK to get

$$\sum_{v':|v'|=1} g_1^{(2)}(0,0;0,v') \le \sum_{v':|v'|=1} \text{Prob}^{(1)}[e_1 \leftrightarrow 0]\text{Prob}^{(2)}[0 \leftrightarrow v'] = s + O(s^2).$$

For a lower bound, (again use Lemma E.2 in the last step)

$$\begin{aligned}
&\sum_{v':|v'|=1} g_1^{(2)}(0,0;0,v') \ge (2d-2) g_1^{(2)}(0,0;0,e_2) \\
\ge\ & (2d-2) \left\langle \left\langle \left\langle \text{I}^{(1)}[e_1 \bullet 0]\, \text{I}^{(2)}[0 \bullet e_2]\, \text{I}[e_2 \notin C_1^{\{e_2,0\}}(0)] \text{I}[e_1 \notin C_0^{\{e_1,0\}}(0)] \right\rangle^{(2)} \right\rangle^{(1)} \right\rangle^{(0)} \\
\ge\ & (2d-2) {p_c}^2\, \text{Prob}^{(1)}[e_2 \notin C_1^{\{e_2,0\}}(0)]\, \text{Prob}^{(0)}[e_1 \notin C_0^{\{e_1,0\}}(0)] \\
=\ & (2d-2) {p_c}^2 [1 - O(s^2)]^2 = s + O(s^2). \qquad \text{(E.82)}
\end{aligned}$$

(3) Third: other cases, i.e. $(v \ne 0)$ or $(v=0; w \ne 0, e_1)$. As we will see, other contributions all sum up to $O(s^2)$. We will prove this by using BK, and also classifying the contributions by possible values of t in the figure.

(3-a) The case where we can take $t = v'$ [in addition to $(v \ne 0)$ or $(v=0, w \ne 0, e_1)$]. We just use BK, to get

$$g_1^{(2)}(0,w;v,v') \le \tau(0,v)\tau(v,v')\tau(e_1,v')\tau(v',w)^2.$$

Now we sum the RHS under the constraint, $(v \ne 0)$ or $(v=0, w \ne 0, e_1)$ [with $|v - v'| = 1$]. We write $v' = v + f$. We classify into three cases.

First, $f = e_1, v = 0$: In this case we also have $w \ne 0, e_1$, and thus

$$\sum_{v=0, f=e_1} \sum_{w \ne 0, e_1} g_1^{(2)}(0,w;v,v+f) \le \tau(0,e_1) \sum_{w \ne 0, e_1} \tau(e_1,w)^2 \le \tau(0,e_1) B(0) = O(s^2).$$

Second, $f = e_1, v \ne 0$:

$$\begin{aligned}
\sum_{v \ne 0, f=e_1} \sum_w g_1^{(2)}(0,w;v,v+f) &\le \tau(0,e_1) \sum_{v \ne 0} \tau(0,v)^2 \sum_w \tau(e_1,w)^2 \\
&= \tau(0,e_1) B(0) B''(0) = O(s^2). \qquad \text{(E.83)}
\end{aligned}$$

Third, $f \neq e_1$:

$$\sum_{|f|=1, f\neq e_1} \sum_w g_1^{(2)}(0,w;v,v+f) \leq \tau(0,e_1) \sum_{|f|=1,f\neq e_1} \sum_w \tau(0,v)\tau(e_1,v+f)\tau(v+f,w)^2$$
$$\leq \tau(0,e_1) \sum_{|f|=1,f\neq e_1} B''(f-e_1)B''(0) = O(s^2). \quad \text{(E.84)}$$

In the last step, we used the fact that now $|f-e_1| \geq 2$.

(3-b) Second, $|t-v'|=1$ case: We write $v' = v+f$, $t = v+f+g$, with $|f|=|g|=1$. We classify into six cases, as follows:

First, $f+g=0, v=0$ case. This is the most complicated. In this case because we have already treated $v=0, w=0, e_1$ cases in (1) and (2), we can assume $w \neq 0, e_1$. We further consider $w=f$ and $w \neq f$ separately. When $w \neq f$, we just use BK,

$$\leq \sum_{|f|=1} \sum_{w\neq f,0} \tau(0,e_1)\tau(0,w)\tau(w,f) = \sum_{|f|=1} \tau(0,e_1)B(e_1) = O(s^2).$$

and when $w=f$ note that now we have

$$\leq \sum_{|f|=1} \mathrm{Prob}^{(1)}[(e_1 \leftrightarrow 0) \circ (f \in \tilde{C}_1^{\{0,f\}})],$$

from the definition of the event E_2. This is bounded using Lemma E.2 by

$$\leq \sum_{|f|=1} p_c\, O(s^2) = O(s^2).$$

Second, $f+g=0, v=e_1$. By definition of the diagram, here we must have disjoint connections from e_1+f to w, from w to e_1, and finally from 0 to $v=e_1$, with the latter connection not via the bond $(0,e_1)$. This leads to the bound

$$\leq \mathrm{Prob}^{(0)}[e_1 \in \tilde{C}_0^{\{0,e_1\}}(0)] \sum_{|f|=1} \sum_w \tau(e_1+f,w)\tau(w,e_1) = O(s^2)(2d)B''(f) = O(s^2).$$

Third, $f+g=0, v\neq 0, e_1$.

$$\leq \sum_{|f|=1} \sum_{v\neq 0,e_1} \sum_w \tau(0,v)\tau(e_1,v)\tau(v,w)\tau(w,v+f) = \sum_{|f|=1} B(e_1)B''(f) = O(s^2).$$

Fourth, $f=e_1$. Since the case $f+g=0$ has been already taken care of, we only consider $g \neq -e_1$ here.

$$\leq \sum_{|g|=1,g\neq -e_1} \sum_{v,w} \tau(0,v)\tau(v,v+e_1+g)\tau(v+e_1+g,e_1)\tau(v+e_1+g,w)\tau(w,v+e_1)$$
$$\leq \sum_{|g|=1,g\neq -e_1} \tau(0,e_1+g)B''(-g)B''(e_1) \leq (2d-1)O(s^2)O(s)^2 = O(s^3). \quad \text{(E.85)}$$

Fifth, $g=e_1$ case is treated quite similarly as above, and is $O(s^3)$.

Sixth, finally when $f,g\neq e_1$, note that we now have $|f+g-e_1| \geq 3$ because $f+g=0$ has already been taken care of. So

$$\leq \sum_{|f|=|g|=1: f,g\neq e_1} B''(f+g-e_1)B''(g)\tau(0,f+g) \leq (2d)^2 O(s^3)O(s)O(s^2) = O(s^4).$$

(3-c) Finally,

$$\begin{aligned}
[|v'-t|>1 \text{ contribution}] &\leq \left[\sum_{t,v,f:|f|=1,|v+f-t|>1} \tau(0,v)\tau(v,t)\tau(t,e_1)\right] \sup_{|y|>1} B''(y) \\
&\leq (2d)T'(e_1) \sup_{|y|>1} B''(y) \leq (2d)O(s)O(s^2) \\
&= O(s^2). \qquad \text{(E.86)}
\end{aligned}$$

Combining (3-a)–(3-c), we see that the contributions from case (3) to the summation $\sum_{w,v,v'} g_1^{(2)}(0,w;v,v')$ do not exceed $O(s^3)$.

Combining cases (1)–(3), we finally have

$$\sum_w g^{(2)}(0,w) = 2s^2 + O(s^3). \qquad \text{(E.87)}$$