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A simple convergence proof for the lace expansion

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Abstract. We use the lace expansion to give a simple proof that the critical two-point function for weakly self-avoiding walk on $\mathbb{Z}^d$ has decay $|x|^{-(d-2)}$ in dimensions $d > 4$. The proof uses elementary Fourier analysis and the Riemann–Lebesgue Lemma.

Résumé. Nous utilisons le développement en lacets afin de montrer que la fonction à deux points de la marche faiblement auto-évitante au point critique et en dimension $d > 4$ décroît comme $|x|^{-(d-2)}$. La preuve en appelle de manière élémentaire à la théorie de Fourier ainsi qu'au Lemme de Riemann–Lebesgue.

1. Introduction and main result

The lace expansion has been used to prove $|x|^{-(d-2)}$ decay for the long-distance behaviour of critical two-point functions in a variety of statistical mechanical models on $\mathbb{Z}^d$ above their upper critical dimensions, including self-avoiding walk for $d > 4$ [3,4,8,10], percolation for $d > 6$ [8,10], lattice trees and lattice animals for $d > 8$ [8,10], the Ising model for $d > 4$ [13], and the φ^4 model for $d > 4$ [4,14]. For weakly self-avoiding walk and oriented percolation in dimensions $d > 4$, local central limit theorems have also been proved [1,2,20]. Related results for long-range models are proved in [6].

Typically, $|k|^{-2}$ behaviour for the Fourier transform of the critical two-point function (near $k = 0$) had been proved first (as in, e.g., [5,15]). However, this does not directly imply $|x|^{-(d-2)}$ behaviour for the inverse Fourier transform; see [12, Example 1.6.2] for a counterexample and [18, Appendix A] for further discussion of this point.

Our purpose here is to use the lace expansion to give a simple proof that the critical two-point function for weakly self-avoiding walk in dimensions $d > 4$ has Gaussian decay $|x|^{-(d-2)}$. Apart from the derivation of the lace expansion, which is well documented in the literature and not repeated here, our proof uses little more than elementary Fourier analysis, the Riemann–Lebesgue Lemma, and the product rule for differentiation. Although the realm of application of our convergence proof for the lace expansion appears to be less general than other methods, its application to weakly self-avoiding walk is strikingly simple and provides a new tool for problems of this genre.

To make the presentation as simple as possible, we restrict attention to the two-point function of the nearest-neighbour weakly self-avoiding walk. For background we refer to [12,15]. We follow the approach in [10] apart from one key ingredient which is significantly simplified.

Let $D : \mathbb{Z}^d \to \mathbb{R}$ be given by $D(x) = \frac{1}{2d}$ if $\|x\|_1 = 1$ and otherwise $D(x) = 0$. Let D^{*n} denote the n-fold convolution of D with itself. For $n \in \mathbb{N}$, let $\mathcal{W}_n(x)$ denote the set of n-step walks from 0 to x, i.e., the set of $\omega = (\omega(0), \omega(1), \ldots, \omega(n))$ with each $\omega(i) \in \mathbb{Z}^d$, $\omega(0) = 0$, $\omega(n) = x$, and $\|\omega(i) - \omega(i-1)\|_1 = 1$ for $1 \le i \le n$. The set $\mathcal{W}_0(x)$ consists of the zero-step walk $\omega(0) = 0$ when $x = 0$, and otherwise it is the empty set. We write $\Omega = 2d$ for the degree of the nearest-neighbour graph. The simple random walk two-point function is defined, for $z \in [0, 1/\Omega]$, by

$$C_z(x) = \sum_{n=0}^{\infty} \sum_{\omega \in \mathcal{W}_n(x)} z^n = \sum_{n=0}^{\infty} (z\Omega)^n D^{*n}(x). \tag{1.1}$$

The Green function is $C_{1/\Omega}(x)$.

For $\omega \in \mathcal{W}_n(x)$ and $0 \leq s < t \leq n$, we define

$$U_{st}(\omega) = \begin{cases} -1 & (\omega(s) = \omega(t)), \\ 0 & (\text{otherwise}). \end{cases} \tag{1.2}$$

Given $\beta \in (0,1)$, $z \geq 0$, and $x \in \mathbb{Z}^d$, the *weakly self-avoiding walk two-point function* is then defined by

$$G_z(x) = \sum_{n=0}^{\infty} \sum_{\omega \in \mathcal{W}_n(x)} z^n \prod_{0 \leq s < t \leq n} \left(1 + \beta U_{st}(\omega)\right). \tag{1.3}$$

The *susceptibility* is defined by $\chi(z) = \sum_{x \in \mathbb{Z}^d} G_z(x)$. A standard subadditivity argument implies the existence of $z_c = z_c(\beta) \geq z_c(0) = 1/\Omega$ such that $\chi(z)$ is finite if and only if $z \in [0, z_c)$; also $\chi(z) \geq z_c/(z_c - z)$ so $\chi(z_c) = \infty$ (see, e.g., [15, Theorem 2.3]). In particular, $G_z(x)$ is finite if $z \in [0, z_c)$; in fact it decays exponentially in x. We will prove the following theorem. The constant a_d in the theorem is $a_d = \frac{d \Gamma(\frac{d-2}{2})}{2 \pi^{d/2}}$.

Theorem 1.1. *Let $d > 4$, and let $\beta > 0$ be sufficiently small. There is a constant $c_d = a_d(1 + O(\beta))$ such that*

$$G_{z_c}(x) = c_d \frac{1}{|x|^{d-2}} + o\left(\frac{1}{|x|^{d-2}}\right). \tag{1.4}$$

For $d > 5$, the error term is improved to $o(|x|^{-(d-1)})$.

In [17], the method of proof of Theorem 1.1 is extended to analyse the near-critical two-point function. Namely, it is proved in [17] that for $d > 4$ and for β sufficiently small, there are constants $\kappa_0 > 0$ and $\kappa_1 \in (0, 1)$ such that for all $z \in (0, z_c)$ and all $x \in \mathbb{Z}^d$,

$$G_z(x) \leq \kappa_0 \frac{1}{1 \vee |x|^{d-2}} e^{-\kappa_1 (z_c - z)^{1/2} |x|}. \tag{1.5}$$

The estimate (1.5) is applied in [17] to prove existence of a "plateau" for the weakly self-avoiding walk two-point function on a large discrete torus in dimensions $d > 4$.

An alternate proof of Theorem 1.1 is given in [3], based on Banach algebras and a fixed-point theorem. That proof avoids explicit use of the Fourier transform, though it does rely on an expansion of the Green function $C_{1/\Omega}(x)$ which is proved using the Fourier transform. Theorem 1.1 is proved for the strictly self-avoiding walk (the case $\beta = 1$) in [8], and for spread-out strictly self-avoiding walk in [10]. Thus Theorem 1.1 is not new or best possible; our goal here is to present a new and simple method of proof rather than to obtain a new result. A sample consequence of Theorem 1.1 is that the bubble condition holds for $d > 4$, and this implies the matching upper bound $\chi(z) \leq O((z_c - z)^{-1})$ (see [15, Theorem 2.3]).

We use the Fourier transform. Let $\mathbb{T}^d = (\mathbb{R}/2\pi\mathbb{Z})^d$ denote the continuum torus of period 2π. For a summable function $f : \mathbb{Z}^d \to \mathbb{C}$ we define its Fourier transform by

$$\hat{f}(k) = \sum_{x \in \mathbb{Z}^d} f(x) e^{ik \cdot x} \quad \left(k \in \mathbb{T}^d\right). \tag{1.6}$$

The inverse Fourier transform is

$$f(x) = \int_{\mathbb{T}^d} \hat{f}(k) e^{-ik \cdot x} \frac{dk}{(2\pi)^d} \quad \left(x \in \mathbb{Z}^d\right). \tag{1.7}$$

2. Lace expansion

The lace expansion was introduced by Brydges and Spencer [5] to prove that the weakly self-avoiding walk is diffusive in dimensions $d > 4$. In the decades since 1985, the lace expansion has been adapted and extended to a broad range of models and results.

For the weakly self-avoiding walk, the lace expansion [5,12,15] produces an explicit formula for the $\mathbb{Z}^d$-symmetric function $\Pi_z : \mathbb{Z}^d \to \mathbb{R}$ which satisfies, for $z \in [0, z_c)$,

$$G_z(x) = \delta_{0,x} + z\Omega(D * G_z)(x) + (\Pi_z * G_z)(x) \quad \left(x \in \mathbb{Z}^d\right), \tag{2.1}$$

or equivalently,

$$\hat{G}_z(k) = \frac{1}{1 - z\Omega\hat{D}(k) - \hat{\Pi}_z(k)} \quad (k \in \mathbb{T}^d). \tag{2.2}$$

Let $\delta : \mathbb{Z}^d \to \mathbb{R}$ denote the Kronecker delta $\delta(x) = \delta_{0,x}$. Then $\hat{\delta}(k) = 1$. We define

$$F_z = \delta - z\Omega D - \Pi_z, \qquad \hat{F}_z = 1 - z\Omega\hat{D} - \hat{\Pi}_z. \tag{2.3}$$

Then

$$(G_z * F_z)(x) = \delta_{0,x}, \qquad \hat{G}_z(k) = \frac{1}{\hat{F}_z(k)}. \tag{2.4}$$

For simple random walk, the above equations simplify to $C_z * A_z = \delta$ and $\hat{C}_z = 1/[1 - z\Omega\hat{D}]$ with $A_z = \delta - z\Omega D$.

3. Proof of main result

3.1. *Diagrammatic estimate*

As in many applications of the lace expansion, we use a bootstrap argument. We define the *bootstrap function*

$$b(z) = \sup_{x \in \mathbb{Z}^d} \frac{G_z(x)}{C_{1/\Omega}(x)} \quad (z \in [0, z_c]). \tag{3.1}$$

The bootstrap function can be seen to be finite and continuous in $z \in [0, z_c)$, using the fact that $G_z(x)$ is continuous and decays exponentially for large $|x|$. We do not know *a priori* that $G_{z_c}(x)$ is finite. By definition, $b(z) \leq 1$ for $z \in [0, \frac{1}{\Omega}]$. The next proposition gives consequences of the assumption that $b(z) \leq 3$. We will not need to know more about the function Π_z than Proposition 3.1, so we do not give its explicit formula here. The formula can be found in [5,12,15].

Proposition 3.1. *Let $d > 4$ and let β be sufficiently small. Fix $z \in [\frac{1}{\Omega}, z_c]$. If $b(z) \leq 3$ then there is a constant K depending only on d (and on "3") such that*

$$\left|\Pi_z(x)\right| \leq K\beta \frac{1}{1 + |x|^{3(d-2)}} \quad (x \in \mathbb{Z}^d) \tag{3.2}$$

and hence $\hat{\Pi}_z \in C^s(\mathbb{T}^d)$ for any nonnegative integer $s < 2d - 6$, in particular $\hat{\Pi}_z \in C^{d-2}(\mathbb{T}^d)$. In addition, the infrared bound holds, i.e., there exists $c > 0$ (independent of β, z, k) such that

$$\hat{F}_z(k) \geq c|k|^2 \quad (k \in \mathbb{T}^d). \tag{3.3}$$

Proof. The bound (3.2) is a diagrammatic estimate proved via well-developed technology (e.g., [3, (12)]) and we omit its proof. It follows immediately from (3.2) that $|x|^s|\Pi_z(x)|$ is summable for all $s < 2d - 6$ and hence that $\hat{\Pi}_z \in C^s(\mathbb{T}^d)$ for any nonnegative integer $s < 2d - 6$. For (3.3), we use

$$\hat{F}_z(k) = \hat{F}_z(0) + \left[\hat{F}_z(k) - \hat{F}_z(0)\right] = \hat{F}_z(0) + z\Omega\left[1 - \hat{D}(k)\right] + \left[\hat{\Pi}_z(0) - \hat{\Pi}_z(k)\right]. \tag{3.4}$$

The first term on the right-hand side is $\hat{F}_z(0) = \chi(z)^{-1} \geq 0$. By definition, the second term obeys $1 - \hat{D}(k) = d^{-1}\sum_{j=1}^{d}(1 - \cos k_j) \geq \frac{4|k|^2}{\pi^2\Omega}$. By (3.2), the second derivative of $\hat{\Pi}_z(k)$ with respect to k is $O(\beta)$, and (3.3) then follows by a Taylor estimate on $\hat{\Pi}_z(0) - \hat{\Pi}_z(k)$ (by symmetry there is no linear term in k). $\qquad\square$

3.2. *Isolation of leading term*

Following [10], we isolate the leading term by writing G_z as a z-dependent multiple of the random walk two-point function C_μ at a z-dependent value of μ. Let $A_z = \delta - z\Omega D$, $\lambda > 0$ and $\mu \in [0, \frac{1}{\Omega}]$. Since $C_\mu * A_\mu = \delta$ and $G_z * F_z = \delta$,

we have

$$
\begin{aligned}
G_z &= \lambda C_\mu + \delta * G_z - \lambda C_\mu * \delta \\
&= \lambda C_\mu + C_\mu * A_\mu * G_z - \lambda C_\mu * F_z * G_z \\
&= \lambda C_\mu + C_\mu * E_{z,\lambda,\mu} * G_z,
\end{aligned}
\tag{3.5}
$$

with

$$
E_{z,\lambda,\mu} = A_\mu - \lambda F_z. \tag{3.6}
$$

Given $z \in [\frac{1}{\Omega}, z_c)$, we choose $\lambda = \lambda_z$ and $\mu = \mu_z$ in order to achieve

$$
\sum_{x \in \mathbb{Z}^d} E_{z,\lambda_z,\mu_z}(x) = \sum_{x \in \mathbb{Z}^d} |x|^2 E_{z,\lambda_z,\mu_z}(x) = 0, \tag{3.7}
$$

namely (since $\sum_x |x|^2 D(x) = 1$)

$$
\lambda_z = \frac{1}{\hat{F}_z(0) - \sum_x |x|^2 F_z(x)} = \frac{1}{1 - \hat{\Pi}_z(0) + \sum_x |x|^2 \Pi_z(x)}, \tag{3.8}
$$

$$
\mu_z \Omega = 1 - \lambda_z \hat{F}_z(0) = \frac{z\Omega + \sum_x |x|^2 \Pi_z(x)}{\hat{F}_z(0) + z\Omega + \sum_x |x|^2 \Pi_z(x)}. \tag{3.9}
$$

By Proposition 3.1, if we assume $b(z) \le 3$ then the second moment of Π_z is $O(\beta)$, and hence the above formulas are well defined, $\lambda_z = 1 + O(\beta)$, and $\mu_z \Omega \in [0, 1)$. In particular, if $b(z_c) \le 3$ (as we will eventually show to be the case), then, since $\hat{F}_{z_c}(0) = \chi(z_c)^{-1} = 0$, we see from (3.9) that $\mu_{z_c} = 1/\Omega$ is the critical value for C_μ.

With these choices of λ_z, μ_z, we have

$$
G_z = \lambda_z C_{\mu_z} + f_z, \qquad f_z = C_{\mu_z} * E_z * G_z, \tag{3.10}
$$

with

$$
E_z = E_{z,\lambda_z,\mu_z} = (1 - \lambda_z)(\delta - D) - \lambda_z \hat{\Pi}_z(0) D + \lambda_z \Pi_z. \tag{3.11}
$$

By definition,

$$
\hat{f}_z(k) = \hat{C}_{\mu_z}(k) \hat{E}_z(k) \hat{G}_z(k). \tag{3.12}
$$

Roughly, since we have arranged via (3.7) that the Taylor expansion of $\hat{E}_z(k)$ has no constant term or term of order $|k|^2$, we expect it to be of order $\beta|k|^4$. On the other hand, according to the infrared bound, the Fourier transform of $\hat{G}_z(k)$ will be of order $|k|^{-2}$ for small $|k|$. The same is true for $\hat{C}_{\mu_z}(k)$, so $\hat{f}_z(0) = O(\beta)$. We will show that this less singular behaviour of $\hat{f}_z(k)$ translates into better decay than $|x|^{-(d-2)}$ for $f_z(x)$. This will permit the bootstrap argument to be completed by proving $b(z) \le 2$, and the proof will essentially be complete. The details in this rough sketch are given below.

3.3. *The bootstrap*

The bootstrap argument is encapsulated in the following proposition.

Proposition 3.2. *Let $d > 4$ and fix $z \in [\Omega^{-1}, z_c)$. If $b(z) \le 3$ then for β sufficiently small (not depending on z) it is in fact the case that $b(z) \le 2$.*

The next proposition is a replacement for the bound on $C_{\mu_z} * E_z$ in [10, Proposition 1.9] which required a delicate Fourier analysis of C_{μ_z}, and of the bound of [3, Lemma 4] which used the expansion $C_{1/\Omega}(x) = a|x|^{-(d-2)} + b|x|^{-d} + O(|x|^{-(d+2)})$ from [19] which was also proved by careful Fourier analysis.

Proposition 3.3. *Let $d > 4$ and let β be sufficiently small. Let $z \in [\frac{1}{\Omega}, z_c]$. Under the assumption that $b(z) \leq 3$, the derivatives $\nabla^\alpha \hat{f}_z(k)$ obey*

$$\left\| \nabla^\alpha \hat{f}_z \right\|_{L^1(\mathbb{T}^d)} \leq O(\beta) \tag{3.13}$$

provided $|\alpha| \leq d - 2$ when $d = 5$ and $|\alpha| \leq d - 1$ for $d \geq 6$, with constant depending only on d (not on z).

The importance of (3.13) resides in the fact that the smoothness of a function on the torus implies bounds on the decay of its (inverse) Fourier transform. More precisely, it is proved in [7, Corollary 3.3.10] using integration by parts that there is a constant $\kappa_{d,s}$, depending only on the dimension d and the maximal order s of differentiation, such that if $\nabla^\alpha \hat{\psi} \in L^1(\mathbb{T}^d)$ for all multi-indices α with $|\alpha| \leq s$ then

$$\left| \psi(x) \right| \leq \kappa_{d,s} \frac{1}{1 \vee |x|^s} \max_{|\alpha| \in \{0,s\}} \left\| \nabla^\alpha \hat{\psi} \right\|_{L^1(\mathbb{T}^d)}. \tag{3.14}$$

In addition to the quantitative estimate (3.14), it follows from the integrability of $\nabla^\alpha \hat{\psi}$ together with the Riemann–Lebesgue Lemma (e.g., [7, Proposition 3.3.1]) that the inverse Fourier transform of $\nabla^\alpha \hat{\psi}$ vanishes at infinity, so $|x|^s \psi(x) \to 0$ as $|x| \to \infty$.

Proof of Proposition 3.2. Suppose that $b(z) \leq 3$. Let $n_d = d - 2$ for $d = 5$ and $n_d = d - 1$ for $d \geq 6$. By (3.13), as discussed around (3.14), $|f_z(x)| = o(|x|^{-n_d})$ and $|f_z(x)| \leq O(\beta|x|^{-n_d})$, with z-independent constant in the latter bound. Therefore,

$$G_z(x) = \lambda_z C_{\mu_z}(x) + o\left(|x|^{-n_d}\right), \tag{3.15}$$

$$G_z(x) = \lambda_z C_{\mu_z}(x) + O\left(\beta|x|^{-n_d}\right), \tag{3.16}$$

with the constant in (3.16) independent of x and z. As is well known (e.g., [11,19]), $C_{1/\Omega}(x) \sim a_d |x|^{-(d-2)}$. From (3.16), we see that by taking β sufficiently small we can obtain

$$G_z(x) \leq \left(1 + O(\beta)\right) C_{1/\Omega}(x) + O(\beta) C_{1/\Omega}(x) \leq 2 C_{1/\Omega}(x), \tag{3.17}$$

i.e., $b(z) \leq 2$. This completes the proof. $\qquad\square$

Proof of Theorem 1.1. Since $b(z) \leq b(\Omega^{-1}) \leq 1$ for $z \leq \Omega^{-1}$ by definition, it follows from Proposition 3.2 and the continuity of the function b that the interval $(2, 3]$ is forbidden for values of $b(z)$, so $b(z) \leq 2$ for all $z \in [0, z_c)$. By monotone convergence, also $b(z_c) \leq 2$. Thus λ_z approaches a limit $\lambda_{z_c} = 1 + O(\beta)$ as $z \to z_c^-$. Since $b(z_c) \leq 2$, (3.15) holds for $z = z_c$ so $G_{z_c}(x) = \lambda_{z_c} C_{1/\Omega}(x) + o(|x|^{-n_d})$. Thus Theorem 1.1 is proved subject to Proposition 3.3. $\qquad\square$

3.4. *Proof of Proposition 3.3*

It remains only to prove Proposition 3.3. The next lemma is closely related to [10, Lemma 7.2]. It reflects our choice of λ_z, μ_z to achieve (3.7).

Lemma 3.4. *Let $d > 4$ and let β be sufficiently small. Let $z \in [\frac{1}{\Omega}, z_c]$ and suppose that $b(z) \leq 3$. There is a $c_0 > 0$ (independent of z) such that $|\nabla^\alpha \hat{E}_z(k)| \leq c_0 \beta$ for $|\alpha| < 2d - 6$, and moreover*

$$\left| \nabla^\alpha \hat{E}_z(k) \right| \leq c_0 \beta \times \begin{cases} |k|^{4-|\alpha|} & (d > 5), \\ |k|^{4-|\alpha|} \log |k|^{-1} & (d = 5), \end{cases} \quad \left(|\alpha| \leq 3 \right). \tag{3.18}$$

Proof. We first prove that $|\nabla^\alpha \hat{E}_z(k)| \leq c_0 \beta$ for $|\alpha| < 2d - 6$, via the inequality $|\nabla^\alpha \hat{E}_z(k)| \leq \sum_x |x^\alpha E_z(x)|$ together with term-by-term estimation in the formula for E_z in (3.11). Indeed, since $\lambda_z = 1 + O(\beta)$, the contribution from all moments of the term $(1 - \lambda_z)(\delta - D)$ is $O(\beta)$, and since $|\hat{\Pi}_z(0)| \leq O(\beta)$ by (3.2), all moments of $\lambda_z \hat{\Pi}_z(0) D$ are also $O(\beta)$. Finally, the moments of $\lambda_z |\Pi_z|$ with order less than $2d - 6$ are bounded by $O(\beta)$ using (3.2). In the remainder of the proof, we can therefore restrict to $|\alpha| \leq 3$.

Let $g_x(k) = \cos(k \cdot x) - 1 + \frac{(k \cdot x)^2}{2!}$. By symmetry and by (3.7),

$$\hat{E}_z(k) = \sum_{x \in \mathbb{Z}^d} E_z(x) \cos(k \cdot x) = \sum_{x \in \mathbb{Z}^d} E_z(x) g_x(k). \tag{3.19}$$

By explicit computation of the derivatives and elementary properties of sine and cosine,

$$\left| g_x(k) \right| \leq c \big(|k|^4 |x|^4 \wedge \big(1 + |k|^2 |x|^2 \big) \big), \qquad \left| \nabla_i g_x(k) \right| \leq c \big(|k|^3 |x|^4 \wedge \big(|x| + |k| |x|^2 \big) \big), \tag{3.20}$$

$$\left| \nabla_{ij}^2 g_x(k) \right| \leq c \big(|k|^2 |x|^4 \wedge |x|^2 \big), \qquad \left| \nabla_{ijl}^3 g_x(k) \right| \leq c \big(|k| |x|^4 \wedge |x|^3 \big), \tag{3.21}$$

where $\wedge$ denotes minimum. In the upper bound $|\nabla^\alpha \hat{E}_z(k)| \leq \sum_x |E_z(x)| |\nabla^\alpha g_x(k)|$, we estimate the sum over $|x| \leq |k|^{-1}$ using the $|x|^4$ bound on $\nabla^\alpha g_x(k)$, and we estimate the sum over $|x| > |k|^{-1}$ using the other alternative in the minimum. By a term-by-term estimate using (3.11) with (3.2) (as in the previous paragraph), for $|\alpha| \leq 3$ and $d \geq 5$, and for small $|k|$,

$$\sum_{|x| \leq |k|^{-1}} |x|^4 \big| E_z(x) \big| \leq O(\beta) \int_1^{|k|^{-1}} \frac{r^{d-1+4}}{r^{3(d-2)}} \, dr = O(\beta) \int_1^{|k|^{-1}} \frac{1}{r^{2d-9}} \, dr = \begin{cases} O(\beta) & (d > 5), \\ O(\beta \log |k|^{-1}) & (d = 5), \end{cases} \tag{3.22}$$

$$\sum_{|x| > |k|^{-1}} |x|^{|\alpha|} \big| E_z(x) \big| \leq O(\beta) \int_{|k|^{-1}}^\infty \frac{r^{d-1+|\alpha|}}{r^{3(d-2)}} \, dr = O(\beta) \int_{|k|^{-1}}^\infty \frac{1}{r^{2d-5-|\alpha|}} \, dr = O \big(\beta |k|^{2d-6-|\alpha|} \big), \tag{3.23}$$

and the desired result then follows after some bookkeeping. $\qquad \square$

Proof of Proposition 3.3. Let $n_d = d - 2$ for $d = 5$ and $n_d = d - 1$ for $d \geq 6$; then $n_d < 2d - 6$. Our goal is to prove the bound (3.13) on derivatives of $\hat{f}_z$ of order up to n_d. By the infrared bound and Lemma 3.4, for $d > 5$ we have

$$\left| \hat{f}_z(k) \right| \leq |k|^{-2} O(\beta) |k|^4 |k|^{-2} = O(\beta), \tag{3.24}$$

and a similar estimate holds for $d = 5$ with an additional factor $\log |k|^{-1}$ in the upper bound.

Estimation of the L^1 norm of derivatives of $\hat{f}_z(k)$ is an exercise in power counting, as follows. For $|\alpha| \leq n_d$, by the product rule for differentiation $\nabla^\alpha \hat{f}_z(k)$ involves terms

$$\nabla^{\alpha_1} \hat{C}_{\mu_z}(k) \nabla^{\alpha_2} \hat{E}_z(k) \nabla^{\alpha_3} \hat{G}_z(k), \qquad \big(|\alpha_1| + |\alpha_2| + |\alpha_3| = |\alpha| \big). \tag{3.25}$$

By Lemma 3.4, each derivative on $\hat{E}$ up to the fourth order reduces by 1 the original power $|k|^4$ for that factor (an unimportant factor $\log |k|^{-1}$ is present for $d = 5$), and subsequent derivatives do not cause further reduction; the net effect is therefore reduction by $\min\{|\alpha_2|, 4\}$. Similarly, each derivative on $\hat{C}_{\mu_z}$ or $\hat{G}_z$ reduces (worsens) its power $|k|^{-2}$ by 1; we illustrate the idea for $d = 5$, for which $n_d = d - 2 = 3$:

$$\left| \nabla_i \frac{1}{\hat{F}} \right| = \left| \frac{\nabla_i \hat{F}}{\hat{F}^2} \right| \leq c_1 \frac{|k_i|}{|k|^4} \leq c_1 \frac{1}{|k|^3}, \qquad \left| \nabla_i^2 \frac{1}{\hat{F}} \right| \leq \left| \frac{\nabla_i^2 \hat{F}}{\hat{F}^2} \right| + \left| \frac{2(\nabla_i \hat{F})^2}{\hat{F}^3} \right| \leq c_2 \frac{k_i^0}{|k|^4} + c_2 \frac{k_i^2}{|k|^6}, \tag{3.26}$$

$$\left| \nabla_i^3 \frac{1}{\hat{F}} \right| \leq \left| \frac{\nabla_i^3 \hat{F}}{\hat{F}^2} \right| + \left| \frac{6(\nabla_i \hat{F})(\nabla_i^2 \hat{F})}{\hat{F}^3} \right| + \left| \frac{6(\nabla_i \hat{F})^3}{\hat{F}^4} \right| \leq c_3 \frac{k_i^0}{|k|^4} + c_3 \frac{k_i^{1+0}}{|k|^6} + c_3 \frac{k_i^3}{|k|^8}. \tag{3.27}$$

A detail in the above calculation is that $|\nabla_i \hat{F}|$ can be bounded by a multiple of $|\nabla_i \hat{D}| + |\nabla_i \hat{\Pi}|$, with the first term of order k_i by explicit calculation and the second also of order k_i by Taylor's Theorem, symmetry, and the boundedness of $\nabla_i^2 \Pi$.

In the general case, in advancing from one derivative to the next, when the derivative acts on the numerator it either maintains the same power of $|k|$ or reduces it by 1, and if it acts on the denominator then it increases the power of the denominator by $|k|^2$ and increases the power of the numerator by 1; the net result is reduction of the overall power of $|k|$ by at most 1. For $|\alpha| \leq n_d$, the total resulting power is (for small $|k|$) at worst

$$\frac{|k|^{4-\min\{|\alpha_2|, 4\}}}{|k|^{2+|\alpha_1|} |k|^{2+|\alpha_3|}} = \frac{1}{|k|^{|\alpha_1| + \min\{|\alpha_2|, 4\} + |\alpha_3|}} \leq \frac{1}{|k|^{|\alpha|}} \leq \frac{1}{|k|^{n_d}}. \tag{3.28}$$

This is in $L^1(\mathbb{T}^d)$ (also with the additional logarithmic factor when $d = 5$) and the norm in $L^1(\mathbb{T}^d)$ is $O(\beta)$ due to the factor β in the bound on $\hat{E}_z$ in Lemma 3.4. This completes the proof.[1] □

3.5. *Concluding remarks*

(i) The main difference between the above proof and the proofs in [3,8,10] is our avoidance of any need to convert the decay in x-space of factors in a convolution $C_{\mu_z} * E_z * G_z$ into decay of the convolution (which is delicate since, e.g., the convolution of two factors each with decay $|x|^{-(d-2)}$ has worse decay $|x|^{-(d-4)}$ when $d > 4$). In the above proof, we encounter instead the Fourier transform $\hat{C}_{\mu_z} \hat{E}_z \hat{G}_z$ and the corresponding step is handled simply via the product rule for differentiation.

(ii) To control the lace expansion, the analysis in any of [3,8,10] only requires a bound of the form $|\Pi(x)| \leq O(|x|^{-(d+2+\epsilon)})$ for $\epsilon > 0$ (with $\epsilon = 2$ in [3]), whereas the above proof requires the more demanding bound $|\Pi(x)| \leq O(|x|^{-(2d-2+\epsilon)})$ in order for $\hat{\Pi}(k)$ to have derivatives of order $d - 2$. For self-avoiding walk, the upper bound (3.2) has power $3(d - 2) = 2d - 2 + (d - 4)$ which is sufficient. For the Ising and φ^4 models, $\Pi(x)$ also obeys an upper bound $|x|^{-3(d-2)}$ [4,13,14]. However, the above proof appears not to apply to percolation or to lattice trees and lattice animals, where the bound on $\Pi(x)$ is $|x|^{-2(d-2)}$ for percolation and $|x|^{-3(d-2)+d}$ for lattice trees and lattice animals [10]. It would be of interest to understand better why this breakdown occurs and whether there is any possiblity to overcome it in these settings with upper critical dimension equal to 6 or 8.

(iii) With further effort, it may be possible to extend our approach to spread-out models of strictly self-avoiding walk or the Ising or φ^4 models in dimensions $d > 4$ by proving a version of (3.13) in those settings. Possibly this could simplify aspects of the analysis in [4,10,13,14]. However, this is beyond our current scope and we do not draw a conclusion about this question here.

(iv) It is natural to ask whether our approach could be used for the nearest-neighbour strictly self-avoiding walk in dimensions $d \geq 5$, to give an alternate proof of the $|x|^{-(d-2)}$ decay proved in [8]. This certainly could not be done, at present, without a portion of the very sizeable and computer-assisted input from [9] listed in [8, Proposition 1.3]; in fact results beyond those of [9] may be needed to deal with the higher derivatives of $\hat{\Pi}(k)$ encountered here. A further and serious obstacle, as pointed out and overcome with a different method in [8], is that the amplitude $a_d = \frac{d}{2}\pi^{-d/2}\Gamma((d - 2)/2)$ in the asymptotic formula for the critical simple random walk two-point function (and appearing in Theorem 1.1) grows rapidly with d, so the small parameter that facilitates the bootstrap argument is more hidden and more delicate to exploit.

Acknowledgements

References

[1] L. Avena, E. Bolthausen and C. Ritzmann. A local CLT for convolution equations with an application to weakly self-avoiding walks. *Ann. Probab.* **44** (2016) 206–234. MR3456336 https://doi.org/10.1214/14-AOP971

[2] E. Bolthausen and C. Ritzmann. A central limit theorem for convolution equations and weakly self-avoiding walks, 2001. Unpublished document. Available at https://arxiv.org/abs/math/0103218.

[3] E. Bolthausen, R. van der Hofstad and G. Kozma. Lace expansion for dummies. *Ann. Inst. Henri Poincaré B, Probab. Stat.* **54** (2018) 141–153. MR3765883 https://doi.org/10.1214/16-AIHP797

[4] D. C. Brydges, T. Helmuth and M. Holmes. The continuous-time lace expansion. *Commun. Pure Appl. Math.* Available at https://arxiv.org/abs/1905.09605.

[5] D. C. Brydges and T. Spencer. Self-avoiding walk in 5 or more dimensions. *Comm. Math. Phys.* **97** (1985) 125–148. MR0782962

[6] L.-C. Chen and A. Sakai. Critical two-point functions for long-range statistical-mechanical models in high dimensions. *Ann. Probab.* **43** (2015) 639–681. MR3306002 https://doi.org/10.1214/13-AOP843

[7] L. Grafakos. *Classical Fourier Analysis*, 3rd edition. Springer, New York, 2010. MR2445437

[8] T. Hara. Decay of correlations in nearest-neighbor self-avoiding walk, percolation, lattice trees and animals. *Ann. Probab.* **36** (2008) 530–593. MR2393990 https://doi.org/10.1214/009117907000000231

[9] T. Hara and G. Slade. The lace expansion for self-avoiding walk in five or more dimensions. *Rev. Math. Phys.* **4** (1992) 235–327. MR1174248 https://doi.org/10.1142/S0129055X9200008X

[10] T. Hara, R. van der Hofstad and G. Slade. Critical two-point functions and the lace expansion for spread-out high-dimensional percolation and related models. *Ann. Probab.* **31** (2003) 349–408. MR1959796 https://doi.org/10.1214/aop/1046294314

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[1]An earlier proof of Proposition 3.3 was based on Kotani's Theorem [16] rather than via the direct proof given here.

[11] G. F. Lawler and V. Limic. *Random Walk: A Modern Introduction*. Cambridge University Press, Cambridge, 2012. MR2677157 https://doi.org/10.1017/CBO9780511750854

[12] N. Madras and G. Slade. *The Self-Avoiding Walk*. Birkhäuser, Boston, 1993. MR1197356

[13] A. Sakai. Lace expansion for the Ising model. *Comm. Math. Phys.* **272** (2007) 283–344. Correction: A. Sakai. Correct bounds on the Ising lace-expansion coefficients. Available at https://arxiv.org/abs/2003.09856. MR2300246 https://doi.org/10.1007/s00220-007-0227-1

[14] A. Sakai. Application of the lace expansion to the φ^4 model. *Comm. Math. Phys.* **336** (2015) 619–648. MR3322382 https://doi.org/10.1007/s00220-014-2256-x

[15] G. Slade. *The Lace Expansion and Its Applications. Ecole d'Eté de Probabilités de Saint–Flour XXXIV–2004. Lecture Notes in Mathematics* **1879**. Springer, Berlin, 2006. MR2239599

[16] G. Slade Kotani's theorem for the Fourier transform. Unpublished document, 2020. Available at https://arxiv.org/abs/2006.06532.

[17] G. Slade The near-critical two-point function for weakly self-avoiding walk in high dimensions. Preprint, 2020. Available at https://arxiv.org/abs/2008.00080. MR2883859 https://doi.org/10.4171/072-1/9

[18] A. D. Sokal. An alternate constructive approach to the φ_3^4 quantum field theory, and a possible destructive approach to φ_4^4. *Ann. Inst. Henri Poincaré* **37** (1982) 317–398. MR0693644

[19] K. Uchiyama. Green's functions for random walks on $\mathbb{Z}^N$. *Proc. Lond. Math. Soc.* **77** (1998) 215–240. MR1625467 https://doi.org/10.1112/S0024611598000458

[20] R. van der Hofstad and G. Slade. A generalised inductive approach to the lace expansion. *Probab. Theory Related Fields* **122** (2002) 389–430. MR1892852 https://doi.org/10.1007/s004400100175