# HIGH-DIMENSIONAL NEAR-CRITICAL PERCOLATION AND THE TORUS PLATEAU 

By Tom Hutchcroft ${ }^{1, a}$, Emmanuel Michta ${ }^{2, b}$ and Gordon Slade ${ }^{2, c}$<br>${ }^{1}$ Division of Physics, Mathematics and Astronomy, California Institute of Technology, ${ }^{a}$ t.hutchcroft@caltech.edu<br>${ }^{2}$ Department of Mathematics, University of British Columbia, ${ }^{\mathrm{b}}$ michta@math.ubc.ca, ${ }^{\mathrm{c}}$ slade@math.ubc.ca


#### Abstract

We consider percolation on $\mathbb{Z}^{d}$ and on the $d$-dimensional discrete torus, in dimensions $d \geq 11$ for the nearest-neighbour model and in dimensions $d>$ 6 for spread-out models. For $\mathbb{Z}^{d}$ we employ a wide range of techniques and previous results to prove that there exist positive constants $c$ and $C$ such that the slightly subcritical two-point function and one-arm probabilities satisfy $$
\begin{aligned} \mathbb{P}_{p_{c}-\varepsilon}(0 \leftrightarrow x) & \leq \frac{C}{\|x\|^{d-2}} e^{-c \varepsilon^{1 / 2}\|x\|}, \\ \frac{c}{r^{2}} e^{-C \varepsilon^{1 / 2} r} & \leq \mathbb{P}_{p_{c}-\varepsilon}\left(0 \leftrightarrow \partial[-r, r]^{d}\right) \leq \frac{C}{r^{2}} e^{-c \varepsilon^{1 / 2} r} . \end{aligned}
$$

Using this, we prove that throughout the critical window the torus two-point function has a "plateau," meaning that it decays for small $x$ as $\|x\|^{-(d-2)}$ but for large $x$ is essentially constant and of order $V^{-2 / 3}$ where $V$ is the volume of the torus. The plateau for the two-point function leads immediately to a proof of the torus triangle condition, which is known to have many implications for the critical behaviour on the torus, and also leads to a proof that the critical values on the torus and on $\mathbb{Z}^{d}$ are separated by a multiple of $V^{-1 / 3}$ The torus triangle condition and the size of the separation of critical points have been proved previously, but our proofs are different and are direct consequences of the bound on the $\mathbb{Z}^{d}$ two-point function. In particular, we use results derived from the lace expansion on $\mathbb{Z}^{d}$, but in contrast to previous work on high-dimensional torus percolation, we do not need or use a separate torus lace expansion.


## 1. Introduction and results.

1.1. Introduction. Percolation on $\mathbb{Z}^{d}$ has been intensively studied by mathematicians and physicists since the 1950s as a fundamental model of a phase transition. Of particular interest is the universal critical behaviour in the vicinity of the critical value $p_{c}$. From a mathematical perspective, the critical behaviour has been established for certain two-dimensional models, using the breakthroughs enabled by conformal invariance and the Schramm-Loewner evolution [46, 47], and for a wide class of models above the upper critical dimension $d=6$, using the lace expansion [22,26]. The critical behaviour in intermediate dimensions $d=3,4,5,6$ remains a major challenge for probability theory which, at present, appears not to have adequate tools even to approach the problem. Considerable progress has been also made in the understanding of the finite-size scaling associated with critical percolation on a highdimensional discrete torus. In this paper we consider percolation in dimensions $d>6$, both on $\mathbb{Z}^{d}$ and on the torus.

The role of $d=6$ as the upper critical dimension for percolation was first pointed out by Toulouse [49]. The meaning of "upper critical dimension" is that the critical exponents for

[^0]percolation on $\mathbb{Z}^{d}$ in dimensions $d>6$ are predicted to be the same as for percolation on a tree (known as mean-field theory), whereas for $d<6$ they are not. Critical exponents for $d=6$ are predicted to have logarithmic corrections to mean-field behaviour [16]. Various one-sided mean-field bounds for critical exponents, such as $\gamma \geq 1, \beta \leq 1$, and $\delta \geq 2$, have been proven to hold in all dimensions [2, 3]. In addition, results implying that mean-field behaviour cannot apply in dimensions $d<6$ have been obtained in [13, 48] (see also [8]). In an important paper in 1984, Aizenman and Newman [2] identified a condition predicted to be valid for $d>6$-the triangle condition-as a sufficient condition for $\gamma=1$ which is meanfield behaviour for the expected cluster size (also called the susceptibility). Then Barsky and Aizenman [3] proved that the triangle condition also implies that $\beta=1$ (exponent for the percolation probability) and $\delta=2$ (exponent for the magnetisation); see also the recent paper [34] for alternative proofs of these results.

In 1990, Hara and Slade [22] derived their lace expansion for bond percolation and used it to verify the triangle condition for the nearest-neighbour model in sufficiently large dimensions ( $d \geq 19$ is large enough [23]). Later, Fitzner and van der Hofstad [17] extended this to all $d \geq 11$. An extension to $d>6$ has not yet been possible and seems to be impossible without the introduction of some significant new idea, due to the fact that convergence of the lace expansion is proved using a small parameter which is closely related to the triangle diagram and which is not believed to be small in dimensions close to but above 6 . On the other hand, since the critical exponents are predicted to be universal, meaning that they take the same values for any symmetric short-range model in a given dimension $d$, it is natural to introduce models with a parameter that can be taken to be small in any fixed dimension $d>6$. This was accomplished in [22], where the triangle condition was proved for a wide variety of sufficiently spread-out models in any dimension $d>6$. A basic example of a spread-out model is a finite-range model of bond percolation on $\mathbb{Z}^{d}$ with long bonds, not just nearest-neighbour bonds, and the reciprocal of the degree provides a small parameter for convergence of the lace expansion in any dimension $d>6$. Related results for long-range models have also been established in [14, 27].

Over the last 30 years, a large literature on high-dimensional percolation has emerged, using the convergence of the lace expansion as a starting point, typically both for sufficiently spread-out models in dimensions $d>6$ and for the nearest-neighbour model in large enough dimensions. Reviews can be found in [26, 44]. In particular, Hara proved the square root decay of the mass (inverse correlation length) [19]; Hara, van der Hofstad and Slade [21] and Hara [20] proved that the critical two-point function has the Gaussian decay $|x|^{-(d-2)}$; Kozma and Nachmias [37] proved the mean-field behaviour $r^{-2}$ for the one-arm exponent, and Chatterjee and Hanson [11] identified the decay of the critical two-point function in a half-space. We use these results to prove our main results for high-dimensional percolation on $\mathbb{Z}^{d}$. Following the methodology of [31], we also use the OSSS theory of decision trees [42], whose application to statistical mechanical models was pioneered by Duminil-Copin, Raoufi, and Tassion [15], to obtain a new differential inequality which facilitates the transfer of certain estimates at the critical point to estimates at nearby subcritical points.

Our results for $\mathbb{Z}^{d}$ consist of an upper bound of the form $|x|^{-(d-2)} \exp \left[-c\left|p-p_{c}\right|^{1 / 2}|x|\right]$, for the slightly subcritical two-point function, and upper and lower bounds of the form $r^{-2} \exp \left[-c\left|p-p_{c}\right|^{1 / 2} r\right]$ for the (extrinsic) one-arm probability. We stress for the avoidance of doubt that the inclusion of these sharp exponential factors for $p<p_{c}$, with the square root in the exponent, requires substantial new ideas and is not a minor extension of the previous results.

In a separate line of research initiated by Borgs, Chayes, van der Hofstad, Slade, and Spencer in [5, 6], the critical behaviour of percolation on a discrete $d$-dimensional torus has been studied in depth, both for the nearest-neighbour model with $d$ sufficiently large and for
sufficiently spread-out models in dimensions $d>6$. There is a triangle condition for the torus (and indeed for general high-dimensional transient graphs) which implies that percolation on the torus behaves in many respects like the Erdős-Rényi random graph. In particular, the notion of a critical point, which is valid for $\mathbb{Z}^{d}$, is replaced by the notion of a critical scaling window of $p$ values. These ideas are developed in [5, 24-26, 29] and are based on the verification of the triangle condition in high dimensions via a separate lace expansion on the torus as opposed to on $\mathbb{Z}^{d}$ [6].

Our first result for the torus is a proof that the torus two-point function has a "plateau." The plateau refers to the fact that the torus two-point function, within and slightly below the critical window, decays for small $x$ like its $\mathbb{Z}^{d}$ counterpart, before levelling off at a constant value for large $x$. Related plateaux have been proven to exist for simple random walk (the lattice Green function) for $d>2$ [45, 51, 52], for weakly self-avoiding walk for $d>4$ [45], and partially for the Ising model for $d>4$ [43]. As we show, the plateau for the torus percolation two-point function is highly effective for the analysis of torus percolation (a similar situation applies for weakly self-avoiding walk on a torus for $d>4$ [40]). In particular, it directly gives a proof of the torus triangle condition, a proof that throughout the critical window the torus susceptibility is of the order of the cube root of the torus volume, and a proof that the $\mathbb{Z}^{d}$ critical value lies in the critical window for the torus. The triangle condition was proved previously via a separate lace expansion on the torus [6], which we do not need; the behaviour of the torus susceptibility was obtained previously in [5], while the verification that the $\mathbb{Z}^{d}$ critical value lies in the window was the main topic of [24, 25]. Our work establishes these results directly by applying results on $\mathbb{Z}^{d}$ rather than via a separate torus lace expansion.
1.2. The models. Let $\mathbb{G}=(\mathbb{V}, \mathbb{B})$ be a finite or infinite graph with vertex set $\mathbb{V}$ and edge (bond) set $\mathbb{B}$. Given $p \in[0,1]$, we consider independent and identically distributed Bernoulli random variables associated to each bond $b \in \mathbb{B}$, taking the value "open" with probability $p$ and the value "closed" with probability $1-p$. We denote the probability of an event $E$ by $\mathbb{P}_{p}(E)$ and the expectation of a random variable $X$ by $\mathbb{E}_{p} X$.

We consider four different choices of $\mathbb{G}$ :
(i) Nearest-neighbour model on $\mathbb{Z}^{d}: \mathbb{V}=\mathbb{Z}^{d}$ and $\mathbb{B}$ consists of all pairs $\{x, y\}$ with $\| x-$ $y \|_{1}=1$. We assume that $d \geq 11$.
(ii) Spread-out model on $\mathbb{Z}^{d}: \mathbb{V}=\mathbb{Z}^{d}$ and $\mathbb{B}$ consists of all pairs $\{x, y\}$ with $\|x-y\|_{1} \leq L$, for some (large) fixed $L>1$. We assume that $d>6$ and $L$ is sufficiently large depending on $d$.
(iii) Nearest-neighbour model on the torus $\mathbb{T}_{r}^{d}: \mathbb{V}=(\mathbb{Z} / r \mathbb{Z})^{d}$ for (large) period $r>2$ and $\mathbb{B}$ consists of all pairs $\|x-y\|_{1}=1$ with addition $\bmod r$. We assume that $d \geq 11$. We write $V=r^{d}$ for the volume of the torus and are interested in the limit $r \rightarrow \infty$.
(iv) Spread-out model on the torus $\mathbb{T}_{r}^{d}: \mathbb{V}=(\mathbb{Z} / r \mathbb{Z})^{d}$ for (large) period $r>2 L$ with (large) fixed $L>1$ and $\mathbb{B}$ consists of all pairs $\|x-y\|_{1} \leq L$ with addition mod $r$. We assume that $d>6$ and $L$ is sufficiently large depending on $d$.

Notation: We set $\mathbb{N}=\{1,2, \ldots\}$. We use $c, C$ for positive constants that can vary from line to line. We write $f \sim g$ to mean $\lim f / g=1, f \preceq g$ to mean $f \leq C g, f \succeq g$ to mean $g \preceq f$, and $f \asymp g$ to mean that $f \preceq g \preceq f$, where we require that all constants depend only on the dimension $d$ and the spread-out parameter $L$. Constants depending on additional parameters will be denoted by subscripts so that, for example, " $f(n, \lambda) \asymp_{\lambda} g(n, \lambda)$ for every $n \geq 1$ and $\lambda>0$ " means that, for each $\lambda>0$, there exist positive constants $c_{\lambda}, C_{\lambda}$ such that $c_{\lambda} g(n, \lambda) \leq f(n, \lambda) \leq C_{\lambda} g(n, \lambda)$ for every $n \geq 1$. For $a, b \in \mathbb{R}$, we write $a \vee b=\max \{a, b\}$. We write $\Lambda_{r}^{d}=[-r, r]^{d} \cap \mathbb{Z}^{d}$ for the box of side length $2 r+1$ in $\mathbb{Z}^{d}$, omitting the $d$ when it is unambiguous to do so. The boundary $\partial \Lambda_{r}^{d}$ of $\Lambda_{r}^{d}$ consists of the points $x \in \mathbb{Z}^{d}$ with $\|x\|_{\infty}=r$. To avoid dividing by zero, we use the Japanese bracket notation $\langle x\rangle:=\|x\|_{\infty} \vee 1$
for $x \in \mathbb{Z}^{d}$. Our notational convention is that objects on the torus have a label $\mathbb{T}$, so the twopoint function on the torus is written as $\tau_{p}^{\mathbb{T}}(x)$. Generally, objects without the torus label are for $\mathbb{Z}^{d}$.

For $\mathbb{Z}^{d}$ the restrictions on the dimension $d$ and the range $L$ described in (i) and (ii) above are so that previous lace expansion results can be applied. We apply these $\mathbb{Z}^{d}$ results to the torus under the same restriction. We apply existing results, obtained via the lace expansion for $\mathbb{Z}^{d}$, and do not need to revisit or further develop the expansion itself (nor do we use a separate torus expansion as in [6]). More precisely, our results hold for any $d>6$ and $L \geq 1$ such that the two-point function $\tau_{p}(x):=\mathbb{P}_{p}(0 \leftrightarrow x)$ satisfies

$$
\begin{equation*}
\tau_{p_{c}}(x) \asymp\langle x\rangle^{-d+2} . \tag{T}
\end{equation*}
$$

Here and throughout the paper, we write $p_{c}$ for the critical value for percolation on $\mathbb{Z}^{d}$. The estimate ( T ) was proven to hold in settings (i) and (ii) above by Hara, van der Hofstad, and Slade [20, 21] and Fitzner and van der Hofstad [17]. Our results also rely crucially on those of Kozma and Nachmias [37] and Chatterjee and Hanson [11], who worked under the same assumptions. We will refer to (i) and (ii) collectively as high-dimensional percolation on $\mathbb{Z}^{d}$ and to (iii) and (iv) as high-dimensional percolation on the torus.
1.3. Results for $\mathbb{Z}^{d}$. The two-point function. Our first result, and main tool for all our further results, concerns the transition from exponential to power-law decay for the two-point function.

THEOREM 1.1. Let $d>6$, and suppose that $(\mathrm{T})$ holds on $\mathbb{Z}^{d}$. There exist positive constants $c$ and $C$ such that

$$
\begin{equation*}
\tau_{p}(x) \leq \frac{C}{\langle x\rangle^{d-2}} \exp \left[-c\left(p_{c}-p\right)^{1 / 2}\langle x\rangle\right] \tag{1.1}
\end{equation*}
$$

for every $p \in\left(0, p_{c}\right]$ and $x \in \mathbb{Z}^{d}$.
Theorem 1.1 is a partial substantiation, via a one-sided bound, of the generally unproven guiding principle in the scaling theory for critical phenomena in statistical mechanical models on $\mathbb{Z}^{d}$ that two-point functions near a critical point generically have decay of the form

$$
\begin{equation*}
\tau_{p}(x) \approx \frac{1}{\langle x\rangle^{d-2+\eta}} g(|x| / \xi(p)) \tag{1.2}
\end{equation*}
$$

in some reasonable meaning for " $\approx$," when $\langle x\rangle$ is of roughly the same order as the correlation length $\xi(p)$ and $p$ is close to its critical value $p_{c}$. The universal critical exponent $\eta$ depends on the dimension, the correlation length $\xi(p) \approx\left(1-p / p_{c}\right)^{-v}$ diverges as $p \rightarrow p_{c}$ with a dimension-dependent universal critical exponent $v$, and $g$ is a function with rapid decay. In high dimensions, $\eta=0$ and $v=\frac{1}{2}$. The role of (1.2) in the derivation of scaling relations between critical exponents, such as Fisher's relation $\gamma=(2-\eta) \nu$, can be found in [18], Section 9.2.

Let us now summarise how Theorem 1.1 compares to previous results. For highdimensional percolation on $\mathbb{Z}^{d},(\mathrm{~T})$ is known in the more precise asymptotic form

$$
\begin{equation*}
\tau_{p_{c}}(x) \sim A_{\tau} \frac{1}{\langle x\rangle^{d-2}} \quad(x \rightarrow \infty) \tag{1.3}
\end{equation*}
$$

for some positive constant $A_{\tau}$ with an explicit error estimate [20, 21]. For all dimensions $d \geq 2$ and for $p<p_{c}$, there is exponential decay, in the sense that the mass (or inverse correlation length)

$$
\begin{equation*}
m(p)=-\lim _{n \rightarrow \infty} \frac{1}{n} \log \tau_{p}\left(n e_{1}\right)=-\sup _{n \geq 1} \frac{1}{n} \log \tau_{p}\left(n e_{1}\right), \tag{1.4}
\end{equation*}
$$

is strictly positive for $p<p_{c}$ [18]. In fact, more is known, in general, and it is shown in [18], Proposition 6.47), that there is a constant $c$ such that

$$
\begin{equation*}
c p^{d} \frac{1}{\|x\|_{1}^{4 d(d-1)}} e^{-m(p)\|x\|_{1}} \leq \tau_{p}(x) \leq e^{-m(p)\|x\|_{\infty}} \tag{1.5}
\end{equation*}
$$

for all $d \geq 2, p \in[0,1]$, and $x \neq 0$. Hara [19] proved that in high-dimensional percolation on $\mathbb{Z}^{d}$ the mass satisfies the asymptotic formula

$$
\begin{equation*}
m(p) \sim A_{m}\left(p_{c}-p\right)^{1 / 2} \quad\left(p \rightarrow p_{c}^{-}\right) \tag{1.6}
\end{equation*}
$$

for some positive constant $A_{m}$. With (1.5) this immediately implies that there is a positive constant $c$ such that

$$
\begin{equation*}
\tau_{p}(x) \leq \exp \left[-c\left(p_{c}-p\right)^{1 / 2}\|x\|_{\infty}\right] \tag{1.7}
\end{equation*}
$$

for every $p<p_{c}$ and $x \in \mathbb{Z}^{d}$. Theorem 1.1 improves this bound by a polynomial term which is believed to be sharp. No such estimate on the slightly subcritical two-point function had previously been proven for high-dimensional percolation on $\mathbb{Z}^{d}$.

For weakly self-avoiding walk in dimension $d>4$, a result analogous to Theorem 1.1 was proved only recently in [45]; that proof does not extend to percolation, and our methods are different and do not extend to weakly self-avoiding walk. On the basis of the asymptotic behaviour for the lattice Green function presented in [41], we believe that the precise asymptotic behaviour of the subcritical two-point function for fixed $p<p_{c}$ and for $d>6$ takes the form

$$
\begin{equation*}
\tau_{p}(x) \sim A_{p, \hat{x}} m(p)^{(d-3) / 2} \frac{1}{|x|_{p}^{(d-1) / 2}} e^{-m(p)|x|_{p}} \quad\left(|x|_{p} \rightarrow \infty\right) \tag{1.8}
\end{equation*}
$$

with $|\cdot|_{p}$ a $p$-dependent norm on $\mathbb{R}^{d}$ (not the $\ell_{p}$ norm) which interpolates monotonically between the $\ell_{1}$ and $\ell_{2}$ norms, as $p$ increases over the interval ( $0, p_{c}$ ), and with an amplitude $A_{p, \hat{x}}$ that approaches a nonzero constant (independent of the direction $\hat{x}$ ) as $p \uparrow p_{c}$. The polynomial factors in (1.8) can be rearranged as $\left(m(p)|x|_{p}\right)^{(d-3) / 2}|x|_{p}^{-(d-2)}$, so when $|x|_{p}$ is comparable to the correlation length $m(p)^{-1}$ the conjectured asymptotic estimate (1.8) becomes consistent with (1.1).

REMARK. It is impossible for the bound (1.1) to hold for all $x$ if we replace $c\left(p_{c}-\right.$ $p)^{1 / 2}$ by $m(p)$ in the exponent. To see this, we recall that the Ornstein-Zernike decay $\tau_{p}(n, 0, \ldots, 0) \sim C_{p} n^{-(d-1) / 2} e^{-m(p) n}$ was proved by Campanino, Chayes and Chayes [10] for $p<p_{c}$ in dimensions $d \geq 2$ (but without the control conjectured in (1.8) for the $p$ dependence of the constant $C_{p}$ as $p \rightarrow p_{c}$ ). From this, by taking $n$ large, we see that (1.1) can only hold if the constant $c$ is such that $c\left(p_{c}-p\right)^{1 / 2}$ is strictly smaller than $m(p)$.

The one-arm probability. Our second main result for percolation on $\mathbb{Z}^{d}$ concerns the probability that the cluster of the origin has a large radius in slightly subcritical percolation. Recall that $\Lambda_{r}=\Lambda_{r}^{d}=[-r, r]^{d} \cap \mathbb{Z}^{d}$ is the box of side length $2 r+1$ and $\partial \Lambda_{r}$ is its boundary.

THEOREM 1.2. Let $d>6$, and suppose that $(\mathrm{T})$ holds on $\mathbb{Z}^{d}$. There exist positive constants $c$ and $C$ such that

$$
\begin{equation*}
\frac{c}{r^{2}} \exp \left(-C\left(p_{c}-p\right)^{1 / 2} r\right) \leq \mathbb{P}_{p}\left(0 \leftrightarrow \partial \Lambda_{r}\right) \leq \frac{C}{r^{2}} \exp \left(-c\left(p_{c}-p\right)^{1 / 2} r\right) \tag{1.9}
\end{equation*}
$$

for every $p_{c} / 2 \leq p \leq p_{c}$ and $r \geq 1$.

The restriction $p \geq p_{c} / 2$ appearing in Theorem 1.2 is only needed for the lower bound, and $p_{c} / 2$ could be replaced by any constant strictly between 0 and $p_{c}$. The $p=p_{c}$ case of this theorem was proven by Kozma and Nachmias [37] and is used crucially in our proof. By (1.7) and a union bound,

$$
\begin{equation*}
\mathbb{P}_{p}\left(0 \leftrightarrow \partial \Lambda_{r}\right) \leq 2 d(2 r+1)^{d-1} \exp \left(-c\left(p_{c}-p\right)^{1 / 2} r\right) \tag{1.10}
\end{equation*}
$$

for every $p<p_{c}$ and $r \geq 2$; the content of the upper bound of Theorem 1.2 is to identify the correct polynomial prefactor. Similar theorems have been established for the distribution of the volume and intrinsic radius of slightly subcritical percolation clusters in [33], Section 4; these proofs apply to arbitrary transitive graphs, satisfying the triangle condition, and are much easier to prove than Theorem 1.2; see also [33], Section 5, for an overview of what is expected to hold for slightly supercritical percolation in high dimensions.

REMARK. Shortly after this paper first appeared on the arXiv, we learned of independent work of Chatterjee, Hanson, and Sosoe [12] containing an alternative proof of the subcritical one-arm estimate of Theorem 1.2. Their work is largely orthogonal to ours, using very different methods to establish Theorem 1.2 and not considering the slightly subcritical two-point function or finite-size scaling on the torus. Their work also establishes several further new results on the chemical distance and the number of spanning clusters in a box, which we do not study here. Six months later, a third proof of the upper bound of Theorem 1.2 appeared in [50].
1.4. Results for the torus. Percolation on the high-dimensional torus has received much attention in recent years [5, 6, 24-26, 29], with considerable related work on hypercube percolation including [7, 26, 28, 30]. That work has concentrated on the torus susceptibility and on questions with a flavour like those in the literature on the Erdős-Rényi random graph, such as the cluster size distribution. It was based on a torus triangle condition and required a lace expansion analysis directly on the torus with the focus on an intrinsically defined torus critical point which was later related to the critical point $p_{c}$ for $\mathbb{Z}^{d}$. In the following we analyse torus percolation in the vicinity of $p_{c}$ directly, with our main tool being the nearcritical bound on the $\mathbb{Z}^{d}$ two-point function provided by Theorem 1.1. At the end of this section, we will discuss the intrinsically defined critical point and the torus triangle condition.

Our principal focus here is on the torus two-point function $\tau_{p}^{\mathbb{T}}(x):=\mathbb{P}_{p}^{\mathbb{T}}(0 \leftrightarrow x)$ (for $x \in$ $\mathbb{T}_{r}^{d}$ ) and its "plateau." Despite the substantial progress on high-dimensional torus percolation, a detailed analysis of the behaviour of the torus two-point function $\tau_{p}^{\mathbb{T}}(x)$, within and below the critical window of width $V^{-1 / 3}$ about $p_{c}$, has been missing until now. A sizeable physics literature for related models, such as the Ising model (in dimensions $d>4$ ), predicts the existence of a "plateau" for the torus two-point function, namely, that within the critical window $\tau_{p}^{\mathbb{T}}(x)$ decays like the $\mathbb{Z}^{d}$ two-point function $\tau_{p_{c}}(x)$ for a certain volume-dependent range of $x$ values, but beyond this range $\tau_{p}^{\mathbb{T}}(x)$ levels off at an approximately constant value which exceeds $\tau_{p_{c}}(x)$. For the Ising model this is discussed, for example, in [43, 51, 52] and references therein. Different behaviour is expected for free boundary conditions and has recently been proved for the Ising model in [9]. The plateau has recently been proven to exist for the simple random walk two-point function (lattice Green function) on the torus in all dimensions $d>2[45,51]$ and for weakly self-avoiding walk on the torus in dimensions $d>4$ [45]. The plateau is applied in an essential way to analyse the weakly self-avoiding walk on a torus in dimensions $d>4$ in [41]. The differences between free, bulk, and periodic boundary conditions for percolation have been emphasised in [1], where the focus is on the maximal cluster size rather than the two-point function plateau; see also [26], Section 13.6.

Before stating our results on the torus two-point function, let us first recall some relevant background on the susceptibility. Let $\chi(p):=\sum_{x \in \mathbb{Z}^{d}} \tau_{p}(x)$ be the $\mathbb{Z}^{d}$ susceptibility which is known [2, 17, 22] to satisfy the mean-field asymptotics

$$
\begin{equation*}
\chi(p) \asymp \frac{1}{1-p / p_{c}} \quad\left(p \rightarrow p_{c}^{-}\right) \tag{1.11}
\end{equation*}
$$

for high-dimensional percolation on $\mathbb{Z}^{d}$.
By using the estimate (1.1) on the $\mathbb{Z}^{d}$ two-point function, we prove the following theorem. For notational convenience we sometimes evaluate a $\mathbb{Z}^{d}$ two-point function at a point $x \in \mathbb{T}_{r}^{d}$ with the understanding that in this case we regard $x$ as a point in $[-r / 2, r / 2)^{d} \cap \mathbb{Z}^{d}$. This occurs in the statement of Theorem 1.3.

THEOREM 1.3 (The two-point function plateau). Let $d>6$, and suppose that (T) holds on $\mathbb{Z}^{d}$ :

- Below the scaling window: There exist positive constants $c_{1}$ and $C_{1}$, depending only on $d$ and $L$, such that

$$
\begin{equation*}
\tau_{p}^{\mathbb{T}}(x) \leq \tau_{p}(x)+C_{1} \frac{\chi(p)}{V} \exp \left[-c_{1} m(p) r\right] \tag{1.12}
\end{equation*}
$$

for every $r>2$, every $x \in \mathbb{T}_{r}^{d}$, and every $p \in\left[0, p_{c}\right)$. Moreover, there exist positive constants $A_{1}, A_{2}, c_{2}$, and $M$ such that

$$
\begin{equation*}
\tau_{p}^{\mathbb{T}}(x) \geq \tau_{p}(x)+c_{2} \frac{\chi(p)}{V} \tag{1.13}
\end{equation*}
$$

for every $r>2$, every $x \in \mathbb{T}_{r}^{d}$ with $\|x\|_{\infty}>M$, and every $p \in\left[p_{c}-A_{1} V^{-2 / d}, p_{c}-\right.$ $\left.A_{2} V^{-1 / 3}\right]$.

- Inside the scaling window: For each $0<\delta \leq 1$ and $A>0$, there exist positive constants $r_{0}$ and $C_{3}$, depending only on $d, L, \delta$, and $A$, such that

$$
\begin{equation*}
\tau_{p}^{\mathbb{T}}(x) \leq(1+\delta) \tau_{p_{c}}(x)+\frac{C_{3}}{V^{2 / 3}} \tag{1.14}
\end{equation*}
$$

for every $p \in\left[0, p_{c}+A V^{-1 / 3}\right], r>r_{0}$, and $x \in \mathbb{T}_{r}^{d}$. In fact, the upper bound (1.14) holds for $p \leq p_{c}$ with $\delta=0$. In addition, there exists a positive constant $M$, depending only on $d$ and $L$, and a positive $c_{3}$, depending on $d, L$, and $A$, such that

$$
\begin{equation*}
\tau_{p}^{\mathbb{T}}(x) \geq(1-\delta) \tau_{p_{c}}(x)+\frac{c_{3}}{V^{2 / 3}} \tag{1.15}
\end{equation*}
$$

for every $r>r_{0}$, every $x \in \mathbb{T}_{r}^{d}$ with $\|x\|_{\infty}>M$, and every $p \in\left[p_{c}-A V^{-1 / 3}, p_{c}+\right.$ $A V^{-1 / 3}$ ].

With a choice $A \geq A_{2}$, the above theorem gives upper and lower bounds on the two-point function throughout the range $p_{c}-A_{1} V^{-2 / d} \leq p \leq p_{c}+A V^{-1 / 3}$. Below the window the bounds involve $\chi(p) / V$, which is infinite at $p_{c}$, whereas within the window this constant term is replaced by $V^{-2 / 3}$.

The upper bound of (1.12) is essentially an immediate consequence of (1.1). The lower bound also uses (1.1) but requires a more involved argument inspired by the method used for weakly self-avoiding walk in [45]. For the estimates inside the scaling window, we also use the critical one-arm result of Kozma and Nachmias [37].

We emphasise that the susceptibility $\chi(p)$, appearing in Theorem 1.3 , is the susceptibility for $\mathbb{Z}^{d}$, not for the torus. The upper bound (1.14) at $p=p_{c}$ and with $\delta=0$ is proved in [29], Theorem 1.7; we complement the upper bound with a lower bound of the same order and extend these bounds through the entire scaling window. The following corollary provides a more compact though less precise version of (1.14)-(1.15).

Corollary 1.4. Let $d>6$, and suppose that ( T ) holds on $\mathbb{Z}^{d}$. Then, for all $A>0$, there exists $r_{0}$ such that if $r>r_{0}$, then

$$
\begin{equation*}
\tau_{p}^{\mathbb{T}}(x) \asymp_{A} \frac{1}{\langle x\rangle^{d-2}}+\frac{1}{V^{2 / 3}} \tag{1.16}
\end{equation*}
$$

for every $x \in \mathbb{T}_{r}^{d}$ and every $p \in\left[p_{c}-A V^{-1 / 3}, p_{c}+A V^{-1 / 3}\right]$.
Proof. The upper bound follows immediately from the upper bound (1.14) together with our assumption (T).

The lower bound also follows immediately from (1.15) if $\|x\|_{\infty}>M$, once we assume that $r$ is sufficiently large that $A V^{-1 / 3} \leq A_{1} V^{-2 / d}$. If instead $\|x\|_{\infty} \leq M$, then it is joined to the origin by a path of length at most $M d$. Therefore, if we choose $r$ large enough that $A V^{-1 / 3} \leq p_{c} / 2$, then we have that $\tau_{p}^{\mathbb{T}}(x) \geq\left(p_{c} / 2\right)^{M d} \geq\left(p_{c} / 2\right)^{M d}\langle x\rangle^{-(d-2)}$. If we take $r$ large enough that $V^{-2 / 3}<M^{d-2}$, then for $\|x\|_{\infty} \leq M$, we also find that

$$
\begin{equation*}
\tau_{p}^{\mathbb{T}}(x) \geq \frac{1}{2}\left(\frac{p_{c}}{2}\right)^{M d}\left(\frac{1}{\langle x\rangle^{d-2}}+\frac{1}{V^{2 / 3}}\right) \tag{1.17}
\end{equation*}
$$

This completes the proof.
Consequently, for $\langle x\rangle^{d-2}<V^{2 / 3}$, we have $\tau_{p}^{\mathbb{T}}(x) \asymp\langle x\rangle^{-(d-2)}$, whereas for $\langle x\rangle^{d-2}>$ $V^{2 / 3}$, we have $\tau_{p}^{\mathbb{T}}(x) \asymp V^{-2 / 3}$. This is the plateau: the torus two-point function levels off at an approximately constant value once $x$ is large enough.

There is, in fact, a hierarchy of plateaux extending (1.16). Consider $p=p_{c}-V^{-a}$ with $a \in\left(\frac{2}{d}, \frac{1}{3}\right]$. By (1.11) $\chi(p) \asymp V^{a}$ and by (1.6) $m(p) r \asymp V^{\frac{1}{d}-\frac{a}{2}} \rightarrow 0$ as $r \rightarrow \infty$. For such $p$ the plateau effect occurs as soon as $\langle x\rangle^{d-2} \succeq V^{1-a}$. When $a=\frac{2}{d}$, the constant terms in (1.12)-(1.13) are of order $r^{-(d-2)}$ which is the smallest order that $\tau_{p}(x)$ can achieve for $x \in \mathbb{T}_{r}^{d}$. If $a<\frac{2}{d}$, then $m(p) r \rightarrow \infty$, and the plateau effect is absent.

The torus susceptibility is defined by $\chi^{\mathbb{T}}(p)=\sum_{x \in \mathbb{T}_{r}^{d}} \tau_{p}^{\mathbb{T}}(x)$. The following corollary of Theorem 1.3 shows that $\chi^{\mathbb{T}}(p) \asymp V^{1 / 3}$ for $p$ in the scaling window. It shows that the correct transfer of the bounds on the two-point in Theorem 1.3 from below the window into the window is achieved by replacing the $\mathbb{Z}^{d}$ susceptibility by the torus susceptibility. The corollary reproduces a result of $[5,6]$ via quite different methods and without any torus lace expansion.

Corollary 1.5. Let $d>6$, and suppose that $(\mathrm{T})$ holds on $\mathbb{Z}^{d}$. For any $A>0$, there exists $r_{0}$ such that

$$
\begin{equation*}
\chi^{\mathbb{T}}(p) \asymp_{A} V^{1 / 3} \tag{1.18}
\end{equation*}
$$

for all $r>r_{0}$ and all $p \in\left[p_{c}-A V^{-1 / 3}, p_{c}+A V^{-1 / 3}\right]$.
PROOF. This follows immediately by summation of (1.16) over $x \in \mathbb{T}_{r}^{d}$. Indeed, summation of $\tau_{p_{c}}(x) \asymp\langle x\rangle^{-(d-2)}$ over the torus yields $r^{2}=V^{2 / d}$ which is smaller when $d>6$ than the sum of the constant term which is $V \cdot V^{-2 / 3}=V^{1 / 3}$.

The cube-root divergence in the volume, given by Corollary 1.5 for periodic boundary conditions, should be contrasted with the situation for free boundary conditions. For free boundary conditions, it is a corollary of the bounds on the finite-volume two-point function in [11], Theorem 1.2, that in our setting of high-dimensional percolation the susceptibility at $p=p_{c}$ diverges as $r^{2}$ rather than $r^{d / 3}$. This is one setting where the controversy in the physics literature detailed, for example, in [52], is rigorously resolved.

Next, we discuss the torus triangle condition. In general, on a finite graph there is no unique definition of a critical value because the critical behaviour extends over a scaling window of $p$ values. In the above we have used the $\mathbb{Z}^{d}$ critical value $p_{c}$ as our reference point for the torus with large volume. In [5], (1.7), instead, an intrinsically defined critical value $p_{\mathbb{T}}$ is defined to be the unique solution, given a fixed $\lambda \in\left[V^{-1 / 3}, V^{2 / 3}\right]$, of the equation

$$
\begin{equation*}
\chi^{\mathbb{T}}\left(p_{\mathbb{T}}\right)=\lambda V^{1 / 3} \tag{1.19}
\end{equation*}
$$

We are interested in large volume $V$, so given any $\lambda>0$ eventually we do have $\lambda \in$ $\left[V^{-1 / 3}, V^{2 / 3}\right.$ ] and $p_{\mathbb{T}}$ is then well defined. Of course, $p_{\mathbb{T}}$ depends on $\lambda$, but only slightly, and, as we show below, $p_{\mathbb{T}}$ is an effective critical point no matter which $\lambda$ is chosen.

The torus triangle diagram is

$$
\begin{equation*}
\mathrm{T}_{p}^{\mathbb{T}}(x)=\sum_{u, v \in \mathbb{T}_{r}^{d}} \tau_{p}^{\mathbb{T}}(u) \tau_{p}^{\mathbb{T}}(v-u) \tau_{p}^{\mathbb{T}}(x-v) \tag{1.20}
\end{equation*}
$$

and it attains its maximum value when $x=0$ by [2], Lemma 3.3. The torus triangle condition is the statement that $\mathbb{T}_{p \mathbb{T}}^{\mathbb{T}}(x)$ is bounded by a constant independent of $r$ and $x$. Extensive consequences of the torus triangle condition are derived in [5, 24-26]. These consequences often require the stronger assumption that

$$
\begin{equation*}
\mathrm{T}_{p_{\mathbb{T}}}^{\mathbb{T}}(x) \leq \mathbb{1}(x=0)+a_{0} \tag{1.21}
\end{equation*}
$$

for $x \in \mathbb{T}_{r}^{d}$ and some small context-dependent constant $a_{0}$. We refer to the condition (1.21) as the $a_{0}$-strong torus triangle condition. The (strong) torus triangle condition is proved in [6], using a finite-graph version of the lace expansion, under the assumption that either $d$ is very large or $d>6$ and $L$ is large. We do not need or use the finite-graph lace expansion in this paper and give an alternate proof of the torus triangle condition based on Theorem 1.3. Our proof of the $a_{0}$-strong torus triangle condition relies also on its $\mathbb{Z}^{d}$ counterpart, namely, that

$$
\begin{equation*}
\mathrm{T}_{p_{c}}(x)=\sum_{u, v \in \mathbb{Z}^{d}} \tau_{p_{c}}(u) \tau_{p_{c}}(v-u) \tau_{p_{c}}(x-v) \leq \mathbb{1}(x=0)+a_{0} . \tag{1.22}
\end{equation*}
$$

Given any $a_{0}>0$, the $a_{0}$-strong triangle condition (1.22) is proved in [22] if $d \geq d_{0}$ for the nearest-neighbour model, and if $d>6$ and $L \geq L_{0}$ for the spread-out model, with $d_{0}$ and $L_{0}$ sufficiently large depending on $a_{0}$.

THEOREM 1.6. Let $d>6$, and suppose that ( T ) holds on $\mathbb{Z}^{d}$ :

- $p_{\mathbb{T}}$ in scaling window: There exists $\lambda_{0}>0$ such that, for any $\lambda \in\left(0, \lambda_{0}\right]$, there exist constants $C_{1}, r_{0}$ (both depending on $\lambda$ ) such that if we define $p_{\mathbb{T}}=p_{\mathbb{T}}(\lambda)$ via $\chi^{\mathbb{T}}\left(p_{\mathbb{T}}\right)=$ $\lambda V^{1 / 3}$, then $\left|p_{\mathbb{T}}-p_{c}\right| \leq C_{1} V^{-1 / 3}$ for every $r>r_{0}$.
- Torus triangle condition: With $\lambda_{0}, r_{0}$ as above, there exists $C_{2}$ such that $\mathbb{T}_{p_{\mathbb{T}}(\lambda)}^{\mathbb{T}}(x) \leq C_{2}$ for every $\lambda \in\left(0, \lambda_{0}\right], r>r_{0}$ and $x \in \mathbb{T}_{r}^{d}$.
- Strong torus triangle condition: Fix any $a_{0}>0$, and assume further that the $\frac{1}{2} a_{0}$-strong triangle condition (1.22) holds on $\mathbb{Z}^{d}$. There exists $\lambda_{1}>0$ (depending on $a_{0}$ ) such that $\mathrm{T}_{p_{\mathbb{T}}(\lambda)}^{\mathbb{T}}(x) \leq \mathbb{1}(x=0)+a_{0}$, for every $\lambda \in\left(0, \lambda_{1}\right], r>r_{0}\left(\lambda, a_{0}\right)$, and $x \in \mathbb{T}_{r}^{d}$.

The critical window for the torus $\mathbb{T}_{r}^{d}$ was defined in [5] (see [5], Theorem 1.3) to consist of the values of $p$ lying within distance of order $V^{-1 / 3}$ from $p_{\mathbb{T}}$. It was not proven at that time that the critical value $p_{c}$ for $\mathbb{Z}^{d}$ lies in the window, and it was not until several years later that this fact was proved [25]. Theorem 1.6 gives an alternate proof. In Theorem 1.3 we have defined the window as being centred at $p_{c}$ rather than at $p_{\mathbb{T}}$. The fact that $\left|p_{c}-p_{\mathbb{T}}\right| \preceq V^{-1 / 3}$ indicates that either choice of centre can be used. We emphasise that the analysis of [5, 25] relied on performing a separate lace expansion on the torus to establish the torus triangle condition, which our proof bypasses.
1.5. About the proof. We now give a brief overview of the structure of the paper and the proofs of our main theorems:

- In Sections 2 and 3, we prove our results concerning near-critical percolation on $\mathbb{Z}^{d}$, Theorems 1.1 and 1.2. Both theorems rely on the notion of pioneer edges which are those through which the cluster of the origin enters some halfspace for the first time:
- In Section 2 we formulate an estimate on the expected number of pioneer edges for a hyperplane, Theorem 2.3, which we then show to imply Theorems 1.1 and 1.2 in conjunction with the critical two-point estimate (T) and the critical one-arm results of Kozma and Nachmias [37].
- In Section 3 we prove our main estimate on pioneer edges, Theorem 2.3. This proof has two components: First, in Section 3.1 we apply diagrammatic methods utilising the halfspace two-point function estimates of Chatterjee and Hanson [11] to prove at criticality that the expected number of pioneers for the hyperplane $\left\{x: x_{1}=n\right\}$ is bounded by a constant independent of $n$ (Proposition 3.1). Using this together with the aforementioned one-arm estimates of Kozma and Nachmias [37], we deduce a power-law tail bound with exponent $2 / 3$ on the total number of pioneer edges at criticality (Lemma 3.10). Second, in Section 3.2 we use the theory of decision trees and the OSSS inequality, which we review in Section 3.2.1, to obtain a differential inequality applying to the distribution of the total number of pioneers (Lemma 3.11). This differential inequality is of the same form as that obtained for the distribution of the radius by Menshikov [39] (see also [15]) and for the distribution of volume by Hutchcroft [31]. Using this differential inequality together with our results on the distribution of the number of pioneers at $p_{c}$, we deduce sharp estimates on the distribution of the total number of pioneers at $p_{c}-\varepsilon$ (Proposition 3.9) and conclude the proof of Theorem 2.3.
- In Section 4 we apply our $\mathbb{Z}^{d}$ results to prove the case of Theorem 1.3 in which $p$ lies below the scaling window. This eventually leads us to easily derive in Section 4.4 the (two versions of the) torus triangle condition presented in Theorem 1.6. In order to apply the $\mathbb{Z}^{d}$ result of Theorem 1.1 to the torus, we crucially rely on a coupling of percolation on $\mathbb{Z}^{d}$ and on the torus that was first introduced by Benjamini and Schramm [4] and developed extensively in the works of Heydenreich and van der Hofstad [24, 25]. The massive decay of $\tau_{p}(x)$ for $p<p_{c}$ then directly gives the upper-bound (1.12) while the lower bound requires fine control of diagrammatic estimates and thus occupies the bulk of this section. Much of this work follows the same general strategy used to analyse weakly self-avoiding walk on the torus in [40] but differs in the details.
- Finally, in Section 5 we prove the part of Theorem 1.3 in which $p$ lies inside the scaling window. The relevant lower bounds are easy consequences of the "below the scaling window" estimates since $\tau_{p}^{\mathbb{T}}(x)$ is monotone in $p$. The upper bounds are proven first at $p_{c}$ using the coupling between $\mathbb{Z}^{d}$ and $\mathbb{T}_{r}^{d}$ percolation as well as the extrinsic one-arm result from Kozma and Nachmias [37] using a variation on the methods of van der Hofstad and Sapozhnikov [29]. We then extend the result for $p \in\left(p_{c}, p_{c}+A V^{-1 / 3}\right.$ ] by the combination of an elementary coupling of Bernoulli percolation at different probabilities and of the input of the intrinsic one-arm exponent controlled in Kozma and Nachmias [36].

2. Near-critical percolation: Proof of Theorems 1.1-1.2. In this section we prove Theorems 1.1-1.2, subject to Theorem 2.3, which concerns the expected number of pioneer edges.
2.1. Pioneer edges. For each $n \in \mathbb{Z}$, let $S_{n}$ be the hyperplane $\left\{\left(y_{1}, \ldots, y_{d}\right) \in \mathbb{Z}^{d}: y_{1}=\right.$ $n\}$, and let $H_{n}$ be the halfspace $H_{n}=\left\{\left(y_{1}, \ldots, y_{d}\right) \in \mathbb{Z}^{d}: y_{1} \geq n\right\}$. We will often write $H=$ $H_{0}$ to lighten notation.


Fig. 1. Pioneer edges of a cluster in $\mathbb{Z}^{2}$. Edges that are pioneers with respect to the distinguished vertex $x$ are thick and red, while other edges are thin and black.

DEFINITION 2.1. Given $x \in \mathbb{Z}^{d}$, we call an edge $\{y, z\} \in \mathbb{B}$ an $x$-pioneer if $x_{1}<z_{1}$, $y_{1}<z_{1},\{y, z\}$ is open, and $x$ is connected to $y$ by an open path contained in the half-space $\left\{\left(w_{1}, \ldots, w_{d}\right): w_{1}<z_{1}\right\}$. That is, $\{y, z\}$ is an $x$-pioneer if $z$ lies to the right of $x$ and there exists an open path starting at $x$ whose last edge is $\{y, z\}$ with the path lying strictly to the left of $z$ at every previous time. For each $x \in \mathbb{Z}^{d}$ and $n \geq 1$, we define $\mathcal{P}_{x}(n)$ to be the set of $x$-pioneers $\{y, z\}$ with $y_{1}<x_{1}+n \leq z_{1}$ and define $\mathcal{P}_{x}=\bigcup_{n \geq 1} \mathcal{P}_{x}(n)$ to be the set of all $x$-pioneers; see Figure 1 .

Since we are only interested in finite-range models, we have that $\mathcal{P}_{x}(n) \cap \mathcal{P}_{x}(m)=\varnothing$ when $m \geq n+L$ and hence that $\frac{1}{L} \sum_{n \geq 1}\left|\mathcal{P}_{x}(n)\right| \leq\left|\mathcal{P}_{x}\right| \leq \sum_{n \geq 1}\left|\mathcal{P}_{x}(n)\right|$. For each $p \in[0,1]$ and $n \geq 1$, we define $P_{p}(n)=\mathbb{E}_{p}\left|\mathcal{P}_{0}(n)\right|$ which may be infinite. We begin by noting that $P_{p}(n)$ satisfies the following elementary submultiplicativity property.

LEMMA 2.2. $\quad P_{p}(n+m) \leq P_{p}(n) \max _{0 \leq i \leq L-1} P_{p}(m-i)$ for every $p \in[0,1]$ and $n \geq 1$ and $m \geq L$.

It is convenient to simplify the inequality of Lemma 2.2, as follows. Let $e_{1}=(1,0, \ldots, 0)$ be the basis vector in the positive horizontal direction. First, we observe that if $A_{m}$ denotes the event that each of the $m$ unit-length edges of the horizontal path connecting 0 to $m e_{1}$ are open, then

$$
\begin{align*}
\mathbb{E}_{p}\left|\mathcal{P}_{0}(n+m)\right| & \geq \mathbb{E}_{p}\left[\left|\mathcal{P}_{0}(n+m)\right| \mathbb{1}\left(A_{m}\right)\right] \geq \mathbb{E}_{p}\left[\left|\mathcal{P}_{m e_{1}}(n)\right| \mathbb{1}\left(A_{m}\right)\right]  \tag{2.1}\\
& \geq \mathbb{E}_{p}\left|\mathcal{P}_{m e_{1}}(n)\right| p^{m}=\mathbb{E}_{p}\left|\mathcal{P}_{0}(n)\right| p^{m}
\end{align*}
$$

where we used the Harris-FKG inequality for the third inequality. A complementary bound can be obtained by a similar argument, with the result that

$$
\begin{equation*}
p^{m} P_{p}(n) \leq P_{p}(n+m) \leq p^{-m} P_{p}(n) \tag{2.2}
\end{equation*}
$$

for every $0<p \leq 1$ and $n, m \geq 1$. Therefore, by Lemma 2.2 we obtain the simplified submuliplicative inequality

$$
\begin{equation*}
P_{p}(n+m) \leq p^{-L+1} P_{p}(n) P_{p}(m) \tag{2.3}
\end{equation*}
$$

for every $0<p \leq 1$ and $n, m \geq 1$.
Proof of Lemma 2.2. Suppose that $\{y, z\} \in \mathcal{P}_{0}(n+m)$ and that $y_{1}<z_{1}$ so that there exists a simple open path $\gamma$, starting at 0 , that has $\{y, z\}$ as its last edge and lies strictly to the left of $z$ at every previous step. Letting $\{a, b\}$ be the first edge crossed by $\gamma$ as it enters $H_{n}$ for the first time (where $a_{1}<b_{1}$ ), we see that the portion of $\gamma$ up to and including the edge $\{a, b\}$
and the portion of $\gamma$ after this edge are disjoint witnesses for the events that $\{a, b\} \in \mathcal{P}_{0}(n)$ and that $\{y, z\} \in \mathcal{P}_{b}\left(n+m-b_{1}\right)$. It follows by a union bound and the BK inequality that

$$
\begin{align*}
P_{p}(n+m) & =\sum_{y_{1}<z_{1}} \mathbb{P}_{p}\left(\{y, z\} \in \mathcal{P}_{0}(n+m)\right)  \tag{2.4}\\
& \leq \sum_{a_{1}<b_{1}} \mathbb{P}_{p}\left(\{a, b\} \in \mathcal{P}_{0}(n)\right) \sum_{y_{1}<z_{1}} \mathbb{P}_{p}\left(\{y, z\} \in \mathcal{P}_{0}\left(n+m-b_{1}\right)\right) . \tag{2.5}
\end{align*}
$$

Now, if $a_{1}<b_{1}$ and $\{a, b\} \in \mathcal{P}_{0}(n)$, then we must have that $n \leq b_{1} \leq n+L-1$ and the claim follows by translation-invariance.

Theorems 1.1 and 1.2 will both be deduced from the following theorem.
THEOREM 2.3. Let $d>6$, and suppose that $(\mathrm{T})$ holds. There exist positive constants $c$ and $C$ such that

$$
\begin{equation*}
\exp \left[-C\left(p_{c}-p\right)^{1 / 2} n\right] \preceq P_{p}(n) \preceq \exp \left[-c\left(p_{c}-p\right)^{1 / 2} n\right] \tag{2.6}
\end{equation*}
$$

for every $n \geq 1$ and $p_{c} / 2 \leq p \leq p_{c}$.
The lower bound of Theorem 2.3 is an easy consequence of (2.3), as follows. First, by Fekete's lemma $-\lim _{n \rightarrow \infty} \frac{1}{n} \log P_{p}(n)$ is well-defined, as an element of $[-\infty, \infty]$, and satisfies

$$
\begin{equation*}
-\lim _{n \rightarrow \infty} \frac{1}{n} \log P_{p}(n)=\sup _{n \geq 1}-\frac{1}{n} \log \left[p^{-L+1} P_{p}(n)\right] \tag{2.7}
\end{equation*}
$$

for every $0<p \leq 1$. Also, $\mathcal{P}_{0}(n)$ must be nonempty on the event that 0 is connected to $n e_{1}$ and hence by Markov's inequality $\tau_{p_{c}}\left(n e_{1}\right) \leq P_{p}(n)$ for every $n \geq 1$. Using this together with (1.6), one may verify that the exponential decay rate of $P_{p}(n)$ is equal to the mass $m(p)$, whenever $p<p_{c}$, and hence that

$$
\begin{equation*}
p^{L-1} \exp [-m(p) n] \leq P_{p}(n) \leq \exp [-m(p) n+o(n)] \tag{2.8}
\end{equation*}
$$

as $n \rightarrow \infty$ for each fixed $p<p_{c}$, where the subexponential correction in the upper bound may depend on the value of $p<p_{c}$. (Indeed, explicit upper bounds of this form can be deduced from (1.7) by direct summation.) This is, of course, consistent with Theorem 2.3 since $m(p) \asymp\left|p-p_{c}\right|^{1 / 2}$ as $p \uparrow p_{c}$ in the high-dimensional setting [20]. Theorem 2.3 eliminates the subexponential term from this upper bound for high-dimensional models at the cost of replacing $m(p)$ with $c m(p)$ for some positive constant $c$. This will be used to obtain the sharp control of the subexponential terms in Theorems 1.1 and 1.2.

### 2.2. Proof of Theorems 1.1-1.2.

2.2.1. Proof of Theorem 1.1 and the upper bound of Theorem 1.2. We now show how Theorem 2.3 easily implies Theorem 1.1 and the upper bound of Theorem 1.2.

Proof of Theorem 1.1, given Theorem 2.3. Note that it is enough to prove the result for $p \in\left[p_{c} / 2, p_{c}\right]$ so that Theorem 2.3 applies. Indeed, for $p \leq p_{c} / 2$, we can simply use monotonicity to bound $\tau_{p}(x) \leq \tau_{p_{c} / 2}(x)$ and deduce a bound of the desired form (with a smaller constant in the exponential).

We may assume without loss of generality that the point $x \in \mathbb{Z}^{d}$ satisfies $x_{1}=\langle x\rangle \geq 4 L \geq$ 1. Let $n=\left\lfloor x_{1} / 2\right\rfloor$. Suppose that the origin is connected to $x$ by some simple open path $\gamma$, and let $\{a, b\}$ be the edge that is crossed by $\gamma$, as it enters the halfspace $H_{n}$ for the first time,
with $a_{1}<b_{1}$. Then the portion of $\gamma$ up to and including the edge $\{a, b\}$ and the portion of $\gamma$ after this edge are disjoint witnesses for the events $\{a, b\} \in \mathcal{P}_{0}(n)$ and $\{b \leftrightarrow x\}$. Thus, we have by a union bound and the BK inequality that

$$
\begin{align*}
\mathbb{P}_{p}(0 \leftrightarrow x) & \leq \sum_{a_{1}<b_{1}} \mathbb{P}_{p}\left(\{a, b\} \in \mathcal{P}_{0}(n)\right) \mathbb{P}_{p}(b \leftrightarrow x) \\
& \leq \sum_{a_{1}<b_{1}} \mathbb{P}_{p}\left(\{a, b\} \in \mathcal{P}_{0}(n)\right) \cdot\langle x-b\rangle^{-d+2} \tag{2.9}
\end{align*}
$$

for every $0 \leq p \leq p_{c}$, where we used (T) in the second inequality. Now, if $\{a, b\} \in \mathcal{P}_{0}(n)$, then we must have that $n \leq b_{1} \leq n+L-1$ and hence that $\langle x-b\rangle \geq x_{1}-b_{1} \geq x_{1} / 4$ so that there exists a positive constant $c$ such that

$$
\begin{align*}
\mathbb{P}_{p}(0 \leftrightarrow x) & \preceq\langle x\rangle^{-d+2} \sum_{a_{1}<b_{1}} \mathbb{P}_{p}\left(\{a, b\} \in \mathcal{P}_{0}(n)\right)=\langle x\rangle^{-d+2} P_{p}(n)  \tag{2.10}\\
& \preceq\langle x\rangle^{-d+2} \exp \left[-c\left(p_{c}-p\right)^{1 / 2}\langle x\rangle\right]
\end{align*}
$$

by Theorem 2.3. This completes the proof of Theorem 1.1.
Proof of upper bound of Theorem 1.2, given Theorem 2.3. Recall that $H_{n}$ denotes the halfspace $\left\{y \in \mathbb{Z}^{d}: y_{1} \geq n\right\}$. It suffices by symmetry to prove that there exists a positive constant $c$ such that

$$
\begin{equation*}
\mathbb{P}_{p}\left(0 \leftrightarrow H_{2 n}\right) \preceq \frac{1}{n^{2}} \exp \left[-c\left(p_{c}-p\right)^{1 / 2} n\right] \tag{2.11}
\end{equation*}
$$

for every $n \geq 0$ and $p_{c} / 2 \leq p \leq p_{c}$. Let $n \geq 2 L \geq 1$, and suppose that the origin is connected to the halfspace $H_{2 n}$ by some simple open path $\gamma$. Letting $\{a, b\}$ with $a_{1}<b_{1}$ be the edge that is crossed by $\gamma$, as it enters the halfspace $H_{n}$ for the first time, we observe that the portion of $\gamma$ up to and including the edge $\{a, b\}$ and the portion of $\gamma$ after this edge are disjoint witnesses for the events $\{a, b\} \in \mathcal{P}_{0}(n)$ and $\left\{b \leftrightarrow H_{2 n}\right\}$. Thus, we have by a union bound and the BK inequality that

$$
\begin{equation*}
\mathbb{P}_{p}\left(0 \leftrightarrow H_{2 n}\right) \leq \sum_{a_{1}<b_{1}} \mathbb{P}_{p}\left(\{a, b\} \in \mathcal{P}_{0}(n)\right) \mathbb{P}_{p}\left(b \leftrightarrow H_{2 n}\right) \tag{2.12}
\end{equation*}
$$

Since $b_{1} \leq n+L-1 \leq 3 n / 2$, we deduce by the main result of [37] (i.e., the $p=p_{c}$ case of Theorem 1.2) that

$$
\begin{align*}
\mathbb{P}_{p}\left(0 \leftrightarrow H_{2 n}\right) & \preceq n^{-2} \sum_{a_{1}<b_{1}} \mathbb{P}_{p}\left(\{a, b\} \in \mathcal{P}_{0}(n)\right)  \tag{2.13}\\
& =n^{-2} P_{p}(n) \preceq n^{-2} \exp \left[-c\left(p_{c}-p\right)^{1 / 2} n\right]
\end{align*}
$$

as claimed, where we used Theorem 2.3 in the final inequality.
2.2.2. Proof of lower bound of Theorem 1.2. In this section we apply Theorem 2.3 to prove the lower bound of Theorem 1.2. We give the proof for the nearest-neighbour model, the general finite-range proof being similar but requiring more involved notation.

We begin with some definitions. Recall that $S_{r}$ denotes the hyperplane $\left\{x \in \mathbb{Z}^{d}: x_{1}=r\right\}$ for each $r \in \mathbb{Z}$. For each $-\infty \leq n \leq m \leq \infty$, let $S_{n, m}$ denote the slab $S_{n, m}:=\bigcup_{i=n}^{m} S_{i}$. For each $r \geq 0$, let $X_{r}$ be the number of points in the hyperplane $S_{r}$ that are connected to the origin by an open path lying within the halfspace $S_{-\infty, r}$, and let $Y_{r} \leq X_{r}$ be the number of points in the hyperplane $S_{r}=\left\{x \in \mathbb{Z}^{d}: x_{1}=r\right\}$ that are connected to the origin by an open path lying within the slab $S_{-r, r}$. Since we are working with nearest-neighbour models, every
edge in $\mathcal{P}_{0}(r+1)$ must be of the form $\{(r, x),(r+1, x)\}$ for some $x \in \mathbb{Z}^{d-1}$, and the edge $\{(r, x),(r+1, x)\}$ belongs to $\mathcal{P}_{0}(r+1)$ if and only if it is open and $(r, x)$ is connected to 0 inside the halfspace lying to the left of $(r, x)$. From this it follows that

$$
\begin{equation*}
\mathbb{E}_{p} X_{r}=\frac{1}{p} \mathbb{E}_{p}\left|\mathcal{P}_{0}(r+1)\right| \tag{2.14}
\end{equation*}
$$

for every $r \geq 0$ and $0<p \leq 1$.
PRoof of Lower bound of Theorem 1.2. Let $r \geq 1$. The lower bound we wish to prove asserts that

$$
\begin{equation*}
\mathbb{P}_{p}\left(0 \leftrightarrow \partial \Lambda_{r}\right) \geq \frac{c}{r^{2}} \exp \left(-C\left(p_{c}-p\right)^{1 / 2} r\right) \tag{2.15}
\end{equation*}
$$

Since $\left\{Y_{r}>0\right\} \subset\left\{0 \leftrightarrow \partial \Lambda_{r}\right\}$, it suffices to prove that the above lower bound holds with instead $\mathbb{P}_{p}\left(Y_{r}>0\right)$ on the left-hand side. We will prove this via the Cauchy-Schwarz inequality

$$
\begin{equation*}
\mathbb{P}_{p}\left(Y_{r}>0\right) \geq \frac{\left(\mathbb{E}_{p} Y_{r}\right)^{2}}{\mathbb{E}_{p}\left[Y_{r}^{2}\right]} \tag{2.16}
\end{equation*}
$$

together with suitable estimates on the first and second moments of $Y_{r}$.
It follows from (2.14) and Theorem 2.3 that there exist positive constants $c$ and $C$ such that

$$
\begin{equation*}
\exp \left[-C\left(p_{c}-p\right)^{1 / 2} r\right] \preceq \mathbb{E}_{p} X_{r} \preceq \exp \left[-c\left(p_{c}-p\right)^{1 / 2} r\right] \tag{2.17}
\end{equation*}
$$

for every $p_{c} / 2 \leq p \leq p_{c}$ and $r \geq 0$. We write $\{x \stackrel{A}{\leftrightarrow} y\}$ to mean that $x$ and $y$ are connected by an open path using only vertices of $A$. Observe that for each $r \geq 1$ and $x \in S_{r}$, we have the inclusion of sets

$$
\begin{equation*}
\left\{0 \stackrel{S_{-\infty, r}}{\longleftrightarrow} x\right\} \backslash\left\{0 \stackrel{S_{-r, r}}{\longleftrightarrow} x\right\} \subseteq \bigcup_{y \in S_{-r}}\left\{0 \stackrel{S_{-r, r}}{\longleftrightarrow} y\right\} \circ\left\{y \stackrel{S_{-\infty, r}}{\longleftrightarrow} x\right\} \tag{2.18}
\end{equation*}
$$

Indeed, if the event on the left-hand side of this inclusion holds, $\gamma$ is an open path connecting 0 and $x$ in $S_{-\infty, r}$, and $y$ is the first point of $S_{-r}$ visited by $\gamma$ then the portions of $\gamma$ before and after visiting $y$ are disjoint witnesses for the events $\left\{0 \stackrel{S_{-r, r}}{\longleftrightarrow} y\right\}$ and $\left\{y \stackrel{S_{-\infty, r}}{\longleftrightarrow} x\right\}$, as claimed. It follows by a union bound, the BK inequality, and translation and reflection symmetry that

$$
\begin{equation*}
\mathbb{E}_{p} X_{r} \leq \mathbb{E}_{p} Y_{r}+\mathbb{E}_{p} Y_{r} \cdot \mathbb{E}_{p} X_{2 r} \tag{2.19}
\end{equation*}
$$

for every $0 \leq p \leq 1$ and $r \geq 0$. Applying the estimate (2.17), it follows that $\mathbb{E}_{p} Y_{r} \asymp \mathbb{E}_{p} X_{r}$ for every $p_{c} / 2 \leq p \leq p_{c}$ and $r \geq 0$ and hence that

$$
\begin{equation*}
\exp \left[-C\left(p_{c}-p\right)^{1 / 2} r\right] \leq \mathbb{E}_{p} Y_{r} \preceq \exp \left[-c\left(p_{c}-p\right)^{1 / 2} r\right] \tag{2.20}
\end{equation*}
$$

for every $p_{c} / 2 \leq p \leq p_{c}$ and $r \geq 0$.
We turn now to the second moment of the random variable $Y_{r}$. Suppose that $x$ and $y$ are two points in $S_{r}$, both of which are connected to the origin in $S_{-r, r}$. There must exist a point $z \in S_{-r, r}$ such that the events $\left\{0 \stackrel{S_{-r, r}}{\longleftrightarrow} z\right\},\left\{z \stackrel{S_{-r, r}}{\longleftrightarrow} x\right\}$, and $\left\{z \stackrel{S_{-r, r}}{\longleftrightarrow} y\right\}$ all occur disjointly. It follows by a union bound and the BK inequality that

$$
\begin{equation*}
\mathbb{E}_{p}\left[Y_{r}^{2}\right] \leq \sum_{k=-r}^{r} \sum_{z \in S_{k}} \mathbb{P}_{p}\left(0 \stackrel{S_{-r, r}}{\longleftrightarrow} z\right) \sum_{x, y \in S_{r}} \mathbb{P}_{p}\left(z \stackrel{S_{-r, r}}{\longleftrightarrow} x\right) \mathbb{P}_{p}\left(z \stackrel{S_{-r, r}}{\longleftrightarrow} y\right) \tag{2.21}
\end{equation*}
$$

and hence by (2.17) that

$$
\begin{equation*}
\mathbb{E}_{p}\left[Y_{r}^{2}\right] \preceq \sum_{k=-r}^{r} \sum_{z \in S_{k}} \mathbb{P}_{p}\left(0 \stackrel{S_{-r, r}}{\longleftrightarrow} z\right) \exp \left[-2 c\left(p_{c}-p\right)^{1 / 2}(r-k)\right] \tag{2.22}
\end{equation*}
$$

for every $p_{c} / 2 \leq p \leq p_{c}$. Our next goal is to bound the resulting sum over $z$ for each $-r \leq$ $k \leq r$. Suppose that $z \in S_{k}$ for some $-r \leq k \leq r$, and suppose that the origin is connected to $z$ by a simple open path in $S_{-r, r}$. By considering the right-most point that this path visits, we see that there must exist $0 \leq a \leq r$ and $w \in S_{a}$ such that the events $\left\{0 \stackrel{S_{-r, a}}{\longleftrightarrow} w\right\}$ and $\left\{w \stackrel{S_{-r, a}}{\longleftrightarrow} z\right\}$ occur disjointly. Thus, applying a union bound and the BK inequality again as above, we obtain that

$$
\begin{align*}
\sum_{z \in S_{k}} \mathbb{P}_{p}\left(0 \stackrel{S_{-r, r}}{\longleftrightarrow} z\right) & \leq \sum_{a=k \vee 0}^{r} \sum_{w \in S_{a}} \sum_{z \in S_{k}} \mathbb{P}_{p}\left(0 \stackrel{S_{-r, a}}{\longleftrightarrow} w\right) \mathbb{P}_{p}\left(w \stackrel{S_{-r, a}}{\longleftrightarrow} z\right)  \tag{2.23}\\
& \leq \sum_{a=k \vee 0}^{r} \sum_{w \in S_{a}} \sum_{z \in S_{k}} \mathbb{P}_{p}\left(0 \stackrel{S_{-\infty, a}}{\longleftrightarrow} w\right) \mathbb{P}_{p}\left(w \stackrel{S_{-\infty, a}}{\longleftrightarrow} z\right),
\end{align*}
$$

and a further application of (2.17) gives that

$$
\begin{align*}
\sum_{z \in S_{k}} \mathbb{P}_{p}\left(0 \stackrel{S_{-r, r}}{\longleftrightarrow} z\right) & \preceq \sum_{a=k \vee 0}^{r} \exp \left[-c\left(p_{c}-p\right)^{1 / 2} a-c\left(p_{c}-p\right)^{1 / 2}(a-k)\right]  \tag{2.24}\\
& \preceq r \exp \left[-c\left(p_{c}-p\right)^{1 / 2}|k|\right]
\end{align*}
$$

for every $p_{c} / 2 \leq p \leq p_{c}, r \geq 1$, and $-r \leq k \leq r$. Putting these estimates together, we obtain that

$$
\begin{align*}
\mathbb{E}_{p}\left[Y_{r}^{2}\right] & \preceq r \sum_{k=-r}^{r} \exp \left[-2 c\left(p_{c}-p\right)^{1 / 2}(r-k)-c\left(p_{c}-p\right)^{1 / 2}|k|\right]  \tag{2.25}\\
& \preceq r^{2} \exp \left[-c\left(p_{c}-p\right)^{1 / 2} r\right]
\end{align*}
$$

for every $p_{c} / 2 \leq p \leq p_{c}$ and $r \geq 1$. Putting this together with the lower bound of (2.20), we obtain

$$
\begin{equation*}
\mathbb{P}_{p}\left(Y_{r}>0\right) \geq \frac{\left(\mathbb{E}_{p} Y_{r}\right)^{2}}{\mathbb{E}_{p}\left[Y_{r}^{2}\right]} \succeq \frac{1}{r^{2}} \exp \left[-(2 C-c)\left(p_{c}-p\right)^{1 / 2} r\right] \tag{2.26}
\end{equation*}
$$

for every $p_{c} / 2 \leq p \leq p_{c}$ and $r \geq 1$. This completes the proof.
3. Expected number of pioneers: Proof of Theorem 2.3. In this section we complete the proof of Theorems 1.1-1.2 by proving Theorem 2.3.

### 3.1. The expected number of critical pioneers.

3.1.1. The critical case of Theorem 2.3. In this section we prove the $p=p_{c}$ case of Theorem 2.3.

Proposition 3.1. Let $d>6$, and suppose that $(\mathrm{T})$ holds. There exist positive constants $c$ and $C$ such that $c \leq P_{p_{c}}(n) \leq C$ for every $n \geq 1$.

Note that the lower bound $P_{p_{c}}(n) \geq p_{c}^{L-1}$ holds in every dimension by taking $p \uparrow p_{c}$ in the estimate (2.8) above; the main content of the proposition is that a matching upper bound holds in the high-dimensional case.

To ease notation, we will prove the upper bound of Proposition 3.1 only for nearestneighbour percolation. The general proof for finite-range models is very similar but substantially more involved, as one must introduce various additional summations to most calculations. This assumption will be in force for the remainder of Section 3.1. We write $P(n)=P_{p_{c}}(n)$ and $\mathbb{P}=\mathbb{P}_{p_{c}}$ to lighten notation. Recall that $H$ denotes the half-space $\left\{(n, x): n \geq 0, x \in \mathbb{Z}^{d-1}\right\}$ and that the edge $\{(n-1, x),(n, x)\}$ belongs to $\mathcal{P}_{0}(n)$ if and only if it is open and $(n-1, x)$ is connected to 0 inside the halfspace lying to the left of $(n-1, x)$. We again write $\{x \stackrel{A}{\hookrightarrow} y\}$ to mean that $x$ and $y$ are connected by an open path, using only vertices of $A$. By (2.14)

$$
\begin{equation*}
P(n)=\mathbb{E}\left|\mathcal{P}_{0}(n)\right|=\sum_{x \in \mathbb{Z}^{d-1}} p_{c} \cdot \mathbb{P}\left((0,0) \stackrel{S_{-\infty, n-1}}{\longleftrightarrow}(n-1, x)\right) . \tag{3.1}
\end{equation*}
$$

By translation and reflection symmetry, this gives (with term-by-term equality)

$$
\begin{equation*}
P(n)=\sum_{x \in \mathbb{Z}^{d-1}} p_{c} \cdot \mathbb{P}((0, x) \stackrel{H}{\longleftrightarrow}(n-1,0)) . \tag{3.2}
\end{equation*}
$$

A further translation by $(0,-x)$, followed by replacement of $-x$ by $x$, gives

$$
\begin{equation*}
P(n)=\sum_{x \in \mathbb{Z}^{d-1}} p_{c} \cdot \mathbb{P}((0,0) \stackrel{H}{\leftrightarrow}(n-1, x)) . \tag{3.3}
\end{equation*}
$$

This equality makes it convenient for us to consider, for each $n \geq 0$, the quantity

$$
\begin{equation*}
\bar{P}(n):=\frac{1}{p_{c}} P(n+1)=\sum_{x \in \mathbb{Z}^{d-1}} \mathbb{P}((0, x) \stackrel{H}{\leftrightarrow}(n, 0))=\sum_{x \in \mathbb{Z}^{d-1}} \mathbb{P}((0,0) \stackrel{H}{\leftrightarrow}(n, x)), \tag{3.4}
\end{equation*}
$$

instead of $P(n)$ itself. A very similar proof to that of Lemma 2.2 yields that $\bar{P}$ is submultiplicative in the sense that $\bar{P}(n+m) \leq \bar{P}(n) \bar{P}(m)$ for each $n, m \geq 0$.

In order to upper bound $P(n)$, we will prove a complementary supermultiplicative-type estimate on $\bar{P}(n)$ via diagrammatic methods. For each $n \geq 0$, define

$$
\begin{equation*}
\bar{P}^{*}(n)=\max _{0 \leq k \leq n} \bar{P}(k) \tag{3.5}
\end{equation*}
$$

We deduce Proposition 3.1 from the following two estimates.
Proposition 3.2. Let $d>6$, and suppose that $(\mathrm{T})$ holds. There exist positive constants $c>0$ and $\ell \in \mathbb{N}$ such that

$$
\begin{equation*}
\bar{P}^{*}(2 n+\ell) \geq c \bar{P}^{*}(n)^{2} \tag{3.6}
\end{equation*}
$$

for every $n \geq 0$.
LEMMA 3.3. Let $d>6$, and suppose that $(\mathrm{T})$ holds. Then $\bar{P}^{*}(n) \preceq \log (n+2)$ for every $n \geq 0$.

We now show how Proposition 3.1 follows from Proposition 3.2 and Lemma 3.3. In brief, Proposition 3.2 implies that if $\bar{P}^{*}$ is unbounded, then it must grow exponentially rapidly. This contradicts Lemma 3.3, so $\bar{P}^{*}$ must be bounded, as desired.

Proof of Proposition 3.1, given Proposition 3.2 and Lemma 3.3. Let $c>0$ and $\ell \in \mathbb{N}$ be the constants from Proposition 3.2. If there exists $n \geq \ell$ such that $\bar{P}^{*}(n) \geq 2 / c$, then we have by induction that

$$
\begin{equation*}
\bar{P}^{*}\left(3^{k} n\right) \geq \bar{P}^{*}\left(2 \cdot 3^{k-1} n+\ell\right) \geq \frac{1}{c} 2^{2^{k}} \tag{3.7}
\end{equation*}
$$

for every $k \geq 1$. This contradicts Lemma 3.3, and so we must in fact have that $\bar{P}^{*}(n)<2 / c$ for every $n \geq \ell$ and hence for every $n \geq 0$, as claimed.

We now prove Lemma 3.3 which was used above in the proof of Proposition 3.1 and which will also be used in the proof of Proposition 3.2. The proof is based on the upper bounds

$$
\begin{align*}
& \mathbb{P}(x \stackrel{H}{\hookrightarrow} y) \preceq\langle x-y\rangle^{-d+1} \quad \text { for every } x \in \mathbb{Z}^{d} \text { with } x_{1}=0 \text { and every } y \in H,  \tag{3.8}\\
& \mathbb{P}(x \stackrel{H}{\longleftrightarrow} y) \preceq\langle x-y\rangle^{-d} \quad \text { for every } x, y \in \mathbb{Z}^{d} \text { with } x_{1}=y_{1}=0 . \tag{3.9}
\end{align*}
$$

of Chatterjee and Hanson [11], Theorems 7.2 and 1.1(b), as well as their lower bound [11], Theorem 1.1(b),

$$
\begin{equation*}
\mathbb{P}(x \stackrel{H}{\longleftrightarrow} y) \succeq\langle x-y\rangle^{-d+1} \tag{3.10}
\end{equation*}
$$

$$
\text { for every } x \in \mathbb{Z}^{d} \text { with } x_{1}=0 \text { and every } y \in H \text { with }\langle y-x\rangle \leq 2 y_{1} .
$$

The above bounds are valid for $d>6$, assuming that (T) holds.
REMARK 3.4. For $d>2$ and given $x, y \in H$ with $y=\left(y_{1}, \ldots, y_{d}\right)$, we set $\bar{y}=$ $\left(-y_{1}, y_{2}, \ldots, y_{d}\right)$. By the method of images (see, e.g., [38], Proposition 8.1.1), the half-space lattice Green function is given by $G_{H}(x, y)=G(x, y)-G(x, \bar{y})$ where the unrestricted lattice Green function $G(x, y)$ is asymptotic to a multiple of $|x-y|^{2-d}$. It is natural to assume that the critical two-point function has the same behaviour, which suggests an extended version

$$
\begin{equation*}
\mathbb{P}(x \stackrel{H}{\longleftrightarrow} y) \preceq \frac{\left(x_{1}+1\right)\left(y_{1}+1\right)}{\langle x-y\rangle^{d}} \quad \text { for every } x, y \in H \tag{3.11}
\end{equation*}
$$

of the Chatterjee-Hanson bounds, which we believe to be sharp when $x_{1} \vee y_{1} \leq K\langle x-y\rangle$ for some fixed $K>0$. If this bound were proven, it would be possible to deduce Proposition 3.1 directly by summation. Although [12], Theorem 6, proves a strengthened form of the Chatterjee-Hanson half-space two-point function estimate, the strengthened version is not sharp when both points lie near the boundary, and it remains an open problem to improve the estimate for the half-space two-point function to an extent where it could be used to prove our critical pioneers estimate Proposition 3.1 via direct summation.

Proof of Lemma 3.3. It suffices to prove that $\bar{P}(n) \preceq \log (n+2)$ for each $n \geq 0$. Let $R=(n+1)^{d}$. By (3.4) and (3.8),

$$
\begin{align*}
\bar{P}(n) & =\sum_{x \in \mathbb{Z}^{d-1}} \mathbb{P}((0, x) \stackrel{H}{\leftrightarrow}(n, 0)) \\
& \preceq \sum_{x \in \Lambda_{n}^{d-1}}(n+1)^{-d+1}+\sum_{x \in \Lambda_{R}^{d-1} \backslash \Lambda_{n}^{d-1}}\langle x\rangle^{-d+1}+\sum_{x \in \mathbb{Z}^{d-1} \backslash \Lambda_{R}^{d-1}} \mathbb{P}((n, 0) \stackrel{H}{\leftrightarrow}(0, x)) . \tag{3.12}
\end{align*}
$$

To control the final term, we use the Harris-FKG inequality, (3.9) and (3.10) to obtain that

$$
\begin{align*}
& \sum_{x \in \mathbb{Z}^{d-1} \backslash \Lambda_{R}^{d-1}} \mathbb{P}((0, x) \stackrel{H}{\longleftrightarrow}(n, 0)) \\
\leq & \sum_{x \in \mathbb{Z}^{d-1} \backslash \Lambda_{R}^{d-1}} \mathbb{P}((0,0) \stackrel{H}{\longleftrightarrow}(n, 0))^{-1} \cdot \mathbb{P}((0,0) \stackrel{H}{\leftrightarrow}(0, x))  \tag{3.13}\\
& \sum_{x \in \mathbb{Z}^{d-1} \backslash \Lambda_{R}^{d-1}}(n+1)^{d-1}\langle x\rangle^{-d} .
\end{align*}
$$

Putting these bounds together and using that $\left|\left\{x \in \mathbb{Z}^{d-1}:\langle x\rangle=r\right\}\right|=O\left((r+1)^{d-2}\right)$ for every $r \geq 0$, we deduce that

$$
\begin{align*}
\bar{P}(n) & =\sum_{x \in \mathbb{Z}^{d-1}} \mathbb{P}((0, x) \stackrel{H}{\longleftrightarrow}(n, 0)) \\
& \preceq 1+\sum_{r=n}^{R} r^{-1}+\sum_{r=R}^{\infty}(n+1)^{d-1} r^{-2}  \tag{3.14}\\
& \preceq 1+\log \frac{R+1}{n+1}+\frac{(n+1)^{d-1}}{R} \preceq \log (n+2),
\end{align*}
$$

and the proof is complete.
3.1.2. Proof of Proposition 3.2. In this section we prove Proposition 3.2. As a first step, we make the following definition.

DEFINITION 3.5. Let $e_{1}=(1,0, \ldots, 0)$ be the unit vector in the horizontal direction. Recall that, for each $k \in \mathbb{Z}, S_{k}$ denotes the hyperplane $S_{k}=\left\{(k, x): x \in \mathbb{Z}^{d-1}\right\}=\left\{x \in \mathbb{Z}^{d}\right.$ : $\left.x_{1}=k\right\}$ and $H_{k}$ denotes the halfspace $H_{k}=\bigcup_{i \geq k} S_{i}$. Given $0 \leq k<n, x \in S_{k}$ and $y \in S_{n}$, we say that $x$ is a good pivotal vertex for the event $\left\{0 \stackrel{H_{0}}{\longleftrightarrow} y\right\}$ if the following hold:

1. The edge $\left\{x, x+e_{1}\right\}$ is open.
2. 0 is connected to $x$ in $H_{0}$ off of the edge $\left\{x, x+e_{1}\right\}$.
3. $x+e_{1}$ is connected to $y$ in $H_{k+1}$.
4. 0 is not connected to $y$ in $H_{0}$ off of the edge $\left\{x, x+e_{1}\right\}$.

We claim that if 0 is connected to $y$ in $H_{0}$, then for each $0 \leq k<n$ there is at most one good pivotal vertex $x \in S_{k}$ for the event $\left\{0 \stackrel{H_{0}}{\longleftrightarrow} y\right\}$. Indeed, if $x$ is a good pivotal vertex, then any open path from 0 to $y$ in $H_{0}$ must pass through the edge $\left\{x, x+e_{1}\right\}$. If $x, z \in S_{k}$ were distinct good pivotal vertices, then there would exist simple open paths $\gamma_{1}$ and $\gamma_{2}$ connecting 0 to $y$ in $H_{0}$ such that $\gamma_{1}$ visits $S_{k}$ for the last time at $x$ and $\gamma_{2}$ visits $S_{k}$ for the last time at $z$. The concatenation of the portion of $\gamma_{1}$, up until its visit to $z$ with the portion of $\gamma_{2}$ after it visits $z$, would, therefore, be an open simple path connecting 0 and $y$ in $H_{0}$ that avoids $x$, contradicting the assumption that $x$ is a good pivotal vertex.

The fact that there is at most one good pivotal vertex implies by (3.4) that

$$
\begin{align*}
\bar{P}(n) & =\sum_{y \in S_{n}} \mathbb{P}\left(0 \stackrel{H_{0}}{\longleftrightarrow} y\right) \geq \sum_{x \in S_{k}} \sum_{y \in S_{n}} \mathbb{P}\left(0 \stackrel{H_{0}}{\longleftrightarrow} y, x \text { a good pivotal vertex for this event }\right)  \tag{3.15}\\
& =\frac{p_{c}}{1-p_{c}} \sum_{x \in S_{k}} \sum_{y \in S_{n}} \mathbb{P}\left(0 \stackrel{H_{0}}{\longleftrightarrow} x, x \stackrel{H_{0}}{\longleftrightarrow} x+e_{1}, \text { and } x+e_{1} \stackrel{H_{k+1}}{\longleftrightarrow} y\right)
\end{align*}
$$

for every $0 \leq k<n$. By symmetry we have equivalently that

$$
\begin{equation*}
\bar{P}(n+k) \geq \frac{p_{c}}{1-p_{c}} \sum_{y \in S_{n}} \sum_{x \in S_{-k}} \mathbb{P}\left(x \stackrel{H_{-k}}{\longleftrightarrow} 0,0 \stackrel{H_{-k}}{\longleftrightarrow} e_{1}, \text { and } e_{1} \stackrel{H_{1}}{\longleftrightarrow} y\right) \tag{3.16}
\end{equation*}
$$

for every $n \geq 1$ and $k \geq 0$.
To make use of this inequality, we will first prove the following lemma. Like many results in high-dimensional percolation, its proof relies on a bound on the open triangle diagram

$$
\begin{equation*}
\mathrm{T}_{p}(x)=\sum_{y, z \in \mathbb{Z}^{d}} \tau_{p}(y) \tau_{p}(z-y) \tau_{p}(x-z) \tag{3.17}
\end{equation*}
$$

at the critical value $p=p_{c}$. The triangle diagram was introduced by Aizenman and Newman in 1984 [2], and the finiteness of $\mathrm{T}_{p_{c}}(x)$ was proved in [22] for sufficiently large $d$ for the nearest-neighbour model and for $d>6$ for sufficiently spread-out models and extended in [17] to the nearest-neighbour model in dimensions $d \geq 11$. Although historically the proof of (T) relied on this finiteness of the triangle diagram, a posteriori ( T ) yields (for $d>6$ )

$$
\begin{equation*}
\mathrm{T}_{p_{c}}(x) \preceq \sum_{y, z \in \mathbb{Z}^{d}}\langle y\rangle^{2-d}\langle z-y\rangle^{2-d}\langle x-z\rangle^{2-d} \preceq\langle x\rangle^{6-d} \tag{3.18}
\end{equation*}
$$

via the elementary convolution estimate [21], Proposition 1.7.
Indeed, [21], Proposition 1.7, states more generally that, for each $a, b>0$ with $a+b<d$, there exists a constant $C=C(d, a, b)$ such that

$$
\begin{equation*}
\sum_{y \in \mathbb{Z}^{d}}\langle y\rangle^{a-d}\langle x-y\rangle^{b-d} \leq C\langle x\rangle^{a+b-d} \tag{3.19}
\end{equation*}
$$

for every $x \in \mathbb{Z}^{d}$, and it follows by applying this estimate twice that, for each $a, b, c>0$ with $a+b+c<d$, there exists a constant $C=C(d, a, b, c)$ such that

$$
\begin{equation*}
\sum_{y, z \in \mathbb{Z}^{d}}\langle y\rangle^{a-d}\langle z-y\rangle^{b-d}\langle x-z\rangle^{c-d} \leq C\langle x\rangle^{a+b+c-d} \tag{3.20}
\end{equation*}
$$

for every $x \in \mathbb{Z}^{d}$. The following proof will, in fact, apply (3.19) with $a, b, c=2+\varepsilon$ rather than the usual triangle estimate (3.18).

Lemma 3.6. Let $d>6$, and suppose that $(\mathrm{T})$ holds. There exists a positive constant $\ell$ such that

$$
\begin{equation*}
\sum_{x \in S_{-n}} \sum_{y \in S_{n+\ell}} \mathbb{P}\left(x \stackrel{H_{-n}}{\longleftrightarrow} 0,0 \stackrel{H_{-n}}{\longleftrightarrow} \ell e_{1}, \ell e_{1} \stackrel{H_{\ell}}{\longleftrightarrow} y\right) \geq \frac{1}{2} \bar{P}(n)^{2} \tag{3.21}
\end{equation*}
$$

for every $n \geq 0$ such that $\bar{P}(n)=\bar{P}^{*}(n)$.

Proof. Fix $n \geq 0$. We follow a variation on the strategy of [36], Lemma 3.2, illustrated in Figure 2. Let $K_{0, n}$ denote the cluster of 0 in $H_{-n}$, and let $\mathscr{C}_{0}$ be the set of finite connected subsets of $\mathbb{Z}^{d}$ containing 0 . By conditioning on $K_{0, n}$, we see that

$$
\begin{equation*}
\mathbb{P}\left(x \stackrel{H_{-n}}{\longleftrightarrow} 0,0 \stackrel{H_{-n}}{\longleftrightarrow} \ell e_{1}, \ell e_{1} \stackrel{H_{\ell}}{\longleftrightarrow} y\right)=\sum_{\substack{A \in \mathscr{C}_{0} \\ A \ni x}} \mathbb{P}\left(K_{0, n}=A, \ell e_{1} \stackrel{H_{\ell}}{\longleftrightarrow} y\right) \mathbb{1}\left(\ell e_{1} \notin A\right) \tag{3.22}
\end{equation*}
$$

for each $\ell \geq 1, x \in S_{-n}$, and $y \in S_{n+\ell}$. Note, moreover, that if $A \in \mathscr{C}_{0}$ is such that $y \notin A$, then

$$
\begin{equation*}
\mathbb{P}\left(K_{0, n}=A, \ell e_{1} \stackrel{H_{\ell}}{\longleftrightarrow} y\right) \mathbb{1}\left(\ell e_{1} \notin A\right)=\mathbb{P}\left(K_{0, n}=A, \ell e_{1} \stackrel{H_{\ell}}{\longleftrightarrow} y \text { off } A\right), \tag{3.23}
\end{equation*}
$$



FIg. 2. Schematic illustration of the diagrammatic estimate used to prove Lemma 3.6. The squiggly red line indicates that 0 and $\ell e_{1}$ are not connected by an open path in the half-space $H_{-n}$. To prove the lemma, it suffices to prove that the second diagrammatic sum on the right-hand side is much smaller than the first when the separation parameter $\ell$ is large.
where we write " $\ell e_{1} \stackrel{H_{\ell}}{\longleftrightarrow} y$ off $A$ " to mean that there is an open path from $\ell e_{1}$ to $y$ in $H_{\ell}$ that does not visit any vertex of $A$, including at its endpoints. Since the events $\left\{K_{0, n}=A\right\}$ and $\left\{\ell e_{1} \stackrel{H_{\ell}}{\longleftrightarrow} y\right.$ off $\left.A\right\}$ depend on disjoint sets of edges (namely, those edges with at least one endpoint in $A$ and those edges with neither endpoint in $A$ ), these two events are independent, and we deduce that

$$
\begin{equation*}
\mathbb{P}\left(K_{0, n}=A, \ell e_{1} \stackrel{H_{\ell}}{\longleftrightarrow} y\right) \mathbb{1}\left(\ell e_{1} \notin A\right)=\mathbb{P}\left(K_{0, n}=A\right) \mathbb{P}\left(\ell e_{1} \stackrel{H_{\ell}}{\longleftrightarrow} y \text { off } A\right) . \tag{3.24}
\end{equation*}
$$

Next, we observe that

$$
\begin{equation*}
\mathbb{P}\left(\ell e_{1} \stackrel{H_{l}}{\longleftrightarrow} y \text { off } A\right)=\mathbb{P}\left(\ell e_{1} \stackrel{H_{\ell}}{\longleftrightarrow} y\right)-\mathbb{P}\left(\ell e_{1} \stackrel{H_{\ell}}{\longleftrightarrow} y \text { only via } A\right), \tag{3.25}
\end{equation*}
$$

where we write " $\ell e_{1} \stackrel{H_{l}}{\longleftrightarrow} y$ only via $A$ " to mean that there is an open path from $\ell e_{1}$ to $y$ in $H_{\ell}$ but every such path must visit a vertex of $A$. (This holds, in particular, if $\ell e_{1}$ is connected to $y$ in $H_{\ell}$ and belongs to the set $A$.) It follows that

$$
\mathbb{P}\left(x \stackrel{H_{-n}}{\longleftrightarrow} 0,0 \stackrel{H_{-n}}{\longleftrightarrow} \ell e_{1}, \ell e_{1} \stackrel{H_{\ell}}{\longleftrightarrow} y\right)=\sum_{\substack{A \in \mathscr{C}_{0} \\ A \ni x}} \mathbb{P}\left(K_{0, n}=A\right) \mathbb{P}\left(\ell e_{1} \stackrel{H_{\ell}}{\longleftrightarrow} y\right)
$$

$$
\begin{equation*}
-\sum_{\substack{A \in \mathscr{C}_{0} \\ A \ni x}} \mathbb{P}\left(K_{0, n}=A\right) \mathbb{P}\left(\ell e_{1} \stackrel{H_{\ell}}{\longleftrightarrow} y \text { only via } A\right) \tag{3.26}
\end{equation*}
$$

and hence that

$$
\mathbb{P}\left(x \stackrel{H_{-n}}{\longleftrightarrow} 0,0 \stackrel{H_{-n}}{\longleftrightarrow} \ell e_{1}, \ell e_{1} \stackrel{H_{\ell}}{\longleftrightarrow} y\right)=\mathbb{P}\left(x \stackrel{H_{-n}}{\longleftrightarrow} 0\right) \mathbb{P}\left(\ell e_{1} \stackrel{H_{\ell}}{\longleftrightarrow} y\right)
$$

$$
\begin{equation*}
-\sum_{\substack{A \in \mathscr{C}_{0} \\ A \ni x}} \mathbb{P}\left(K_{0, n}=A\right) \mathbb{P}\left(\ell e_{1} \stackrel{H_{\ell}}{\longleftrightarrow} y \text { only via } A\right) \tag{3.27}
\end{equation*}
$$

for every $\ell \geq 1, x \in S_{-n}$, and $y \in S_{n+\ell}$.

Our goal is to prove that the sum over $x \in S_{-n}$ and $y \in S_{n+\ell}$ of the left-hand side of (3.27) is bounded below by $\frac{1}{2} \bar{P}^{*}(n)^{2}$, assuming that $\bar{P}(n)=\bar{P}^{*}(n)$. For the first term on the right-hand side, it follows from (3.4) that

$$
\begin{equation*}
\sum_{x \in S_{-n}} \sum_{y \in S_{n+\ell}} \mathbb{P}\left(x \stackrel{H_{-n}}{\longleftrightarrow} 0\right) \mathbb{P}\left(\ell e_{1} \stackrel{H_{\ell}}{\longleftrightarrow} y\right)=\bar{P}(n)^{2} . \tag{3.28}
\end{equation*}
$$

It, therefore, suffices to prove that we can choose $\ell$ large in order to obtain

$$
\begin{equation*}
\sum_{x \in S_{-n}} \sum_{y \in S_{n+\ell}} \sum_{\substack{A \in \mathscr{C}_{0} \\ A \ni x}} \mathbb{P}\left(K_{0, n}=A\right) \mathbb{P}\left(\ell e_{1} \stackrel{H_{\ell}}{\longleftrightarrow} y \text { only via } A\right) \leq \frac{1}{2} \bar{P}(n)^{2} \tag{3.29}
\end{equation*}
$$

for every $n \geq 0$ such that $\bar{P}(n)=\bar{P}^{*}(n)$. The remainder of the proof is devoted to establishing (3.29).

As a first step, we observe by the BK inequality that

$$
\begin{align*}
\mathbb{P}\left(\ell e_{1} \stackrel{H_{\ell}}{\longleftrightarrow} y \text { only via } A\right) & \leq \sum_{a \in A} \mathbb{P}\left(\left\{\ell e_{1} \stackrel{H_{\ell}}{\longleftrightarrow} a\right\} \circ\left\{a \stackrel{H_{\ell}}{\longleftrightarrow} y\right\}\right)  \tag{3.30}\\
& \leq \sum_{a \in A} \mathbb{P}\left(\ell e_{1} \stackrel{H_{\ell}}{\longleftrightarrow} a\right) \mathbb{P}\left(a \stackrel{H_{\ell}}{\longleftrightarrow} y\right)
\end{align*}
$$

for every $\ell \geq 1$ and $y \in S_{n+\ell}$. Indeed, if the event on the left-hand side occurs, then there must exist a simple open path connecting $\ell e_{1}$ to $y$ in $H_{\ell}$ that passes through $A$ at some point $a$, and the portions of this path before and after visiting $a$ are disjoint witnesses for the events $\left\{\ell e_{1} \stackrel{H_{\ell}}{\longleftrightarrow} a\right\}$ and $\left\{a \stackrel{H_{\ell}}{\longleftrightarrow} y\right\}$. It follows that

$$
\begin{align*}
& \sum_{\substack{A \in \mathscr{C}_{0} \\
A \ni x}} \mathbb{P}\left(K_{0, n}=A\right) \mathbb{P}\left(\ell e_{1} \stackrel{H_{\ell}}{\longleftrightarrow} y \text { only via } A\right) \\
& \quad \leq \sum_{\substack{A \in \mathscr{C}_{0} \\
A \ni x}} \mathbb{P}\left(K_{0, n}=A\right) \sum_{a \in A} \mathbb{P}\left(\ell e_{1} \stackrel{H_{\ell}}{\longleftrightarrow} a\right) \mathbb{P}\left(a \stackrel{H_{\ell}}{\longleftrightarrow} y\right)  \tag{3.31}\\
& \quad=\sum_{a \in H_{\ell}} \mathbb{P}\left(0 \stackrel{H_{-n}}{\longleftrightarrow} x, 0 \stackrel{H_{-n}}{\longleftrightarrow} a\right) \mathbb{P}\left(\ell e_{1} \stackrel{H_{\ell}}{\longleftrightarrow} a\right) \mathbb{P}\left(a \stackrel{H_{\ell}}{\longleftrightarrow} y\right)
\end{align*}
$$

for each $\ell \geq 1, x \in S_{-n}$, and $y \in S_{n+\ell}$. Now, if 0 is connected to both $x$ and $a$ in $H_{-n}$, there must exist $z \in H_{-n}$ such that the events $\left\{0 \stackrel{H_{-n}}{\longleftrightarrow} z\right\},\left\{z \stackrel{H_{-n}}{\longleftrightarrow} x\right\}$, and $\left\{z \stackrel{H_{-n}}{\longleftrightarrow} a\right\}$ all occur disjointly, so it follows by the BK inequality that

$$
\begin{equation*}
\mathbb{P}\left(0 \stackrel{H_{-n}}{\longleftrightarrow} x, 0 \stackrel{H_{-n}}{\longleftrightarrow} a\right) \leq \sum_{z \in H_{-n}} \mathbb{P}\left(0 \stackrel{H_{-n}}{\longleftrightarrow} z\right) \mathbb{P}\left(z \stackrel{H_{-n}}{\longleftrightarrow} x\right) \mathbb{P}\left(z \stackrel{H_{-n}}{\longleftrightarrow} a\right) . \tag{3.32}
\end{equation*}
$$

We insert (3.32) into (3.31) and insert the result into (3.29). The sums over $x$ and $y$ can then be performed explicitly, since these variables each appear in just one factor. For the sum over $x$, we use the fact that, for $z \in S_{j}$ with $j \geq-n$, we have

$$
\begin{equation*}
\sum_{x \in S_{-n}} \mathbb{P}\left(z \stackrel{H_{-n}}{\longleftrightarrow} x\right)=\bar{P}(n+j) . \tag{3.33}
\end{equation*}
$$

For the sum over $y$, we use that

$$
\begin{align*}
\sum_{y \in S_{r}} \mathbb{P}\left(0 \stackrel{H_{-m}}{\longleftrightarrow} y\right) & \leq \sum_{y \in S_{r}} \sum_{k=-(r \wedge 0)}^{m} \sum_{w \in S_{-k}} \mathbb{P}\left(\left\{0 \stackrel{H_{-k}}{\longleftrightarrow} w\right\} \circ\left\{w \stackrel{H_{-k}}{\longleftrightarrow} y\right\}\right)  \tag{3.34}\\
& \leq \sum_{k=-(r \wedge 0)}^{m} \bar{P}(k) \bar{P}(r+k)
\end{align*}
$$

for every $m \geq 0$ and $r \geq-m$, which follows by decomposing a simple open path from 0 to $y$, according to its left-most point, and using the BK inequality. The result is

$$
\begin{align*}
& \sum_{x \in S_{-n}} \sum_{y \in S_{n+\ell}} \sum_{\substack{A \in \mathscr{C}_{0} \\
A \ni x}} \mathbb{P}\left(K_{0, n}=A\right) \mathbb{P}\left(\ell e_{1} \stackrel{H_{\ell}}{\longleftrightarrow} y \text { only via } A\right) \\
& \leq \sum_{i=\ell}^{\infty} \sum_{a \in S_{i}} \sum_{j=-n}^{\infty} \sum_{z \in S_{j}} \mathbb{P}(0 \leftrightarrow z) \bar{P}(n+j) \mathbb{P}(z \leftrightarrow a) \mathbb{P}\left(\ell e_{1} \stackrel{H_{\ell}}{\longleftrightarrow} a\right) \\
& \quad \times \sum_{k=0 \vee(i-n-\ell)}^{i-\ell} \bar{P}(k) \bar{P}(n+\ell-i+k)  \tag{3.35}\\
& \quad \leq \bar{P}^{*}(n)^{2} \sum_{i=\ell}^{\infty} \sum_{a \in S_{i}} \sum_{j=-n}^{\infty} \sum_{z \in S_{j}} \mathbb{P}(0 \leftrightarrow z) \mathbb{P}(z \leftrightarrow a) \mathbb{P}\left(\ell e_{1} \stackrel{H_{\ell}}{\longleftrightarrow} a\right)(i+1) \bar{P}(j \vee 0) \bar{P}^{*}(i),
\end{align*}
$$

where in the last step we used $\bar{P}(k) \leq \bar{P}^{*}(i)$ and $\bar{P}(n+\ell-i+k) \leq \bar{P}^{*}(n)$ for $k \leq i-\ell$ as well as the submultiplicative property of $\bar{P}$ to see that $\bar{P}(n+j) \leq \bar{P}^{*}(n) \bar{P}(j \vee 0)$.

To estimate the right-hand side of (3.35), we use Lemma 3.3 to bound $\bar{P}(j \vee 0)$ and $\bar{P}^{*}(i)$, and (T) to bound $\mathbb{P}(0 \leftrightarrow z)$ and $\mathbb{P}(z \leftrightarrow a)$. Also, we use the half-space estimate (3.8) to see that

$$
\begin{equation*}
\mathbb{P}\left(\ell e_{1} \stackrel{H_{\ell}}{\longleftrightarrow} a\right)(i+1) \bar{P}^{*}(i) \preceq\left\langle\ell e_{1}-a\right\rangle^{-d+1}(i+1) \log (i \vee 2) . \tag{3.36}
\end{equation*}
$$

Here $i \geq \ell$, so $i \leq \ell+\left\langle\ell e_{1}-a\right\rangle$ and, therefore,

$$
\begin{align*}
& \mathbb{P}\left(\ell e_{1} \stackrel{H_{\ell}}{\longleftrightarrow} a\right)(i+1) \bar{P}^{*}(i) \\
& \quad \preceq\left\langle\ell e_{1}-a\right\rangle^{-d+2} \log \left(\left\langle\ell e_{1}-a\right\rangle \vee 2\right)+\ell \log (\ell \vee 2)\left\langle\ell e_{1}-a\right\rangle^{-d+1}  \tag{3.37}\\
& \quad \preceq\left\langle\ell e_{1}-a\right\rangle^{-d+2+1 / 4}+\ell^{5 / 4}\left\langle\ell e_{1}-a\right\rangle^{-d+1}
\end{align*}
$$

Thus, with the left-hand side of our goal (3.29) temporarily written as $T_{n, \ell}$, using the crude bound $\bar{P}(j \vee 0) \preceq \log (j \vee 2) \preceq\langle z\rangle^{1 / 4}$ yields that

$$
\begin{align*}
T_{n, \ell} \leq & \bar{P}^{*}(n)^{2} \sum_{i=\ell}^{\infty} \sum_{a \in S_{i}} \sum_{j=-n}^{\infty} \sum_{z \in S_{j}}\langle z\rangle^{-d+2+1 / 4}\langle z-a\rangle^{-d+2}\left\langle\ell e_{1}-a\right\rangle^{-d+2+1 / 4}  \tag{3.38}\\
& +\ell^{5 / 4} \bar{P}^{*}(n)^{2} \sum_{i=\ell}^{\infty} \sum_{a \in S_{i}} \sum_{j=-n}^{\infty} \sum_{z \in S_{j}}\langle z\rangle^{-d+2+1 / 4}\langle z-a\rangle^{-d+2}\left\langle\ell e_{1}-a\right\rangle^{-d+1}
\end{align*}
$$

from which for $d \geq 7$ (3.20) yields

$$
\begin{align*}
T_{n, \ell} & \preceq \bar{P}^{*}(n)^{2}\left(\left\langle\ell e_{1}\right\rangle^{-d+6+1 / 2}+\ell^{5 / 4}\left\langle\ell e_{1}\right\rangle^{-d+5+1 / 4}\right) \\
& \preceq \ell^{-d+6+1 / 2} P^{*}(n)^{2} \leq \ell^{-1 / 2} P^{*}(n)^{2} . \tag{3.39}
\end{align*}
$$



FIG. 3. Schematic illustrations of the events $\mathscr{B}_{i}(x, y)(l e f t)$ and $\mathscr{C}_{i}(x, y)($ right $)$. The blue edges represent those that are forced to be open in $\omega^{i-1}$. The squiggly red line indicates that $(\ell-i) e_{1}$ and $(\ell+1-i) e_{1}$ lie in distinct clusters in the half-space $H_{-n}$.

Since this bound holds uniformly over $n \geq 0$ and $\ell \geq 1$ and since the prefactor $\ell^{-1 / 2}$ tends to zero as $\ell \rightarrow \infty$, we deduce that there exists a constant $\ell$ such that

$$
\begin{equation*}
T_{n, \ell} \leq \frac{1}{2} \bar{P}^{*}(n)^{2} \tag{3.40}
\end{equation*}
$$

This proves (3.29) and, therefore, completes the proof.
Finally, we deduce Proposition 3.2 from (3.16) and Lemma 3.6. In preparation for this and inspired by [32], Section 4, we define three events and prove a lemma relating them, as follows. Fix any $n \geq 0, x \in S_{-n}$, and $y \in S_{n+\ell}$, where $\ell$ is fixed as in Lemma 3.6. We define the event

$$
\begin{equation*}
\mathscr{A}(x, y)=\left\{x \stackrel{H_{-n}}{\longleftrightarrow} 0,0 \stackrel{H_{-n}}{\longleftrightarrow} \ell e_{1}, \ell e_{1} \stackrel{H_{l}}{\longleftrightarrow} y\right\} . \tag{3.41}
\end{equation*}
$$

Then (3.21) can be rewritten more compactly as

$$
\begin{equation*}
\sum_{x \in S_{-n}} \sum_{y \in S_{n+\ell}} \mathbb{P}(\mathscr{A}(x, y)) \geq \frac{1}{2} \bar{P}(n)^{2} \tag{3.42}
\end{equation*}
$$

for every $n \geq 0$ such that $\bar{P}(n)=\bar{P}^{*}(n)$.
Let $\eta$ be the left-directed horizontal geodesic connecting $\ell e_{1}$ to 0 , and for each $1 \leq i \leq \ell$, let $\eta_{i}$ be the $i$ th edge crossed by $\eta$. Given a Bernoulli bond percolation configuration $\omega$ on $\mathbb{Z}^{d}$, let $\omega^{i}$ be the configuration obtained from $\omega$ by setting

$$
\omega^{i}(e)= \begin{cases}1, & e \in\left\{\eta_{j}: 1 \leq j \leq i\right\}  \tag{3.43}\\ \omega(e), & e \notin\left\{\eta_{j}: 1 \leq j \leq i\right\}\end{cases}
$$

In particular, $\omega^{0}=\omega$. For each $1 \leq i \leq \ell$, let $\mathscr{B}_{i}(x, y)$ be the event that 0 and $\ell e_{1}$ are connected in $H_{-n}$ in $\omega^{i}$ but not in $\omega^{i-1}, 0$ is connected to $x$ in $H_{-n}$ in $\omega^{i-1}$, and $\ell e_{1}$ is connected to $y$ in $H_{\ell}$ in $\omega^{i-1}$. Finally, for each $1 \leq i \leq \ell$ let

$$
\begin{align*}
\mathscr{C}_{i}(x, y)= & \left\{x \stackrel{H_{-n}}{\longleftrightarrow}(\ell-i) e_{1},(\ell-i) e_{1} \stackrel{H_{-n}}{\longleftrightarrow}(\ell+1-i) e_{1},\right.  \tag{3.44}\\
& \left.(\ell+1-i) e_{1} \stackrel{H_{\ell+1-i}}{\longleftrightarrow} y\right\} .
\end{align*}
$$

The events $\mathscr{B}_{i}(x, y)$ and $\mathscr{C}_{i}(x, y)$ are depicted in Figure 3.
Lemma 3.7. With the above setup and with $p=p_{c}$,

$$
\begin{equation*}
\mathbb{P}(\mathscr{A}(x, y)) \leq \sum_{i=1}^{\ell} p_{c}^{-i+1} \mathbb{P}\left(\mathscr{C}_{i}(x, y)\right) \tag{3.45}
\end{equation*}
$$

Proof. Given a configuration $\omega$, let $i$ be minimal such that 0 and $\ell e_{1}$ are connected in $\omega^{i}$. When the event $\mathscr{A}(x, y)$ holds, $i$ cannot be zero and hence must be between 1 and $\ell$. Since the clusters of 0 and $\ell e_{1}$ are both larger in $\omega^{i-1}$ than they are in $\omega$, we must have that 0 is connected to $x$ in $H_{-n}$ in $\omega^{i-1}$ and that $\ell e_{1}$ is connected to $y$ in $H_{\ell}$ in $\omega^{i-1}$ which means that $\mathscr{B}_{i}(x, y)$ holds. It follows that

$$
\begin{equation*}
\mathscr{A}(x, y) \subseteq \bigcup_{i=1}^{\ell} \mathscr{B}_{i}(x, y) . \tag{3.46}
\end{equation*}
$$

Since we also have the inclusion of events $\mathscr{C}_{i}(x, y) \supseteq \mathscr{B}_{i}(x, y) \cap\left\{\omega\left(\eta_{j}\right)=1\right.$ for every $1 \leq$ $j \leq i-1\}$ and since the two events on the right of this inclusion are independent, we have that

$$
\begin{equation*}
\mathbb{P}\left(\mathscr{C}_{i}(x, y)\right) \geq p_{c}^{i-1} \mathbb{P}\left(\mathscr{B}_{i}(x, y)\right) \tag{3.47}
\end{equation*}
$$

With (3.46) this completes the proof.

Proof of Proposition 3.2. It suffices to prove that there exist positive constants $c>0$ and $\ell \in \mathbb{N}$ such that $\bar{P}^{*}(2 n+\ell) \geq c \bar{P}^{*}(n)^{2}$ for every $n \geq 0$. Let $\ell$ be as in Lemma 3.6, and suppose that $n \geq 0$ has $\bar{P}(n)=\bar{P}^{*}(n)$. Constants in this proof are permitted to depend on $\ell$. In view of (3.42), the desired inequality will follow, for such $n$, if we show that

$$
\begin{equation*}
\sum_{x \in S_{-n}} \sum_{y \in S_{n+\ell}} \mathbb{P}(\mathscr{A}(x, y)) \preceq(\ell+1) \bar{P}^{*}(2 n+\ell-1) . \tag{3.48}
\end{equation*}
$$

However, this is, in fact, sufficient for general $n \geq 0$, since we may take $0 \leq n^{\prime} \leq n$ such that $\bar{P}\left(n^{\prime}\right)=\bar{P}^{*}(n)$ to then deduce that

$$
\begin{equation*}
\bar{P}^{*}(2 n+\ell) \geq \bar{P}^{*}\left(2 n^{\prime}+\ell\right) \succeq \bar{P}\left(n^{\prime}\right)^{2}=\bar{P}^{*}(n)^{2} \tag{3.49}
\end{equation*}
$$

for every $n \geq 0$, as claimed.
It remains to prove (3.48). Since both sides of the inequality are positive and the right-hand side is finite by Lemma 3.3, it suffices to consider the case $n \geq 1$. By Lemma 3.7

$$
\begin{equation*}
\mathbb{P}(\mathscr{A}(x, y)) \leq \sum_{i=1}^{\ell} \mathbb{P}\left(\mathscr{C}_{i}(x, y)\right) \tag{3.50}
\end{equation*}
$$

By translation invariance applied to the event $\mathscr{C}_{i}(x, y)$, this gives

$$
\sum_{x \in S_{-n}} \sum_{y \in S_{n+\ell}} \mathbb{P}(\mathscr{A}(x, y))
$$

$$
\begin{equation*}
\preceq \sum_{i=1}^{\ell} \sum_{x \in S_{-n-\ell+i}} \sum_{y \in S_{n+i-1}} \mathbb{P}\left(x \stackrel{H_{-n-\ell+i}}{\longleftrightarrow} 0,0 \stackrel{H_{-n-\ell+i}}{\longleftrightarrow} e_{1}, e_{1} \stackrel{H_{1}}{\longleftrightarrow} y\right) . \tag{3.51}
\end{equation*}
$$

Using the assumption that $n \geq 1$, we have by (3.16) that the right-hand side of (3.51) is bounded above by

$$
\begin{equation*}
\sum_{i=1}^{\ell} \frac{1-p_{c}}{p_{c}} \bar{P}(2 n+\ell-i) \preceq(\ell+1) \bar{P}^{*}(2 n+\ell-1) . \tag{3.52}
\end{equation*}
$$

This proves (3.48) and, therefore, completes the proof.

### 3.2. Proof of Theorem 2.3.

3.2.1. Randomised algorithms and the OSSS inequality. Our deduction of Theorem 2.3 from Proposition 3.1 relies crucially on the OSSS inequality of O'Donnell, Saks, Schramm, and Servedio [42], which we now briefly review. This inequality has recently been recognised as a powerful and flexible tool in the study of critical and near-critical percolation models, following the breakthrough work of Duminil-Copin, Raoufi, and Tassion [15]. We build, in particular, on the techniques developed to apply this inequality to prove inequalities between critical exponents in [31].

Let $E$ be a countable set. Informally, a decision tree is a deterministic procedure for querying the values of $\omega \in\{0,1\}^{E}$ that starts by querying the value of some fixed element of $E$ and chooses which element of $E$ to query at each subsequent step as a function of the values it has already observed. Formally, a decision tree is a function $T:\{0,1\}^{E} \rightarrow E^{\mathbb{N}}$ from subsets of $E$ to infinite $E$-valued sequences $T=\left(T_{1}, T_{2}, \ldots\right)$ such that $T_{1}(\omega)=e_{1}$ for some $e_{1} \in E$ not depending on $\omega$ and such that, for each $n \geq 2$, there exists a function $S_{n}:(E \times\{0,1\})^{n-1} \rightarrow E$ such that

$$
\begin{equation*}
T_{n}(\omega)=S_{n}\left[\left(T_{i}, \omega\left(T_{i}\right)\right)_{i=1}^{n-1}\right] \tag{3.53}
\end{equation*}
$$

where we think of $T_{n}(\omega)$ as the element of $E$ that is queried at time $n$ when given $\omega$ as an input to the procedure.

Let $\mu$ be a probability measure on $\{0,1\}^{E}$, and let $\omega$ be a random variable with law $\mu$. For each decision tree $T$ and $n \geq 1$, we define $\mathcal{F}_{n}(T)$ to be the $\sigma$-algebra generated by the random variables $\left\{T_{i}(\omega): 1 \leq i \leq n\right\}$ and define $\mathcal{F}(T)=\bigcup_{n \geq 1} \mathcal{F}_{n}(T)$. We say that $T$ computes a measurable function $f:\{0,1\}^{E} \rightarrow \mathbb{R}$ if $f(\omega)$ is measurable with respect to the $\mu$-completion of the $\sigma$-algebra $\mathcal{F}(T)$. This is equivalent by Lévy's $0-1$ law to the statement that

$$
\begin{equation*}
\mu\left[f(\omega) \mid \mathcal{F}_{n}(T)\right] \underset{n \rightarrow \infty}{\longrightarrow} f(\omega), \quad \mu \text {-a.s. } \tag{3.54}
\end{equation*}
$$

To allow for exploration algorithms that are naturally described as parallel rather than serial algorithms, it is convenient to introduce the slightly more general notion of decision forests. A decision forest is defined to be a collection of decision trees $F=\left\{T^{i}: i \in I\right\}$ indexed by a countable set $I$. Given a decision forest $F=\left\{T^{i}: i \in I\right\}$ and a probability measure $\mu$ on $\{0,1\}^{E}$, we let $\mathcal{F}(F)$ be the smallest $\sigma$-algebra containing all of the $\sigma$-algebras $\mathcal{F}\left(T^{i}\right)$ and say that a measurable function $f:\{0,1\}^{E} \rightarrow \mathbb{R}$ is computed by $F$ if it is measurable with respect to the $\mu$-completion of the $\sigma$-algebra $\mathcal{F}(F)$.

Let $E$ be a countable set, let $\mu$ be a probability measure on $E$, and let $F=\left\{T^{i}: i \in I\right\}$ be a decision forest on $E$. For each $e \in E$, we define the revealment probability

$$
\begin{equation*}
\delta_{e}(F, \mu)=\mu\left(\text { there exists } i \in I \text { and } n \geq 1 \text { such that } T_{n}^{i}(\omega)=e\right) \tag{3.55}
\end{equation*}
$$

so that $\delta_{e}(F, \mu)$ is the probability that the status of $e$ is ever queried when implementing the decision forest $F$ on a sample from the measure $\mu$. Finally, we define for each probability measure $\mu$ on $\{0,1\}^{E}$ and each pair of measurable functions $f, g:\{0,1\}^{E} \rightarrow \mathbb{R}$ the quantity

$$
\begin{equation*}
\operatorname{CoVr}_{\mu}[f, g]=(\mu \otimes \mu)\left[\left|f\left(\omega_{1}\right)-g\left(\omega_{2}\right)\right|\right]-\mu\left[\left|f\left(\omega_{1}\right)-g\left(\omega_{1}\right)\right|\right] \tag{3.56}
\end{equation*}
$$

where $\omega_{1}, \omega_{2}$ are drawn independently from the measure $\mu$. Thus, if $f$ and $g$ are $\{0,1\}$ valued, then

$$
\begin{align*}
\operatorname{CoVr}_{\mu}[f, g] & =2 \operatorname{Cov}_{\mu}[f, g] \\
& =2 \mu(f(\omega)=g(\omega)=1)-2 \mu(f(\omega)=1) \mu(g(\omega)=1) \tag{3.57}
\end{align*}
$$

We are now ready to state the version of the OSSS inequality that we will use which is a special case of [31], Corollary 2.4.

THEOREM 3.8 (OSSS for decision forests). Let $E$ be a finite or countably infinite set, and let $\mu$ be a product measure on $\{0,1\}^{E}$. Then for every pair of measurable, $\mu$-integrable functions $f, g:\{0,1\}^{E} \rightarrow \mathbb{R}$ and every decision forest $F$ computing $g$, we have that

$$
\begin{equation*}
\sum_{e \in E} \delta_{e}(F, \mu) \operatorname{Cov}_{\mu}[f, \omega(e)] \geq \frac{1}{2}\left|\operatorname{CoVr}_{\mu}[f, g]\right| \tag{3.58}
\end{equation*}
$$

See [15] for an extension of the OSSS inequality to monotonic measures, such as the law of the Fortuin-Kasteleyn random-cluster model.
3.2.2. Differential inequalities for Dini derivatives. In order to discuss how the OSSS inequality leads to differential inequalities in the infinite-volume setting (without any need for finite-volume approximation and limit), it is convenient to introduce the notion of Dini derivatives; see, for example, [35] for further background. The lower-right Dini derivative of a function $f:[a, b] \rightarrow \mathbb{R}$ is defined to be

$$
\begin{equation*}
\left(\frac{d}{d x}\right)_{+} f(x)=\liminf _{\varepsilon \downarrow 0} \frac{f(x+\varepsilon)-f(x)}{\varepsilon} \tag{3.59}
\end{equation*}
$$

for each $x \in[a, b)$. In our setting it is a classical and elementary fact [18], Theorem 2.34, that if $A$ is an event depending on at most finitely many edges, then $\mathbb{P}_{p}(A)$ is a polynomial in $p$ with derivative

$$
\begin{equation*}
\frac{d}{d p} \mathbb{P}_{p}(A)=\frac{1}{p(1-p)} \sum_{e \in E} \operatorname{Cov}[\omega(e), \mathbb{1}(A)] \tag{3.60}
\end{equation*}
$$

If $A$ is an increasing event depending possibly on infinitely many edges, we still have the lower-right Dini derivative bound

$$
\begin{equation*}
\left(\frac{d}{d p}\right)_{+} \mathbb{P}_{p}(A) \geq \frac{1}{p(1-p)} \sum_{e \in E} \operatorname{Cov}[\omega(e), \mathbb{1}(A)] \tag{3.61}
\end{equation*}
$$

A detailed proof of this is given in [31], Proposition 2.1. Thus, the OSSS inequality allows us to prove lower bounds on derivatives of increasing events by exhibiting decision forests that compute these events and have small maximum revealment.

Lower bounds on the lower-right Dini derivatives of monotone functions can often be used in much the same way as bounds on the classical derivative of a differentiable function. For example, the usual logarithmic derivative formula

$$
\begin{equation*}
\left(\frac{d}{d x}\right)_{+} \log f(x)=\frac{1}{f(x)}\left(\frac{d}{d x}\right)_{+} f(x) \tag{3.62}
\end{equation*}
$$

remains valid. Also, if $f:[a, b] \rightarrow \mathbb{R}$ is increasing, then

$$
\begin{equation*}
f(b)-f(a) \geq \int_{a}^{b}\left(\frac{d}{d x}\right)_{+} f(x) d x \tag{3.63}
\end{equation*}
$$

Since every measurable function has measurable Dini derivatives [35], Theorem 3.6.5, the above integral is well defined.
3.2.3. Slightly subcritical pioneers: Proof of Theorem 2.3. Our goal in this section is to study the distribution of the total number of 0-pioneers $\left|\mathcal{P}_{0}\right|$ in critical and slightly subcritical percolation. The main result is the following proposition which strengthens Theorem 2.3.

Proposition 3.9. Let $d>6$, and suppose that $(\mathrm{T})$ holds. There exists a positive constant $c$ such that

$$
\begin{equation*}
\mathbb{E}_{p_{c}-\varepsilon}\left|\mathcal{P}_{0}\right| \preceq \varepsilon^{-1 / 2} \quad \text { and } \quad \mathbb{P}_{p_{c}-\varepsilon}\left(\left|\mathcal{P}_{0}\right| \geq k\right) \preceq k^{-2 / 3} \exp \left[-c \varepsilon^{3 / 2} k\right] \tag{3.64}
\end{equation*}
$$

for every $0 \leq \varepsilon \leq p_{c} / 2$ and $k \geq 1$.
The exponential tail bound on $\left|\mathcal{P}_{0}\right|$ is not needed for the proofs of the main theorems but is included since it may be of independent interest. Before proving this proposition, we show how it implies Theorem 2.3.

Proof of Theorem 2.3, given Proposition 3.9. Recall that $P_{p}(n)=\mathbb{E}_{p}\left|\mathcal{P}_{0}(n)\right|$. We have already observed that the lower bound of Theorem 2.3 holds, and we have already proved the desired upper bound when $p=p_{c}$ in Proposition 3.1. It, therefore, suffices to prove that there exists a positive constant $c$ such that

$$
\begin{equation*}
P_{p}(n) \preceq \exp \left[-c\left(p_{c}-p\right)^{1 / 2} n\right] \tag{3.65}
\end{equation*}
$$

for every $n \geq 1$ and $p_{c} / 2 \leq p<p_{c}$. Fix $p$ in this interval. As discussed below, Definition 2.1, $\mathcal{P}_{0}(n) \cap \mathcal{P}_{0}(m)=\varnothing$ when $|n-m| \geq L$ and hence by Proposition 3.9

$$
\begin{equation*}
\frac{1}{N} \sum_{n=1}^{N} P_{p}(n) \leq \frac{L}{N} \mathbb{E}_{p}\left|\mathcal{P}_{0}\right| \preceq \frac{1}{\left(p_{c}-p\right)^{1 / 2} N} \tag{3.66}
\end{equation*}
$$

for every $N \geq 1$. It follows that there exists a constant $C$ such that there exists $n_{p}$ with $1 \leq n_{p} \leq C\left(p_{c}-p\right)^{-1 / 2}$ such that $P_{p}\left(n_{p}\right) \leq p^{L-1} / 2$. It follows inductively by the submultiplicativity estimate (2.3) that $P_{p}\left(k n_{p}\right) \leq p^{L-1} 2^{-k}$ for every $k \geq 1$. Since we also have that $P_{p}(n) \leq P_{p_{c}}(n) \preceq 1$, it follows by another application of (2.3) that $P_{p}\left(k n_{p}+r\right) \preceq 2^{-k}$ for every $k \geq 1$ and $0 \leq r<n_{p}$. If now we write arbitrary $n$ as $n=\left\lfloor\frac{n}{n_{p}}\right\rfloor n_{p}+r$, then we see that the above gives $P_{p}(n) \preceq 2^{-\left\lfloor n / n_{p}\right\rfloor}$, and the desired exponential estimate follows from the upper bound $n_{p} \preceq\left(p_{c}-p\right)^{-1 / 2}$.

To begin the proof of Proposition 3.9, we first note that Proposition 3.1, together with the result of Kozma and Nachmias [37] that Theorem 1.2 holds for $p=p_{c}$, yield the following important corollary describing the distribution of the total number of pioneers at criticality.

LEMMA 3.10. Let $d>6$, and suppose that $(\mathrm{T})$ holds. Then $\mathbb{P}_{p_{c}}\left(\left|\mathcal{P}_{x}\right| \geq k\right) \preceq k^{-2 / 3}$ for every $x \in \mathbb{Z}^{d}$ and $k \geq 1$.

Proof. It suffices to consider the case $x=0$. Let $n, k \geq 1$, where $n$ is a parameter we will optimise over shortly. By Markov's inequality

$$
\begin{align*}
\mathbb{P}_{p_{c}}\left(\left|\mathcal{P}_{0}\right| \geq k\right) \leq & \mathbb{P}_{p_{c}}\left(\left|\mathcal{P}_{0}\right| \geq k \text { and } 0 \text { is not connected to } H_{n}\right)  \tag{3.67}\\
& +\mathbb{P}_{p_{c}}\left(0 \text { is connected to } H_{n}\right) \\
\leq & \frac{1}{k} \mathbb{E}_{p_{c}}\left[\sum_{i=0}^{n}\left|\mathcal{P}_{0}(i)\right|\right]+\mathbb{P}_{p_{c}}\left(0 \text { is connected to } H_{n}\right) \preceq \frac{n}{k}+\frac{1}{n^{2}} \tag{3.68}
\end{align*}
$$

for every $k, n \geq 1$, where the first bound follows from Proposition 3.1 and the second follows from the aforementioned result of Kozma and Nachmias. The claim follows by taking $n=$ $\left\lceil k^{1 / 3}\right\rceil$.

We now apply the OSSS inequality to deduce Proposition 3.9 from Lemma 3.10. We follow closely the proof of [31], Theorem 1.1. For each $p \in[0,1]$ and $h \geq 0$, we write $\mathbb{P}_{p, h}$ for the
law of the pair $(\omega, \mathcal{G})$ where $\omega$ is distributed as Bernoulli- $p$ bond percolation and $\mathcal{G}$ is a ghost field independent of $\omega$, that is, a random subset of the edge set $\mathbb{B}$ in which each edge is included independently at random with inclusion probability $1-e^{-h}$. We call an edge green if it belongs to $\mathcal{G}$. (While it is more standard to consider ghost fields to be random sets of vertices, it is more convenient for our purposes to take them to be random sets of edges.) As a first and key step, we use the OSSS inequality to prove a differential inequality.

Lemma 3.11. The differential inequality

$$
\begin{equation*}
\left(\frac{d}{d p}\right)_{+} \log \mathbb{P}_{p}\left(\left|\mathcal{P}_{0}\right| \geq k\right) \geq \frac{1}{2 p(1-p)}\left[\frac{k\left(1-e^{-1}\right)}{\sum_{i=0}^{k} \mathbb{P}_{p}\left(\left|\mathcal{P}_{0}\right| \geq i\right)}-1\right] \tag{3.69}
\end{equation*}
$$

holds for every $k \geq 1$ and $0<p<1$.

Proof. By (3.61) and (3.62),

$$
\begin{equation*}
\left(\frac{d}{d p}\right)_{+} \log \mathbb{P}_{p}\left(\left|\mathcal{P}_{0}\right| \geq k\right) \geq \frac{1}{\mathbb{P}_{p}\left(\left|\mathcal{P}_{0}\right| \geq k\right)} \frac{1}{p(1-p)} \sum_{e \in \mathbb{B}} \operatorname{Cov}\left[\omega(e), \mathbb{1}\left(\left|\mathcal{P}_{0}\right| \geq k\right)\right] \tag{3.70}
\end{equation*}
$$

where $\mathbb{B}$ is the set of edges. It, therefore, suffices to prove that

$$
\begin{equation*}
\sum_{e \in \mathbb{B}} \operatorname{Cov}\left[\omega(e), \mathbb{1}\left(\left|\mathcal{P}_{0}\right| \geq k\right)\right] \geq \frac{1}{2}\left[\frac{k\left(1-e^{-1}\right)}{\sum_{i=0}^{k} \mathbb{P}_{p}\left(\left|\mathcal{P}_{0}\right| \geq i\right)}-1\right] \mathbb{P}_{p}\left(\left|\mathcal{P}_{0}\right| \geq k\right) \tag{3.71}
\end{equation*}
$$

For this we will use Theorem 3.8.
For the setup for Theorem 3.8, we let $(\omega, \mathcal{G})$ have law $\mathbb{P}_{p, h}$, where we think of $\mathbb{P}_{p, h}$ as a product measure on $\{0,1\}^{\mathbb{B} \times\{\text { perc,ghost }\}}$. Consider the two Boolean functions

$$
\begin{equation*}
f(\omega, \mathcal{G})=f(\omega)=\mathbb{1}\left(\left|\mathcal{P}_{0}\right| \geq k\right) \quad \text { and } \quad g(\omega, \mathcal{G})=\mathbb{1}\left(\mathcal{P}_{0} \cap \mathcal{G} \neq \varnothing\right) \tag{3.72}
\end{equation*}
$$

We say that an edge is horizontal if its endpoints have distinct first coordinates. We can determine the value of $g$ by first revealing the value of the ghost field at each horizontal edge and then exploring the cluster of each green horizontal edge in the halfspace lying strictly to the left of its rightmost endpoint. This exploration process can be encoded as a decision forest $F=\left\{T^{e}: e \in \mathbb{B}\right\}$ in which the decision tree $T^{e}$ first queries the status of the ghost field at the edge $e$, halting if it discovers that $\mathcal{G}(e)=0$. If the decision tree discovers that $\mathcal{G}(e)=1$, it next checks whether $e$ is open in $\omega$, halting if it is closed and otherwise exploring the cluster of the leftmost endpoint of $e$ in the halfspace lying strictly to the left of the rightmost endpoint of $e$; see the proof of [31], Proposition 3.1, to see how such a decision forest may be defined formally. This decision forest clearly computes $g$.

Its revealments satisfy

$$
\begin{align*}
& \delta_{e, \operatorname{perc}}\left(F, \mathbb{P}_{p, h}\right)  \tag{3.73}\\
& \quad \leq \mathbb{P}_{p, h}(e \in \mathcal{G} \text { or at least one of the endpoints of } e \text { has a pioneer in } \mathcal{G}), \\
& \delta_{e, \text { ghost }}\left(F, \mathbb{P}_{p, h}\right)=1 \tag{3.74}
\end{align*}
$$

for each $e \in \mathbb{B}$. We can bound the revealment probabilities of edges by the union bound

$$
\begin{align*}
\delta_{e, \text { perc }}\left(F, \mathbb{P}_{p, h}\right) & \leq \mathbb{P}_{p, h}(e \in \mathcal{G})+2 \mathbb{P}_{p, h}(0 \text { has a pioneer in } \mathcal{G}) \\
& =1-e^{-h}+2 \mathbb{E}_{p, h}\left[1-e^{-h\left|\mathcal{P}_{0}\right|}\right] \tag{3.75}
\end{align*}
$$

It, therefore, follows from the OSSS inequality Theorem 3.8 that

$$
\operatorname{Cov}[f, g]=\frac{1}{2}|\operatorname{CoVr}[f, g]|
$$

$$
\begin{align*}
& \leq \sum_{e \in \mathbb{B}} \delta_{e, \text { perc }}\left(F, \mathbb{P}_{p, h}\right) \operatorname{Cov}[f, \omega(e)]+\sum_{e \in \mathbb{B}} \delta_{e, \text { ghost }}\left(F, \mathbb{P}_{p, h}\right) \operatorname{Cov}[f, \mathcal{G}(e)] \\
& =\sum_{e \in \mathbb{B}} \delta_{e, \text { perc }}\left(F, \mathbb{P}_{p, h}\right) \operatorname{Cov}[f, \omega(e)]  \tag{3.76}\\
& \leq\left(1-e^{-h}+2 \mathbb{E}_{p, h}\left[1-e^{-h\left|\mathcal{P}_{0}\right|}\right]\right) \sum_{e \in \mathbb{B}} \operatorname{Cov}[f, \omega(e)],
\end{align*}
$$

where we used that $f(\omega, \mathcal{G})=f(\omega)$ is independent of the ghost field $\mathcal{G}$ in the equality on the second line. On the other hand, we can also compute that

$$
\begin{align*}
\operatorname{Cov}[f, g] & =\mathbb{P}_{p, h}\left(\left|\mathcal{P}_{0}\right| \geq k,\left|\mathcal{P}_{0} \cap \mathcal{G}\right| \geq 1\right)-\mathbb{P}_{p}\left(\left|\mathcal{P}_{0}\right| \geq k\right) \mathbb{P}_{p, h}\left(\left|\mathcal{P}_{0} \cap \mathcal{G}\right| \geq 1\right) \\
& =\mathbb{E}_{p}\left[\left(1-e^{-h\left|\mathcal{P}_{0}\right|}\right) \mathbb{1}\left(\left|\mathcal{P}_{0}\right| \geq k\right)\right]-\mathbb{E}_{p}\left[1-e^{-h\left|\mathcal{P}_{0}\right|}\right] \mathbb{P}_{p}\left(\left|\mathcal{P}_{0}\right| \geq k\right)  \tag{3.77}\\
& \geq\left(1-e^{-h k}\right) \mathbb{P}_{p}\left(\left|\mathcal{P}_{0}\right| \geq k\right)-\mathbb{E}_{p}\left[1-e^{-h\left|\mathcal{P}_{0}\right|}\right] \mathbb{P}_{p}\left(\left|\mathcal{P}_{0}\right| \geq k\right)
\end{align*}
$$

so that

$$
\begin{equation*}
\sum_{e \in \mathbb{B}} \operatorname{Cov}[f, \omega(e)] \geq \frac{\left(1-e^{-h k}\right)-\mathbb{E}_{p}\left[1-e^{-h\left|\mathcal{P}_{0}\right|}\right]}{1-e^{-h}+2 \mathbb{E}_{p}\left[1-e^{-h\left|\mathcal{P}_{0}\right|}\right]} \mathbb{P}_{p}\left(\left|\mathcal{P}_{0}\right| \geq k\right) \tag{3.78}
\end{equation*}
$$

for every $k \geq 1,0 \leq p \leq 1$, and $h \geq 0$. The claimed inequality (3.71) follows by taking $h=1 / k$ and using the elementary fact that

$$
\begin{equation*}
1-e^{-1 / k}+\mathbb{E}_{p}\left[1-e^{-\left|\mathcal{P}_{0}\right| / k}\right] \leq \frac{1}{k}+\frac{1}{k} \mathbb{E}_{p}\left[\min \left\{k,\left|\mathcal{P}_{0}\right|\right\}\right]=\frac{1}{k} \sum_{i=0}^{k} \mathbb{P}_{p}\left(\left|\mathcal{P}_{0}\right| \geq i\right) \tag{3.79}
\end{equation*}
$$

This completes the proof.
Proof of Proposition 3.9. We now analyse the differential inequality (3.69) to prove the desired slightly subcritical bounds. We begin with the proof of the inequality $\mathbb{E}_{p}\left|\mathcal{P}_{0}\right| \preceq$ $\left(p_{c}-p\right)^{-1 / 2}$. Since $\mathbb{P}_{p}\left(\left|\mathcal{P}_{0}\right| \geq k\right)$ is an increasing function of $p$, we have by Lemma 3.11 and Lemma 3.10 that there exist positive constants $c_{1}$ and $C_{1}$ such that

$$
\begin{align*}
\left(\frac{d}{d p}\right)_{+} \log \mathbb{P}_{p}\left(\left|\mathcal{P}_{0}\right| \geq k\right) & \geq \frac{1}{2 p(1-p)}\left[\frac{k\left(1-e^{-1}\right)}{C_{1} \sum_{i=0}^{k}(i+1)^{-2 / 3}}-1\right]  \tag{3.80}\\
& \geq \frac{1}{2 p(1-p)}\left[c_{1} k^{2 / 3}-1\right]
\end{align*}
$$

for every $0<p \leq p_{c}$ and $k \geq 1$. Integration of this inequality over the interval [ $p, p_{c}$ ], together with (3.63), shows that there exist positive constants $c_{2}$ and $C_{2}$ such that

$$
\begin{align*}
\mathbb{P}_{p}\left(\left|\mathcal{P}_{0}\right| \geq k\right) & \leq \mathbb{P}_{p_{c}}\left(\left|\mathcal{P}_{0}\right| \geq k\right) \exp \left(-\int_{p}^{p_{c}} \frac{1}{2 q(1-q)}\left[c_{1} k^{2 / 3}-1\right] d q\right)  \tag{3.81}\\
& \leq C_{2} k^{-2 / 3} \exp \left(-c_{2}\left(p_{c}-p\right) k^{2 / 3}\right)
\end{align*}
$$

for every $p_{c} / 2 \leq p \leq p_{c}$ and $k \geq 1$. It follows by calculus that there exists a positive constant $C_{3}$ such that

$$
\begin{equation*}
\mathbb{E}_{p}\left|\mathcal{P}_{0}\right| \leq C_{2} \sum_{k=1}^{\infty} k^{-2 / 3} e^{-c_{2}\left(p_{c}-p\right) k^{2 / 3}} \leq C_{3}\left(p_{c}-p\right)^{-1 / 2} \tag{3.82}
\end{equation*}
$$

for every $p_{c} / 2 \leq p<p_{c}$, as claimed.

Finally, we prove that $\mathbb{P}_{p}\left(\left|\mathcal{P}_{0}\right| \geq k\right) \preceq k^{-2 / 3} \exp \left[-c\left(p_{c}-p\right)^{3 / 2} k\right]$. The case of $p=p_{c}$ has been proved already in Lemma 3.10, so we can restrict attention here to $p_{c} / 2 \leq p<p_{c}$. The differential inequality (3.69) implies the simplified inequality

$$
\begin{equation*}
\left(\frac{d}{d p}\right)_{+} \log \mathbb{P}_{p}\left(\left|\mathcal{P}_{0}\right| \geq k\right) \geq \frac{1}{2 p(1-p)}\left[\frac{k\left(1-e^{-1}\right)}{1+\mathbb{E}_{p}\left|\mathcal{P}_{0}\right|}-1\right] \tag{3.83}
\end{equation*}
$$

We again integrate the above inequality and conclude that there exist positive constants $c_{3}$, $c_{4}$, and $C_{3}$ such that if $0<\varepsilon \leq p_{c} / 4$, then

$$
\begin{align*}
\mathbb{P}_{p_{c}-2 \varepsilon}\left(\left|\mathcal{P}_{0}\right| \geq k\right) & \leq \mathbb{P}_{p_{c}-\varepsilon}\left(\left|\mathcal{P}_{0}\right| \geq k\right) \exp \left(-\int_{p_{c}-2 \varepsilon}^{p_{c}-\varepsilon} \frac{1}{2 q(1-q)}\left[c_{3} \varepsilon^{1 / 2} k-1\right] d q\right)  \tag{3.84}\\
& \leq C_{3} k^{-2 / 3} \exp \left(-c_{4} \varepsilon^{3 / 2} k\right)
\end{align*}
$$

for every $k \geq 1$. This completes the proof.
4. Plateau below the window: Proof of Theorem 1.3. In this section we apply our bound on the slightly subcritical $\mathbb{Z}^{d}$ two-point function from Theorem 1.1 to prove the plateau estimates for the torus two-point function below the scaling window in Theorem 1.3. As a corollary we also prove the torus triangle condition, Theorem 1.6.
4.1. Preliminaries. We start by recording some preliminary estimates that we will use repeatedly in the rest of the section. For each $x \in \mathbb{Z}^{d}$ and $0 \leq p \leq 1$, the $\mathbb{Z}^{d}$ open bubble and open triangle diagrams $\mathrm{B}_{p}(x)$ and $\mathrm{T}_{p}(x)$ are defined by

$$
\begin{align*}
& \mathrm{B}_{p}(x)=\sum_{u \in \mathbb{Z}^{d}} \tau_{p}(u) \tau_{p}(x-u)=\left(\tau_{p} * \tau_{p}\right)(x)  \tag{4.1}\\
& \mathrm{T}_{p}(x)=\sum_{u, v \in \mathbb{Z}^{d}} \tau_{p}(u) \tau_{p}(v-u) \tau_{p}(x-v)=\left(\tau_{p} * \tau_{p} * \tau_{p}\right)(x) \tag{4.2}
\end{align*}
$$

Upper bounds on these two quantities are given in the next lemma. The notation $p_{c}$ always refers to the critical value for $\mathbb{Z}^{d}$.

LEMMA 4.1. Let $d>6$, and suppose that $(\mathrm{T})$ holds on $\mathbb{Z}^{d}$. There exist positive constants $C_{1}, C_{2}$ such that

$$
\begin{align*}
\mathrm{B}_{p}(x) & \leq \frac{C_{1}}{\langle x\rangle^{d-4}} e^{-c_{1} m(p)\|x\|_{\infty}}  \tag{4.3}\\
\mathrm{T}_{p}(x) & \leq \frac{C_{2}}{\langle x\rangle^{d-6}} e^{-c_{1} m(p)\|x\|_{\infty}} \tag{4.4}
\end{align*}
$$

for every $x \in \mathbb{Z}^{d}$ and every $p \leq p_{c}$.
Proof. We insert the bound of Theorem 1.1 into the convolutions defining the bubble and triangle diagrams. By the triangle inequality, the exponential factors are bounded above by an overall factor $e^{-c_{1} m(p)|x|}$. For the powers, let $f(x)=\langle x\rangle^{-(d-2)}$. Since $d>6$, we have by (3.19)-(3.20) that $(f * f)(x) \preceq\langle x\rangle^{-(d-4)}$ and $(f * f * f)(x) \preceq\langle x\rangle^{-(d-6)}$. Together, this gives the desired result.

Observe that if $x \in \mathbb{T}_{r}^{d}$ is regarded as a point in $\left[-\frac{r}{2}, \frac{r}{2}\right)^{d} \cap \mathbb{Z}^{d}$, then $\langle x+r u\rangle \asymp r\langle u\rangle$ uniformly in nonzero $u \in \mathbb{Z}^{d}$ since

$$
\begin{equation*}
\|x+r u\|_{\infty} \geq\|r u\|_{\infty}-\frac{r}{2} \geq\|r u\|_{\infty}-\frac{1}{2}\|r u\|_{\infty}=\frac{1}{2}\|r u\|_{\infty} \tag{4.5}
\end{equation*}
$$

and

$$
\begin{equation*}
\|x+r u\|_{\infty} \leq \frac{r}{2}+\|r u\|_{\infty} \leq \frac{1}{2}\|r u\|_{\infty}+\|r u\|_{\infty}=\frac{3}{2}\|r u\|_{\infty} . \tag{4.6}
\end{equation*}
$$

The following elementary lemma will be useful with $v=c m(p)$.
Lemma 4.2. Let $r \geq 2, a>0$, and $v>0$. Then

$$
\begin{equation*}
\sum_{u \in \mathbb{Z}^{d}: u \neq 0} \frac{1}{\|x+r u\|_{\infty}^{d-a}} e^{-\nu\|x+r u\|_{\infty}} \preceq_{a} \frac{1}{\nu^{a} r^{d}} e^{-\frac{1}{4} \nu r} \tag{4.7}
\end{equation*}
$$

for every $x \in \mathbb{T}_{r}^{d} \equiv\left[-\frac{r}{2}, \frac{r}{2}\right)^{d} \cap \mathbb{Z}^{d}$.
Proof. Let $a>0$. It follows from (4.5) that for any nonzero $u \in \mathbb{Z}^{d}$ and $r \geq 2,\langle x+$ $r u\rangle \geq \frac{1}{2}\|r u\|_{\infty}$ and thus

$$
\begin{aligned}
\sum_{u \neq 0} \frac{1}{\|x+r u\|_{\infty}^{d-a}} e^{-\nu\langle x+r u\rangle} & \leq \sum_{u \neq 0} \frac{1}{\left(\frac{1}{2}\|r u\|_{\infty}\right)^{d-a}} e^{-\frac{1}{2} \nu\|r u\|_{\infty}} \\
& \leq 2^{d-a} e^{-\frac{1}{4} \nu r} \sum_{N=1}^{\infty} \sum_{u:\|u\|_{\infty}=N} \frac{1}{\|r u\|_{\infty}^{d-a}} e^{-\frac{1}{4} \nu\|r u\|_{\infty}} \\
& \preceq_{a} r^{a-d} e^{-\frac{1}{4} \nu r} \sum_{N=1}^{\infty} N^{d-1-d+a} e^{-\frac{1}{4} \nu r N}
\end{aligned}
$$

We bound the sum on the right-hand side by an integral to obtain an upper bound which is a constant multiple of

$$
\begin{equation*}
r^{a-d} e^{-\frac{1}{4} v r} \int_{1}^{\infty} u^{a-1} e^{-\frac{1}{4} v r u} d u=\frac{1}{v^{a} r^{d}} e^{-\frac{1}{4} \nu r} \int_{\nu r}^{\infty} t^{a-1} e^{-t / 4} d t \tag{4.9}
\end{equation*}
$$

The integral is uniformly bounded since $a>0$. This concludes the proof.
REMARK 4.3. Bounds expressed in terms of the mass $m(p)$, such as the one in Lemma 4.2 with $v=c m(p)$, can also be expressed in terms of the susceptibility $\chi(p)$ since

$$
\begin{equation*}
\frac{1}{m(p)^{2}} \preceq \chi(p) \tag{4.10}
\end{equation*}
$$

To prove (4.10), we first fix any $p_{1} \in\left(0, p_{c}\right)$. For $p \leq p_{1}$, since $m$ is decreasing and since $1=\chi(0) \leq \chi(p)$, we have $m(p)^{-2} \leq m\left(p_{1}\right)^{-2} \leq m\left(p_{1}\right)^{-2} \chi(p)$ and the desired upper bound follows for $p \in\left(0, p_{1}\right]$. We can choose $p_{1}$ close enough to $p_{c}$ that $m(p)^{-2}$ and $\chi(p)$ are comparable for $p \in\left(p_{1}, p_{c}\right)$, since both are asymptotic to $\left(1-p / p_{c}\right)^{-1}$. In particular, for $p_{1}$ close enough to $p_{c}$ there exists $C$ such that $m(p)^{-2} \leq C \chi(p)$ for those $p \in\left[p_{1}, p_{c}\right)$.

The following three estimates will be useful.
Lemma 4.4. Let $d>6$, and suppose that (T) holds on $\mathbb{Z}^{d}$. For $r \geq 2, x \in \mathbb{T}_{r}^{d}$, and $0 \leq p<p_{c}$,

$$
\begin{align*}
& \sum_{u \in \mathbb{Z}^{d}} \tau_{p}(x+r u) \leq \tau_{p}(x)+C \frac{\chi(p)}{V} e^{-\frac{c}{4} m(p) r},  \tag{4.11}\\
& \sum_{u \in \mathbb{Z}^{d}} \mathrm{~B}_{p}(x+r u) \leq \mathrm{B}_{p}(x)+C \frac{\chi(p)^{2}}{V} e^{-\frac{c}{4} m(p) r}, \tag{4.12}
\end{align*}
$$

$$
\begin{equation*}
\sum_{u \in \mathbb{Z}^{d}} \mathrm{~T}_{p}(x+r u) \leq \mathrm{T}_{p}(x)+C \frac{\chi(p)^{3}}{V} e^{-\frac{c}{4} m(p) r} \tag{4.13}
\end{equation*}
$$

Proof. For the first inequality, we separate the $u=0$ term from the sum and apply Theorem 1.1, as well as Lemma 4.2 with $a=2$, to obtain that there exist positive constants $c, C_{1}$, and $C_{2}$ such that

$$
\begin{align*}
\sum_{u \in \mathbb{Z}^{d}} \tau_{p}(x+r u) & \leq \tau_{p}(x)+\sum_{u \neq 0} \frac{C_{1}}{\langle x+r u\rangle^{d-2}} e^{-c m(p)\|x+r u\|_{\infty}}  \tag{4.14}\\
& \leq \tau_{p}(x)+C_{2} \frac{\chi(p)}{V} e^{-\frac{c}{4} m(p) r} \tag{4.15}
\end{align*}
$$

For the bubble and triangle diagrams, in place of Theorem 1.1 we instead use the bounds of Lemma 4.1 , which modify the power $d-2$ in the above inequality to $d-4$ for the bubble and $d-6$ for the triangle. We then apply Lemma 4.2 and Remark 4.3 with $a=4$ and with $a=6$ to complete the proof.
4.2. Upper bound on the torus two-point function. PROOF OF (1.12). The proof is as in [45]. For each $0 \leq p<p_{c}$ and $x \in \mathbb{T}_{r}^{d} \equiv\left[-\frac{r}{2}, \frac{r}{2}\right)^{d} \cap \mathbb{Z}^{d}$, we define

$$
\begin{equation*}
\psi_{r, p}(x)=\sum_{u \in \mathbb{Z}^{d}: u \neq 0} \tau_{p}(x+r u) \tag{4.16}
\end{equation*}
$$

which is finite only when $p<p_{c}$. It follows by a simple coupling argument, originating in the work of Benjamini and Schramm [4] and further developed in [24], Proposition 2.1, that

$$
\begin{equation*}
\tau_{p}^{\mathbb{T}}(x) \leq \tau_{p}(x)+\psi_{r, p}(x) \tag{4.17}
\end{equation*}
$$

for every $r>2, x \in \mathbb{T}_{r}^{d}$ and $0 \leq p \leq 1$. By Lemma 4.4

$$
\begin{equation*}
\psi_{r, p}(x) \leq C \frac{\chi(p)}{V} e^{-\frac{c}{4} m(p) r} \tag{4.18}
\end{equation*}
$$

and with (4.17) this immediately yields the upper bound (1.12).
4.3. Lower bound on the torus two-point function below the window. We now turn to the proof of the lower bound (1.13) on the torus two-point function for $p$ below the scaling window. This proof is model dependent and although it follows the general strategy used for weakly self-avoiding walk in [45], it differs in details.

We seek a lower bound of the form $r^{-d} \chi$ for the difference

$$
\begin{equation*}
\psi_{r, p}^{\mathbb{T}}(x)=\tau_{p}^{\mathbb{T}}(x)-\tau_{p}(x) \quad\left(x \in \mathbb{T}^{d}\right) \tag{4.19}
\end{equation*}
$$

To this end, we first make the decomposition

$$
\begin{equation*}
\psi_{r, p}^{\mathbb{T}}(x)=\psi_{r, p}(x)-\left(\psi_{r, p}(x)-\psi_{r, p}^{\mathbb{T}}(x)\right) \tag{4.20}
\end{equation*}
$$

We can then deduce the lower bound (1.13) as an immediate consequence of the following two lemmas.

LEMmA 4.5. Let $d>6$, and suppose that $(\mathrm{T})$ holds on $\mathbb{Z}^{d}$. There exist positive constants $A_{2}$ and $c_{\psi}$ such that if $r>2$ and $p_{c}-A_{2} r^{-2} \leq p<p_{c}$, then

$$
\begin{equation*}
\psi_{r, p}(x) \geq c_{\psi} \frac{\chi(p)}{V} \tag{4.21}
\end{equation*}
$$

for every $x \in \mathbb{T}_{r}^{d}$.

Lemma 4.6. Let $d>6$, and suppose that $(\mathrm{T})$ holds on $\mathbb{Z}^{d}$. Let $c_{\psi}$ be as in Lemma 4.5. There exist positive constants $A_{1}$ and $M$ such that if $r>2$ and $0 \leq p \leq p_{c}-A_{1} V^{-1 / 3}$, then

$$
\begin{equation*}
\psi_{r, p}(x)-\psi_{r, p}^{\mathbb{T}}(x) \leq \frac{1}{2} c_{\psi} \frac{\chi(p)}{V} \tag{4.22}
\end{equation*}
$$

for every $x \in \mathbb{T}_{r}^{d}$ with $\|x\|_{\infty} \geq M$.
Proof of (1.13) subject to Lemmas 4.5 and 4.6. By definition,

$$
\begin{equation*}
\tau_{p}^{\mathbb{T}}(x)=\tau_{p}(x)+\psi_{r, p}(x)-\left[\psi_{r, p}(x)-\psi_{r, p}^{\mathbb{T}}(x)\right] \tag{4.23}
\end{equation*}
$$

By Lemmas 4.5 and 4.6, if $\|x\|_{\infty} \geq M$ and $p_{c}-A_{2} r^{-2} \leq p \leq p_{c}-A_{1} V^{-1 / 3}$, then we have the lower bound

$$
\begin{equation*}
\tau_{p}^{\mathbb{T}}(x) \geq \tau_{p}(x)+c_{\psi} \frac{\chi(p)}{V}-\frac{1}{2} c_{\psi} \frac{\chi(p)}{V}=\tau_{p}(x)+\frac{1}{2} c_{\psi} \frac{\chi(p)}{V} \tag{4.24}
\end{equation*}
$$

which is the desired estimate.
4.3.1. Proof of Lemma 4.5. In this section we prove the lower bound on $\psi_{r, p}(x)$ stated in Lemma 4.5. We begin with the following simple observation.

LEMMA 4.7. Let $d>6$, and suppose that $(\mathrm{T})$ holds on $\mathbb{Z}^{d}$. The inequality

$$
\begin{equation*}
\tau_{p_{c}}(x)-\tau_{p}(x) \preceq \frac{p_{c}-p}{\langle x\rangle^{d-4}} \tag{4.25}
\end{equation*}
$$

holds for every $p_{c} / 2 \leq p \leq p_{c}$ and $x \in \mathbb{Z}^{d}$.
Proof. The proof of (4.25) is a consequence of the following standard differential inequality (cf. [2]). Let $\tau_{p}^{n}(x)=\mathbb{P}\left(0 \leftrightarrow x\right.$ inside $\left.[-n, n]^{d}\right)$. It is easy to see that, for every $n>0, \tau_{p}^{n}(x)$ is differentiable in $p$ and that $\tau_{p}^{n}(x) \rightarrow \tau_{p}(x)$ as $n \rightarrow \infty$. Let $p \in\left[\frac{1}{2} p_{c}, p_{c}\right]$. By Russo's Formula and the BK inequality (with the sum over the undirected bonds in $[-n, n]^{d}$ ), we have

$$
\frac{d}{d p} \tau_{p}^{n}(x)=\frac{1}{p} \sum_{\{u, v\}} \mathbb{P}_{p}\left(\{u, v\} \text { is pivotal for } 0 \leftrightarrow x \text { inside }[-n, n]^{d},\{u, v\} \text { is open }\right)
$$

$$
\begin{align*}
& \leq \frac{1}{p} \sum_{\{u, v\}} \mathbb{P}_{p}\left(\left\{0 \leftrightarrow u \text { inside }[-n, n]^{d}\right\} \circ\left\{u \leftrightarrow x \text { inside }[-n, n]^{d}\right\}\right)  \tag{4.26}\\
& \preceq\left(\tau_{p} * \tau_{p}\right)(x)
\end{align*}
$$

It follows by monotonicity in $p$ and the bound on the bubble from Lemma 4.1 that

$$
\begin{equation*}
\frac{d}{d p} \tau_{p}^{n}(x) \preceq\left(\tau_{p_{c}} * \tau_{p_{c}}\right)(x) \preceq \frac{1}{\langle x\rangle^{d-4}} . \tag{4.27}
\end{equation*}
$$

Integration of (4.27) over [ $p, p_{c}$ ], followed by the limit as $n \rightarrow \infty$, gives (4.25).
We now apply Lemma 4.7 to complete the proof of Lemma 4.5.
Proof of Lemma 4.5. Let $x \in \mathbb{T}_{r}^{d}$. To obtain a lower bound on $\psi_{r, p}$, we may sum in (4.16) over only those $u \in \mathbb{Z}^{d}$ with $\|u\|_{\infty} \leq R$ with $R \geq 1$ a large number depending on $r$
and $p_{c}-p$ to be chosen shortly. By ( T ) and (4.25), there exist positive constants $c_{1}$ and $C_{1}$ such that, for every $y \in \mathbb{Z}^{d}$,

$$
\begin{equation*}
\tau_{p}(y)=\tau_{p_{c}}(y)-\left(\tau_{p_{c}}(y)-\tau_{p}(y)\right) \geq \frac{c_{1}}{\langle y\rangle^{d-2}}-\frac{C_{1}\left(p_{c}-p\right)}{\langle y\rangle^{d-4}} \tag{4.28}
\end{equation*}
$$

With this, together with (4.5)-(4.6), we see that there exist positive constants $c_{2}, C_{2}, C_{3}$ such that

$$
\sum_{u \in \mathbb{Z}^{d}: u \neq 0} \tau_{p}(x+r u) \geq \sum_{1 \leq\|u\|_{\infty} \leq R} \tau_{p}(x+r u)
$$

$$
\begin{align*}
& \geq \frac{2}{3} c_{1} \sum_{1 \leq\|u\|_{\infty} \leq R} \frac{1}{\|r u\|_{\infty}^{d-2}}-2 C_{1}\left(p_{c}-p\right) \sum_{1 \leq\|u\|_{\infty} \leq R} \frac{1}{\|r u\|_{\infty}^{d-4}}  \tag{4.29}\\
& \geq \frac{c_{2}}{r^{d-2}} R^{2}-\frac{C_{2}\left(p_{c}-p\right)}{r^{d-4}} R^{4}=\frac{c_{2}}{r^{d-2}} R^{2}\left(1-C_{3}\left(p_{c}-p\right) r^{2} R^{2}\right)
\end{align*}
$$

for every $r, R \geq 1$ and $p_{c} / 2 \leq p<p_{c}$. Now we choose $R^{2}=\left(2 C_{3}\left(p_{c}-p\right) r^{2}\right)^{-1}$ and require $p_{c}-p \leq A_{2} r^{-2}$ with $A_{2}$ chosen small enough for $R$ to be indeed greater than 1. This gives

$$
\begin{equation*}
\sum_{u \in \mathbb{Z}^{d}: u \neq 0} \tau_{p}(x+r u) \geq \frac{c_{2} R^{2}}{2 r^{d-2}}=\frac{c_{3}}{\left(p_{c}-p\right) r^{d}} \succeq \frac{\chi(p)}{V} \tag{4.30}
\end{equation*}
$$

for every $r \geq 2$ and $p \in\left[p_{c}-A_{2} r^{-2}, p_{c}\right)$, and completes the proof.
4.3.2. Proof of Lemma 4.6. We now prove Lemma 4.6, which states that there exist constants $M$ and $A_{1}$ such that if $p \leq p_{c}-A_{1} V^{-1 / 3}$, then

$$
\begin{equation*}
\psi_{r, p}(x)-\psi_{r, p}^{\mathbb{T}}(x) \leq \frac{1}{2} c_{\psi} r^{-d} \chi(p) \tag{4.31}
\end{equation*}
$$

for every $x \in \mathbb{T}_{r}^{d}$ with $\|x\|_{\infty} \geq M$, where $c_{\psi}$ is the constant from Lemma 4.5. In order to prove this, we will prove the following more general inequality.

LEMmA 4.8. Let $d>6$, and suppose that $(\mathrm{T})$ holds on $\mathbb{Z}^{d}$. There exists a constant $C$ such that the inequality

$$
\begin{equation*}
\psi_{r, p}(x)-\psi_{r, p}^{\mathbb{T}}(x) \leq C \frac{\chi(p)}{V}\left(\mathrm{~T}_{p}(x)+\frac{\chi(p)^{3}}{V}\right) \tag{4.32}
\end{equation*}
$$

holds for every $r>2, x \in \mathbb{T}_{r}^{d}$, and $p<p_{c}$.
Proof of Lemma 4.6, given Lemma 4.8. By taking $p \leq p_{c}-A_{1} V^{-1 / 3}$, we see from the bound on the susceptibility in (1.11) that $\chi(p) \preceq A_{1}^{-1} V^{1 / 3}$, so the term $V^{-1} \chi(p)^{3}$ can be made as small as desired by taking $A_{1}$ sufficiently large. By (4.4) the triangle term $\mathrm{T}_{p}(x)$ can be made as small as desired by taking $\|x\|_{\infty} \geq M$ with $M$ sufficiently large. Thus, we can choose the constants $A_{1}, M$ in such a way that the right-hand side of (4.32) is at most $\frac{1}{2} c_{\psi} \chi(p) / V$. This gives the desired inequality (4.31).

We turn now to the proof of Lemma 4.8. We build upon the coupling of percolation on $\mathbb{Z}^{d}$ and $\mathbb{T}_{r}^{d}$ developed by Heydenreich and van der Hofstad [24], Proposition 2.1. With this coupling they proved that

$$
\begin{align*}
\psi_{r, p}(x)-\psi_{r, p}^{\mathbb{T}}(x) \leq & \frac{1}{2} \sum_{u \in \mathbb{Z}^{d}} \sum_{v \neq u} \mathbb{P}(0 \leftrightarrow x+r u, x+r v)  \tag{4.33}\\
& +\sum_{u \in \mathbb{Z}^{d}} \mathbb{P}\left(\{0 \leftrightarrow x+r u\} \cap\{0 \leftrightarrow \underset{\mathbb{T}}{ } x\}^{c}\right)
\end{align*}
$$

for every $x \in \mathbb{T}_{r}^{d}$, where $\{x \underset{\mathbb{T}}{\overleftrightarrow{ }} y\}$ denotes the event that $x$ is connected to $y$ by an open path in $\mathbb{T}_{r}^{d}$ in the coupling (see [24], (5.4)).

The proof of Lemma 4.8 is immediate, using the following two lemmas to bound the two terms in (4.33). In the first lemma, there is room to spare by a factor $\chi$ in the last term; this is consistent with [24]. Also, we see the bubble rather than the triangle which again has room to spare.

LEmmA 4.9. Let $d>6$, and suppose that $(\mathrm{T})$ holds on $\mathbb{Z}^{d}$. The inequality

$$
\begin{equation*}
\sum_{u \in \mathbb{Z}^{d}} \sum_{v \neq u} \mathbb{P}(0 \leftrightarrow x+r u, x+r v) \preceq \frac{\chi}{V}\left(\mathrm{~B}_{p}(x)+\frac{\chi^{2}}{V}\right) \tag{4.34}
\end{equation*}
$$

holds for every $0 \leq p<p_{c}, r>2$, and $x \in \mathbb{T}_{r}^{d}$.
Proof. We use $x, y$ for torus points, $u, v, w$ for translating points in $\mathbb{Z}^{d}$, and, for clarity, write the two-point function as $\tau(u, v)$ in place of the usual $\tau_{p}(v-u)$. By the BK inequality,

$$
\begin{align*}
& \sum_{u \in \mathbb{Z}^{d}} \sum_{v \neq u} \mathbb{P}(0 \leftrightarrow x+r u, x+r v) \\
& \quad \leq \sum_{z, u \in \mathbb{Z}^{d}} \sum_{v \neq u} \tau(0, z) \tau(z, x+r u) \tau(z, x+r v) \\
& =\sum_{y \in \mathbb{T}_{r}^{d}} \sum_{w \in \mathbb{Z}^{d}} \tau(0, x+y+r w) \sum_{u \in \mathbb{Z}^{d}} \tau(y, r(u-w)) \sum_{v \neq u} \tau(y, r(v-w))  \tag{4.35}\\
& \quad=\sum_{y \in \mathbb{T}_{r}^{d}} \sum_{w \in \mathbb{Z}^{d}} \tau(0, x+y+r w) \sum_{u \in \mathbb{Z}^{d}} \tau(y, r u) \sum_{v \neq u} \tau(y, r v),
\end{align*}
$$

where in the third line we replaced $z$ by $x+y+r w$ and in the fourth we replaced $u$ by $u+w$ and $v$ by $v+w$. For the sum over $v$, it follows from Lemma 4.4 that

$$
\begin{align*}
\sum_{v \neq u} \tau(y, r v) & =\sum_{v \neq u} \tau(y, r v)\left(\mathbb{1}_{u=0}+\mathbb{1}_{u \neq 0}\right) \\
& =\mathbb{1}_{u=0} \sum_{v \neq 0} \tau(y, r v)+\mathbb{1}_{u \neq 0} \sum_{v \neq u} \tau(y, r v)  \tag{4.36}\\
& \preceq \mathbb{1}_{u=0} \frac{\chi}{V}+\mathbb{1}_{u \neq 0}\left(\tau(0, y)+\frac{\chi}{V}\right) \preceq \mathbb{1}_{u \neq 0} \tau(0, y)+\frac{\chi}{V} .
\end{align*}
$$

This leads, using Lemma 4.4 again, to

$$
\begin{equation*}
\sum_{u \in \mathbb{Z}^{d}} \tau(y, r u) \sum_{v \neq u} \tau(y, r v) \preceq \tau(0, y) \frac{\chi}{V}+\frac{\chi}{V}\left(\tau(0, y)+\frac{\chi}{V}\right) \preceq \frac{\chi}{V}\left(\tau(0, y)+\frac{\chi}{V}\right) . \tag{4.37}
\end{equation*}
$$

Thus, we have an upper bound on (4.35), given by

$$
\begin{align*}
& \frac{\chi}{V} \sum_{y \in \mathbb{T}_{r}^{d}} \sum_{w \in \mathbb{Z}^{d}} \tau(0, x+y+r w)\left(\tau(0, y)+\frac{\chi}{V}\right) \\
& \quad=\frac{\chi^{3}}{V^{2}}+\frac{\chi}{V} \sum_{y \in \mathbb{T}_{r}^{d}} \sum_{w \in \mathbb{Z}^{d}} \tau(0, x+y+r w) \tau(0, y) \tag{4.38}
\end{align*}
$$

We extend this last sum over $y$ to all of $\mathbb{Z}^{d}$ and use the inequality for the bubble from Lemma 4.4 to finally get that

$$
\begin{equation*}
\sum_{u \in \mathbb{Z}^{d}} \sum_{v \neq u} \mathbb{P}(0 \leftrightarrow x+r u, x+r v) \preceq \frac{\chi}{V}\left(\mathrm{~B}_{p}(x)+\frac{\chi^{2}}{V}\right), \tag{4.39}
\end{equation*}
$$

as claimed.
LEMMA 4.10. Let $d>6$, and suppose that $(\mathrm{T})$ holds on $\mathbb{Z}^{d}$. The estimate

$$
\begin{equation*}
\sum_{u \in \mathbb{Z}^{d}} \mathbb{P}\left(\{0 \leftrightarrow x+r u\} \cap\{0 \overleftrightarrow{\mathbb{T}} x\}^{c}\right) \leq \frac{\chi(p)}{V}\left(T_{p}(x)+\frac{\chi(p)^{3}}{V}\right) \tag{4.40}
\end{equation*}
$$

holds for every $0 \leq p<p_{c}, r>2$, and $x \in \mathbb{T}_{r}^{d}$.
Proof. Our starting point is the set inclusion

$$
\begin{align*}
\{0 \leftrightarrow x+r u\} \cap\{0 \overleftrightarrow{\mathbb{T}} x\}^{c} \subseteq & \bigcup_{z \in \mathbb{Z}^{d}} \bigcup_{a \in \mathbb{T}_{r}^{d}} \bigcup_{v_{1}, v_{2} \in \mathbb{Z}^{d}: v_{1} \neq v_{2}}\{0 \leftrightarrow z\} \circ\left\{z \leftrightarrow a+r v_{1}\right\}  \tag{4.41}\\
& \circ\left\{z \leftrightarrow a+r v_{2}\right\} \circ\left\{a+r v_{2} \leftrightarrow x+r u\right\}
\end{align*}
$$

for every $x \in \mathbb{T}_{r}^{d}$ and $u \in \mathbb{Z}^{d}$ which arises from the coupling of torus and $\mathbb{Z}^{d}$ percolation in [24], Proposition 2.1. We use $a, b, x, y, z$ for torus points and use $u, v, w$ for translating vectors. It follows from the set inclusion (4.41), together with a union bound and the BK inequality, that

$$
\begin{align*}
& \mathbb{P}\left(\{0 \leftrightarrow x+r u\} \cap\left\{0 \leftrightarrow \overleftrightarrow{\mathbb{T}}^{x}\right\}^{c}\right) \\
& \quad \leq \sum_{a \in \mathbb{T}_{r}^{d}} \sum_{z, v_{1} \in \mathbb{Z}^{d}} \sum_{v_{2} \neq v_{1}} \tau_{p}(z) \tau_{p}\left(a+r v_{2}-z\right) \tau_{p}\left(a+r v_{1}-z\right) \tau_{p}\left(x+r u-a-r v_{2}\right) \tag{4.42}
\end{align*}
$$

We translate to more convenient vertices, as follows. First, we write the $\mathbb{Z}^{d}$ point $a+r v_{2}-$ $x$ uniquely as a torus point $y$ plus $r v$ with $v \in \mathbb{Z}^{d}$ and, similarly for the others, to obtain

$$
\begin{align*}
a+r v_{2}-x & =y+r v, \quad y \in \mathbb{T}_{r}^{d}, v \in \mathbb{Z}^{d} \\
a+r v_{1}-x & =y+r v+r v^{\prime}, \quad v^{\prime}=v_{2}-v_{1} \neq 0  \tag{4.43}\\
z-x & =y+z^{\prime}+r u^{\prime}, \quad z^{\prime} \in \mathbb{T}_{r}^{d}, u^{\prime} \in \mathbb{Z}^{d}
\end{align*}
$$

This gives

$$
\sum_{u \in \mathbb{Z}^{d}} \mathbb{P}\left(\{0 \leftrightarrow x+r u\} \cap\{0 \overleftrightarrow{\mathbb{T}} x\}^{c}\right)
$$

$$
\begin{align*}
\leq & \sum_{y, z^{\prime} \in \mathbb{T}_{r}^{d}} \sum_{v, u^{\prime} \in \mathbb{Z}^{d}} \tau_{p}\left(x+y+z^{\prime}+r u^{\prime}\right) \tau_{p}\left(-z^{\prime}+r\left(v-u^{\prime}\right)\right)  \tag{4.44}\\
& \times \sum_{v^{\prime} \neq 0} \tau_{p}\left(-z^{\prime}+r\left(v^{\prime}+v-u^{\prime}\right)\right) \sum_{u \in \mathbb{Z}^{d}} \tau_{p}(-y+r(u-v)) .
\end{align*}
$$

We bound the sums over $u$ and $v^{\prime}$ with Lemma 4.4 and obtain

$$
\begin{equation*}
\sum_{u \in \mathbb{Z}^{d}} \tau_{p}(-y+r(u-v))=\sum_{u \in \mathbb{Z}^{d}} \tau_{p}(-y+r u) \leq \tau_{p}(y)+C \frac{\chi(p)}{V} \tag{4.45}
\end{equation*}
$$

$$
\begin{equation*}
\sum_{v^{\prime} \neq 0} \tau_{p}\left(-z^{\prime}+r\left(v^{\prime}+v-u^{\prime}\right)\right)=\sum_{v^{\prime} \neq v-u^{\prime}} \tau_{p}\left(-z^{\prime}+r v^{\prime}\right) \leq \tau_{p}\left(z^{\prime}\right) \mathbb{1}_{v \neq u^{\prime}}+C \frac{\chi(p)}{V} \tag{4.46}
\end{equation*}
$$

Then we perform the sum over $v$, which after translating $v$ by $u^{\prime}$, is bounded similarly using

$$
\begin{align*}
& \sum_{v \in \mathbb{Z}^{d}} \tau_{p}\left(-z^{\prime}+r\left(v-u^{\prime}\right)\right)\left(\tau_{p}\left(z^{\prime}\right) \mathbb{1}_{v \neq u^{\prime}}+C \frac{\chi(p)}{V}\right) \\
& \quad=\sum_{v \in \mathbb{Z}^{d}} \tau_{p}\left(-z^{\prime}+r v\right)\left(\tau_{p}\left(z^{\prime}\right) \mathbb{1}_{v \neq 0}+C \frac{\chi(p)}{V}\right)  \tag{4.47}\\
& \quad \leq C \frac{\chi(p)}{V} \tau_{p}\left(z^{\prime}\right)+C \frac{\chi(p)}{V} \sum_{v \in \mathbb{Z}^{d}} \tau_{p}\left(-z^{\prime}+r v\right) \\
& \quad \leq \frac{\chi(p)}{V} \tau_{p}\left(z^{\prime}\right)+\frac{\chi(p)^{2}}{V^{2}}
\end{align*}
$$

This leads to

$$
\begin{align*}
& \sum_{u \in \mathbb{Z}^{d}} \mathbb{P}\left(\{0 \leftrightarrow x+r u\} \cap\{0 \overleftrightarrow{\mathbb{T}} x\}^{c}\right)  \tag{4.48}\\
& \quad \preceq \frac{\chi(p)}{V} \sum_{y, z^{\prime} \in \mathbb{T}_{r}^{d}} \sum_{u^{\prime} \in \mathbb{Z}^{d}} \tau_{p}\left(x+y+z^{\prime}+r u^{\prime}\right)\left(\tau_{p}(y)+\frac{\chi(p)}{V}\right)\left(\tau_{p}\left(z^{\prime}\right)+\frac{\chi(p)}{V}\right)
\end{align*}
$$

We expand out the brackets and recognise that the term containing the product $\tau_{p}(y) \tau_{p}\left(z^{\prime}\right)$ obeys

$$
\begin{align*}
& \sum_{u^{\prime} \in \mathbb{Z}^{d}} \sum_{y, z^{\prime} \in \mathbb{T}_{r}^{d}} \tau_{p}\left(x+y+z^{\prime}+r u^{\prime}\right) \tau_{p}\left(z^{\prime}\right) \tau_{p}(y) \\
& \quad=\sum_{u^{\prime} \in \mathbb{Z}^{d}} \sum_{y, z^{\prime} \in \mathbb{T}_{r}^{d}} \tau_{p}\left(z^{\prime}\right) \tau_{p}\left(y+z^{\prime}-z^{\prime}\right) \tau_{p}\left(x+y+z^{\prime}+r u^{\prime}\right)  \tag{4.49}\\
& \quad \leq \sum_{u^{\prime} \in \mathbb{Z}^{d}} \mathrm{~T}_{p}\left(x+r u^{\prime}\right) \preceq \mathrm{T}_{p}(x)+\frac{\chi(p)^{3}}{V}
\end{align*}
$$

Meanwhile, the two terms containing exactly one of $\tau_{p}(y)$ or $\tau_{p}\left(z^{\prime}\right)$ are equal and can be expressed as

$$
\frac{\chi(p)}{V} \sum_{u^{\prime} \in \mathbb{Z}^{d}} \sum_{y, z^{\prime} \in \mathbb{T}_{r}^{d}} \tau_{p}\left(x+y+z^{\prime}+r u^{\prime}\right) \tau_{p}\left(z^{\prime}\right)
$$

$$
\begin{equation*}
=\frac{\chi(p)}{V} \sum_{w \in \mathbb{Z}^{d}} \sum_{z^{\prime} \in \mathbb{T}_{r}^{d}} \tau_{p}\left(x+z^{\prime}+w\right) \tau_{p}\left(z^{\prime}\right) \leq \frac{\chi(p)^{3}}{V} \tag{4.50}
\end{equation*}
$$

where we extended the sum over $z^{\prime}$ to all of $\mathbb{Z}^{d}$ in the last inequality. Finally, the term not containing either $\tau_{p}(y)$ or $\tau_{p}\left(z^{\prime}\right)$ can be expressed as

$$
\begin{align*}
& \frac{\chi(p)^{2}}{V^{2}} \sum_{y, z^{\prime} \in \mathbb{T}_{r}^{d}} \sum_{u^{\prime} \in \mathbb{Z}^{d}} \tau_{p}\left(x+y+z^{\prime}+r u^{\prime}\right) \\
& \quad=\frac{\chi(p)^{2}}{V^{2}} \sum_{z^{\prime} \in \mathbb{T}_{r}^{d}} \sum_{w \in \mathbb{Z}^{d}} \tau_{p}\left(x+z^{\prime}+w\right)=\frac{\chi(p)^{3}}{V} . \tag{4.51}
\end{align*}
$$

Summation of these contributions gives

$$
\begin{equation*}
\sum_{u \in \mathbb{Z}^{d}} \mathbb{P}\left(\{0 \leftrightarrow x+r u\} \cap\{0 \overleftrightarrow{\mathbb{T}} x\}^{c}\right) \preceq \frac{\chi(p)}{V}\left(\mathrm{~T}_{p}(x)+\frac{\chi(p)^{3}}{V}\right), \tag{4.52}
\end{equation*}
$$

and the proof is complete.
Proof of Lemma 4.8. Lemmas 4.9 and 4.10 give bounds on the two terms on the right-hand side of (4.33), namely,

$$
\begin{equation*}
\psi_{r, p}(x)-\psi_{r, p}^{\mathbb{T}}(x) \preceq \frac{\chi}{V}\left(\mathrm{~B}_{p}(x)+\frac{\chi^{2}}{V}\right)+\frac{\chi}{V}\left(\mathrm{~T}_{p}(x)+\frac{\chi^{3}}{V}\right) \tag{4.53}
\end{equation*}
$$

Since the bubble is bounded above by the triangle and the susceptibility is at least 1 , this gives the desired estimate

$$
\begin{equation*}
\psi_{r, p}(x)-\psi_{r, p}^{\mathbb{T}}(x) \preceq \frac{\chi}{V}\left(\mathrm{~T}_{p}(x)+\frac{\chi^{3}}{V}\right) \tag{4.54}
\end{equation*}
$$

and the proof is complete.
4.4. The torus triangle condition: Proof of Theorem 1.6. To conclude this section, we show how the torus plateau leads to easy proofs that $p_{\mathbb{T}}$ lies in the scaling window and that the torus triangle condition holds.

Proof of Theorem 1.6. Fix $\varepsilon>0$ sufficiently small that $\varepsilon^{-1} \geq A_{2}$, where $A_{2}$ is as in Theorem 1.3. Recalling from (1.11) that $\chi \asymp\left(p_{c}-p\right)^{-1}$ and setting $p_{0}=p_{c}-\varepsilon^{-1} V^{-1 / 3}$, we have that $\chi\left(p_{0}\right) \asymp \varepsilon V^{1 / 3}$. On the other hand, for sufficiently large $r$ (depending on $M$ ) the lower bound of (1.13) applies to give

$$
\begin{equation*}
\chi^{\mathbb{T}}\left(p_{0}\right) \succeq \sum_{x \in \mathbb{T}_{r}^{d}:\|x\|_{\infty}>M} V^{-1} \chi\left(p_{0}\right) \succeq\left(V-(2 M+1)^{d}\right) V^{-1} \chi\left(p_{0}\right) \succeq \varepsilon V^{1 / 3} \tag{4.55}
\end{equation*}
$$

Since we also have by the coupling that $\chi^{\mathbb{T}} \leq \chi$, it follows that there exist positive constants $c_{1}$ and $C_{2}$ such that

$$
\begin{equation*}
c_{1} \varepsilon V^{1 / 3} \leq \chi^{\mathbb{T}}\left(p_{0}\right) \leq \chi\left(p_{0}\right) \leq C_{2} \varepsilon V^{1 / 3} \tag{4.56}
\end{equation*}
$$

A second application of (1.11) yields that there exists a constant $C_{3} \geq 1$ such that if we define $p_{1}=p_{c}-C_{3} \varepsilon^{-1} V^{-1 / 3}$, then $\chi^{\mathbb{T}}\left(p_{1}\right) \leq \chi\left(p_{1}\right) \leq c_{1} \varepsilon V^{1 / 3}$. It follows by the intermediate value theorem that if we define $\lambda=\lambda(\varepsilon)=c_{1} \varepsilon$, then the $p_{\mathbb{T}}$ defined by $\chi^{\mathbb{T}}\left(p_{\mathbb{T}}\right)=\lambda V^{1 / 3}$ (which does exist if $r$ exceeds some value $r_{0}(\lambda)$ ) satisfies $p_{1} \leq p_{\mathbb{T}} \leq p_{0}$ and hence that $0 \leq$ $p_{c}-p_{\mathbb{T}} \preceq \varepsilon^{-1} V^{-1 / 3}$. With the choice $\lambda_{0}=c_{1} A_{2}^{-1}$, this concludes the proof that $p_{\mathbb{T}}=p_{\mathbb{T}}(\lambda)$ lies in the scaling window if $\lambda \in\left(0, \lambda_{0}\right]$ and $r>r_{0}(\lambda)$.

Let $p<p_{c}$ and $x \in \mathbb{T}_{r}^{d}$. The open torus triangle diagram is defined by

$$
\begin{equation*}
\mathrm{T}_{p}^{\mathbb{T}}(x)=\sum_{y, z \in \mathbb{T}_{r}^{d}} \tau_{p}^{\mathbb{T}}(y) \tau_{p}^{\mathbb{T}}(z-y) \tau_{p}^{\mathbb{T}}(x-z) \tag{4.57}
\end{equation*}
$$

and (4.17) then implies that

$$
\begin{equation*}
\mathrm{T}_{p}^{\mathbb{T}}(x) \leq \sum_{y, z \in \mathbb{T}_{r}^{d}} \sum_{u, v, w \in \mathbb{Z}^{d}} \tau_{p}(y+r u) \tau_{p}(z-y+r v) \tau_{p}(x-z+r w) \tag{4.58}
\end{equation*}
$$

We replace the index $v$ by $v^{\prime}-u$ and then replace $w$ by $w^{\prime}-v^{\prime}$. The above right-hand side becomes (after setting $y^{\prime}=y+r u$ and $z^{\prime}=z+r v^{\prime}$ )

$$
\begin{equation*}
\sum_{y^{\prime}, z^{\prime}, w^{\prime} \in \mathbb{Z}^{d}} \tau_{p}\left(y^{\prime}\right) \tau_{p}\left(z^{\prime}-y^{\prime}\right) \tau_{p}\left(x-z^{\prime}+r w^{\prime}\right)=\sum_{w^{\prime} \in \mathbb{Z}^{d}} \mathrm{~T}_{p}\left(x+r w^{\prime}\right) \tag{4.59}
\end{equation*}
$$

with $\mathrm{T}_{p}\left(x+r w^{\prime}\right)$ the open $\mathbb{Z}^{d}$ triangle diagram. By Lemma 4.4 this gives that

$$
\begin{equation*}
\mathrm{T}_{p}^{\mathbb{T}}(x) \leq \mathrm{T}_{p}(x)+C_{1} \frac{\chi(p)^{3}}{V} \leq \mathrm{T}_{p_{c}}(x)+C_{1} \frac{\chi(p)^{3}}{V} \tag{4.60}
\end{equation*}
$$

With $p_{\mathbb{T}}$ as in the previous paragraph, this implies that

$$
\begin{equation*}
\mathrm{T}_{p_{\mathbb{T}}}^{\mathbb{T}}(x) \leq \mathrm{T}_{p_{c}}(x)+C_{1} C_{2}^{3} \varepsilon^{3} \tag{4.61}
\end{equation*}
$$

With the bound $\varepsilon \leq A_{2}^{-1}$, this concludes the proof of the torus triangle condition.
It remains to prove that the $a_{0}$-strong version of the triangle condition holds when its $\mathbb{Z}^{d}$ counterpart holds with $\frac{1}{2} a_{0}$. But under this assumption it follows from (4.61) that

$$
\begin{equation*}
\mathrm{T}_{p_{\mathbb{T}}}^{\mathbb{T}}(x) \leq \mathbb{1}(x=0)+\frac{1}{2} a_{0}+C_{1} C_{2}^{3} \varepsilon^{3} \tag{4.62}
\end{equation*}
$$

We require that $\varepsilon^{3} \leq \varepsilon_{0}^{3}=a_{0}\left(2 C_{1} C_{2}^{3}\right)^{-1}$. With $\lambda \leq \lambda_{1}=c_{1} \varepsilon_{0}$, the right-hand side of (4.62) is at most $\mathbb{1}(x=0)+a_{0}$, and the proof is complete.
5. Plateau within the scaling window: Proof of Theorem 1.3. We now turn to the part of Theorem 1.3 concerning the case that $p$ lies within the scaling window of the torus. The window consists of $p$ values with $\left|p-p_{c}\right| \leq A V^{-1 / 3}$ with $A$ arbitrary but fixed.
5.1. Lower bound in the window. We begin by proving the lower bound (1.15), which follows simply from the monotonicity of $\tau_{p}^{\mathbb{T}}$ in $p$, the lower bound below the window, and the comparison of $\tau_{p}$ with $\tau_{p_{c}}$ provided by Lemma 4.7.

Proof of (1.15). Let $A_{1}$ and $A_{2}$ be the constants from the "below the scaling window" part of Theorem 1.3. Fix $A>0$. It suffices by monotonicity of the torus two-point function to prove the claimed estimate in the case that $A \geq A_{1}$ and $p=p_{c}-A V^{-1 / 3}$. We denote this value of $p$ by $p^{\prime}$. Thus, there exists a constant $r_{0}$, depending on $A$, such that $A V^{-1 / 3} \leq$ $A_{2} r^{-2}=A_{2} V^{-2 / d} l e p_{c} / 2$ for every $r \geq r_{0}$.

By (1.11), $\chi\left(p^{\prime}\right) \geq c_{\chi} A^{-1} V^{1 / 3}$ for some constant $c_{\chi}$ depending only on $d$ and $L$. It then follows from (1.13) that there exists a constant $M$ (depending on $d$ and $L$ ) such that

$$
\begin{equation*}
\tau_{p^{\prime}}^{\mathbb{T}}(x) \geq \tau_{p^{\prime}}(x)+\frac{c_{2} c_{\chi}}{A V^{2 / 3}} \tag{5.1}
\end{equation*}
$$

for every $x \in \mathbb{T}_{r}^{d}$ with $\|x\|_{\infty} \geq M$. By Lemma 4.7 (with our assumption that $r \geq r_{0}$ guaranteeing that $p^{\prime} \geq p_{c} / 2$ ), we have moreover that

$$
\begin{equation*}
\tau_{p_{c}}(x)-\tau_{p^{\prime}}(x) \preceq \frac{A}{V^{1 / 3}\langle x\rangle^{d-4}} \preceq \frac{A r^{2}}{V^{1 / 3}} \tau_{p_{c}}(x) \tag{5.2}
\end{equation*}
$$

Since the prefactor $A r^{2} V^{-1 / 3}$ tends to zero, as $r \rightarrow \infty$, it follows from (5.1)-(5.2) that, for each $\delta>0$, there exists an $r_{1} \geq r_{0}$, depending on $A$, such that

$$
\begin{equation*}
\tau_{p^{\prime}}^{\mathbb{T}}(x) \geq(1-\delta) \tau_{p_{c}}(x)+\frac{c_{2} c_{\chi}}{A V^{2 / 3}} \tag{5.3}
\end{equation*}
$$

for every $x \in \mathbb{T}_{r}^{d}$ with $\|x\|_{\infty} \geq M$ and every $r \geq r_{1}$. This completes the proof.
5.2. Upper bound in the window. It remains to prove the upper bound. To do so, we first consider the case $p=p_{c}$. We then extend the upper bound to the window ( $p_{c}, p_{c}+A V^{-1 / 3}$ ] by proving that, in this window, the two-point function changes only up to a multiplicative factor (that can be chosen to be arbitrarily close to one) plus an additive constant term of order $V^{-2 / 3}$.

At $p_{c}$ the upper bound is not new and was proven previously in [29], Theorem 1.7. However, our proof, which is based on the extrinsic (Euclidean) distance, seems more direct than that of [29] where the intrinsic distance was used. Our proof relies on the extrinsic one-arm exponent estimate

$$
\begin{equation*}
\mathbb{P}_{p_{c}}\left(0 \leftrightarrow \partial \Lambda_{\ell}\right) \asymp \frac{1}{\ell^{2}} \tag{5.4}
\end{equation*}
$$

of Kozma and Nachmias [37] (i.e., the $p=p_{c}$ case of Theorem 1.2).
Proposition 5.1. Let $d>6$, and suppose that $(\mathrm{T})$ holds on $\mathbb{Z}^{d}$. There is a $C>0$ such that

$$
\begin{equation*}
\tau_{p_{c}}^{\mathbb{T}}(x) \leq \tau_{p_{c}}(x)+C V^{-2 / 3} \tag{5.5}
\end{equation*}
$$

for all $r>2$ and all $x \in \mathbb{T}_{r}^{d}$.
Proof. Fix some large and positive integer $M$. Since $\mathbb{Z}^{d}$ covers the torus $\mathbb{T}_{r}^{d}$, every path in $\mathbb{T}_{r}^{d}$ can be lifted to a path in $\mathbb{Z}^{d}$ that is unique up to the choice of starting point. For $x, y \in \mathbb{T}_{r}^{d}$, we define $E_{\geq \ell}(x, y)$ to be the event that $x$ and $y$ are connected by a simple $\mathbb{T}_{r}^{d}$ path that lifts to a $\mathbb{Z}^{d}$-path of diameter greater than or equal to $\ell$, and we define $E_{\leq \ell}(x, y)$ similarly. In addition, we define $A_{\geq \ell}(x)$ to be the event that there exists some simple $\mathbb{T}_{r}^{d}$ path, starting from $x$, that lifts to a $\mathbb{Z}^{d}$-path of diameter at least $\ell$. We have trivially that $\{0 \underset{\mathbb{T}}{\overleftrightarrow{\leftrightarrow}} x\}=E_{\leq 3 M r}(0, x) \cup E_{\geq 3 M r}(0, x)$ so that

$$
\begin{equation*}
\mathbb{P}_{p_{c}}^{\mathbb{T}}(0 \underset{\mathbb{T}}{\overleftrightarrow{ }} x) \leq \mathbb{P}_{p_{c}}^{\mathbb{T}}\left(E_{\leq 3 M r}(0, x)\right)+\mathbb{P}_{p_{c}}^{\mathbb{T}}\left(E_{\geq 3 M r}(0, x)\right) \tag{5.6}
\end{equation*}
$$

Note that on $E_{\geq 3 M r}(0, x)$, the events $A_{\geq M r}(0)$ and $A_{\geq M r}(x)$ must occur disjointly. Also, in the coupling between torus and $\mathbb{Z}^{d}$ percolation we have

$$
\begin{equation*}
A_{\geq \ell}(0) \subset\left\{0 \underset{\mathbb{Z}}{\overleftrightarrow{ }} \partial \Lambda_{\ell}\right\} \quad \text { and } \quad A_{\geq \ell}(x) \subset\left\{x \underset{\mathbb{Z}}{\overleftrightarrow{\leftrightarrow}} x+\partial \Lambda_{\ell}\right\} \tag{5.7}
\end{equation*}
$$

where we recall that $\Lambda_{\ell}=[-\ell, \ell]^{d} \cap \mathbb{Z}^{d}$. Thus, by the BK inequality on the torus,

$$
\begin{align*}
\mathbb{P}_{p_{c}}^{\mathbb{T}}\left(E_{\geq 3 M r}(0, x)\right) & \leq \mathbb{P}_{p_{c}}^{\mathbb{T}}\left(A_{\geq M r}(0) \circ A_{\geq M r}(x)\right) \\
& \leq \mathbb{P}_{p_{c}}^{\mathbb{T}}\left(A_{M r}(0)\right) \mathbb{P}_{p_{c}}^{\mathbb{T}}\left(A_{M r}(x)\right) \leq \mathbb{P}_{p_{c}}\left(0 \leftrightarrow \partial \Lambda_{M r}\right)^{2} \tag{5.8}
\end{align*}
$$

Using the one-arm upper bound of (5.4), this gives

$$
\begin{equation*}
\mathbb{P}_{p_{c}}^{\mathbb{T}}\left(E_{\geq 3 M r}(0, x)\right) \preceq \frac{1}{M^{4} r^{4}} . \tag{5.9}
\end{equation*}
$$

For the first term of (5.6), we simply use the coupling and a union bound to see that

$$
\begin{align*}
\mathbb{P}_{p_{c}}^{\mathbb{T}}\left(E_{\leq 3 M r}(0, x)\right) & \leq \mathbb{P}_{p_{c}}\left(\bigcup_{\|u\|_{\infty} \leq 3 M}\{0 \leftrightarrow x+r u\}\right) \\
& \leq \tau_{p_{c}}(x)+\sum_{1 \leq\|u\|_{\infty} \leq 3 M} \tau_{p_{c}}(x+r u) . \tag{5.10}
\end{align*}
$$

The latter sum can be bounded above by an integral over the $d$-dimensional box of radius $3 M$, namely,

$$
\begin{equation*}
\sum_{1 \leq\|u\|_{\infty} \leq 3 M} \tau_{p_{c}}(x+r u) \preceq \int_{\|u\|_{\infty} \leq 3 M} \frac{1}{\langle r u\rangle^{d-2}} d u \preceq M^{2} r^{-(d-2)} . \tag{5.11}
\end{equation*}
$$

Together, these bounds imply that there exists a constant $C$ such that

$$
\begin{equation*}
\tau_{p_{c}}^{\mathbb{T}}(x) \leq \tau_{p_{c}}(x)+C M^{2} r^{-(d-2)}+C M^{-4} r^{-4} \tag{5.12}
\end{equation*}
$$

The choice $M=r^{(d-6) / 6}$ gives the desired upper bound $\tau_{p_{c}}(x)+C V^{-2 / 3}$ at $p=p_{c}$.
Next, we prove an upper bound at the top of the scaling window. For $p \in\left(p_{c}, p_{c}+\right.$ $A V^{-1 / 3}$ ], we use the intrinsic distance $d_{\mathrm{int}}$ which is the graph distance on the percolation configuration. If $x, y$ are not connected in the configuration, then $d_{\text {int }}(x, y)=\infty$. Given a percolation configuration, we define the (random) intrinsic ball centred at $x$ and of radius $\ell$ by

$$
\begin{equation*}
B_{\mathrm{int}}(x, \ell)=\left\{y \in \mathbb{T}_{r}^{d}: d_{\mathrm{int}}(x, y) \leq \ell\right\} \tag{5.13}
\end{equation*}
$$

Thus,

$$
\begin{equation*}
\{x \leftrightarrow y \text { by a path of length } \leq \ell\}=\left\{y \in B_{\text {int }}(x, \ell)\right\} . \tag{5.14}
\end{equation*}
$$

We write the boundary of the intrinsic ball as

$$
\begin{equation*}
\partial B_{\mathrm{int}}(x, \ell)=B_{\mathrm{int}}(x, \ell) \backslash B_{\mathrm{int}}(x, \ell-1)=\left\{y \in \mathbb{T}_{r}^{d}: d_{\mathrm{int}}(x, y)=\ell\right\} \tag{5.15}
\end{equation*}
$$

Given a subset $g$ of edges of the edge set $\mathbb{B}$ of $\mathbb{T}_{r}^{d}$ or $\mathbb{Z}^{d}$, we define $B_{\text {int }}^{g}(x, \ell)$ similarly as $B_{\text {int }}(x, \ell)$, except that the intrinsic distance from $x$ to $y$ is determined only using paths consisting of edges of $g$.

Kozma and Nachmias [36] computed the asymptotic behaviour of the critical instrinsic one-arm probability in high dimensions to be

$$
\begin{equation*}
\mathbb{P}_{p_{c}}\left(\partial B_{\mathrm{int}}(0, \ell) \neq \varnothing\right) \asymp \frac{1}{\ell} \tag{5.16}
\end{equation*}
$$

for every $\ell \geq 1$. In fact, they also proved an extension of the upper bound of (5.16) involving the intrinsic ball restricted to a subgraph $g$; their proof also extends immediately to the torus, as explained in [29], Theorem 2.1(i), and implies, in particular, that

$$
\begin{equation*}
\max _{g \subset \mathbb{B}\left(\mathbb{T}_{r}^{d}\right)} \mathbb{P}_{p_{c}}^{\mathbb{T}}\left(\partial B_{\mathrm{int}}^{g}(0, \ell) \neq \varnothing\right) \preceq \frac{1}{\ell} \tag{5.17}
\end{equation*}
$$

for every $r, \ell \geq 1$. (An important remark is that the proof of (5.17) does not require the lace expansion on $\mathbb{T}_{r}^{d}$ but, instead, uses results for $\mathbb{Z}^{d}$ along with the coupling of torus and $\mathbb{Z}^{d}$ percolation-this is discussed in greater detail in the verification of [25], Theorem 4.1(b). As such, there is no circular reasoning here nor an appeal to any result obtained via the torus lace expansion.)

We first isolate two estimates in the following lemma, whose proof uses a standard coupling of percolation at different values of $p$.

LEMMA 5.2. For $0 \leq p<q \leq 1, \ell \geq 1$, and $g \subset \mathbb{B}\left(\mathbb{T}_{r}^{d}\right)$,

$$
\begin{align*}
\mathbb{P}_{q}^{\mathbb{T}}\left(\partial B_{\mathrm{int}}^{g}(0, \ell) \neq \varnothing\right) & \leq\left(\frac{q}{p}\right)^{\ell} \mathbb{P}_{p}^{\mathbb{T}}\left(\partial B_{\mathrm{int}}^{g}(0, \ell) \neq \varnothing\right)  \tag{5.18}\\
\mathbb{P}_{q}^{\mathbb{T}}\left(x \in B_{\mathrm{int}}(0, \ell)\right) & \leq\left(\frac{q}{p}\right)^{\ell} \mathbb{P}_{p}^{\mathbb{T}}\left(x \in B_{\mathrm{int}}(0, \ell)\right) \tag{5.19}
\end{align*}
$$

Proof. Let $0 \leq p<q \leq 1$. We begin with (5.18). Given a subset $g$ of the edge set of $\mathbb{T}_{r}^{d}$, we write $R_{g}(\ell)=\left\{\partial \bar{B}_{\text {int }}^{g}(0, \ell) \neq \varnothing\right\}$. We use the standard coupling of percolation configurations via uniform random variables assigned to each edge of the torus (see [18], p. 11); these uniform random variables are defined on some probability space $(\Omega, \mathcal{A}, \mathbb{Q})$. We write $\eta_{p}^{\mathbb{T}}$ for the induced percolation configuration. Since $R_{g}(\ell)$ depends on the edges inside $\mathbb{T}_{r}^{d}$ only, hence on finitely many edges, we can write

$$
\begin{align*}
\mathbb{Q}\left(\eta_{p}^{\mathbb{T}} \in R_{g}(\ell), \eta_{q}^{\mathbb{T}} \in R_{g}(\ell)\right) & =\sum_{\omega \in\{0,1\}^{\mathbb{T}} r_{r}^{d}} \mathbb{Q}\left(\eta_{p}^{\mathbb{T}} \in R_{g}(\ell), \eta_{q}^{\mathbb{T}}=\omega, \omega \in R_{g}(\ell)\right) \\
& =\sum_{\omega \in R_{g}(\ell)} \mathbb{Q}\left(\eta_{p}^{\mathbb{T}} \in R_{g}(\ell) \mid \eta_{q}^{\mathbb{T}}=\omega\right) \mathbb{Q}\left(\eta_{q}^{\mathbb{T}}=\omega\right) \tag{5.20}
\end{align*}
$$

Since the above left-hand side is at most $\mathbb{Q}\left(\eta_{p}^{\mathbb{T}} \in R_{g}(\ell)\right)$, (5.18) follows once we prove that

$$
\begin{equation*}
\mathbb{Q}\left(\eta_{p}^{\mathbb{T}} \in R_{g}(\ell) \mid \eta_{q}^{\mathbb{T}}=\omega\right) \geq\left(\frac{p}{q}\right)^{\ell} \tag{5.21}
\end{equation*}
$$

To prove (5.21), we first observe that on a specific configuration $\omega \in R_{g}(\ell)$, there exists inside $\omega$ a deterministic path of open edges, starting from 0 , of length $\ell$. A fortiori, on the event $\left\{\eta_{q}^{\mathbb{T}}=\omega\right\}$ there exist $\ell$ independent uniform random variables $U_{1}, \ldots, U_{\ell}$ attached to these open edges such that $U_{i} \leq q$ for all $1 \leq i \leq \ell$. For $\left\{\eta_{p}^{\mathbb{T}} \in R_{g}(\ell)\right\}$ to occur, it is enough that $U_{i} \leq p$ for all $1 \leq i \leq \ell$ which gives

$$
\begin{equation*}
\mathbb{Q}\left(\eta_{p}^{\mathbb{T}} \in R_{g}(\ell) \mid \eta_{q}^{\mathbb{T}}=\omega\right) \geq \mathbb{Q}\left(\bigcap_{i=1}^{\ell}\left\{U_{i} \leq p\right\} \mid \eta_{q}^{\mathbb{T}}=\omega\right) \tag{5.22}
\end{equation*}
$$

Since $U_{1}, \ldots, U_{\ell}$ are independent of the other uniform random variables, the above righthand side is equal to

$$
\begin{equation*}
\mathbb{Q}\left(\bigcap_{i=1}^{\ell}\left\{U_{i} \leq p\right\} \mid \eta_{q}^{\mathbb{T}}=\omega\right)=\mathbb{Q}\left(\bigcap_{i=1}^{\ell}\left\{U_{i} \leq p\right\} \mid \bigcap_{i=1}^{\ell}\left\{U_{i} \leq q\right\}\right)=\left(\frac{p}{q}\right)^{\ell}, \tag{5.23}
\end{equation*}
$$

which proves (5.21) and hence completes the proof of (5.18).
The proof of (5.19) is almost identical, using the fact that on $\left\{x \in B_{\text {int }}(0, \ell)\right\}$ there must exist a path of length less than or equal to $\ell$ connecting 0 to $x$. Then keeping the uniform random variables along this path open upon reducing $q$ to $p$ gives the result.

We now prove (1.14) in the following proposition.
Proposition 5.3. Let $d>6$, and suppose that (T) holds on $\mathbb{Z}^{d}$. Fix $\delta \in(0,1]$. For all $A>0$, there exists a constant $C_{A}$ such that

$$
\begin{equation*}
\tau_{p}^{\mathbb{T}}(x) \leq(1+\delta) \tau_{p_{c}}(x)+C_{A} \delta^{-2} V^{-2 / 3} \tag{5.24}
\end{equation*}
$$

for all $r>2, x \in \mathbb{T}_{r}^{d}$ and $p \in\left(p_{c}, p_{c}+A V^{-1 / 3}\right]$.
Proof. Let $x \in \mathbb{T}_{r}^{d}, A>0$ and $\delta>0$. By the monotonicity of $\tau_{p}^{\mathbb{T}}(x)$ in $p$ and the independence of the upper bound on $p$, it suffices to prove (5.24) for $p=p_{c}+\varepsilon$ with $\varepsilon=A V^{-1 / 3}$. We set $\gamma=\lfloor\delta / \varepsilon\rfloor$ and begin with the decomposition

$$
\begin{align*}
\mathbb{P}_{p_{c}+\varepsilon}^{\mathbb{T}}(0 \leftrightarrow x) & =\mathbb{P}_{p_{c}+\varepsilon}^{\mathbb{T}}\left(x \in \partial B_{\text {int }}(0, \ell) \text { for some } \ell\right)  \tag{5.25}\\
& \leq \mathbb{P}_{p_{c}+\varepsilon}^{\mathbb{T}}\left(x \in B_{\text {int }}(0,3 \gamma)\right)+\mathbb{P}_{p_{c}+\varepsilon}^{\mathbb{T}}\left(x \in \partial B_{\text {int }}(0, \ell) \text { for some } \ell \geq 3 \gamma\right)
\end{align*}
$$

For the first term in (5.25), we use (5.19) and Proposition 5.1 to see that

$$
\begin{align*}
\mathbb{P}_{p_{c}+\varepsilon}^{\mathbb{T}}\left(x \in B_{\mathrm{int}}(0,3 \gamma)\right) & \leq\left(\frac{p_{c}+\varepsilon}{p_{c}}\right)^{3 \gamma} \mathbb{P}_{p_{c}}^{\mathbb{T}}\left(x \in B_{\mathrm{int}}(0,3 \gamma)\right) \\
& \leq e^{3 \gamma \varepsilon / p_{c}} \mathbb{P}_{p_{c}}^{\mathbb{T}}(0 \leftrightarrow x)  \tag{5.26}\\
& \leq e^{3 \delta / p_{c}}\left(\tau_{p_{c}}(x)+C V^{-2 / 3}\right)
\end{align*}
$$

We consider now the second term in (5.25). For this we argue as in the proof of [29], (1.11) (see also [36], Lemma 2.6). We first observe that on the event $\left\{x \in \partial B_{\text {int }}(0, \ell)\right.$ for some $\ell \geq$ $3 \gamma\}$, the two events $\left\{\partial B_{\text {int }}(0, \gamma) \neq \varnothing\right\}$ and $\left\{\partial B_{\text {int }}(x, \gamma) \neq \varnothing\right\}$ must both occur. However, these two events do not necessarily occur disjointly. Indeed, in order for $d_{\text {int }}(0, y)$ to be large there must not only be a long open path from 0 to $y$ but also no shorter path; the sets of closed edges guaranteeing that no such short path exists may be shared in the common realisation of the two events. On the other hand, the set of vertices in $B_{\text {int }}(0, \gamma)$ and in $B_{\text {int }}(x, \gamma)$ are disjoint. To deal with this situation, we define $G$ to be the random graph whose edge set consists of all edges which touch $B_{\text {int }}(x, \gamma-1)$ (the vertex set of $G$ consists of the vertices incident to at least one edge in this set); these are exactly the edges needed to determine the random set $B_{\mathrm{int}}(x, \gamma)$. From now on we identify subgraphs of the torus with subsets of $\mathbb{B}\left(\mathbb{T}_{r}^{d}\right)$ and write $g^{c}$ for the complement of a subgraph $g \subseteq \mathbb{B}\left(\mathbb{T}_{r}^{d}\right)$. Since $B_{\mathrm{int}}(0, \gamma)$ and $B_{\mathrm{int}}(x, \gamma)$ are disjoint, we see that

$$
\begin{align*}
& \mathbb{P}_{p_{c}+\varepsilon}^{\mathbb{T}}\left(x \in \partial B_{\mathrm{int}}(0, \ell) \text { for some } \ell \geq 3 \gamma\right) \\
& \quad \leq \mathbb{P}_{p_{c}+\varepsilon}^{\mathbb{T}}\left(\partial B_{\mathrm{int}}^{G^{c}}(0, \gamma) \neq \varnothing, \partial B_{\mathrm{int}}(x, \gamma) \neq \varnothing\right)  \tag{5.27}\\
& \quad=\sum_{g \subset \mathbb{B}\left(\mathbb{T}_{r}^{d}\right)} \mathbb{P}_{p_{c}+\varepsilon}^{\mathbb{T}}\left(\partial B_{\mathrm{int}}^{g}(0, \gamma) \neq \varnothing, \partial B_{\mathrm{int}}(x, \gamma) \neq \varnothing, G^{c}=g\right) .
\end{align*}
$$

By definition of $G$, the events $\left\{\partial B_{\mathrm{int}}^{g}(0, \gamma) \neq \varnothing\right\}$ and $\left\{G^{c}=g, \partial B_{\mathrm{int}}(x, \gamma) \neq \varnothing\right\}$ depend on different edges (namely, those of $g$ and $g^{c}$, respectively), and hence

$$
\begin{align*}
& \mathbb{P}_{p_{c}+\varepsilon}^{\mathbb{T}}\left(x \in \partial B_{\mathrm{int}}(0, \ell) \text { for some } \ell \geq 3 \gamma\right) \\
& \quad \leq \sum_{g \subset \mathbb{B}\left(\mathbb{T}_{r}^{d}\right)} \mathbb{P}_{p_{c}+\varepsilon}^{\mathbb{T}}\left(\partial B_{\mathrm{int}}^{g}(0, \gamma) \neq \varnothing\right) \mathbb{P}_{p_{c}+\varepsilon}^{\mathbb{T}}\left(\partial B_{\mathrm{int}}(x, \gamma) \neq \varnothing, G^{c}=g\right) \\
& \quad \leq \max _{g \subset \mathbb{B}\left(\mathbb{T}_{r}^{d}\right)} \mathbb{P}_{p_{c}+\varepsilon}^{\mathbb{T}}\left(\partial B_{\mathrm{int}}^{g}(0, \gamma) \neq \varnothing\right) \mathbb{P}_{p_{c}+\varepsilon}^{\mathbb{T}}\left(\partial B_{\mathrm{int}}(x, \gamma) \neq \varnothing\right)  \tag{5.28}\\
& \quad \leq \max _{g \subset \mathbb{B}\left(\mathbb{T}_{r}^{d}\right)} \mathbb{P}_{p_{c}+\varepsilon}^{\mathbb{T}}\left(\partial B_{\mathrm{int}}^{g}(0, \gamma) \neq \varnothing\right)^{2} .
\end{align*}
$$

Now, we have by (5.18) with $p=p_{c}$ and $q=p_{c}+\varepsilon$ and by (5.17) that

$$
\begin{equation*}
\mathbb{P}_{p_{c}+\varepsilon}^{\mathbb{T}}\left(\partial B_{\mathrm{int}}^{g}(0, \gamma) \neq \varnothing\right) \leq\left(\frac{p_{c}+\varepsilon}{p_{c}}\right)^{\gamma} \mathbb{P}_{p_{c}}^{\mathbb{T}}\left(\partial B_{\mathrm{int}}^{g}(0, \gamma) \neq \varnothing\right) \leq\left(\frac{p_{c}+\varepsilon}{p_{c}}\right)^{\gamma} \frac{1}{\gamma} \tag{5.29}
\end{equation*}
$$

and hence that

$$
\begin{equation*}
\mathbb{P}_{p_{c}+\varepsilon}^{\mathbb{T}}\left(x \in \partial B_{\text {int }}(0, \ell) \text { for some } \ell \geq 3 \gamma\right) \preceq \frac{1}{\gamma^{2}}\left(\frac{p_{c}+\varepsilon}{p_{c}}\right)^{2 \gamma} \preceq \frac{\varepsilon^{2}}{\delta^{2}} e^{2 \delta / p_{c}} . \tag{5.30}
\end{equation*}
$$

Altogether, by (5.26) and (5.30) we, therefore, have

$$
\begin{equation*}
\tau_{p}^{\mathbb{T}}(x) \leq e^{3 \delta / p_{c}} \tau_{p_{c}}(x)+C \delta^{-2} e^{3 \delta / p_{c}} V^{-2 / 3} \tag{5.31}
\end{equation*}
$$

Finally, we replace $e^{3 \delta / p_{c}}$ by $1+\delta^{\prime}$, and we may obtain in this way any $\delta^{\prime} \in(0,1]$. This gives the desired result, and the proof is complete.

Funding. This work was carried out primarily while TH was a Senior Research Associate at the University of Cambridge, during which time he was supported by ERC starting grant 804166 (SPRS).

The work of EM and GS was supported in part by NSERC of Canada.

## REFERENCES

[1] Aizenman, M. (1997). On the number of incipient spanning clusters. Nuclear Phys. B 485 551-582. MR1431856 https://doi.org/10.1016/S0550-3213(96)00626-8
[2] Aizenman, M. and Newman, C. M. (1984). Tree graph inequalities and critical behavior in percolation models. J. Stat. Phys. 36 107-143. MR0762034 https://doi.org/10.1007/BF01015729
[3] Barsky, D. J. and Aizenman, M. (1991). Percolation critical exponents under the triangle condition. Ann. Probab. 19 1520-1536. MR1127713
[4] Benjamini, I. and Schramm, O. (1996). Percolation beyond $Z^{d}$, many questions and a few answers. Electron. Commun. Probab. 1 71-82. MR1423907 https://doi.org/10.1214/ECP.v1-978
[5] Borgs, C., Chayes, J. T., van der Hofstad, R., Slade, G. and Spencer, J. (2005). Random subgraphs of finite graphs. I. The scaling window under the triangle condition. Random Structures Algorithms 27 137-184. MR2155704 https://doi.org/10.1002/rsa. 20051
[6] Borgs, C., Chayes, J. T., van der Hofstad, R., Slade, G. and Spencer, J. (2005). Random subgraphs of finite graphs. II. The lace expansion and the triangle condition. Ann. Probab. 33 1886-1944. MR2165583 https://doi.org/10.1214/009117905000000260
[7] Borgs, C., Chayes, J. T., van der Hofstad, R., Slade, G. and Spencer, J. (2006). Random subgraphs of finite graphs. III. The phase transition for the $n$-cube. Combinatorica 26 395-410. MR2260845 https://doi.org/10.1007/s00493-006-0022-1
[8] Borgs, C., Chayes, J. T., Kesten, H. and Spencer, J. (1999). Uniform boundedness of critical crossing probabilities implies hyperscaling. Random Structures Algorithms 15 368-413.
[9] Camia, F., Jiang, J. and Newman, C. M. (2021). The effect of free boundary conditions on the Ising model in high dimensions. Probab. Theory Related Fields 181 311-328. MR4341075 https://doi.org/10.1007/s00440-021-01041-9
[10] Campanino, M., Chayes, J. T. and Chayes, L. (1991). Gaussian fluctuations of connectivities in the subcritical regime of percolation. Probab. Theory Related Fields 88 269-341. MR1100895 https://doi.org/10.1007/BF01418864
[11] Chatterjee, S. and Hanson, J. (2020). Restricted percolation critical exponents in high dimensions. Comm. Pure Appl. Math. 73 2370-2429. MR4156622 https://doi.org/10.1002/cpa. 21938
[12] Chatterjee, S., Hanson, J. and Sosoe, P. (2021). Subcritical connectivity and some exact tail exponents in high dimensional percolation. Preprint. Available at https://arxiv.org/pdf/2107.14347.
[13] Chayes, J. T. and Chayes, L. (1987). On the upper critical dimension of Bernoulli percolation. Comm. Math. Phys. 113 27-48. MR0918403
[14] CHEN, L.-C. and SAKAI, A. (2015). Critical two-point functions for long-range statistical-mechanical models in high dimensions. Ann. Probab. 43 639-681. MR3306002 https://doi.org/10.1214/13-AOP843
[15] Duminil-Copin, H., Raoufi, A. and Tassion, V. (2019). Sharp phase transition for the random-cluster and Potts models via decision trees. Ann. of Math. (2) 189 75-99. MR3898174 https://doi.org/10.4007/ annals.2019.189.1.2
[16] Essam, J. W., Gaunt, D. S. and Guttmann, A. J. (1978). Percolation theory at the critical dimension. J. Phys. A: Math. Gen. 11 1983-1990.
[17] Fitzner, R. and van der Hofstad, R. (2017). Mean-field behavior for nearest-neighbor percolation in $d>$ 10. Electron. J. Probab. 22 Paper No. 43 . MR3646069 https://doi.org/10.1214/17-EJP56
[18] Grimmett, G. (1999). Percolation, 2nd ed. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences] 321. Springer, Berlin. MR1707339 https://doi.org/10.1007/ 978-3-662-03981-6
[19] HARA, T. (1990). Mean-field critical behaviour for correlation length for percolation in high dimensions. Probab. Theory Related Fields 86 337-385. MR1069285 https://doi.org/10.1007/BF01208256
[20] Hara, T. (2008). Decay of correlations in nearest-neighbor self-avoiding walk, percolation, lattice trees and animals. Ann. Probab. 36 530-593. MR2393990 https://doi.org/10.1214/009117907000000231
[21] Hara, T., van der Hofstad, R. and Slade, G. (2003). Critical two-point functions and the lace expansion for spread-out high-dimensional percolation and related models. Ann. Probab. 31 349-408. MR1959796 https://doi.org/10.1214/aop/1046294314
[22] Hara, T. and Slade, G. (1990). Mean-field critical behaviour for percolation in high dimensions. Comm. Math. Phys. 128 333-391. MR1043524
[23] Hara, T. and Slade, G. (1994). Mean-field behaviour and the lace expansion. In Probability and Phase Transition (Cambridge, 1993) (G. Grimmett, ed.). NATO Adv. Sci. Inst. Ser. C: Math. Phys. Sci. 420 87-122. Kluwer Academic, Dordrecht. MR1283177
[24] Heydenreich, M. and van der Hofstad, R. (2007). Random graph asymptotics on high-dimensional tori. Comm. Math. Phys. 270 335-358. MR2276449 https://doi.org/10.1007/s00220-006-0152-8
[25] Heydenreich, M. and Van der Hofstad, R. (2011). Random graph asymptotics on high-dimensional tori II: Volume, diameter and mixing time. Probab. Theory Related Fields 149 397-415. Correction: Probab. Theory Related Fields, 175:1183-1185, (2019). MR2776620 https://doi.org/10.1007/ s00440-009-0258-y
[26] Heydenreich, M. and van der Hofstad, R. (2017). Progress in High-Dimensional Percolation and Random Graphs. CRM Short Courses. Springer, Cham. MR3729454
[27] Heydenreich, M., van der Hofstad, R. and Sakai, A. (2008). Mean-field behavior for long- and finite range Ising model, percolation and self-avoiding walk. J. Stat. Phys. 132 1001-1049. MR2430773 https://doi.org/10.1007/s10955-008-9580-5
[28] Van der Hofstad, R. and Nachmias, A. (2017). Hypercube percolation. J. Eur. Math. Soc. (JEMS) 19 725-814. MR3612867 https://doi.org/10.4171/JEMS/679
[29] VAN DER HOFSTAD, R. and SAPOZhnikov, A. (2014). Cycle structure of percolation on high-dimensional tori. Ann. Inst. Henri Poincaré Probab. Stat. 50 999-1027. MR3224297 https://doi.org/10.1214/ 13-AIHP565
[30] Hulshof, T. and Nachmias, A. (2020). Slightly subcritical hypercube percolation. Random Structures Algorithms 56 557-593. MR4060356 https://doi.org/10.1002/rsa. 20853
[31] Hutchcroft, T. (2020). New critical exponent inequalities for percolation and the random cluster model. Probab. Math. Phys. 1 147-165. MR4408005 https://doi.org/10.2140/pmp.2020.1.147
[32] Hutchcroft, T. (2020). Locality of the critical probability for transitive graphs of exponential growth. Ann. Probab. 48 1352-1371. MR4112717 https://doi.org/10.1214/19-AOP1395
[33] Hutchcroft, T. (2022). Slightly supercritical percolation on nonamenable graphs I: The distribution of finite clusters. Proc. Lond. Math. Soc. Online first. Available at https://doi.org/10.1112/plms. 12474.
[34] Hutchcroft, T. (2022). On the derivation of mean-field percolation critical exponents from the triangle condition. J. Stat. Phys. 189 Paper No. 6. MR4462652 https://doi.org/10.1007/s10955-022-02967-7
[35] Kannan, R. and Krueger, C. K. (1996). Advanced Analysis on the Real Line. Universitext. Springer, New York. MR1390758
[36] Kozma, G. and Nachmias, A. (2009). The Alexander-Orbach conjecture holds in high dimensions. Invent. Math. 178 635-654. MR2551766 https://doi.org/10.1007/s00222-009-0208-4
[37] Kozma, G. and Nachmias, A. (2011). Arm exponents in high dimensional percolation. J. Amer. Math. Soc. 24 375-409. MR2748397 https://doi.org/10.1090/S0894-0347-2010-00684-4
[38] Lawler, G. F. and Limic, V. (2010). Random Walk: A Modern Introduction. Cambridge Studies in Advanced Mathematics 123. Cambridge Univ. Press, Cambridge. MR2677157 https://doi.org/10.1017/ CBO9780511750854
[39] Menshikov, M. V. (1986). Coincidence of critical points in percolation problems. Sov. Math., Dokl. 33 856-859.
[40] Michta, E. and Slade, G. (2021). Weakly self-avoiding walk on a high-dimensional torus. Preprint. Available at https://arxiv.org/pdf/2107.14170.
[41] Michta, E. and Slade, G. (2022). Asymptotic behaviour of the lattice Green function. ALEA Lat. Am. J. Probab. Math. Stat. 19 957-981. MR4448687 https://doi.org/10.30757/alea.v19-38
[42] O’Donnell, R., Saks, M., Schramm, O. and Servedio, R. A. (2005). Every decision tree has an influential variable. In 46th Annual IEEE Symposium on Foundations of Computer Science (FOCS'05) 31-39. IEEE, New York.
[43] Papathanakos, V. (2006). Finite-Size Effects in High-Dimensional Statistical Mechanical Systems: The Ising Model with Periodic Boundary Conditions. Ph.D. thesis, Princeton Univ.
[44] SLade, G. (2006). The Lace Expansion and Its Applications. Lecture Notes in Math. 1879. Springer, Berlin. Lectures from the 34th Summer School on Probability Theory held in Saint-Flour, July 6-24, 2004, Edited and with a foreword by Jean Picard. MR2239599
[45] SLAdE, G. (2020). The near-critical two-point function and the torus plateau for weakly self-avoiding walk in high dimensions. Preprint. Available at https://arxiv.org/pdf/2008.00080.
[46] Smirnov, S. (2001). Critical percolation in the plane: Conformal invariance, Cardy's formula, scaling limits. C. R. Acad. Sci. Paris Sér. I Math. 333 239-244. MR1851632 https://doi.org/10.1016/ S0764-4442(01)01991-7
[47] Smirnov, S. and Werner, W. (2001). Critical exponents for two-dimensional percolation. Math. Res. Lett. 8 729-744. MR1879816 https://doi.org/10.4310/MRL.2001.v8.n6.a4
[48] TASAKI, H. (1987). Hyperscaling inequalities for percolation. Comm. Math. Phys. 113 49-65. MR0918404
[49] Toulouse, G. (1974). Perspectives from the theory of phase transitions. Nuovo Cimento 23B 234-240.
[50] Vanneuville, H. (2022). Sharpness of Bernoulli percolation via couplings. Preprint. Available at https: //arxiv.org/pdf/2201.08223.pdf.
[51] Zhou, Z., Grimm, J., Deng, Y. and Garoni, T. M. (2020). Random-length random walks and finite-size scaling on high-dimensional hypercubic lattices I: Periodic boundary conditions. Preprint. Available at https://arxiv.org/pdf/2008.00913.
[52] Zhou, Z., Grimm, J., Fang, S., Deng, Y. and Garoni, T. M. (2018). Random-length random walks and finite-size scaling in high dimensions. Phys. Rev. Lett. 121185701.


[^0]:    Received July 2021; revised June 2022.
    MSC2020 subject classifications. Primary 60K35, 82B43; secondary 05C80, 82B27.
    Key words and phrases. Percolation, lace expansion, two-point function, one-arm exponent, triangle condition, torus plateau.

