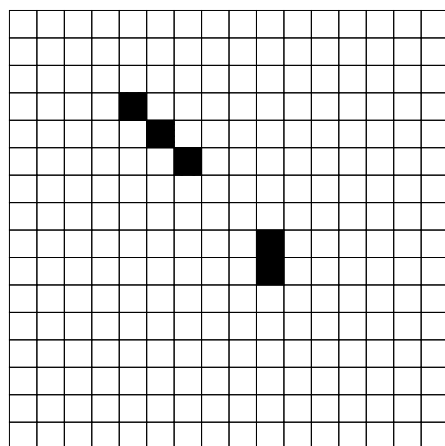


Roland Bauerschmidt, David C. Brydges, and
Gordon Slade

Introduction to a renormalisation group method

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Preface

This book provides an introduction to a mathematically rigorous renormalisation group method which is inspired by Kenneth Wilson's original ideas from the early 1970s, for which he was awarded the 1982 Nobel Prize in Physics. The method has been developed and applied over the past ten years in a series of papers authored by various subsets of the present authors, along with Martin Lohmann, Alexandre Tomberg and Benjamin Wallace.

We present the general setting of the problems in critical phenomena that have been addressed by the method, with focus on the 4-dimensional $|\phi|^4$ spin system and the 4-dimensional continuous-time weakly self-avoiding walk. We give a self-contained analysis of the 4-dimensional *hierarchical* $|\phi|^4$ model, which is simpler than its Euclidean counterpart but still reveals many of the ideas and techniques of the renormalisation group method. We comment on, and give detailed references for, the extension of the method to the Euclidean setting in Appendix A. The book is intended to be a starting point for a reader who may not have prior knowledge of the renormalisation group method.

The book originated from lecture notes that were prepared for courses at several summer schools. Subsequently the lecture notes were significantly developed and rewritten. The courses were given at:

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Cambridge, UK
Damariscotta, ME
Vancouver, BC

Roland Bauerschmidt
David C. Brydges
Gordon Slade

June 28, 2019

Roland Bauerschmidt
Department of Pure Mathematics and Mathematical Statistics
University of Cambridge
Centre for Mathematical Sciences
Wilberforce Road
Cambridge, CB3 0WB, UK
rb812@cam.ac.uk

David C. Brydges
Department of Mathematics
University of British Columbia
Vancouver, BC, Canada V6T 1Z2
db5d@math.ubc.ca

Gordon Slade
Department of Mathematics
University of British Columbia
Vancouver, BC, Canada V6T 1Z2
slade@math.ubc.ca

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Notation

Throughout this book, we use the following notational conventions.

- $x = o(y)$ means that $x/y \rightarrow 0$ as $y \rightarrow y_0$, where y_0 is supplied by the context.
- $x = O(y)$ means that there exist C, δ such that $|x/y| \leq C$ for $|y - y_0| < \delta$, where y_0 is supplied by the context.
- $x = O_z(y)$ means that $x = O(y)$ as $y \rightarrow y_0$ with z fixed, where y_0 is supplied by the context.
- $A \sim B$ means $A = B(1 + o(1))$.
- $A \asymp B$ means $C^{-1}A \leq B \leq CA$ for a universal constant $C > 0$.
- $A \propto B$ means $A = cB$ for some constant $c > 0$ (which can depend on parameters).
- For $x = (x_i)_{i \in I}$ and $y = (y_i)_{i \in I}$ we write $(x, y) = \sum_{i \in I} x_i y_i$, where the index set I is supplied by the context.

Some commonly used symbols are listed in the index.

Part I

Spin systems and critical phenomena

Chapter 1

Spin systems

1.1 Critical phenomena and the renormalisation group

The subject of critical phenomena and phase transitions has fascinated mathematicians for over half a century. Interest in these topics is now as great as ever, and models such as percolation, the Ising model, self-avoiding walk, dimer systems, and others, are prominent in mathematical physics, in probability theory, and in combinatorics. The physically relevant and mathematically most interesting aspects of the subject centre on universal quantities such as critical exponents. These exponents describe the large-scale behaviour of a system of strongly dependent random variables as a parameter governing the strength of dependence, such as temperature, varies near a critical value at which long-range correlations suddenly appear. The critical exponents are independent of many details of how a model is defined, and for this reason models which are crude in their treatment of local interactions can nevertheless provide accurate information about the large-scale behaviour of real physical systems.

An extensive but incomplete mathematical theory of 2-dimensional critical phenomena has been obtained in recent decades, particularly with the advent of the Schramm-Loewner Evolution at the turn of the century. In high dimensions, namely dimensions $d > 4$ for spin systems and self-avoiding walk, there is a well-developed theory of mean-field behaviour, based on techniques including reflection positivity, differential inequalities, and the lace expansion. The physically most relevant dimension, $d = 3$, has proved intractable to date and remains an outstanding challenge to mathematicians.

The upper critical dimension, $d = 4$, is borderline in the sense that mean-field theory applies in dimensions $d > 4$, but not $d < 4$, and typically this borderline behaviour involves logarithmic corrections to mean-field scaling. Dimension 4 is also the reference for the ε -expansion, which has provided heuristic results in dimension 3 by viewing $d = 3$ as $d = 4 - \varepsilon$ with $\varepsilon = 1$. This book concerns a method for analysing 4-dimensional critical phenomena and proving existence of logarithmic

corrections to scaling. The method has also been applied to lower dimensions via a version of the ε -expansion for long-range models.

In the physics literature, critical phenomena are understood via the renormalisation group method developed by Kenneth G. Wilson in the early 1970s. Wilson received the 1982 Nobel Prize in Physics for this development. Inspiring early references include [85, 156]. Although Wilson’s renormalisation group method is now part of the standard toolbox of theoretical physics, there remain serious challenges to place it on a firm mathematical and non-perturbative foundation. This book presents a renormalisation group method, developed by the authors, which is applicable to the 4-dimensional n -component $|\phi|^4$ spin system and to the 4-dimensional continuous-time weakly self-avoiding walk. The latter is treated rigorously as a supersymmetric “ $n = 0$ ” version of the former. To simplify the setting, we present the method in the context of the 4-dimensional n -component hierarchical $|\phi|^4$ model. Discussion of the self-avoiding walk is deferred to Chapter 11.

Extensions of the methods used in this book can be found in [18–20, 25, 26, 118, 143, 144] (for $n \geq 0$). Alternate approaches to the 4-dimensional $|\phi|^4$ model using block spin renormalisation can be found in [95, 96, 104, 108] (for $n = 1$), and using phase space expansion methods in [82] (for $n = 1$). We make no attempt to provide a thorough review of the many ways in which renormalisation group methods have been applied in mathematical physics. The low-temperature phase has been studied, e.g., in [14, 16]. Renormalisation group methods have recently been applied to gradient field models in [5], to the Coulomb gas in [79, 80], to interacting dimers in [100], and to symmetry breaking in low temperature many-boson systems in [15]. The books [32, 122, 133, 139] provide different approaches to the renormalisation group, and [101] contains useful background.

Two paramount features of critical phenomena are scale invariance and universality. The renormalisation group method exploits the scale invariance to explain universality. This is done via a multi-scale analysis, in which a system studied at a particular scale is represented by an effective Hamiltonian. Scales are analysed sequentially, leading to a map that takes the Hamiltonian at one scale to a Hamiltonian at the next scale. Advancing the scale gives rise to a dynamical system defined by this map. Scale invariance occurs at a fixed point of the map, and different fixed points correspond to different universality classes. The analysis of the dynamical system at and near the fixed point provides a means to compute universal quantities such as critical exponents. In the physics literature, the analysis is typically performed in a perturbative fashion, without control of remainder terms. A mathematically rigorous treatment requires full control of nonperturbative aspects as well.

This book presents a self-contained and complete renormalisation group analysis of the 4-dimensional n -component hierarchical $|\phi|^4$ model. We have set up the analysis in a fashion parallel to that of its Euclidean counterpart in [18, 20]; the Euclidean version involves additional ingredients which make its analysis more involved. In Appendix A, we indicate the main differences and provide references for the Euclidean analysis.

A *spin system* is a collection of random variables, called spins, which we denote $(\varphi_x)_{x \in \Lambda}$ or $(\sigma_x)_{x \in \Lambda}$. In the examples we discuss, the spins are vectors in $\mathbb{R}^n$. The spins are indexed by a set Λ , which we initially assume to be finite, but large, and ultimately we are interested in the infinite volume limit $\Lambda \uparrow \mathbb{Z}^d$. The distribution on spin configurations is specified in terms of an energy $H(\varphi)$ or $H(\sigma)$. We discuss four examples of spin systems in this chapter: the Ising model, the mean-field model, the Gaussian free field, and the $|\varphi|^4$ model.

1.2 Ising model

The prototypical example of a spin system is the Ising model, which is defined as follows. Given a finite box $\Lambda \subset \mathbb{Z}^d$, an Ising configuration is $\sigma = (\sigma_x)_{x \in \Lambda}$, $\sigma_x \in \{-1, 1\}$, as depicted in Figure 1.1. With e one of the $2d$ unit vectors in $\mathbb{Z}^d$, we define the discrete gradient and Laplacian of a function $f : \mathbb{Z}^d \rightarrow \mathbb{C}$ by

$$(\nabla^e f)_x = f_{x+e} - f_x, \quad (\Delta f)_x = -\frac{1}{2} \sum_{e:|e|=1} \nabla^{-e} \nabla^e f_x = \sum_{e:|e|=1} \nabla^e f_x. \quad (1.2.1)$$

An energy is associated to each configuration σ by

$$H_{0,\Lambda}(\sigma) = \frac{1}{4} \sum_{e:|e|=1} \sum_{x \in \Lambda} (\nabla^e \sigma)_x^2, \quad (1.2.2)$$

together with a boundary contribution fixing the spins on the outer boundary of Λ . Let $E^{(2)}$ be the set of edges $\{x, y\}$ where x, y are nearest neighbour lattice sites. The energy (1.2.2) is twice the number of edges in $E^{(2)}$ whose spins disagree. Up to an additive constant, it can also be written as $-\sum_{\{x,y\} \in E^{(2)}} \sigma_x \sigma_y$.

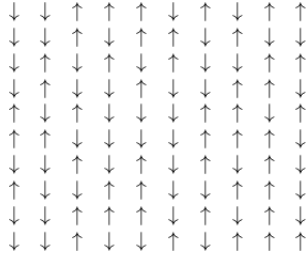
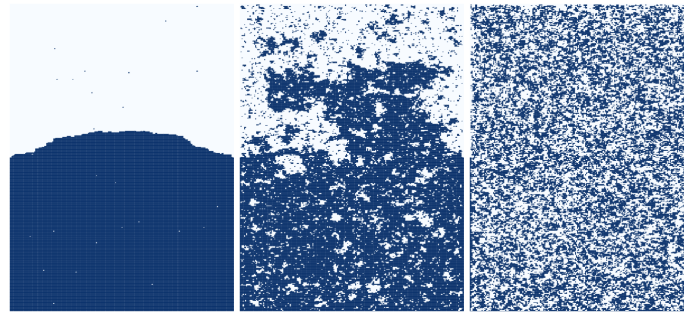


Fig. 1.1 A configuration of the Ising model.

The probability of a configuration σ is given by the finite-volume Gibbs measure

$$P_{T,\Lambda}(\sigma) \propto e^{-H_{0,\Lambda}(\sigma)/T} \prod_{x \in \Lambda} (\delta_{\sigma_x, +1} + \delta_{\sigma_x, -1}), \quad (1.2.3)$$

where T represents *temperature*, and where the constant of proportionality is such that $P_{T,\Lambda}$ is a probability measure. The interaction is *ferromagnetic*: configurations with more neighbouring spins aligned are energetically favourable (lower energy) and have higher probability. The configurations with all spins $+1$ or all spins -1 have the lowest energy. For higher energies there is a larger number of configurations realising that energy, leading to a greater weight—or *entropy*—of these in the probability measure. The competition of energy and entropy, whose relative weight is controlled by the temperature, leads to a phase transition at a critical temperature T_c . For $T < T_c$, the dominant mechanism is the minimising of energy, while for $T > T_c$, it is the effect of entropy that dominates. Typical configurations look dramatically different depending on whether T is below, at, or above the critical temperature T_c ; see Figure 1.2.



Low temperature $T < T_c$ Critical temperature $T = T_c$ High temperature $T > T_c$

Fig. 1.2 Typical configurations of the 2-dimensional Ising model, with boundary spins fixed white for the top half and dark for the bottom half.

To model the effect of an *external magnetic field* $h \in \mathbb{R}$, the Hamiltonian becomes

$$H_{h,\Lambda}(\sigma) = H_{0,\Lambda}(\sigma) - h \sum_{x \in \Lambda} \sigma_x = \frac{1}{4} \sum_e \sum_{x \in \Lambda} (\nabla^e \sigma)_x^2 - h \sum_{x \in \Lambda} \sigma_x. \quad (1.2.4)$$

Associated to this Hamiltonian, there is again a finite-volume Gibbs measure with $H_{0,\Lambda}$ replaced by $H_{h,\Lambda}$ in (1.2.3). The infinite-volume *Gibbs measure* $P_{h,T}$ is defined to be the limit of the measures $P_{h,T,\Lambda}$ as $\Lambda \uparrow \mathbb{Z}^d$. There is work to do to show existence of the limit, which may depend on boundary conditions and fail to be unique. Expectation with respect to $P_{h,T}$ is denoted $\langle \cdot \rangle_{h,T}$. See, e.g., [88, 99, 141] for details about Gibbs measures.

The *magnetisation* is defined by $M(h, T) = \langle \sigma_0 \rangle_{h,T}$, and the *spontaneous magnetisation* is $M_+(T) = \lim_{h \downarrow 0} M(h, T)$. The phase transition for the Ising model is illustrated in Figure 1.3. Above the critical temperature T_c , the spontaneous magnetisation is zero, whereas below T_c it is positive. The slope of the magnetisation $M(h, T)$ at $h = 0$ is called the *magnetic susceptibility*; it diverges as $T \downarrow T_c$. More

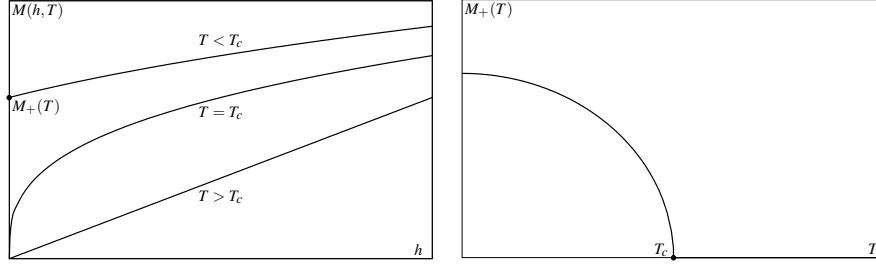


Fig. 1.3 Critical behaviour of the magnetisation.

precisely, for $T \geq T_c$, we define:

$$\text{two-point function: } \tau_{0x}(T) = \langle \sigma_0 \sigma_x \rangle_{0,T}, \quad (1.2.5)$$

$$\text{correlation length: } \xi(T)^{-1} = - \lim_{n \rightarrow \infty} n^{-1} \log \tau_{0,ne_1}(T), \quad (1.2.6)$$

$$\text{susceptibility: } \chi(T) = \sum_{x \in \mathbb{Z}^d} \tau_{0x}(T) = \left. \frac{\partial}{\partial h} M(h, T) \right|_{h=0}. \quad (1.2.7)$$

In (1.2.6), $e_1 = (1, 0, \dots, 0)$ is a unit vector in $\mathbb{Z}^d$. The most subtle and interesting behaviour occurs at and near the phase transition, where the spins develop strong and non-trivial correlations. The scaling of these can be described in terms of various critical exponents, as follows:

$$\chi(T) \sim A_1 (T - T_c)^{-\gamma} \quad (T \downarrow T_c), \quad (1.2.8)$$

$$\xi(T) \sim A_2 (T - T_c)^{-\nu} \quad (T \downarrow T_c), \quad (1.2.9)$$

$$\tau_{0x}(T_c) \sim A_3 |x|^{-(d-2+\eta)} \quad (|x| \rightarrow \infty), \quad (1.2.10)$$

$$M(h, T_c) \sim A_4 h^{1/\delta} \quad (h \downarrow 0), \quad (1.2.11)$$

$$M_+(T) \sim A_5 (T_c - T)^\beta \quad (T \uparrow T_c). \quad (1.2.12)$$

The critical exponents are conjectured to obey certain *scaling relations*, an example of which is *Fisher's relation* $\gamma = (2 - \eta)\nu$. The critical exponents are predicted to be universal. This means that they should depend primarily on the dimension d and not on fine details of how the model is formulated. For example, the exponents are predicted to be the same on the square or triangular or hexagonal lattices for $d = 2$. The main mathematical problem for the Ising model, and for spin systems more generally, is to provide rigorous proof of the existence and universality of the critical exponents. The following is an informal summary of what has been achieved so far.

There has been great success for the case of $d = 2$. For the square lattice $\mathbb{Z}^2$, it has been proved that the critical temperature is given by $T_c^{-1} = \frac{1}{2} \log(1 + \sqrt{2})$, and that the critical exponents $\gamma, \beta, \delta, \eta, \nu$ exist and take the values $\gamma = \frac{7}{4}$, $\beta = \frac{1}{8}$, $\delta = 15$,

$\eta = \frac{1}{4}$, $\nu = 1$. In addition, the law of the interface curve in the middle picture in Figure 1.2 is the Schramm–Loewner Evolution SLE_3 . References for these theorems include [29, 60, 61, 129].

In dimensions $d > 4$, also much is known. The critical exponents $\gamma, \beta, \delta, \eta$ exist and take the values $\gamma = 1$, $\beta = \frac{1}{2}$, $\delta = 3$, $\eta = 0$. These exponents have the same values as for the Ising model defined on the complete graph, which is called the *Curie–Weiss* or *mean-field* Ising model. Precise statements and proofs of these facts can be found in [7, 9, 89, 137]. We discuss the mean-field Ising model in more detail in Section 1.4.

Logarithmic corrections to mean-field behaviour are predicted for $d = 4$ [37, 113, 151], and it is known that there cannot be corrections which are larger than logarithmic [9, 10]. It remains an open problem to prove the precise behaviour for $d = 4$, and in this book we address some closely related problems concerning the $|\varphi|^4$ model. For the hierarchical Ising model in dimension 4, a rigorous renormalisation group analysis is presented in [106].

Only recently has it been proved that the spontaneous magnetisation vanishes at the critical temperature for $\mathbb{Z}^3$ [8]. It remains a major open problem to prove the existence of critical exponents for $d = 3$. In the physics literature, the conformal bootstrap has been used to compute exponents to high accuracy [77].

1.3 Spin systems and universality

The Ising model is only one example of a large class of spin systems. A general class of $O(n)$ -symmetric ferromagnetic spin models can be defined as follows.

Let Λ be a finite set, and let $\beta_{xy} = \beta_{yx}$ be nonnegative spin-spin *coupling constants* indexed by $\Lambda \times \Lambda$. A spin configuration consists of a spin $\varphi_x \in \mathbb{R}^n$ for each $x \in \Lambda$, and can be considered either as a map $\varphi : \Lambda \rightarrow \mathbb{R}^n$ or as an element $\varphi \in \mathbb{R}^{n\Lambda}$. The *bulk energy* of the spin configuration φ is

$$H(\varphi) = \frac{1}{4} \sum_{x, y \in \Lambda} \beta_{xy} |\varphi_x - \varphi_y|^2 + \sum_{x \in \Lambda} h \cdot \varphi_x. \quad (1.3.1)$$

The constant vector h represents an external magnetic field, which may be zero. For a given reference measure μ on $\mathbb{R}^n$ called the *single-spin distribution*, a probability measure on spin configurations is defined by the expectation

$$\langle F \rangle \propto \int_{\mathbb{R}^{n\Lambda}} F(\varphi) e^{-H(\varphi)} \prod_{x \in \Lambda} \mu(d\varphi_x). \quad (1.3.2)$$

The assumption $\beta_{xy} \geq 0$ is the assumption that the model is *ferromagnetic*: it encourages spin alignment. When μ is absolutely continuous it is usually convenient to instead take μ equal to the Lebesgue measure and equivalently add a potential to the energy, i.e.,

$$H(\varphi) = \frac{1}{4} \sum_{x,y \in \Lambda} \beta_{xy} |\varphi_x - \varphi_y|^2 + \sum_{x \in \Lambda} h \cdot \varphi_x + \sum_{x \in \Lambda} w(\varphi_x). \quad (1.3.3)$$

We associate to β the Laplacian matrix Δ_β , which acts on scalar fields $f : \Lambda \rightarrow \mathbb{R}$ by

$$(\Delta_\beta f)_x = \sum_{y \in \Lambda} \beta_{xy} (f_y - f_x). \quad (1.3.4)$$

For the case where $\beta_{xy} = \mathbb{1}_{x \sim y}$ is the indicator that x and y are nearest neighbours in $\mathbb{Z}^d$, this recovers the standard Laplacian of (1.2.1). For vector-valued fields $f = (f^1, \dots, f^n)$ the Laplacian acts component-wise, i.e., $(\Delta_\beta f)^i = \Delta_\beta f^i$. Then we can rewrite $H(\varphi)$ as

$$H(\varphi) = \frac{1}{2} \sum_{x \in \Lambda} \varphi_x \cdot (-\Delta_\beta) \varphi_x + \sum_{x \in \Lambda} h \cdot \varphi_x + \sum_{x \in \Lambda} w(\varphi_x). \quad (1.3.5)$$

Boundary terms can be included in the energy as well.

Examples are given by the following choices of μ and w . Since μ and w provide redundant freedom in the specification of the model, we either specify μ and then assume that $w = 0$, or we specify w and then assume that μ is the Lebesgue measure.

- Ising model: $n = 1$ and $\mu = \delta_{+1} + \delta_{-1}$.
- $O(n)$ model: μ is the uniform measure on $S^{n-1} \subset \mathbb{R}^n$.
- Gaussian free field (GFF): $w(\varphi_x) = m^2 |\varphi_x|^2$ with $m^2 \geq 0$.
- $|\varphi|^4$ model: $w(\varphi_x) = \frac{1}{4} g |\varphi_x|^4 + \frac{1}{2} v |\varphi_x|^2$ with $g > 0$ and $v \in \mathbb{R}$.

The $O(n)$ model is the Ising model when $n = 1$, and it is also called the rotator model for $n = 2$, and the classical Heisenberg model for $n = 3$.

Examples for the choice of interaction β are:

- Mean-field interaction: $\beta_{xy} = \beta / |\Lambda|$ for all $x, y \in \Lambda$.
- Nearest-neighbour interaction: $\Lambda \subset \mathbb{Z}^d$ and $\beta_{xy} = \beta \mathbb{1}_{x \sim y}$.
- Finite-range interaction: $\Lambda \subset \mathbb{Z}^d$ and $\beta_{xy} = \beta \mathbb{1}_{|x-y| \leq R}$ for some $R \geq 1$.
- Long-range interaction: $\Lambda \subset \mathbb{Z}^d$ and $\beta_{xy} \asymp |x-y|^{-(d+\alpha)}$ for some $\alpha \in (0, 2)$.
- Hierarchical interaction: discussed in detail in Chapter 4.

In appropriate limits $|\Lambda| \rightarrow \infty$, the above models typically undergo phase transitions as their respective parameters are varied. As in the example of the Ising model, the critical behaviour can be described by critical exponents. The *universality conjecture* for critical phenomena asserts that the critical behaviour of spin models is the same within very general symmetry classes.

The symmetry class is determined by the number of components n , corresponding to the symmetry group $O(n)$, and the class of coupling constants. For example, in $\mathbb{Z}^d$, the same critical behaviour is predicted when the spin-spin coupling β has any finite range, or bounded variance $\sum_{x \in \mathbb{Z}^d} |x|^2 \beta_{0x}$ (in infinite volume), as long as μ or w has appropriate regularity and growth properties. Also, the same critical behaviour is predicted for the $O(n)$ and $|\varphi|^4$ models. A general proof of the universality conjecture is one of the major open problems of statistical mechanics.

In the remainder of this chapter, we consider three of the above examples: the mean-field model, the Gaussian free field, and the $|\varphi|^4$ model. For both the mean-field model and the Gaussian free field, a complete analysis can be carried out. We present specific instructive cases that illustrate the general phenomena. The $|\varphi|^4$ model is a *generic* case, on which much of the remainder of this book is focussed.

1.4 Mean-field model

1.4.1 Critical behaviour of the mean-field model

Let $n \geq 1$ be an integer, and let $\Lambda = \{0, 1, \dots, N-1\}$ be a finite set. As mentioned in the previous section, the mean-field model corresponds to the choice $\beta_{xy} = \beta/N$ for the coupling constants. With this choice, the Laplacian of (1.3.4) is given by

$$-\Delta_\beta = \beta P \quad \text{with} \quad P = \text{Id} - Q, \quad (1.4.1)$$

where Id denotes the $N \times N$ identity matrix and Q is the constant matrix with entries $Q_{xy} = N^{-1}$. Note that P and Q are orthogonal projections with $P + Q = \text{Id}$. The energy of the mean-field $O(n)$ model is then given by

$$H(\sigma) = \frac{1}{2} \sum_{x \in \Lambda} \sigma_x \cdot (-\Delta_\beta \sigma)_x + \sum_{x \in \Lambda} h \cdot \sigma_x. \quad (1.4.2)$$

The finite-volume expectation is defined by

$$\langle F \rangle_{\beta, h, N} \propto \int_{(S^{n-1})^N} F(\sigma) e^{-H(\sigma)} \prod_{x \in \Lambda} \mu(d\sigma_x), \quad (1.4.3)$$

where the single-spin distribution μ is the uniform measure on the sphere $S^{n-1} \subset \mathbb{R}^n$. In particular, for $n = 1$, the sphere S^{n-1} is the set $\{-1, +1\}$ and we have the mean-field Ising model, or Curie–Weiss model. In terms of the temperature variable T used in our discussion of the Ising model in Section 1.2, here β is the inverse temperature $\beta = 1/T$.

The mean-field Ising model is a canonical example which is discussed in many books on statistical mechanics, including [28, 78, 88]. It is important for various reasons: it is an example where nontrivial critical behaviour can be worked out exactly and completely including computation of critical exponents, its critical exponents have been proven to give bounds on the critical exponents of other models, and its critical exponents are proven or predicted to give the same values as other models in dimensions $d > 4$.

What makes the mean-field model more tractable is its lack of geometry. Apart from an unimportant volume-dependent constant that is independent of the spin configuration, the energy can be rewritten in terms of the mean spin $\bar{\sigma} = N^{-1} \sum_x \sigma_x$

as

$$H(\sigma) = -\frac{1}{2} \frac{\beta}{N} \sum_{x,y} \sigma_x \cdot \sigma_y + \sum_x h \cdot \sigma_x + \text{const} = N \left(-\frac{1}{2} \beta \bar{\sigma} \cdot \bar{\sigma} + h \cdot \bar{\sigma} \right) + \text{const}. \quad (1.4.4)$$

Thus H is actually a function only of the mean spin. This is the origin of the name “mean-field” model.

The susceptibility and magnetisation are defined by

$$M(\beta, h) = \lim_{N \rightarrow \infty} \langle \sigma_0 \rangle_{\beta, h, N}, \quad (1.4.5)$$

$$\chi(\beta, h) = \frac{\partial M}{\partial h}(\beta, h). \quad (1.4.6)$$

For the results we focus on the Ising case $n = 1$, but we present the set-up for the general $O(n)$ model. We will prove the following theorem, which shows that the critical exponents γ, δ, β (for the susceptibility, the vanishing of the magnetisation at the critical point, and the spontaneous magnetisation) take the *mean-field values* $\gamma = 1$, $\delta = 3$, $\bar{\beta} = \frac{1}{2}$. We have written $\bar{\beta}$ for the critical exponent of the spontaneous magnetisation rather than β as in (1.2.12), since here β represents the inverse temperature. The theorem also shows that the critical value of β is $\beta_c = 1$.

Theorem 1.4.1. *Let $\beta_c = 1$.*

(i) *The spontaneous magnetisation obeys*

$$M_+(\beta) \begin{cases} > 0 & (\beta > \beta_c) \\ = 0 & (\beta \leq \beta_c), \end{cases} \quad (1.4.7)$$

and

$$M_+(\beta) \sim (3(\beta - \beta_c))^{1/2} \quad (\beta \downarrow \beta_c). \quad (1.4.8)$$

(ii) *The magnetisation obeys*

$$M(\beta_c, h) \sim (3h)^{1/3} \quad (h \downarrow 0). \quad (1.4.9)$$

(iii) *The susceptibility is finite for $\beta < \beta_c$ for any h , and also for $\beta > \beta_c$ if $h \neq 0$, and*

$$\chi(\beta, 0) = \frac{1}{\beta_c - \beta} \quad (\beta < \beta_c), \quad \chi(\beta, 0_+) \sim \frac{1}{2(\beta - \beta_c)} \quad (\beta \downarrow \beta_c). \quad (1.4.10)$$

1.4.2 Renormalised measure

We start with the following elementary lemma.

Lemma 1.4.2. *Let $\Delta_\beta = -\beta P$ be the mean-field Laplacian. There is a constant $c > 0$ such that*

$$e^{-\frac{1}{2}(\sigma, -\Delta_\beta \sigma)} = c \int_{\mathbb{R}^n} e^{-\frac{\beta}{2}(\varphi - \sigma, \varphi - \sigma)} d\varphi \quad (\sigma \in (\mathbb{R}^n)^N), \quad (1.4.11)$$

where we identify $\varphi \in \mathbb{R}^n$ as a constant vector $(\varphi, \dots, \varphi) \in (\mathbb{R}^n)^N$, and the parentheses denote the inner product on $(\mathbb{R}^n)^N$.

Proof. Let $\bar{\sigma} = N^{-1} \sum_x \sigma_x$ denote the average spin. We can regard both $\bar{\sigma}$ and φ as constant vectors in $(\mathbb{R}^n)^N$. By the discussion around (1.4.1), $Q\sigma = \bar{\sigma}$, and $P = \text{Id} - Q$ projects onto the orthogonal complement of the subspace of constant fields. Therefore,

$$\begin{aligned} (\varphi - \sigma, \varphi - \sigma) &= (\varphi - \sigma, Q(\varphi - \sigma)) + (\varphi - \sigma, P(\varphi - \sigma)) \\ &= N|\varphi - \bar{\sigma}|^2 + (\sigma, P\sigma). \end{aligned} \quad (1.4.12)$$

We take the exponential $\exp(-\frac{1}{2}\beta(\cdot))$ of both sides and integrate over $\varphi \in \mathbb{R}^n$. The term involving $(\sigma, P\sigma)$ factors out of the integral and gives the desired left-hand side of (1.4.11), and the remaining integral is seen to be independent of σ after making the change of variables $\varphi \mapsto \varphi + \bar{\sigma}$. ■

The identity (1.4.11) allows us to decompose the measure of the mean-field model ν on $(S^{n-1})^N$ into two measures, which we call the renormalised measure and the fluctuation measure.

The *renormalised measure* ν_r is a measure on $\mathbb{R}^n$ defined as follows. For $\varphi \in \mathbb{R}^n$, we define the *renormalised potential* by

$$V(\varphi) = -\log \int_{S^{n-1}} e^{-\frac{\beta}{2}(\varphi - \sigma) \cdot (\varphi - \sigma) + h \cdot \sigma} \mu(d\sigma). \quad (1.4.13)$$

The renormalised measure is then defined by the expectation

$$\mathbb{E}_{\nu_r}(G) \propto \int_{\mathbb{R}^n} G(\varphi) e^{-NV(\varphi)} d\varphi. \quad (1.4.14)$$

The *fluctuation measure* is a measure on $(S^{n-1})^N$ but of simpler form than the original $O(n)$ measure. It is a product measure that depends on the renormalised field $\varphi \in \mathbb{R}^n$, and is defined by

$$\mathbb{E}_{\mu_\varphi}(F) = \frac{1}{e^{-NV(\varphi)}} \int_{(S^{n-1})^N} F(\sigma) \prod_{x \in \Lambda} e^{-\frac{\beta}{2}(\varphi - \sigma_x) \cdot (\varphi - \sigma_x) + h \cdot \sigma_x} \mu(d\sigma_x). \quad (1.4.15)$$

Lemma 1.4.3. *The mean-field measure (1.4.3) has the decomposition*

$$\langle F \rangle_{\beta, h, N} = \mathbb{E}_{\nu_r}(\mathbb{E}_{\mu_\varphi}(F)) \quad \text{for } F : (S^{n-1})^N \rightarrow \mathbb{R}. \quad (1.4.16)$$

Proof. The proof is just a matter of substituting in definitions and using (1.4.11):

$$\begin{aligned}
\langle F \rangle_{\beta, h, N} &\propto \int_{(S^{n-1})^N} F(\sigma) e^{-\frac{1}{2}(\sigma, (-\Delta_\beta)\sigma) + (h, \sigma)} \prod_{x \in \Lambda} \mu(d\sigma_x) \\
&\propto \int_{\mathbb{R}^n} \int_{(S^{n-1})^N} F(\sigma) \prod_{x \in \Lambda} e^{-\frac{\beta}{2}(\varphi - \sigma_x) \cdot (\varphi - \sigma_x) + h \cdot \sigma_x} \mu(d\sigma_x) d\varphi \\
&= \int_{\mathbb{R}^n} e^{-NV(\varphi)} \mathbb{E}_{\mu_\varphi}(F) d\varphi \\
&\propto \mathbb{E}_{V_r}(\mathbb{E}_{\mu_\varphi}(F)).
\end{aligned} \tag{1.4.17}$$

Since $\mathbb{E}_V(1) = 1 = \mathbb{E}_{V_r}(\mathbb{E}_{\mu_\varphi}(1))$, the proportional relation becomes an identity. ■

The above decomposition of the measure into a fluctuation measure and a renormalised measure can be seen as a toy example of the idea of renormalisation. This is further discussed in Example 2.1.12.

1.4.3 Magnetisation and susceptibility: Proof of Theorem 1.4.1

To compute the magnetisation, we need the observable $F(\sigma) = \sigma_0$. Let

$$G(\varphi) = \mathbb{E}_{\mu_\varphi}(\sigma_0) = \frac{1}{e^{-V(\varphi)}} \int_{S^{n-1}} \sigma_0 e^{-\frac{\beta}{2}(\varphi - \sigma_0) \cdot (\varphi - \sigma_0) + h \cdot \sigma_0} \mu(d\sigma_0). \tag{1.4.18}$$

Then (1.4.16) and (1.4.14) imply that

$$\langle \sigma_0 \rangle_{\beta, h, N} = \mathbb{E}_{V_r}(G(\varphi)) = \frac{\int_{\mathbb{R}^n} G(\varphi) e^{-NV(\varphi)} d\varphi}{\int_{\mathbb{R}^n} e^{-NV(\varphi)} d\varphi}. \tag{1.4.19}$$

The right-hand side is a finite-dimensional integral, with dimension n independent of the number of vertices N . Therefore Laplace's Principle can be applied to study the limit as $N \rightarrow \infty$. The following exercise is an instance of Laplace's Principle; for much more on this kind of result see [158].

Theorem 1.4.4. *Let $V : \mathbb{R}^n \rightarrow \mathbb{R}$ be continuous with unique global minimum at $\varphi_0 \in \mathbb{R}^n$. Assume that $\int_{\mathbb{R}^n} e^{-V} d\varphi$ is finite and that $\{\varphi \in \mathbb{R}^n : V(\varphi) \leq V(\varphi_0) + 1\}$ is compact. Then for any bounded continuous function $g : \mathbb{R}^n \rightarrow \mathbb{R}$,*

$$\lim_{N \rightarrow \infty} \frac{\int_{\mathbb{R}^n} g(\varphi) e^{-NV(\varphi)} d\varphi}{\int_{\mathbb{R}^n} e^{-NV(\varphi)} d\varphi} = g(\varphi_0). \tag{1.4.20}$$

Exercise 1.4.5. Prove Theorem 1.4.4. [Solution]

Let $G(\varphi) = \mathbb{E}_{\mu_\varphi}(\sigma_0)$ be as above. The critical points φ of the renormalised potential V satisfy

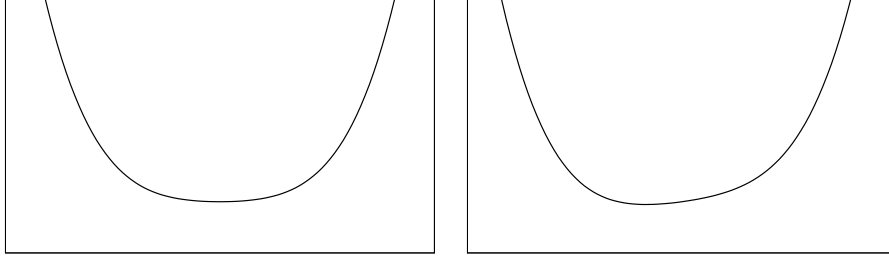


Fig. 1.4 The renormalised potential for $\beta < \beta_c$ with $h = 0$ (left) and $h \neq 0$ (right). The renormalised potential is convex and the minimum is assumed at a unique point in both cases.

$$0 = \nabla V(\varphi) = \beta(\varphi - G(\varphi)), \quad \text{i.e., } \varphi = G(\varphi). \quad (1.4.21)$$

The following lemma gives properties of V for the case $n = 1$. See Figure 1.4 for part (ii) and Figure 1.5 for part (iii).

Lemma 1.4.6. *Let $n = 1$ and set $\beta_c = n = 1$. Then the renormalised potential V and the function G are given by*

$$V(\varphi) = \frac{\beta}{2} \varphi^2 - \log \cosh(\beta \varphi + h) + \text{const}, \quad G(\varphi) = -\frac{\partial V}{\partial h} = \tanh(\beta \varphi + h). \quad (1.4.22)$$

As a consequence:

- (i) For $h \neq 0$, V has a unique minimum $\varphi_0(\beta, h)$ with the same sign as h .
- (ii) For $\beta \leq \beta_c$, V is convex, the unique minimum of V tends to 0 as $h \rightarrow 0$, and $V''(\varphi) \geq \beta(1 - \beta/\beta_c)$ for any $h \in \mathbb{R}$.
- (iii) For $\beta > \beta_c$, V is non-convex, the minima of V are $\pm r$ for some $r = r(\beta) > 0$ if $h = 0$, and as $h \downarrow 0$ the unique minimum converges to $+r$ or $-r$.
- (iv) The minimum $\varphi_0(\beta, h)$ is differentiable in h whenever $h \neq 0$ or $\beta < \beta_c$.

Proof. This is a direct computation. Note that when $n = 1$ the integrals in (1.4.13) and (1.4.18) are just sums over two terms $\sigma = \pm 1$, each with measure $\frac{1}{2}$. ■

Proof of Theorem 1.4.1. For $h \neq 0$ or $\beta \leq \beta_c$, denote by $\varphi_0(\beta, h)$ the unique minimum of V . By Theorem 1.4.4 and (1.4.21), the magnetisation is given by

$$M(\beta, h) = \lim_{N \rightarrow \infty} \langle \sigma_0 \rangle_{\beta, h, N} = \lim_{N \rightarrow \infty} \mathbb{E}_{V_r}(G(\varphi)) = G(\varphi_0(\beta, h)) = \varphi_0(\beta, h). \quad (1.4.23)$$

The susceptibility is by definition given by

$$\chi(\beta, h) = \frac{\partial M}{\partial h}(\beta, h) = \frac{\partial \varphi_0}{\partial h}(\beta, h). \quad (1.4.24)$$

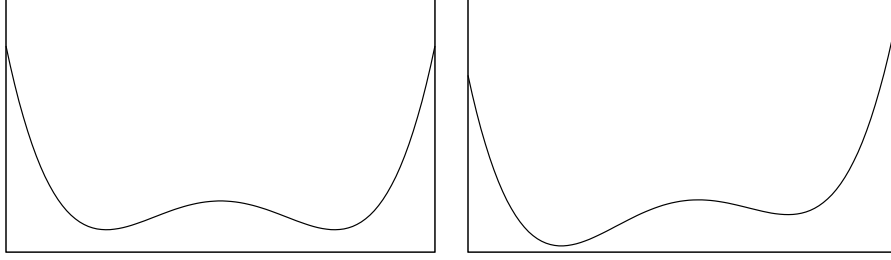


Fig. 1.5 The renormalised potential for $\beta > \beta_c$ with $h = 0$ (left) and $h \neq 0$ (right). For $h \neq 0$ the minimum is unique, while for $h = 0$ there are two minima for $n = 1$ and a set of minima with $O(n)$ symmetry for general n .

(i) Lemma 1.4.6 implies $\varphi_0(\beta, 0_+) = 0$ if $\beta \leq \beta_c = 1$ and $\varphi_0(\beta, 0_+) > 0$ if $\beta > \beta_c$. Since also $\varphi_0(\beta, 0_+) \rightarrow 0$ as $\beta \rightarrow \beta_c$, the asymptotics $\tanh(x) = x - \frac{1}{3}x^3 + o(x^3)$ imply

$$\begin{aligned}\varphi_0(\beta, 0_+) &= \tanh(\beta \varphi_0(\beta, 0_+)) \\ &= \beta \varphi_0(\beta, 0_+) - \frac{1}{3}(\beta \varphi_0(\beta, 0_+))^3 + o(\beta \varphi_0(\beta, 0_+))^3,\end{aligned}\quad (1.4.25)$$

and therefore $\varphi_0 = \varphi_0(\beta, 0_+)$ satisfies

$$(\beta - 1)\varphi_0 = \frac{1}{3}(\beta \varphi_0)^3 + o(\beta \varphi_0)^3. \quad (1.4.26)$$

Using $\varphi_0(\beta, 0_+) > 0$ for $\beta > 1$, the claim follows by dividing by $\varphi_0/3$ and taking the square root:

$$\varphi_0^2 \sim 3 \frac{\beta - 1}{\beta^3} \sim 3(\beta - \beta_c) \quad (\beta \downarrow \beta_c). \quad (1.4.27)$$

(ii) Similarly, if $\beta = 1$ and $h > 0$,

$$\varphi_0 = \tanh(\varphi_0 + h) = \varphi_0 + h - \frac{1}{3}(\varphi_0 + h)^3 + o(\varphi_0 + h)^3 \quad (1.4.28)$$

implies

$$\varphi_0 \sim (3h)^{1/3} \quad (h \downarrow 0). \quad (1.4.29)$$

(iii) Note that $0 = V'_{\beta,h}(\varphi_0(\beta, h))$ implies

$$0 = \frac{\partial^2}{\partial h \partial \varphi} V_{\beta,h}(\varphi_0(\beta, h)) + \frac{\partial^2}{\partial \varphi^2} V_{\beta,h}(\varphi_0(\beta, h)) \frac{\partial \varphi_0}{\partial h}(\beta, h). \quad (1.4.30)$$

Using that

$$\frac{\partial^2}{\partial h \partial \varphi} V_{\beta, h}(\varphi) = -\beta(1 - \tanh^2(\beta \varphi + h)), \quad (1.4.31)$$

$$\frac{\partial^2}{\partial \varphi^2} V_{\beta, h}(\varphi) = \beta - \beta^2(1 - \tanh^2(\beta \varphi + h)), \quad (1.4.32)$$

and $\varphi_0 = \tanh(\beta \varphi_0 + h)$, therefore

$$\frac{\partial \varphi_0}{\partial h}(\beta, h) = \frac{1}{-\beta + (1 - \varphi_0(\beta, h)^2)^{-1}}. \quad (1.4.33)$$

This implies

$$\chi(\beta, 0) = \frac{1}{-\beta + (1 - \varphi_0(\beta, 0)^2)^{-1}} = \frac{1}{1 - \beta} = \frac{1}{\beta_c - \beta} \quad (\beta < \beta_c), \quad (1.4.34)$$

$$\chi(\beta, 0_+) \sim \frac{1}{-\beta + (1 - 3(\beta - 1))^{-1}} \sim \frac{1}{1 - \beta + 3(\beta - 1)} = \frac{1}{2(\beta - \beta_c)} \quad (\beta > \beta_c), \quad (1.4.35)$$

as claimed. $\blacksquare$

We conclude this section with two exercises concerning the extension of some of the above ideas from $n = 1$ to $n > 1$.

Exercise 1.4.7. Let $n = 3$. Show that

$$V(\varphi) = \frac{\beta}{2} |\varphi|^2 - \log \left(\frac{\sinh(|\beta \varphi + h|)}{|\beta \varphi + h|} \right) + \frac{\beta}{2}, \quad (1.4.36)$$

where $V(\varphi)$ was defined in (1.4.13). [\[Solution\]](#)

Exercise 1.4.8. Extend the results of Lemma 1.4.6 to $n > 1$. Let $\beta_c = n$.

- (i) For $\beta \leq \beta_c$, the effective potential V is convex and the minimum of V tends to 0 as $h \rightarrow 0$. Moreover, $\text{Hess} V(\varphi) \geq \beta(1 - \beta/\beta_c)$ for any $h \in \mathbb{R}^n$.
- (ii) For $\beta > \beta_c$, the effective potential V is non-convex.

Hint: [\[76, Theorem D.2\]](#) is helpful. [\[Solution\]](#)

1.5 Gaussian free field and simple random walk

Another fundamental example of a spin system is the Gaussian free field (GFF). The GFF is a spin system whose distribution is Gaussian. In this section, we indicate that its critical behaviour can be computed directly, and establish its connection to the simple random walk. We also introduce the bubble diagram, whose behaviour provides an indication of the special role of dimension 4.

1.5.1 Gaussian free field

Let Λ be a finite set, and let $\beta = (\beta_{xy})_{x,y \in \Lambda}$ be non-negative coupling constants with $\beta_{xy} = \beta_{yx}$. As in (1.3.5), given a spin field $\varphi : \Lambda \rightarrow \mathbb{R}^n$, and given $m^2 > 0$, we define

$$H(\varphi) = \frac{1}{2}(\varphi, (-\Delta_\beta + m^2)\varphi). \quad (1.5.1)$$

We then use H to define a probability measure on field configurations via specification of the expectation

$$\langle F \rangle \propto \int_{(\mathbb{R}^n)^\Lambda} F(\varphi) e^{-H(\varphi)} \prod_{x \in \Lambda} d\varphi_x, \quad (1.5.2)$$

where the integration is with respect to Lebesgue measure on $(\mathbb{R}^n)^\Lambda$.

Definition 1.5.1. An n -component *Gaussian free field* (GFF) with mass $m > 0$ on Λ is a field distributed according to the above measure. An example of particular interest is the case where Λ is a finite approximation to $\mathbb{Z}^d$, and $\beta_{xy} = 1_{x \sim y}$. Then Δ_β is the discrete Laplace operator and we simply write Δ .

Exercise 1.5.2. Show that $(\varphi, -\Delta_\beta \varphi) \geq 0$ for all $\varphi \in \mathbb{R}^\Lambda$. In particular, $(\varphi, (-\Delta_\beta + m^2)\varphi) \geq m^2(\varphi, \varphi) > 0$ for all $\varphi \neq 0$, i.e., $-\Delta_\beta + m^2$ is strictly positive definite if $m^2 > 0$ (and thus so is $(-\Delta_\beta + m^2)^{-1}$). If $\mathbb{1}$ is the constant function on Λ , defined by $\mathbb{1}_x = 1$ for all $x \in \Lambda$ then $-\Delta_\beta \mathbb{1} = 0$ and

$$(-\Delta_\beta + m^2)^{-1} \mathbb{1} = m^{-2} \mathbb{1}. \quad (1.5.3)$$

[Solution]

Definition 1.5.1 can be restated to say that the GFF is defined as the Gaussian field on $\mathbb{R}^{n\Lambda}$ with covariance given by

$$\langle \varphi_x^i \varphi_y^j \rangle = \delta_{ij} (-\Delta_\beta + m^2)^{-1}_{xy}. \quad (1.5.4)$$

For the particular case mentioned in Definition 1.5.1, for which the Laplacian is the standard one on a subset $\Lambda \subset \mathbb{Z}^d$, we write the covariance as

$$C_{xy;\Lambda}(m^2) = (-\Delta^{(\Lambda)} + m^2)^{-1}_{xy}. \quad (1.5.5)$$

See Chapter 2 for a detailed introduction to Gaussian fields. Rather than taking Λ as a subset of $\mathbb{Z}^d$, we can instead take it to be a discrete d -dimensional torus. The use of a torus avoids issues concerning boundary conditions and also preserves translation invariance. For $m^2 > 0$ and for all dimensions $d > 0$, it can be proved that in the limit as the period of the torus goes to infinity, the limit

$$C_{xy}(m^2) = \lim_{\Lambda \uparrow \mathbb{Z}^d} C_{xy;\Lambda}(m^2) \quad (1.5.6)$$

exists and is given in terms of the Laplacian Δ on $\mathbb{Z}^d$ by

$$C_{xy}(m^2) = (-\Delta + m^2)_{xy}^{-1}. \quad (1.5.7)$$

In addition, for $d > 2$ it can be proved that the limit $C_{xy}(0) = \lim_{m^2 \downarrow 0} C_{xy}(m^2)$ exists. The restriction to $d > 2$ is a reflection of the fact that simple random walk on $\mathbb{Z}^d$ is transient if and only if $d > 2$.

As in the corresponding definitions for the Ising model in (1.2.5)–(1.2.7), we define

$$\text{two-point function: } \delta_{ij} C_{xy}(m^2), \quad (1.5.8)$$

$$\text{correlation length: } \xi(m^2)^{-1} = -\lim_{n \rightarrow \infty} n^{-1} \log C_{0,ne_1}(m^2), \quad (1.5.9)$$

$$\text{susceptibility: } \chi(m^2) = \sum_{x \in \mathbb{Z}^d} C_{0x}(m^2). \quad (1.5.10)$$

For the two-point function we allow $m^2 \geq 0$, whereas for the correlation length and susceptibility we restrict to $m^2 > 0$. The susceptibility diverges at the *critical value* $m^2 = 0$. The relations

$$\chi(m^2) = m^{-2} \quad (m^2 > 0), \quad (1.5.11)$$

$$\xi(m^2) \sim m^{-1} \quad (m^2 \downarrow 0), \quad (1.5.12)$$

$$C_{0x}(0) = (-\Delta)_{0x}^{-1} \sim c(d)|x|^{-(d-2+\eta)} \quad (|x| \rightarrow \infty), \quad (1.5.13)$$

respectively follow from (1.5.3), from [121, Theorem A.2], and from a standard fact about the lattice Green function $(-\Delta)^{-1}$ (see, e.g., [114]). The above relations show that the critical exponents for the GFF assume the values

$$\gamma = 1, \quad \nu = \frac{1}{2}, \quad \eta = 0. \quad (1.5.14)$$

These are mean-field values, although the exponents ν and η involve the geometry of $\mathbb{Z}^d$ and therefore are somewhat unnatural for the mean-field model. The fact that $\gamma = (2 - \eta)\nu$ is an instance of Fisher's relation.

1.5.2 Simple random walk

The GFF is intimately related to the simple random walk. In this section, we make contact between the two models in the case of $\mathbb{Z}^d$.

Given $d > 0$ and $x, y \in \mathbb{Z}^d$, an n -step walk on $\mathbb{Z}^d$ from x to y is a sequence $\omega = (x = x_0, x_1, \dots, x_{n-1}, x_n = y)$ of neighbouring points ($|x_i - x_{i-1}| = 1$). We write $|\omega| = n$ for the length of ω , and write $\mathcal{W}(x, y)$ for the set of all walks from x to y . Let V be a complex diagonal $\mathbb{Z}^d \times \mathbb{Z}^d$ matrix whose elements obey $\text{Re} v_x \geq c > 0$ for some positive c . We define the simple random walk two-point function by

$$W_{xy}^{(V)} = \sum_{\omega \in \mathcal{W}(x,y)} \prod_{j=0}^{|\omega|} \frac{1}{2d + v_{\omega_j}}. \quad (1.5.15)$$

The positivity condition on V ensures that the right-hand side converges. For the special case where V has constant diagonal elements m^2 , we write

$$W_{xy}^{(m^2)} = \sum_{\omega \in \mathcal{W}(x,y)} \prod_{j=0}^{|\omega|} \frac{1}{2d + m^2}. \quad (1.5.16)$$

The next lemma shows that W_{xy} is related to the covariance of the GFF.

Lemma 1.5.3. *For $d > 0$ and a diagonal matrix V with $\text{Rev}_x \geq c > 0$,*

$$W_{xy}^{(V)} = (-\Delta + V)_{xy}^{-1}. \quad (1.5.17)$$

In particular,

$$W_{xy}^{(m^2)} = C_{xy}(m^2) = (-\Delta + m^2)_{xy}^{-1}. \quad (1.5.18)$$

Proof. We separate the contribution of the zero-step walk, and for walks taking at least one step we condition on the last step, to obtain

$$W_{xy}^{(V)} = \frac{1}{2d + v_y} \delta_{xy} + \frac{1}{2d + v_y} \sum_{e: |e|=1} W_{x,y+e}^V. \quad (1.5.19)$$

We multiply through by $2d + v_y$ and rearrange the terms to obtain

$$(-\Delta W^{(V)})_{xy} + v_y W_{xy}^{(V)} = \delta_{xy}, \quad (1.5.20)$$

which can be restated as $(-\Delta + V)W^{(V)} = I$, and the proof is complete. $\blacksquare$

With respect to the uniform measure on n -step walks started at x , let $p_n(x, y)$ denote the probability that an n -step walk started at x ends at y . Equation (1.5.16) can be rewritten as

$$W_{xy}^{(m^2)} = \sum_{n=0}^{\infty} p_n(x, y) \frac{(2d)^n}{(2d + m^2)^{n+1}} = (-\Delta + m^2)_{xy}^{-1}. \quad (1.5.21)$$

When $m^2 > 0$, the sum in (1.5.21) is finite in all dimensions. When $m^2 = 0$, $\sum_{n=0}^{\infty} p_n(x, y)$ is the Green function for simple random walk, which is finite if and only if $d > 2$ (see Exercise 1.5.5). Since $\sum_{y \in \mathbb{Z}^d} p_n(x, y) = 1$, an alternate perspective on (1.5.11) is provided by

$$\sum_{x \in \mathbb{Z}^d} W_{xy}^{(m^2)} = \sum_{n=0}^{\infty} \frac{(2d)^n}{(2d + m^2)^{n+1}} = \frac{1}{m^2}. \quad (1.5.22)$$

The central limit theorem asserts that the distribution of p_n is asymptotically Gaussian, and the functional central limit theorem asserts that the scaling limit of simple random walk is Brownian motion. For random walk, *universality* is the statement that the critical exponents and limiting distribution remains the same, not only for simple random walk, but for any random walk composed of i.i.d. steps X_i having mean zero and finite variance.

1.5.3 The bubble diagram

The *bubble diagram* plays a key role in identifying the special role of dimension 4 in critical phenomena. It is defined by

$$B_{m^2} = \sum_{x \in \mathbb{Z}^d} (C_{0x}(m^2))^2, \quad (1.5.23)$$

with $C_{0x}(m^2) = (-\Delta + m^2)^{-1}$ as in (1.5.7). The Fourier transform is useful for the analysis of the bubble diagram.

The Fourier transform of an absolutely summable function $f : \mathbb{Z}^d \rightarrow \mathbb{C}$ is defined by

$$\hat{f}(k) = \sum_{x \in \mathbb{Z}^d} f_x e^{ik \cdot x} \quad (k \in [-\pi, \pi]^d). \quad (1.5.24)$$

The inverse transform is given by

$$f_x = (2\pi)^{-d} \int_{[-\pi, \pi]^d} \hat{f}(k) e^{-ik \cdot x} \quad (x \in \mathbb{Z}^d). \quad (1.5.25)$$

With respect to the Fourier transform, $-\Delta$ acts as a multiplication operator with multiplication by

$$\lambda(k) = 4 \sum_{j=1}^d \sin^2(k_j/2) \quad (k \in [-\pi, \pi]^d). \quad (1.5.26)$$

This means that

$$(-\Delta \hat{f})(k) = \lambda(k) \hat{f}(k), \quad (1.5.27)$$

and hence the Fourier transform of $C_{0x}(m^2)$ is given by

$$\hat{C}_{m^2}(k) = \frac{1}{\lambda(k) + m^2}. \quad (1.5.28)$$

Therefore, by Parseval's formula and (1.5.28),

$$B_{m^2} = \int_{[-\pi, \pi]^d} \frac{1}{(\lambda(k) + m^2)^2} \frac{dk}{(2\pi)^d}. \quad (1.5.29)$$

The logarithmic corrections to scaling for $d = 4$ in Theorem 1.6.1 arise via the logarithmic divergence of the 4-dimensional bubble diagram.

Exercise 1.5.4. Show that $B_0 < \infty$ if and only if $d > 4$, and that, as $m^2 \downarrow 0$,

$$B_{m^2} \sim b_d \times \begin{cases} m^{-(4-d)} & (d < 4) \\ \log m^{-2} & (d = 4), \end{cases} \quad (1.5.30)$$

with $b_1 = \frac{1}{8}$, $b_2 = \frac{1}{4\pi}$, $b_3 = \frac{1}{8\pi}$, $b_4 = \frac{1}{16\pi^2}$. [Solution]

The following exercises review the fact that simple random walk is recurrent in dimensions $d \leq 2$ and transient for $d > 2$, and relate the bubble diagram to intersections of random walks.

Exercise 1.5.5. (i) Let u denote the probability that simple random walk ever returns to the origin. The walk is recurrent if $u = 1$ and transient if $u < 1$. Let N denote the random number of visits to the origin, including the initial visit at time 0. Show that $EN = (1 - u)^{-1}$, so the walk is recurrent if and only if $EN = \infty$.

(ii) Show that

$$EN = \sum_{n=0}^{\infty} p_n(0) = 2d \int_{[-\pi, \pi]^d} \frac{1}{\lambda(k)} \frac{dk}{(2\pi)^d}. \quad (1.5.31)$$

Thus transience is characterised by the integrability of $\hat{C}_0(k) = 1/\lambda(k)$.

(iii) Show that simple random walk is recurrent in dimensions $d \leq 2$ and transient for $d > 2$. [Solution]

Exercise 1.5.6. Let $S^1 = (S_n^1)_{n \geq 0}$ and $S^2 = (S_n^2)_{n \geq 0}$ be two independent simple random walks on $\mathbb{Z}^d$ started at the origin, and let

$$I = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \mathbb{1}_{S_m^1 = S_n^2} \quad (1.5.32)$$

be the random number of intersections of the two walks. Show that

$$EI = (2d)^2 B_0. \quad (1.5.33)$$

Thus EI is finite if and only if $d > 4$. [Solution]

1.6 $|\varphi|^4$ model

1.6.1 Definition of the $|\varphi|^4$ model

As in Section 1.3, the n -component $|\varphi|^4$ model on a set Λ is defined by the expectation

$$\langle F \rangle_{g, \mathbf{v}, \Lambda} = \frac{1}{Z_{g, \mathbf{v}, \Lambda}} \int_{\mathbb{R}^{n\Lambda}} F(\varphi) e^{-H(\varphi)} d\varphi \quad (1.6.1)$$

with

$$H(\varphi) = \frac{1}{2} \sum_{x \in \Lambda} \varphi_x \cdot (-\Delta_\beta \varphi)_x + \sum_{x \in \Lambda} \left(\frac{1}{4} g |\varphi_x|^4 + \frac{1}{2} v |\varphi_x|^2 \right). \quad (1.6.2)$$

Here $g > 0$, $v \in \mathbb{R}$, and $d\varphi = \prod_{x \in \Lambda} d\varphi_x$ is the Lebesgue measure on $(\mathbb{R}^n)^\Lambda$. The *partition function* $Z_{g,v,\Lambda}$ is defined by the condition $\langle 1 \rangle_{g,v,\Lambda} = 1$. An external field h can also be included, but we have omitted it here. We are primarily concerned here with the nearest-neighbour interaction on a d -dimensional discrete torus, for which $\Delta_\beta = \Delta$ is the standard Laplacian. The single-spin distribution is $e^{-(\frac{1}{4}g|\varphi_x|^4 + \frac{1}{2}v|\varphi_x|^2)} d\varphi_x$. For the case $v < 0$, which is our principal interest, we have a double-well potential as depicted for $n = 1$ in Figure 1.6. For $n \geq 2$, it is sometimes called a Mexican hat potential.

With $v = -g\beta$, the single-spin density becomes proportional to $e^{-\frac{1}{4}g(|\varphi_x|^2 - \beta)^2}$. In the limit $g \rightarrow \infty$, this converges to the $O(n)$ model, whose single-spin distribution is the uniform measure on the surface of the sphere of radius $\sqrt{\beta}$ in n dimensions. By rescaling the field by $1/\sqrt{\beta}$, this definition is equivalent to the more usual one, where spins are on the unit sphere and an inverse temperature parameter β multiplies the spin coupling term $\varphi \cdot (-\Delta \varphi)$. Conversely, the $|\varphi|^4$ model can be realised as a limit of $O(n)$ models [72, 142].

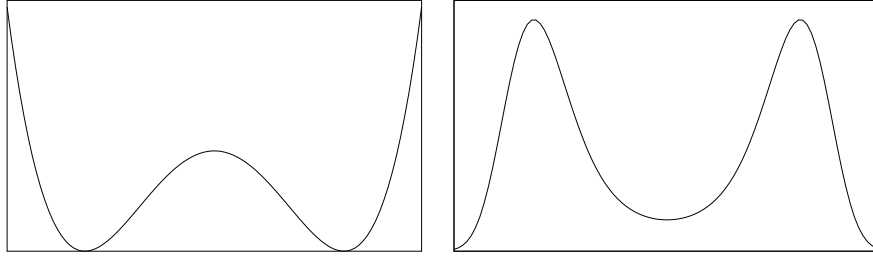


Fig. 1.6 For $n = 1$, the density of the single-spin distribution is shown at right, with its double-well potential at left.

The Ising model Gibbs measure of (1.2.3) is equal to

$$P_{T,\Lambda}(\sigma) \propto e^{-\frac{1}{2T} \sum_{x \in \Lambda} \sigma_x (-\Delta \sigma)_x} \prod_{x \in \Lambda} \frac{1}{2} (\delta_{\sigma_x, 1} + \delta_{\sigma_x, -1}) \quad (1.6.3)$$

Let $\varphi_x = T^{-1/2} \sigma_x$. Then

$$P_{T,\Lambda}(\varphi) \propto e^{-\frac{1}{2} \sum_{x \in \Lambda} \varphi_x (-\Delta \varphi)_x} \prod_{x \in \Lambda} \frac{1}{2} (\delta_{\varphi_x, T^{-1/2}} + \delta_{\varphi_x, -T^{-1/2}}). \quad (1.6.4)$$

Suppose that we replace the single-spin distribution $\frac{1}{2}(\delta_{\sigma_x, T^{-1/2}} + \delta_{\sigma_x, -T^{-1/2}})$ by a smoothed out distribution with two peaks located at $\pm T^{-1/2}$. It may be expected that, as T is decreased, such a model will have a phase transition with the same critical exponents as the Ising model. This is qualitatively similar to the $|\varphi|^4$ model with $\nu < 0$. Now ν plays the role of T , and there is again a phase transition and corresponding critical exponents associated with a (negative) critical value ν_c of ν . Alignment of spins is observed for $\nu < \nu_c$ but not for $\nu > \nu_c$, as illustrated schematically in Figure 1.7. General results on the existence of phase transitions for multi-component spin systems in dimensions $d \geq 3$ are proved in [90]. For $d = 2$, the Mermin–Wagner theorem rules out phase transitions for $n \geq 2$. It is predicted that the $|\varphi|^4$ model is in the same universality class as the $O(n)$ model, for all $n \geq 1$. In particular, the critical exponents of the n -component $|\varphi|^4$ are predicted to be the same as those of the $O(n)$ model.

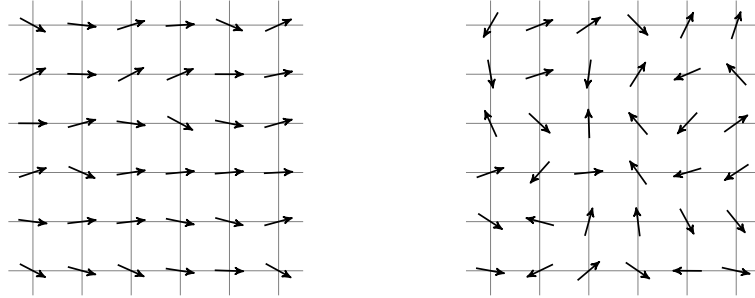


Fig. 1.7 Typical spin configurations for $\nu < \nu_c$ (spins aligned) and for $\nu > \nu_c$ (spins not aligned).

We write $\langle F; G \rangle = \langle FG \rangle - \langle F \rangle \langle G \rangle$ for the covariance of random variables F, G . Five quantities of interest are the *pressure*, the *two-point function*, the *susceptibility*, the *correlation length of order $p > 0$* , and the *specific heat*. These are defined, respectively, as the limits (assuming they exist)

$$p(g, \nu) = \lim_{N \rightarrow \infty} \frac{1}{|\Lambda_N|} \log Z_{g, \nu, \Lambda_N}, \quad (1.6.5)$$

$$\langle \varphi_0^1 \varphi_x^1 \rangle_{g, \nu} = \lim_{N \rightarrow \infty} \langle \varphi_0^1 \varphi_x^1 \rangle_{g, \nu, \Lambda_N} \quad (1.6.6)$$

$$\chi(g, \nu) = \lim_{N \rightarrow \infty} \sum_{x \in \Lambda_N} \langle \varphi_0^1 \varphi_x^1 \rangle_{g, \nu, \Lambda_N}, \quad (1.6.7)$$

$$\xi_p(g, \nu) = \left(\frac{1}{\chi(g, \nu)} \lim_{N \rightarrow \infty} \sum_{x \in \Lambda_N} |x|^p \langle \varphi_0^1 \varphi_x^1 \rangle_{g, \nu, \Lambda_N} \right)^{1/p}, \quad (1.6.8)$$

$$c_H(g, \nu) = \frac{1}{4} \lim_{N \rightarrow \infty} \sum_{x \in \Lambda_N} \langle |\varphi_0|^2; |\varphi_x|^2 \rangle_{g, \nu, \Lambda_N}, \quad (1.6.9)$$

for a sequence of boxes Λ_N approximating $\mathbb{Z}^d$ as $N \rightarrow \infty$. In making the above definitions, we used the fact that $\langle \phi_x \rangle = 0$ for all x due to the $O(n)$ invariance.

In general, the limit defining the pressure has been proved to exist and to be independent of the boundary conditions for the n -component $|\phi|^4$ model for any $d > 0$, $n \geq 1$, $g > 0$ and $v \in \mathbb{R}$ [117]. For $n = 1, 2$, correlation inequalities [83] imply that the pressure is convex, and hence also continuous, in v , and that for the case of free boundary conditions the limit defining the susceptibility exists (possibly infinite) and is monotone non-increasing in v . Proofs are lacking for $n > 2$ due to a lack of correlation inequalities in this case (as discussed, e.g., in [83]), but it is to be expected that these facts known for $n = 1, 2$ are true also for $n > 2$.

1.6.2 Critical exponents of the $|\phi|^4$ model

Dimensions above four

For $d > 4$, the $|\phi|^4$ model has been proven to exhibit mean-field behaviour has been proved. In particular, it is known [7, 89] that for $n = 1, 2$, with $v = v_c + \varepsilon$ and as $\varepsilon \downarrow 0$,

$$\chi(g, v) \asymp \frac{1}{\varepsilon} \quad \text{when } d > 4, \quad n = 1, 2. \quad (1.6.10)$$

The proof is based on correlation inequalities, differential inequalities, and reflection positivity. Also, for $n = 1, 2$, the specific heat does not diverge as $v \downarrow v_c$ [83, 145]. More recently, the lace expansion has been used to prove that for $d > 4$ and small $g > 0$, the critical two-point function has the Gaussian decay

$$\langle \phi_0^1 \phi_x^1 \rangle_{g, v_c} \sim c \frac{1}{|x|^{d-2}} \quad \text{as } |x| \rightarrow \infty, \quad (1.6.11)$$

for $n = 1$ [138] and for $n = 1, 2$ [41]. The above equations are statements that the critical exponents γ, η take their mean-field values $\gamma = 1$ and $\eta = 0$ for $d > 4$.

Dimension four

For dimension $d = 4$, logarithmic corrections to mean-field critical scaling were predicted in [37, 113, 151]. In the early 1980s it was established that the deviation from mean-field scaling is at most logarithmic for $d = 4$, for some quantities including the susceptibility [7, 10, 89]. A number of rigorous results concerning precise critical behaviour of the 4-dimensional case were proved during the 1980s using rigorous renormalisation group methods based on block spins [95, 96, 108] or phase space expansion [82]. The following theorems were proved recently via an approach based on the methods in this book.

Theorem 1.6.1. [18]. For $d = 4$, $n \geq 1$, L large, and $g > 0$ small, there exists $v_c = v_c(g, n) < 0$ such that, with $v = v_c + \varepsilon$ and as $\varepsilon \downarrow 0$,

$$\chi(g, v) \sim A_{g,n} \frac{1}{\varepsilon} (\log \varepsilon^{-1})^{(n+2)/(n+8)}, \quad (1.6.12)$$

$$c_H(g, v) \sim D_{g,n} \times \begin{cases} (\log \varepsilon^{-1})^{(4-n)/(n+8)} & (n < 4) \\ \log \log \varepsilon^{-1} & (n = 4) \\ 1 & (n > 4). \end{cases} \quad (1.6.13)$$

As $g \downarrow 0$, $A_{g,n} \sim ((n+8)g/(16\pi^2))^{(n+2)/(n+8)}$, and $v_c(g, n) \sim -(n+2)gN_4$ (with $N_4 = (-\Delta)_{00}^{-1}$).

Theorem 1.6.2. [26]. For $d = 4$, $n \geq 1$, $p > 0$, L large, and $g > 0$ small (depending on p, n), with $v = v_c + \varepsilon$ and as $\varepsilon \downarrow 0$,

$$\xi_p(g, v) \sim C_{g,n,p} \frac{1}{\varepsilon^{1/2}} (\log \varepsilon^{-1})^{\frac{1}{2}(n+2)/(n+8)}. \quad (1.6.14)$$

Theorem 1.6.3. [144]. For $d = 4$, $n \geq 1$, L large, and $g > 0$ small, as $|x| \rightarrow \infty$,

$$\langle \varphi_0^1 \varphi_x^1 \rangle_{g, v_c} \sim \frac{A'_{g,n}}{|x|^2}, \quad (1.6.15)$$

$$\langle |\varphi_0|^2; |\varphi_x|^2 \rangle_{g, v_c} \sim \frac{nA''_{g,n}}{(\log |x|)^{2(n+2)/(n+8)}} \frac{1}{|x|^4}. \quad (1.6.16)$$

Related further results can be found in [18, 26, 144]. In the above theorems, the infinite-volume limits are taken through a sequence of tori $\Lambda = \Lambda_N = \mathbb{Z}^d / L^N \mathbb{Z}^d$ for sufficiently large L , and it is part of the statements that these limits exist. In Theorem 1.6.3, the left-hand sides refer to the limits taken in the order $\lim_{v \downarrow v_c} \lim_{N \rightarrow \infty}$.

For $n = 1$, Theorem 1.6.3 was proved thirty years earlier, in [95, 96], and the analogue of (1.6.15) was proved for a closely related 1-component model in [82]. In [108], the logarithmic correction $(\log \varepsilon^{-1})^{1/3}$ in (1.6.12) was proved in [108], along with other results including for the correlation length.

This book describes techniques developed to prove the above theorems, with focus on the susceptibility. To keep the focus on the main ideas and avoid further technicalities, we will prove a statement like (1.6.12) for a *hierarchical* version of the $|\varphi|^4$ model; the precise statement is given in Theorem 4.2.1.

Dimensions below four

Dimensions $2 < d < 4$ are studied in the physics literature using expansions in dimension and number of components. In a seminal paper, Wilson and Fisher initiated the study of dimensions below 4 by expanding in small positive $\varepsilon = 4 - d$ [155]. Dimensions above 2 have been studied via expansion in $\varepsilon = d - 2$, and it is also common in the literature to expand in $1/n$ for a large number n of field components.

An alternative to expansion in $\varepsilon = 4 - d$ is to consider long-range interactions decaying with distance r as $r^{-(d+\alpha)}$ with $\alpha \in (0, 2)$ [86, 146]. These models have upper critical dimension 2α , and the ε expansion can be carried out in integer dimensions $d = 1, 2, 3$ by choosing $\alpha = \frac{1}{2}(d + \varepsilon)$. Then $2\alpha = d + \varepsilon$, so d is slightly below the critical dimension when ε is small and positive.

Extensions of Theorems 1.6.1 and 1.6.3 to the long-range setting have been obtained in [118, 143]; see also [1, 3, 39, 52]. In contrast to the above theorems, the long-range results involve a *non-Gaussian* renormalisation group fixed point, with corrections to mean-field scaling that are power law rather than logarithmic. An example of a result of this type is the following theorem. The theorem pertains to the $|\varphi|^4$ model defined with the operator $-\Delta$ in (1.6.2) replaced by the fractional power $(-\Delta)^{\alpha/2}$, with $\alpha = \frac{1}{2}(d + \varepsilon)$ for small $\varepsilon > 0$. The kernel of this operator decays at large distance as $(-\Delta)_{xy}^{\alpha/2} \asymp |x - y|^{-(d+\alpha)}$.

Theorem 1.6.4. [143]. *For $d = 1, 2, 3$, $n \geq 1$, L sufficiently large, and $\varepsilon = 2\alpha - d > 0$ sufficiently small, there exists $\bar{s} \asymp \varepsilon$ such that, for $g \in [\frac{63}{64}\bar{s}, \frac{65}{64}\bar{s}]$, there exists $v_c = v_c(g, n)$ and $C > 0$ such that for $v = v_c + t$ with $t \downarrow 0$, the susceptibility of the long-range model obeys*

$$C^{-1}t^{-(1+\frac{n+2}{n+8}\frac{\varepsilon}{\alpha}-C\varepsilon^2)} \leq \chi(g, v; n) \leq Ct^{-(1+\frac{n+2}{n+8}\frac{\varepsilon}{\alpha}+C\varepsilon^2)}. \quad (1.6.17)$$

This is a statement that the critical exponent γ exists to order ε , with

$$\gamma = 1 + \frac{n+2}{n+8}\frac{\varepsilon}{\alpha} + O(\varepsilon^2). \quad (1.6.18)$$

1.7 Self-avoiding walk

The self-avoiding walk on $\mathbb{Z}^d$ is the uniform probability measure on the set of n -step simple random walk paths on $\mathbb{Z}^d$ with no self-intersections. It is a much studied model of linear polymers [98, 109, 149] and is of independent mathematical interest (see, e.g., [23, 110, 121]). It has long been understood that at a formal (nonrigorous) level, the critical behaviour of the self-avoiding walk is predicted from that of the n -component $|\varphi|^4$ model by setting $n = 0$. For example, the asymptotic formula for the susceptibility of the 4-dimensional $|\varphi|^4$ model given by (1.6.12), namely

$$\chi(g, v) \sim A_{g,n} \frac{1}{\varepsilon} (\log \varepsilon^{-1})^{(n+2)/(n+8)}, \quad (1.7.1)$$

predicts that the susceptibility of the 4-dimensional self-avoiding walk should obey

$$\chi(g, v) \sim A_{g,0} \frac{1}{\varepsilon} (\log \varepsilon^{-1})^{1/4}. \quad (1.7.2)$$

An advantage of the renormalisation group method presented in this book is that it applies equally well to a *supersymmetric* version of the $|\varphi|^4$ model which corresponds exactly (and rigorously) to a model of weakly self-avoiding walk. In particular, (1.7.2) can be proved in this setting [20]. In Chapter 11, we define the supersymmetric version of the $|\varphi|^4$ model and prove its equivalence to the continuous-time weakly self-avoiding walk. This provides a basis for the application of the renormalisation group method. We also comment in Chapter 11 on the sense in which the supersymmetric model corresponds to $n = 0$ components.

Chapter 2

Gaussian fields

In this chapter, we present basic facts about Gaussian integration. Further material can be found in many references, e.g., in [43, 139].

2.1 Gaussian integration

Throughout this chapter, X is a finite set, we write $\mathbb{R}^X = \{\varphi : X \rightarrow \mathbb{R}\}$, and $(\varphi, \psi) = \sum_{x \in X} \varphi_x \psi_x$ for $\varphi, \psi \in \mathbb{R}^X$. We call $\varphi \in \mathbb{R}^X$ a *field*, and a randomly distributed φ is thus a random field. We do not make use of any geometric structure of X here, and only use the fact that $\mathbb{R}^X$ is a finite-dimensional vector space.

Let $C = (C_{xy})_{x,y \in X}$ denote a symmetric positive semi-definite matrix, where *positive semi-definite* means that $(\varphi, C\varphi) \geq 0$ for every $\varphi \in \mathbb{R}^X$. If the inequality is strict for every nonzero φ , we say that C is positive definite. This stronger condition implies that the inverse C^{-1} exists. The following is the higher-dimensional generalisation of the probability measure $\frac{1}{\sqrt{2\pi\sigma}} e^{-x^2/2\sigma} dx$ of a Gaussian random variable with mean 0 and variance σ^2 .

Definition 2.1.1. Let C be positive definite. The centred *Gaussian probability measure* P_C on $\mathbb{R}^X$, with covariance C , is defined by

$$P_C(d\varphi) = \det(2\pi C)^{-\frac{1}{2}} e^{-\frac{1}{2}(\varphi, C^{-1}\varphi)} d\varphi, \quad (2.1.1)$$

where $d\varphi$ is the Lebesgue measure on $\mathbb{R}^X$.

To see that P_C really is a probability measure, it suffices by the spectral theorem to assume that $X = \{1, \dots, n\}$ and that C is diagonal with $C_{ii} = \lambda_i^{-1}$. In this case, as required,

$$\begin{aligned}
\int e^{-\frac{1}{2}(\varphi, C^{-1}\varphi)} d\varphi &= \int e^{-\frac{1}{2\lambda_i} \sum_{i=1}^n \varphi_i^2} \prod_{i=1}^n d\varphi_i \\
&= \prod_{i=1}^n \int e^{-\frac{1}{2\lambda_i} \varphi_i^2} d\varphi_i = \prod_{i=1}^n (2\pi\lambda_i)^{\frac{1}{2}} = \det(2\pi C)^{\frac{1}{2}}. \quad (2.1.2)
\end{aligned}$$

In the case that C is positive semi-definite, but not positive definite, C has a kernel K which is a subspace of $\mathbb{R}^X$. We construct a *degenerate* Gaussian probability measure on $\mathbb{R}^X$ as follows. We set C' equal to the restriction of C to the orthogonal complement $K^\perp$ of K in $\mathbb{R}^X$. By the spectral theorem $K^\perp$ is spanned by eigenvectors of C with positive eigenvalues and therefore is represented by a positive definite matrix in any orthogonal basis for $K^\perp$. We define P_C to be the probability measure on $\mathbb{R}^X$ that is supported on $K^\perp$ and which equals the Gaussian measure $P_{C'}$ when restricted to $K^\perp$. To define this construction concretely, we choose an orthonormal basis of eigenvectors $v_1, \dots, v_n$ in $\mathbb{R}^X$ labelled so that K is spanned by $v_1, \dots, v_k$ for some $k \leq n$ and define

$$P_C(d\varphi) = \det(2\pi C')^{-\frac{1}{2}} e^{-\frac{1}{2}(\varphi(t), C'^{-1}\varphi(t))} \prod_{i \leq k} \delta(dt_i) \prod_{i'=k+1}^n dt_{i'}, \quad (2.1.3)$$

where $\varphi(t) = \sum_{i=1}^n t_i v_i$. Because of the δ factors the random variables (φ, v_i) with $i \leq k$ are a.s. zero according to this probability law. Thus it is straightforward to verify that C continues to be the covariance of φ : e.g., $\text{Var}((\varphi, v_i)) = 0 = (v_i, C v_i)$ for $i = 1, \dots, k$.

Definition 2.1.2. The centred *Gaussian probability measure* P_C on $\mathbb{R}^X$, with covariance C , is defined by Definition 2.1.1 when C is positive definite and by (2.1.3) if C is positive semi-definite. We refer to φ with distribution P_C as a *Gaussian field with covariance* C . The *expectation* of a random variable $F : \mathbb{R}^X \rightarrow \mathbb{R}$ is

$$\mathbb{E}_C F = \int F(\varphi) P_C(d\varphi). \quad (2.1.4)$$

Exercise 2.1.3. Verify the Gaussian integration by parts identity

$$\mathbb{E}_C(F \varphi_x) = \sum_{y \in X} C_{xy} \mathbb{E}_C \left(\frac{\partial F}{\partial \varphi_y} \right), \quad (2.1.5)$$

by writing $\mathbb{E}_C((C^{-1}\varphi)_x F)$ as a derivative (C is invertible when restricted to φ in the support of P_C). [\[Solution\]](#)

Example 2.1.4. The $|\varphi|^4$ model is defined in terms of vector-valued fields $\varphi = (\varphi_x^i)_{x \in \Lambda, i=1, \dots, n}$. These are fields $\varphi \in \mathbb{R}^X$ with the special choice

$$X = n\Lambda = \{(x, i) : x \in \Lambda, i = 1, \dots, n\}. \quad (2.1.6)$$

Given a positive semi-definite matrix $C = (C_{xy})_{x, y \in \Lambda}$, we define an $X \times X$ matrix $(\hat{C}_{(x, i), (y, j)})$ by $\hat{C}_{(x, i), (y, j)} = \delta_{ij} C_{xy}$. We refer to the Gaussian field on $\mathbb{R}^X$ with covari-

ance $\hat{C}$ as the n -component Gaussian field on $\mathbb{R}^A$ with covariance $C = (C_{xy})_{x,y \in A}$. We denote its expectation also by $\mathbb{E}_C$.

Definition 2.1.5. The *convolution* of F with the Gaussian measure P_C is denoted

$$\mathbb{E}_C \theta F(\varphi) = \int F(\varphi + \zeta) P_C(d\zeta) \quad (\varphi \in \mathbb{R}^X), \quad (2.1.7)$$

always assuming the integrals exist. The above defines $\mathbb{E}_C \theta$ as a single operation, but we also view it as the composition of a map $\theta : F \mapsto F(\cdot + \zeta)$ followed by the expectation $\mathbb{E}_C$ which integrates with respect to ζ . The map θ is a homomorphism on the algebra of functions of the field φ .

The following proposition demonstrates an intimate link between Gaussian integration and the Laplace operator

$$\Delta_C = \sum_{x,y \in X} C_{xy} \partial_{\varphi_x} \partial_{\varphi_y}. \quad (2.1.8)$$

Since we are eventually interested in large X (the vertices of a large graph), this Laplace operator acts on a high-dimensional space.

Proposition 2.1.6. For a polynomial $A = A(\varphi)$ in φ of degree at most $2p$,

$$\mathbb{E}_C \theta A = e^{\frac{1}{2} \Delta_C} A = \left(1 + \frac{1}{2} \Delta_C + \cdots + \frac{1}{p! 2^p} \Delta_C^p \right) A. \quad (2.1.9)$$

Proof. Set $v(t, \varphi) = \mathbb{E}_{tC} \theta A(\varphi)$ and $w(t, \varphi) = e^{\frac{1}{2} \Delta_C} A(\varphi)$. It can be seen that v, w are both polynomials in φ of the same degree as A and that both satisfy the heat equation

$$\partial_t u = \frac{1}{2} \Delta_C u, \quad u(0, \varphi) = A(\varphi). \quad (2.1.10)$$

(For v , it is convenient to use $v(t, \varphi) = \int A(\varphi + \sqrt{t} \psi) P_C(d\psi)$ and Gaussian integration by parts.) Since $u = v, w$ are polynomials in φ , the heat equation is equivalent to a finite-dimensional system of linear ODE, with unique solution, and we conclude that $v(t, \cdot) = w(t, \cdot)$ for all $t > 0$. ■

In particular, for a polynomial $A = A(\varphi)$,

$$\mathbb{E}_C A = \mathbb{E}_C \theta A|_{\varphi=0} = e^{\frac{1}{2} \Delta_C} A|_{\varphi=0}, \quad (2.1.11)$$

and thus

$$\mathbb{E}_C(\varphi_x) = 0, \quad \mathbb{E}_C(\varphi_x \varphi_y) = C_{xy}, \quad \mathbb{E}_C(\varphi_x \varphi_y \varphi_u \varphi_v) = C_{xy} C_{uv} + C_{xu} C_{yv} + C_{xv} C_{yu}. \quad (2.1.12)$$

Exercise 2.1.7. By definition, the covariance of random variables F_1, F_2 is

$$\text{Cov}_C(F_1, F_2) = \mathbb{E}_C F_1 F_2 - (\mathbb{E}_C F_1)(\mathbb{E}_C F_2). \quad (2.1.13)$$

By symmetry, $\text{Cov}(\varphi_x^p, \varphi_{x'}^{p'}) = 0$ if $p + p'$ is odd. Show that if $p + p'$ is even then $|\text{Cov}(\varphi_x^p, \varphi_{x'}^{p'})| \leq M_{p,p'} \|C\|^{(p+p')/2}$ where $\|C\| = \max_x C_{xx}$ and $M_{p,p'}$ is a constant depending on p, p' . [Solution]

Proposition 2.1.6 is a version of *Wick's Lemma*; it allows straightforward evaluation of all moments of a Gaussian measure, only in terms of its covariance. The inverse of this formula for expectations of polynomials is *Wick ordering*. The Wick ordering of a polynomial A with respect to a Gaussian measure with covariance C is commonly denoted by $:A:_C$.

Definition 2.1.8. Let $A = A(\varphi)$ be a polynomial. The *Wick ordering* of A with covariance C is

$$:A:_C = e^{-\frac{1}{2}\Delta_C} A. \quad (2.1.14)$$

Thus, essentially by definition,

$$\mathbb{E}_C \theta :A:_C = A. \quad (2.1.15)$$

Note that while the heat semigroup $e^{\frac{1}{2}\Delta_C}$ is contractive on suitable function spaces, and can thus be extended to much more general non-polynomial A , Wick ordering can be interpreted as running the heat equation backwards. For general initial data, this is problematic, but for nice initial data (and polynomials are extremely nice) it is perfectly well-defined. For example, in the proof of Proposition 2.1.6, for polynomials the heat equation is equivalent to a linear ODE, and any linear ODE can be run either forward or backward.

A fundamental property of Gaussian measures is their characterisation by the Laplace transform, also called moment generating function in probability theory.

Proposition 2.1.9. A random field $\varphi \in \mathbb{R}^X$ is Gaussian with covariance C if and only if

$$\mathbb{E}_C(e^{(f, \varphi)}) = e^{\frac{1}{2}(f, Cf)} \quad \text{for all } f \in \mathbb{R}^X. \quad (2.1.16)$$

Proof. Suppose first that C is positive definite. By completion of the square,

$$-\frac{1}{2}(\varphi, C^{-1}\varphi) + (f, \varphi) = -\frac{1}{2}(\varphi - Cf, C^{-1}(\varphi - Cf)) + \frac{1}{2}(f, Cf). \quad (2.1.17)$$

Then (2.1.16) follows by the change of variables $\varphi \mapsto \varphi + Cf$, which leaves the Lebesgue measure invariant. This proves the “only if” direction, and the “if” direction then follows from the fact that the Laplace transform characterises probability measures uniquely [34, p. 390].

If C is positive semi-definite but not positive definite, the Gaussian measure is defined by (2.1.3). The restriction C' of C to the support $K^\perp$ of P_C is invertible, C' and its inverse are isomorphisms of $K^\perp$, and $Cf \in K^\perp$. The reasoning used for the positive definite case thus applies also here. ■

The “only if” direction of Proposition 2.1.9 has the following generalisation which we will use later.

Exercise 2.1.10. For $Z_0 = Z_0(\varphi)$ bounded,

$$\mathbb{E}_C(e^{(f,\varphi)} Z_0(\varphi)) = e^{\frac{1}{2}(f,Cf)} (\mathbb{E}_C \theta Z_0)(Cf) \quad \text{for all } f \in \mathbb{R}^X. \quad (2.1.18)$$

[Solution]

Proposition 2.1.9 also implies the following essential corollary.

Corollary 2.1.11. *Let φ_1 and φ_2 be independent Gaussian fields with covariances C_1 and C_2 . Then $\varphi_1 + \varphi_2$ is a Gaussian field with covariance $C_1 + C_2$. In terms of convolution,*

$$\mathbb{E}_{C_2} \theta \circ \mathbb{E}_{C_1} \theta = \mathbb{E}_{C_1+C_2} \theta. \quad (2.1.19)$$

Proof. By independence, for any $f \in \mathbb{R}^X$,

$$\mathbb{E}(e^{(f,\varphi_1+\varphi_2)}) = \mathbb{E}(e^{(f,\varphi_1)}) \mathbb{E}(e^{(f,\varphi_2)}) = e^{\frac{1}{2}(f,(C_1+C_2)f)}. \quad (2.1.20)$$

By Proposition 2.1.9, $\varphi_1 + \varphi_2$ is Gaussian with covariance $C_1 + C_2$. ■

Corollary 2.1.11 is fundamental for our implementation of the renormalisation group method, whose starting point is a decomposition $C = \sum_{j=1}^N C_j$ of the covariance $C = (-\Delta + m^2)^{-1}$. This allows us to rewrite a Gaussian convolution $\mathbb{E}_C \theta Z_0$, that is difficult to evaluate, into a sequence of convolutions

$$\mathbb{E}_C \theta Z_0 = \mathbb{E}_{C_N} \theta \circ \cdots \circ \mathbb{E}_{C_1} \theta Z_0, \quad (2.1.21)$$

where each expectation on the right-hand side is more tractable.

Example 2.1.12. Let Δ_β be the mean-field Laplacian matrix (1.4.1). Since P and Q are orthogonal projections with $P + Q = \text{Id}$,

$$-\Delta_\beta + m^2 = (\beta + m^2)P + m^2Q. \quad (2.1.22)$$

For $m^2 > 0$, it then follows from the spectral theorem that

$$(-\Delta_\beta + m^2)^{-1} = \frac{1}{\beta + m^2}P + \frac{1}{m^2}Q = \frac{1}{\beta + m^2} + \frac{\beta}{m^2(\beta + m^2)}Q. \quad (2.1.23)$$

The left-hand side is the covariance matrix of a Gaussian field and the two matrices on the right-hand side are each positive definite. This provides a simple example to which (2.1.21) can be applied, with $N = 2$. In fact, Lemma 1.4.2 can be regarded as a limiting case of this fact, where one of the Gaussian measures becomes degenerate in the limit $m^2 \downarrow 0$. For Euclidean or hierarchical models, we use the more elaborate covariance decompositions discussed at length in Chapters 3 and 4.

The following exercise establishes properties of the n -component Gaussian field of Example 2.1.4.

Exercise 2.1.13. Let $C = (C_{xy})_{x,y \in \Lambda}$ be a positive semi-definite matrix on $\mathbb{R}^\Lambda$.

(i) Verify that the components of the corresponding n -component Gaussian field are independent and identically distributed Gaussian fields on Λ with covariance C .

(ii) Let $T \in O(n)$ act on $\mathbb{R}^{n\Lambda}$ by $(T\phi)_x = T\phi_x$ for $x \in \Lambda$, and on $F : \mathbb{R}^{n\Lambda} \rightarrow \mathbb{R}$ by $TF(\phi) = F(T\phi)$. We say that F is $O(n)$ -invariant if $TF = F$ for all $T \in O(n)$. Prove that the n -component Gaussian field is $O(n)$ -invariant, in the sense that for any bounded measurable $F : \mathbb{R}^{n\Lambda} \rightarrow \mathbb{R}$ and $T \in O(n)$,

$$\mathbb{E}_C(F(\phi)) = \mathbb{E}_C(F(T\phi)), \quad \mathbb{E}_C\theta \circ T = T \circ \mathbb{E}_C\theta. \quad (2.1.24)$$

In particular, if F is $O(n)$ -invariant then so is $\mathbb{E}_C\theta F$, and if F_1, F_2 are both $O(n)$ -invariant then so is $\text{Cov}_C(\theta F_1, \theta F_2)$. [\[Solution\]](#)

A second consequence of Proposition 2.1.9 is the following corollary.

Corollary 2.1.14. Let $Y \subset X$. The restriction of P_C to $\mathbb{R}^Y$ is the centred Gaussian probability measure with covariance $C|_{Y \times Y}$.

We are ultimately interested in the infinite-volume limit for the $|\phi|^4$ model. For this, we work with finite sets approximating $\mathbb{Z}^d$, with the aim of obtaining estimates that hold uniformly in the size of the finite set. For Gaussian fields, a construction in infinite volume can be made directly, as a consequence of Corollary 2.1.14.

Exercise 2.1.15. Let S be a possibly infinite set. By definition, an $S \times S$ matrix C is positive definite if $C|_{X \times X}$ is a positive definite matrix for every finite $X \subset S$. Let C be positive definite. Use Corollary 2.1.14 to show that $P_{C|_{X \times X}}$, ($X \subset S$ finite) forms a consistent family of measures. Use the Kolmogorov extension theorem (or the nicer Kolmogorov–Nelson extension theorem [87, Theorem 10.18]) to conclude that there exists a probability measure P_C on $\mathbb{R}^{\mathbb{Z}^d}$ with covariance C . [\[Solution\]](#)

2.2 Cumulants

Definition 2.2.1. Let $A_1, \dots, A_n$ be random variables (not necessarily Gaussian) such that $\mathbb{E}(e^{tA_i}) < \infty$ for t in some neighbourhood of $t = 0$. Their *cumulants*, or *truncated expectations*, are defined by

$$\mathbb{E}(A_1; \dots; A_n) = \frac{\partial^n}{\partial t_1 \dots \partial t_n} \log \mathbb{E}(e^{t_1 A_1 + \dots + t_n A_n}) \Big|_{t_1 = \dots = t_n = 0}. \quad (2.2.1)$$

The truncated expectation of a single random variable is its expectation, and the truncated expectation of a pair of random variables is their covariance:

$$\text{Cov}(A_1, A_2) = \mathbb{E}(A_1; A_2) = \mathbb{E}(A_1 A_2) - \mathbb{E}(A_1)\mathbb{E}(A_2). \quad (2.2.2)$$

The assumption of exponential moments is not necessary to define cumulants. Instead, the logarithm of the expectation on the right-hand side of (2.2.1) may be regarded as a formal power series in t , upon which the derivative acts.

Exercise 2.2.2. Show that the truncated expectations up to order n exist if and only if the expectations of the product of up to n of the A_i exist, and that the latter up to order n determine the truncated expectations up to order n and vice-versa. Hint: Let $I = \{i_1, \dots, i_n\}$. A partition π of I is a collection of disjoint nonempty subsets of I whose union is I . Let $\Pi(I)$ denote the set of all partitions of I . Then if we define $\mu_I = \mathbb{E}(A_{i_1} \cdots A_{i_n})$ and $\kappa_I = \mathbb{E}(A_{i_1}; \dots; A_{i_n})$,

$$\mu_I = \sum_{\pi \in \Pi(I)} \prod_{J \in \pi} \kappa_J. \quad (2.2.3)$$

This system of equations, one for each I , uniquely defines κ_I for all I . [Solution]

The next exercise shows that a collection of random variables is Gaussian if and only if all higher truncated expectations vanish.

Exercise 2.2.3. Use Proposition 2.1.9 and Exercise 2.2.2 to show that a random field φ on X is a Gaussian field with covariance C if and only if for all $p \in \mathbb{N}$ and $x_1, \dots, x_p \in X$,

$$\mathbb{E}(\varphi_{x_1}; \dots; \varphi_{x_p}) = \begin{cases} C_{x_1 x_2} & (p = 2) \\ 0 & (p \neq 2). \end{cases} \quad (2.2.4)$$

[Solution]

In the case of Gaussian fields, with $A_i = A_i(\varphi)$, it is useful to define a convolution version of truncated expectation, by

$$\mathbb{E}_C(\theta A_1; \dots; \theta A_n) = \frac{\partial^n}{\partial t_1 \cdots \partial t_n} \log \mathbb{E}_C \theta(e^{t_1 A_1 + \cdots + t_n A_n}) \Big|_{t_1 = \cdots = t_n = 0}. \quad (2.2.5)$$

In particular,

$$\mathbb{E}_C(\theta A; \theta B) = \text{Cov}_C(\theta A, \theta B), \quad (2.2.6)$$

where

$$\text{Cov}_C(\theta A, \theta B) = \mathbb{E}_C \theta(AB) - (\mathbb{E}_C \theta A)(\mathbb{E}_C \theta B). \quad (2.2.7)$$

If A, B are polynomials, then, by Proposition 2.1.6,

$$\mathbb{E}_C(\theta A; \theta B) = e^{\frac{1}{2} \Delta_C} (AB) - (e^{\frac{1}{2} \Delta_C} A)(e^{\frac{1}{2} \Delta_C} B). \quad (2.2.8)$$

Exercise 2.2.4. For A, B polynomials in φ , let

$$F_C(A, B) = e^{\frac{1}{2} \Delta_C} ((e^{-\frac{1}{2} \Delta_C} A)(e^{-\frac{1}{2} \Delta_C} B)) - AB. \quad (2.2.9)$$

Then $\mathbb{E}_C(\theta A; \theta B) = F_C(\mathbb{E}_C \theta A, \mathbb{E}_C \theta B)$. Show that, if A, B have degree at most p , then

$$F_C(A, B) = \sum_{n=1}^p \frac{1}{n!} \sum_{x_1, y_1} \cdots \sum_{x_n, y_n} C_{x_1, y_1} \cdots C_{x_n, y_n} \frac{\partial^n A}{\partial \varphi_{x_1} \cdots \partial \varphi_{x_n}} \frac{\partial^n B}{\partial \varphi_{y_1} \cdots \partial \varphi_{y_n}}. \quad (2.2.10)$$

[Solution]

Chapter 3

Finite-range decomposition

Our implementation of the renormalisation group method relies on the decomposition of convolution by a Gaussian free field (GFF) into a sequence of convolutions, as in (2.1.21). This requires an appropriate decomposition of the covariance of the Gaussian field into a sum of simpler covariances. Such covariance decompositions, in the context of renormalisation, go back a long way, early examples can be found in [30, 31].

In this chapter, we describe covariance decompositions which have a *finite-range* property. This property is an important ingredient in our renormalisation group method for models defined on the Euclidean lattice [57]. We begin in Section 3.1 by defining the finite-range property and elaborating on (2.1.21) and its role in progressive integration. In Section 3.2, we motivate the finite-range decomposition by first discussing it in the much simpler continuum setting. In Section 3.3, we give a self-contained presentation of a finite-range decomposition of the lattice operator $(-\Delta + m^2)^{-1}$ on $\mathbb{Z}^d$ following the method of [17] (a related earlier method was developed in [48]). This easily gives rise to a finite-range decomposition on the discrete torus, as discussed in Section 3.4.

After this chapter, we do not return to Euclidean models until Appendix A, so in a sense this chapter is a cultural excursion. However, the finite-range decomposition of Proposition 3.3.1 provides a useful motivation for the hierarchical model that becomes our focus after this chapter.

3.1 Progressive integration

Recall from (2.1.21) that a decomposition

$$C = C_1 + \cdots + C_N \tag{3.1.1}$$

of the covariance C provides a way to evaluate a Gaussian expectation progressively, namely,

$$\mathbb{E}_C \theta F = \mathbb{E}_{C_N} \theta \circ \cdots \circ \mathbb{E}_{C_1} \theta F. \quad (3.1.2)$$

This is the point of departure for the renormalisation group method. It allows the left-hand side to be evaluated progressively, one C_j at a time. For this to be useful, the convolutions on the right-hand side need to be more tractable than the original convolution, and therefore useful estimates on the C_j are needed.

In this chapter, we explain a method to decompose the covariance $C = (-\Delta + m^2)^{-1}$ for three different interpretations of the Laplacian: the continuum operator on $\mathbb{R}^d$ (with $m^2 = 0$), the discrete operator on $\mathbb{Z}^d$, and finally the discrete operator on a periodic approximation to $\mathbb{Z}^d$. In each case, we are interested in decompositions with a particular finite-range property.

Definition 3.1.1. Let ζ be a centred Gaussian field on Λ . We say ζ is *finite range* with *range* r if

$$\mathbb{E}_C(\zeta_x \zeta_y) = 0 \quad \text{if } |x - y|_1 > r. \quad (3.1.3)$$

The following exercise demonstrates that the finite-range property has an important consequence for independence.

Exercise 3.1.2. Let φ_x, φ_y be jointly Gaussian random variables which are *uncorrelated*, i.e., $\mathbb{E}(\varphi_x \varphi_y) = 0$. Use Proposition 2.1.9 to show that φ_x and φ_y are *independent*. (For general random variables, independence is a stronger property than being uncorrelated, but for Gaussian random variables the two concepts coincide.) [\[Solution\]](#)

In view of (3.1.2), decomposition of the covariance $C = (-\Delta + m^2)^{-1}$ as $C = \sum_j C_j$, where the matrices C_j are symmetric and positive definite, is equivalent to a decomposition of the GFF φ as

$$\varphi \stackrel{D}{=} \zeta_1 + \cdots + \zeta_N, \quad (3.1.4)$$

where the ζ_j are independent Gaussian fields. Explicitly, for $C = C_1 + C_2$, we have

$$(\mathbb{E}_C \theta F)(\varphi') = \mathbb{E}_C F(\varphi + \varphi') = \mathbb{E}_{C_2} \mathbb{E}_{C_1} F(\zeta_1 + \zeta_2 + \varphi'), \quad (3.1.5)$$

where in the middle the expectation acts on φ , while on the right-hand side each expectation with respect to C_j acts on ζ_j . The fields ζ_j have the finite-range property with range $r = \frac{1}{2}L^j$ if and only if $C_{j;x,y} = 0$ for $|x - y|_1 > \frac{1}{2}L^j$.

3.2 Finite-range decomposition: continuum

In this section, we work frequently with the Fourier transform

$$\hat{f}(p) = \int_{\mathbb{R}^d} f(y) e^{-ipy} dy \quad (3.2.1)$$

of functions $f : \mathbb{R}^d \rightarrow \mathbb{R}$ defined on the continuum. The inverse Fourier transform is

$$f(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \hat{f}(p) e^{ipy} dy. \quad (3.2.2)$$

Definition 3.2.1. A function $f : \mathbb{R}^d \rightarrow \mathbb{R}$ is *positive definite* if it is continuous and has the property that for every integer n and every sequence $(x_1, \dots, x_n)$ of points in $\mathbb{R}^n$ the $n \times n$ matrix $f(x_i - x_j)$ is positive semi-definite.

Exercise 3.2.2. For any $h \in C_c(\mathbb{R}^d, \mathbb{R})$ with $h(-x) = h(x)$, the convolution $h * h$ is positive definite. More generally, if f has Fourier transform obeying $\hat{f} \geq 0$, then f is positive definite. (The converse is also true; this is Bochner's theorem [132, Theorem IX.9].) [Solution]

Proposition 3.2.3. Given $L > 1$ and $\alpha > 0$, there exists $u : \mathbb{R}^d \rightarrow \mathbb{R}$ which is smooth, positive definite, with support in $[-\frac{1}{2}, \frac{1}{2}]^d$, such that

$$|x|^{-\alpha} = \sum_{j \in \mathbb{Z}} L^{-\alpha j} u(L^{-j} x) \quad (x \neq 0). \quad (3.2.3)$$

For $d \neq 2$, and $\alpha = d - 2$, the left-hand side of (3.2.3) is a multiple of the Green function of the Laplace operator $\sum_{i=1}^d \partial_i^2$ on $\mathbb{R}^d$. A similar representation exists for $d = 2$. The right-hand side of (3.2.3) provides a finite-range decomposition of the Green function, in the sense that the j^{th} term vanishes if $|x|_\infty > \frac{1}{2} L^j$. This is an unimportant departure from the definition in terms of $|x|_1$ given below (3.1.4). The scales $j \leq 0$ which appear in the sum are absent for a lattice decomposition. The proof shows that there is considerable flexibility in the choice of the function u .

Proof of Proposition 3.2.3. Choose a function $w \in C_c(\mathbb{R})$ which is not the zero function. By the change of variables $t \mapsto |x|t$,

$$\int_0^\infty t^{-\alpha} w(|x|/t) \frac{dt}{t} = c |x|^{-\alpha}, \quad (3.2.4)$$

with $c = \int_0^\infty t^{-\alpha} w(1/t) \frac{dt}{t}$. After normalising w by multiplication by a constant so that $c = 1$, we obtain

$$|x|^{-\alpha} = \int_0^\infty t^{-\alpha} w(|x|/t) \frac{dt}{t}. \quad (3.2.5)$$

Now choose w with support in $[-\frac{1}{2}, \frac{1}{2}]$ such that $x \mapsto w(|x|)$ is a smooth, positive definite function on $\mathbb{R}^d$. By Exercise 3.2.2, a function w with these properties exists. Given $L > 1$, set

$$u(x) = \int_{1/L}^1 t^{-\alpha} w(|x|/t) \frac{dt}{t}. \quad (3.2.6)$$

It is not hard to check that this is a positive definite function. By change of variables, (3.2.3) holds, and the proof is complete. ■

A statement analogous to Proposition 3.2.3 for the lattice Green function is more subtle. The proof for the continuum exploited in a crucial way two symmetries,

homogeneity and rotation invariance, which are both violated in the discrete case. To motivate and prepare for the construction of the finite-range decomposition for the lattice, we now present another proof of (3.2.3). As in the previous proof, it suffices to show that (3.2.5) holds with w a compactly supported positive definite function. We will create a radial function w whose support is a ball of radius 1 instead of $\frac{1}{2}$; this is an unimportant difference. Our proof exploits a connection with the finite speed of propagation property of hyperbolic equations that originated in [17].

Let $f : \mathbb{R} \rightarrow [0, \infty)$ be such that its Fourier transform is smooth, symmetric, and has support in $[-1, 1]$. We assume that f is not the zero function. By multiplication of f by a constant, we can arrange that

$$\frac{1}{|k|^2} = \int_0^\infty t^2 f(|k|t) \frac{dt}{t} \quad (k \in \mathbb{R}^d, |k| \neq 0). \quad (3.2.7)$$

Indeed, (3.2.7) is just (3.2.4) with $w = f$ and $\alpha = 2$, after change of variables from t to $1/t$. For $d > 2$, the Green function $|x|^{-(d-2)}$ has Fourier transform proportional to $1/|k|^2$. By inverting the Fourier transform, we obtain

$$|x|^{-(d-2)} \propto \int_0^\infty w(t, x) \frac{dt}{t} \quad (3.2.8)$$

where

$$w(t, x) = (2\pi)^{-d} \int_{\mathbb{R}^d} t^2 f(|k|t) e^{ik \cdot x} dk. \quad (3.2.9)$$

Define $w(x) = w(1, x)$. By change of variable, $w(t, x) = t^{-(d-2)} w(x/t)$. In (3.2.8) we have achieved a decomposition like (3.2.5) where $w(t, x)$ has the desired positive definiteness; it remains to prove that $w(x)$ is supported in the unit ball.

By hypothesis the (1-dimensional) Fourier transform $\hat{f}$ is symmetric and $\text{supp } \hat{f} \subset [-1, 1]$. Therefore

$$f(|k|) = (2\pi)^{-1} \int_{-1}^1 \hat{f}(s) \cos(|k|s) ds. \quad (3.2.10)$$

By inserting this into (3.2.9) and setting $t = 1$ we read off the d -dimensional Fourier transform

$$\hat{w}(k) = (2\pi)^{-1} \int_{-1}^1 \hat{f}(s) \cos(|k|s) ds. \quad (3.2.11)$$

That w has support in the unit ball is a consequence of the finite propagation speed of the wave equation, as follows. It suffices to show, for any smooth function u_0 on $\mathbb{R}^d$, that the support of $w * u_0$ is contained in the 1-neighbourhood $\{x \in \mathbb{R}^d \mid \text{dist}(x, \text{supp } u_0) \leq 1\}$ of $\text{supp } u_0$, because we can replace u_0 by the approximate identity $\varepsilon^{-d} u_0(x/\varepsilon)$ and let $\varepsilon \downarrow 0$. Let $u(s, x)$ be the solution to the (d -dimensional) wave equation

$$\frac{\partial^2 u}{\partial s^2} = \Delta u, \quad u(0, x) = u_0(x), \quad \frac{\partial}{\partial s} u(0, x) = 0. \quad (3.2.12)$$

The solution to this equation is

$$u(s, x) = (2\pi)^{-d} \int_{\mathbb{R}^d} \hat{u}_0(k) \cos(s|k|) e^{ik \cdot x} dk. \quad (3.2.13)$$

By combining this with (3.2.11) we have

$$w * u_0(x) = (2\pi)^{-1} \int_{-1}^1 \hat{f}(s) u(s, x) ds. \quad (3.2.14)$$

By the finite propagation speed of the wave equation, the support of $u(s, \cdot)$ is contained in the $|s|$ -neighbourhood of $\text{supp } u_0$. Since the range of the s integral is $s \leq 1$ we have proved that the support of $w * u_0$ is contained in the 1-neighbourhood of $\text{supp } u_0$ as desired.

The formula (3.2.7) generalises to a representation for $|k|^{-\alpha}$ for other values of α by using a different power of t inside the integral, so that finite-range decompositions for $|x|^{\alpha-d}$ can also be constructed by this method. Furthermore, the method applies to the Green function in dimension $d \leq 2$ with the correct interpretation of the domain of function on which the Green function acts.

Exercise 3.2.4. Use the Schwartz–Paley–Wiener Theorem to deduce from (3.2.10) that w has support in the unit ball without referring to the finite propagation speed of the wave equation explicitly. [\[Solution\]](#)

3.3 Finite-range decomposition: lattice

We present a construction of the finite-range decomposition for the lattice Green function which is based on the wave equation perspective of the continuum decomposition explained in (3.2.8)–(3.2.14). The wave equation is now replaced by a discrete wave equation. For the discrete wave equation, the Chebyshev polynomials T_t play a role analogous to the functions $\cos(\sqrt{\cdot}t)$ for the continuous wave equation.

3.3.1 Statement of the decomposition

In this section we state a proposition which provides a decomposition of $(-\Delta_{\mathbb{Z}^d} + m^2)^{-1}$ for all $d > 0$ and $m^2 > 0$. The proposition gives the existence and properties of covariances C_j on $\mathbb{Z}^d$ such that

$$(\Delta_{\mathbb{Z}^d} + m^2)^{-1} = \sum_{j=1}^{\infty} C_j, \quad (3.3.1)$$

where C_j depends on $m^2 > 0$ and the sum converges in the sense of quadratic forms, i.e.,

$$(f, (\Delta_{\mathbb{Z}^d} + m^2)^{-1} f) = \sum_{j=1}^{\infty} (f, C_j f) \quad (f \in \ell^2(\mathbb{Z}^d)). \quad (3.3.2)$$

In particular, by polarisation (choose $f = \delta_x + \delta_y$ and $f = \delta_x - \delta_y$), it also implies convergence of the matrix elements $C_{j;xy}$. The covariances C_j are translation invariant, and have the *finite-range* property that $C_{j;xy} = 0$ if $|x - y|_1 \geq \frac{1}{2}L^j$.

Finite-difference derivatives are defined as follows. For $i = 1, \dots, d$ let e_i be the unit vector $(0, \dots, 1, 0, \dots, 0)$ whose i th component equals 1, and let $e_{-i} = -e_i$ so that, as i ranges over $\{-d, \dots, -1, 1, \dots, d\}$, e_i ranges over the unit vectors in the lattice $\mathbb{Z}^d$. For a function $f : \mathbb{Z}^d \rightarrow \mathbb{R}$ define $\nabla^{e_i} f_x = f_{x+e_i} - f_x$. For a multi-index $\alpha \in \{-d, \dots, d\}^n$ define

$$\nabla^\alpha f = \nabla^{e_{\alpha_1}} \dots \nabla^{e_{\alpha_n}} f. \quad (3.3.3)$$

For example, for $\alpha = (1, -2)$,

$$\nabla^\alpha f_x = (\nabla^{e_1} \nabla^{e_{-2}} f)_x = (\nabla^{e_{-2}} f)_{x+e_1} - (\nabla^{e_{-2}} f)_x = f_{x+e_1-e_2} - f_{x+e_1} - f_{x-e_2} + f_x. \quad (3.3.4)$$

Dependence of C_j on m^2 is captured in terms of the parameter ϑ defined, for $s, t, m^2 \geq 0$ and $j \geq 1$, by

$$\vartheta(t, m^2; s) = \frac{1}{2d + m^2} \left(1 + \frac{m^2 t^2}{2d + m^2} \right)^{-s}, \quad \vartheta_j(m^2; s) = \vartheta(L^j, m^2; s). \quad (3.3.5)$$

Proposition 3.3.1. *Let $d > 0$ and $L > 1$. For all $m^2 > 0$ there exist positive semi-definite matrices $(C_j)_{j \geq 1}$ such that (3.3.1) holds, and such that for all $j \geq 1$,*

$$C_{j;xy} = 0 \quad \text{if } |x - y|_1 \geq \frac{1}{2}L^j \quad (\text{finite-range property}). \quad (3.3.6)$$

The matrix elements $C_{j;xy}$ are functions of $x - y$, are continuous functions of m^2 and have limits as $m^2 \downarrow 0$. Moreover, for all multi-indices α and all $s \geq 0$, there are constants $c_{\alpha,s}$ such that, for all $m^2 \in [0, \infty)$ and $j \geq 1$,

$$|\nabla^\alpha C_{j;xy}| \leq c_{\alpha,s} f_d(L) \vartheta_{j-1}(m^2; s) L^{-(d-2+|\alpha|_1)(j-1)} \quad (\text{scaling estimates}), \quad (3.3.7)$$

with $f_d(L) = 1$ for $d > 2$, $f_2(L) = \log L$, and $f_d(L) = L^{2-d}$ for $d < 2$. The discrete gradients can act either on x or y .

Estimates on derivatives of C_j with respect to m^2 can be found in [17]. We prove Proposition 3.3.1 using the construction of [17]. Finite-range decompositions for the lattice Green function were first constructed in [48], using a different method. Yet another method, which is very general, is used in [4, 42, 59, 135]. Such decompositions have also been obtained for fractional powers of the Laplacian [48, 124, 125, 143].

3.3.2 Integral decomposition

The decomposition we use is structurally similar to that discussed in connection with the wave equation in Section 3.2. Roughly speaking, the Fourier multiplier $|k|^2$ of the continuum Laplacian is replaced by the Fourier multiplier $\lambda(k) = 4 \sum_{j=1}^d \sin^2(k_j/2)$ of the discrete Laplacian given in (1.5.26).

Let f be as in (3.2.7). For $t > 0$, we set

$$f_t^*(x) = \sum_{n \in \mathbb{Z}} f(xt - 2\pi nt) \quad (x \in \mathbb{R}). \quad (3.3.8)$$

Since $\hat{f}$ is smooth, f decays rapidly and therefore the sum on the right-hand side is well-defined for $t > 0$. Moreover, $f_t^* \geq 0$ since $f \geq 0$.

Lemma 3.3.2. For $x \in \mathbb{R} \setminus 2\pi\mathbb{Z}$,

$$\frac{1}{4} \sin^{-2}\left(\frac{1}{2}x\right) = \int_0^\infty t^2 f_t^*(x) \frac{dt}{t}. \quad (3.3.9)$$

Proof. The left-hand side is a meromorphic function on $\mathbb{C}$ with poles at $2\pi\mathbb{Z}$. Its development into partial fractions is (see e.g. [6, p. 204])

$$\frac{1}{4} \sin^{-2}\left(\frac{1}{2}x\right) = \sum_{n \in \mathbb{Z}} (x - 2\pi n)^{-2} \quad (x \in \mathbb{C} \setminus 2\pi\mathbb{Z}). \quad (3.3.10)$$

From (3.2.7) with $|k|$ replaced by $x - 2\pi n$, it follows that

$$\frac{1}{4} \sin^{-2}\left(\frac{1}{2}x\right) = \sum_{n \in \mathbb{Z}} \int_0^\infty t^2 f((x - 2\pi n)t) \frac{dt}{t}. \quad (3.3.11)$$

By hypothesis, f is symmetric, so (3.2.7) holds when $|k|$ in the right-hand side is replaced by the possibly negative $x - 2\pi n$. The order of the sum and the integral can be exchanged, by non-negativity of the integrand, and the proof is complete. ■

For $t > 0$ and $\zeta \in [0, 4]$, we set

$$P_t(\zeta) = f_t^*\left(\arccos\left(1 - \frac{1}{2}\zeta\right)\right). \quad (3.3.12)$$

Since $f_t^* \geq 0$, also $P_t(\zeta) \geq 0$.

Lemma 3.3.3. For $\zeta \in (0, 4)$,

$$\frac{1}{\zeta} = \int_0^\infty t^2 P_t(\zeta) \frac{dt}{t}. \quad (3.3.13)$$

Proof. Let $x = \arccos(1 - \frac{1}{2}\zeta)$, so that $\zeta = 2(1 - \cos x) = 4 \sin^2(\frac{1}{2}x)$. By (3.3.9) and (3.3.12),

$$\frac{1}{\zeta} = \int_0^\infty t^2 f_t^*(x) \frac{dt}{t} = \int_0^\infty t^2 P_t(\zeta) \frac{dt}{t}, \quad (3.3.14)$$

and the proof is complete. $\blacksquare$

We wish to apply (3.3.13) with $\zeta = \lambda(k) + m^2$ for $k \in [-\pi, \pi]^d$, but for m^2 large this choice may not be in $(0, 4)$. Therefore let $M^2 = 2d + m^2$ and set $\zeta = (\lambda(k) + m^2)/M^2$; then $\zeta \in (0, 2]$ provided $m|k| \neq 0$. By (3.3.13),

$$\frac{1}{\lambda(k) + m^2} = \int_0^\infty \hat{w}(t, k) \frac{dt}{t}, \quad (3.3.15)$$

with

$$\hat{w}(t, k) = \frac{t^2}{M^2} P_t \left(\frac{1}{M^2} (\lambda(k) + m^2) \right) \quad (k \in [-\pi, \pi]^d). \quad (3.3.16)$$

Here and below, $k \in [-\pi, \pi]^d$ denotes the Fourier variable of a function defined on the discrete space $\mathbb{Z}^d$ whose points are denoted by $x \in \mathbb{Z}^d$. We use the same letter w to denote the discrete analogue of the function (3.2.9) (which is on the continuum). By (1.5.28), inversion of this d -dimensional discrete Fourier transform gives

$$(-\Delta_{\mathbb{Z}^d} + m^2)_{0x}^{-1} = \int_0^\infty w(t, x) \frac{dt}{t}, \quad (3.3.17)$$

where

$$w(t, x) = \frac{1}{(2\pi)^d} \int_{[-\pi, \pi]^d} \hat{w}(t, k) e^{ik \cdot x} dk \quad (x \in \mathbb{Z}^d). \quad (3.3.18)$$

The identity (3.3.17) is the essential ingredient for the finite-range decomposition. We decompose the integral into intervals $[0, \frac{1}{2}L]$ and $[\frac{1}{2}L^{j-1}, \frac{1}{2}L^j]$ (for $j \geq 2$), and define, for $x \in \mathbb{Z}^d$,

$$C_{1;0x} = \int_0^{\frac{1}{2}L} w(t, x) \frac{dt}{t}, \quad (3.3.19)$$

$$C_{j;0x} = \int_{\frac{1}{2}L^{j-1}}^{\frac{1}{2}L^j} w(t, x) \frac{dt}{t} \quad (j \geq 2). \quad (3.3.20)$$

By (3.3.15), this gives, for $k \in [-\pi, \pi]^d$ and $m^2 \neq 0$,

$$\frac{1}{\lambda(k) + m^2} = \sum_{j=1}^\infty \hat{C}_j(k), \quad (3.3.21)$$

where $\hat{C}_j \geq 0$ is the discrete Fourier transform of C_j . Thus, for any $f \in \ell^2(\mathbb{Z}^d)$,

$$(f, (-\Delta_{\mathbb{Z}^d} + m^2)^{-1} f) = \sum_{j=1}^\infty (f, C_j f), \quad (3.3.22)$$

which proves (3.3.1). Furthermore, by (3.3.16), (3.3.19) and (3.3.20), the inequality $P_t(\zeta) \geq 0$ implies that this decomposition is positive semi-definite.

In Section 3.3.3, we will prove that $P_t(\zeta)$ is a polynomial in ζ of degree at most t . This implies the finite-range property (3.3.6). In fact, by (3.3.16), up to a scalar multiple, $w(t, x - y)$ is the kernel that represents the operator $P_t(M^{-2}(-\Delta + m^2))$, which is then a polynomial in $-\Delta + m^2$ of degree at most t . Since $-\Delta_{xy}$ vanishes unless $|x - y|_1 \leq 1$, it follows that $w(t, x) = 0$ if $|x|_1 > t$. By (3.3.19) and (3.3.20), this gives the finite-range property (3.3.6).

The integration domain for the covariance C_1 differs from the domain for C_j with $j \geq 2$. It is therefore natural to decompose it as $C_0 + C'_1$ with

$$C_{0;0x} = \int_0^1 w(t, x) \frac{dt}{t}, \quad C'_{1;0x} = \int_1^{\frac{1}{2}L} w(t, x) \frac{dt}{t}. \quad (3.3.23)$$

Then C'_1 is of the same form as C_j with $j \geq 2$. We show in Section 3.3.4 that the integral $C_{0;0x}$ can be computed exactly:

$$C_{0;0x} = \frac{1}{2d + m^2} \frac{\hat{f}(0)}{2\pi} \mathbb{1}_{x=0}. \quad (3.3.24)$$

In summary, we have proved that there exist positive semi-definite matrices $(C_j)_{j \geq 1}$ such that (3.3.1) holds as asserted in Proposition 3.3.1 and reduced the finite-range property (3.3.6) to the claim that $P_t(\zeta)$ is a polynomial in ζ of degree at most t .

3.3.3 Chebyshev polynomials

We now obtain properties of P_t defined in (3.3.12). In particular, we show that $P_t(\zeta)$ is a polynomial in ζ of degree at most t . At the end of the section, we discuss parallels with the finite speed of propagation argument in Section 3.2. Now it is the discrete wave equation that is relevant, as is the fact that its fundamental solution can be written in terms of Chebyshev polynomials.

Recall the definition of $P_t(\zeta)$ in (3.3.12). It involves the function f_t^* . By its definition in (3.3.8), f_t^* is periodic with period 2π . By Poisson summation, it can be written in terms of the continuum Fourier transform of f as

$$f_t^*(x) = (2\pi)^{-1} \sum_{p \in \mathbb{Z}} t^{-1} \hat{f}(p/t) \cos(px) \quad (x \in \mathbb{R}). \quad (3.3.25)$$

Exercise 3.3.4. Prove (3.3.25). [Solution]

The Chebyshev polynomials T_p of the first kind are the polynomials of degree $|p|$ defined by

$$T_p(\theta) = \cos(p \arccos(\theta)) \quad (\theta \in [-1, 1], p \in \mathbb{Z}). \quad (3.3.26)$$

Lemma 3.3.5. *For any $t > 0$, when restricted to the interval $\zeta \in [0, 4]$, $P_t(\zeta)$ is a polynomial in ζ , of degree bounded by t .*

Proof. By (3.3.12), (3.3.25), (3.3.26), and $\text{supp}(\hat{f}) \subseteq [-1, 1]$,

$$\begin{aligned} P_t(\zeta) &= \frac{1}{2\pi} \sum_{p \in \mathbb{Z}} t^{-1} \hat{f}(p/t) \cos(p \arccos(1 - \frac{1}{2}\zeta)) \\ &= \frac{1}{2\pi} \sum_{p \in \mathbb{Z} \cap [-t, t]} t^{-1} \hat{f}(p/t) T_p(1 - \frac{1}{2}\zeta). \end{aligned} \quad (3.3.27)$$

This shows that $P_t(\zeta)$ is indeed the restriction of a polynomial in ζ of degree at most t to the interval $\zeta \in [0, 4]$. ■

The following lemma provides an identity and an estimate for the polynomial $P_t(\zeta)$. Note that P_t is constant for $t < 1$, by Lemma 3.3.5.

Lemma 3.3.6. *For any $s \geq 0$, there exists $c_s > 0$ such that, for $\zeta \in [0, 4]$,*

$$P_t(\zeta) = \frac{\hat{f}(0)}{2\pi t} \quad (t < 1), \quad (3.3.28)$$

$$P_t(\zeta) \leq c_s(1 + t^2|\zeta|)^{-s} \quad (t \geq 1). \quad (3.3.29)$$

Proof. Let $\zeta \in [0, 4]$ and set $x = \arccos(1 - \frac{1}{2}\zeta) \in [0, \pi]$. By (3.3.12), $P_t(\zeta) = f_t^*(x)$.

Case $t < 1$. By (3.3.25),

$$P_t(\zeta) = \frac{1}{2\pi} \sum_{p \in \mathbb{Z}} t^{-1} \hat{f}(p/t) \cos(px) = \frac{1}{2\pi} t^{-1} \hat{f}(0), \quad (3.3.30)$$

because the sum reduces to the single term $p = 0$ by the support property of $\hat{f}$ which implies $p \leq t < 1$. This proves (3.3.28).

Case $t \geq 1$. It suffices to consider integers $s \geq 1$. Since $\hat{f}$ is smooth and compactly supported, f decays faster than any inverse power, i.e., for every $s \geq 1$, $|f(x)| = O_s(|x|^{-s})$ as $|x| \rightarrow \infty$. Therefore, by (3.3.8), there exist c'_s, c_s such that for $x \in [0, \pi]$,

$$\begin{aligned} |P_t(\zeta)| &\leq c'_s \sum_{n \in \mathbb{Z}} (1 + t|x - 2\pi n|)^{-4s} \leq c'_s \sum_{n \in \mathbb{Z}} (1 + tx + t\pi|n|)^{-4s} \\ &\leq c'_s(1 + tx)^{-2s} \sum_{n \in \mathbb{Z}} (1 + t\pi|n|)^{-2s} \leq c_s(1 + tx)^{-2s}, \end{aligned} \quad (3.3.31)$$

since the last sum converges. Since $\zeta = 4 \sin^2(\frac{x}{2}) \leq x^2$, we have $x \geq \sqrt{\zeta}$. Therefore,

$$|P_t(\zeta)| \leq c_s(1 + t\sqrt{\zeta})^{-2s} \leq c_s(1 + t^2\zeta)^{-s}, \quad (3.3.32)$$

and the proof is complete. ■

Lemma 3.3.5 can be understood as a consequence of the finite propagation speed of the discrete wave equation

$$u_{p+1} + u_{p-1} - 2u_p = -\zeta u_p, \quad u_0 \text{ given}, \quad u_1 - u_{-1} = 0, \quad (3.3.33)$$

which is analogous to (3.2.12), with derivatives in s replaced by discrete derivatives in p and with $-\Delta$ replaced by ζ . Its solution is given by

$$u_p = T_p(1 - \frac{1}{2}\zeta)u_0. \quad (3.3.34)$$

The Chebyshev polynomials T_p satisfy the recursion relation $T_{p+1}(\theta) + T_{p-1}(\theta) - 2\theta T_p(\theta) = 0$ so that (3.3.34) solves (3.3.33).

The equation (3.3.27) is analogous to (3.2.10) with the continuum wave operator $\cos(\sqrt{-\Delta}s)$ replaced by the fundamental solution $T_p(1 + \frac{1}{2}\Delta)$ to the discrete wave equation.

3.3.4 Proof of Proposition 3.3.1

To complete the proof of Proposition 3.3.1, the main remaining step is to obtain estimates on the function $w(t, x)$ defined in (3.3.18). The next lemma provides the required estimates.

Lemma 3.3.7. *Fix any dimension $d > 0$. For any $x \in \mathbb{Z}^d$, any multi-index α , and any $s \geq 0$, there exists $c_{s,\alpha} \geq 0$ such that*

$$w(t, x) = \frac{t}{2d + m^2} \frac{\hat{f}(0)}{2\pi} \mathbb{1}_{x=0} \quad (t < 1), \quad (3.3.35)$$

$$|\nabla^\alpha w(t, x)| \leq c_{s,\alpha} \vartheta(t, m^2; s) t^{-(d-2+|\alpha|_1)} \quad (t \geq 1). \quad (3.3.36)$$

Proof. Case $t < 1$. By (3.3.16) and (3.3.28),

$$\begin{aligned} w(t, x) &= (2\pi)^{-d} \int_{[-\pi, \pi]^d} \frac{t^2}{M^2} P_t \left(\frac{1}{M^2} (\lambda(k) + m^2) \right) e^{ik \cdot x} dk \\ &= \frac{t}{M^2} \frac{1}{2\pi} \hat{f}(0) \mathbb{1}_{x=0}. \end{aligned} \quad (3.3.37)$$

This proves (3.3.35).

Case $t \geq 1$. Recall from (3.3.5) that

$$\vartheta(t, m^2; s) = \frac{1}{M^2} \left(1 + \frac{m^2 t^2}{M^2} \right)^{-s}. \quad (3.3.38)$$

By definition,

$$\nabla^\alpha w(t, x) = \frac{t^2}{M^2} \frac{1}{(2\pi)^d} \int_{[-\pi, \pi]^d} P_t \left(\frac{1}{M^2} (\lambda(k) + m^2) \right) \nabla^\alpha e^{ik \cdot x} dk. \quad (3.3.39)$$

We use $|\nabla^\alpha e^{ik \cdot x}| \leq C_\alpha |k|^{|\alpha|_1}$, and apply (3.3.29) with $s = s' + s''$ to obtain

$$\left| P_t \left(\frac{1}{M^2} (\lambda(k) + m^2) \right) \right| \leq O(1 + t^2 \lambda(k)/M^2)^{-s'} (1 + tm^2/M^2)^{-s''}. \quad (3.3.40)$$

Elementary calculus shows that $\lambda(k) \asymp |k|^2$ for $k \in [-\pi, \pi]^d$. Also, with s' chosen larger than $|\alpha|_1 + d/2$, we have

$$\int_{[-\pi, \pi]^d} \frac{|k|^{|\alpha|_1}}{(1 + t^2 |k|^2/M^2)^{s'}} dk = O\left((M/t)^{d+|\alpha|_1} \wedge \pi^{|\alpha|_1}\right), \quad (3.3.41)$$

where the first option on the right-hand side arises from extending the domain of integration to $\mathbb{R}^d$ and making the change of variables $k \mapsto (M/t)k$, and the second arises by bounding the integrand by $\pi^{|\alpha|_1}$. With the choice $s'' = s + (d + |\alpha|_1)/2$, it follows that

$$|\nabla^\alpha w(t, x)| \leq O(1 + t^2 m^2/M^2)^{-s-(d+|\alpha|_1)/2} O_\alpha \left((t/M)^{2-d-|\alpha|_1} \wedge (t/M)^2 \right). \quad (3.3.42)$$

For $m^2 \leq 1$, we have $M^2 \asymp 1$ and (3.3.36) follows immediately by choosing the first option in the minimum on the right-hand side. For $m^2 \geq 1$, we have instead $M^2 \asymp m^2$, and by choosing the second option in the minimum we now obtain

$$|\nabla^\alpha w(t, x)| = O_\alpha \left((1 + t^2)^{-s-(d+|\alpha|_1)/2} \frac{t^2}{m^2} \right) = O_\alpha \left(\frac{1}{m^2} (1 + t^2)^{-s} t^{2-d-|\alpha|_1} \right). \quad (3.3.43)$$

This completes the proof for $t \geq 1$. $\blacksquare$

Proof of Proposition 3.3.1. As in (3.3.19)–(3.3.20), we define

$$C_{1;0x} = \int_0^{\frac{1}{2}L} w(t, x) \frac{dt}{t}, \quad (3.3.44)$$

$$C_{j;0x} = \int_{\frac{1}{2}L^{j-1}}^{\frac{1}{2}L^j} w(t, x) \frac{dt}{t} \quad (j \geq 2). \quad (3.3.45)$$

By (3.3.15), this gives

$$\frac{1}{\lambda(k) + m^2} = \sum_{j=1}^{\infty} \hat{C}_j(k), \quad (3.3.46)$$

where $\hat{C}_j \geq 0$ is the discrete Fourier transform of C_j . Thus, for any $f \in \ell^2(\mathbb{Z}^d)$,

$$(f, (-\Delta + m^2)^{-1} f) = \sum_{j=1}^{\infty} (f, C_j f), \quad (3.3.47)$$

which proves (3.3.1). Continuity of $C_{j,0x}$ in the mass m^2 can be seen via an application of the dominated convergence theorem to the integrals (3.3.44)–(3.3.45). Since $w(t, x) = 0$ for $|x|_1 > t$ (as pointed out below (3.3.22)), C_j has the finite-range property (3.3.6).

It remains to prove (3.3.7), which we restate here as

$$|\nabla^\alpha C_{j,xy}| \leq c_{\alpha,s} f_d(L) \vartheta_{j-1}(m^2; s) L^{-(d-2+|\alpha|_1)(j-1)}, \quad (3.3.48)$$

with $f_d(L) = 1$ for $d > 2$, $f_2(L) = \log L$, and $f_d(L) = L^{2-d}$ for $d < 2$, and with

$$\vartheta_{j-1}(m^2; s) = \frac{1}{2d + m^2} \left(1 + \frac{L^{2(j-1)} m^2}{(2d + m^2)} \right)^{-s}. \quad (3.3.49)$$

By (3.3.36) and the change of variables $\tau = L^{j-1} t$,

$$\begin{aligned} |\nabla^\alpha \int_{\frac{1}{2}L^{j-1}}^{\frac{1}{2}L^j} w(t, x-y) \frac{dt}{t}| &\leq c \int_{\frac{1}{2}L^{j-1}}^{\frac{1}{2}L^j} \vartheta(t, m^2; s) t^{-(d-2+|\alpha|_1)} \frac{dt}{t} \\ &\leq c' \vartheta_{j-1}(m^2; s) L^{-(j-1)(d-2+|\alpha|_1)} \int_{\frac{1}{2}}^{\frac{1}{2}L} \tau^{-(d-1+|\alpha|_1)} d\tau, \end{aligned} \quad (3.3.50)$$

where the constants can depend on α, s . The τ integral is bounded by $f_d(L)$ in the worst case $\alpha = 0$. By (3.3.45) the left-hand side equals $|\nabla^\alpha C_{j,xy}|$ for $j \geq 2$, which proves the desired bound for $j \geq 2$.

For $j = 1$ the left-hand side is not equal to $C_{1,xy}$ because the lower bound on the t integral is 1 instead of zero. The above argument does provide the desired estimate on the contribution to C_1 due to integration over $[1, \frac{1}{2}L]$. The remaining contribution to C_1 is C_0 defined in (3.3.23), i.e.,

$$C_{0,0x} = \int_0^1 w(t, x) \frac{dt}{t}. \quad (3.3.51)$$

According to (3.3.35),

$$w(t, x) = \frac{t}{2d + m^2} \frac{\hat{f}(0)}{2\pi} \mathbb{1}_{x=0} \quad (t < 1), \quad (3.3.52)$$

and therefore, as claimed in (3.3.24),

$$C_{0,0x} = \frac{1}{2d + m^2} \frac{\hat{f}(0)}{2\pi} \mathbb{1}_{x=0}. \quad (3.3.53)$$

This contribution to C_1 also obeys (3.3.48) with $j = 1$. Indeed, since $\vartheta_0(m^2, s) \geq 2^{-s}(2d + m^2)^{-1}$, we have

$$|\nabla^\alpha C_{0;0x}| \leq 2^s \vartheta_0(m^2; s) \frac{\hat{f}(0)}{2\pi} |\nabla^\alpha \mathbb{1}_{x=0}|. \quad (3.3.54)$$

This completes the proof. $\blacksquare$

3.4 Finite-range decomposition: torus

For $L > 1$, $N \geq 1$, $m^2 > 0$, and $d > 0$, let $\Lambda_N = \mathbb{Z}^d / L^N \mathbb{Z}^d$ be the d -dimensional discrete torus of period L^N . By (3.3.6), $C_{j;x,y+L^N z} = 0$ for $j < N$, $|x - y| < L^N$, and nonzero $z \in \mathbb{Z}^d$. Therefore,

$$C_{j;x,y} = \sum_{z \in \mathbb{Z}^d} C_{j;x,y+zL^N} \quad (j < N). \quad (3.4.1)$$

Thus we can regard C_j either as a $\mathbb{Z}^d \times \mathbb{Z}^d$ matrix or as a $\Lambda \times \Lambda$ matrix if $j < N$. We also define

$$C_{N,N;x,y} = \sum_{z \in \mathbb{Z}^d} \sum_{j=N}^{\infty} C_{j;x,y+zL^N}. \quad (3.4.2)$$

Since

$$(-\Delta_\Lambda + m^2)^{-1}_{x,y} = \sum_{z \in \mathbb{Z}^d} (-\Delta_{\mathbb{Z}^d} + m^2)^{-1}_{x,y+zL^N}, \quad (3.4.3)$$

it follows from Proposition 3.3.1 that

$$(-\Delta_\Lambda + m^2)^{-1} = \sum_{j=1}^{N-1} C_j + C_{N,N}. \quad (3.4.4)$$

In this finite-range decomposition of the torus covariance, the effect of the torus is concentrated in the term $C_{N,N}$. Estimates on $C_{N,N}$ can be derived from Proposition 3.3.1.

The following is an immediate consequence of (3.4.4).

Corollary 3.4.1. *Let $N \geq 1$, and let φ be the GFF with mass $m > 0$ on Λ_N . There exist independent Gaussian fields ζ_j ($j = 1, \dots, N$), such that $\zeta_j = (\zeta_{j,x})_{x \in X}$ are finite range with range $\frac{1}{2}L^j$ and*

$$\varphi \stackrel{D}{=} \zeta_1 + \dots + \zeta_N. \quad (3.4.5)$$

Proof. This follows from (3.4.4) and Exercise 3.1.2. $\blacksquare$

Chapter 4

The hierarchical model

In Section 4.1, we define a *hierarchical Gaussian field* as a field that satisfies a strengthened version of the finite-range decomposition of Chapter 3. The hierarchical Gaussian free field (hGFF) is a hierarchical field that has comparable large distance behaviour to the lattice Gaussian free field. We explicitly construct a version of it and verify that it indeed has the desired properties. In Section 4.2, we define the hierarchical $|\varphi|^4$ model, and in Theorem 4.2.1 state the counterpart of the asymptotic formula (1.6.12) for the hierarchical model's susceptibility. In Section 4.3, we reformulate the hierarchical $|\varphi|^4$ model as a perturbation of a Gaussian integral, in preparation for its renormalisation group analysis.

4.1 Hierarchical GFF

4.1.1 Hierarchical fields

Periodic boundary conditions are not appropriate for hierarchical fields. Throughout our discussion and analysis of the hierarchical field, Λ_N is a hypercube in $\mathbb{Z}^d$ of side length L^N , with $L > 1$ fixed. As illustrated in Figure 4.1, we partition Λ_N into disjoint blocks of side length L^j , with $0 \leq j \leq N$.

Definition 4.1.1. Fix some origin $0 \in \Lambda_N$. For $0 \leq j \leq N$, $\mathcal{B}_j$ is the set of disjoint blocks B of side length L^j such that $\Lambda_N = \cup_{B \in \mathcal{B}_j} B$ and such that 0 is a corner of some $B \in \mathcal{B}_j$. An element $B \in \mathcal{B}_j$ is called a *block*, or *j-block*. We say that two j -blocks B, B' *do not touch* if any pair of vertices $(x, x') \in B \times B'$ has $|x - x'|_\infty > 1$.

The choice of $\mathcal{B}_j$ depends on the choice of origin, which is fixed arbitrarily once and for all, but our analysis is independent of this choice. The sets $\mathcal{B}_j$ are nested, in the sense that for every j -block $B \in \mathcal{B}_j$ and $k > j$, there is a unique k -block $B' \in \mathcal{B}_k$ such that $B \subset B'$.

By Proposition 3.3.1, the Gaussian fields ζ_j in the finite-range decomposition of Corollary 3.4.1 have the following two properties:

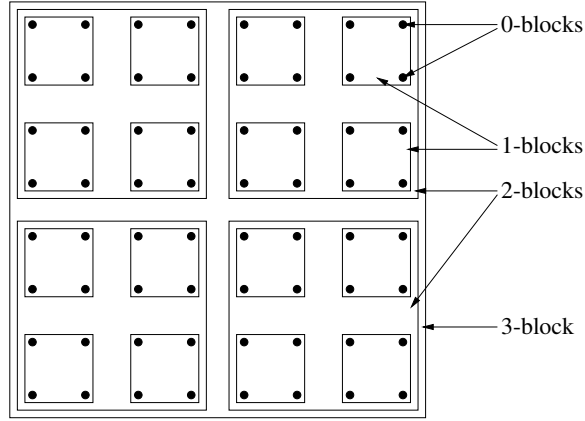


Fig. 4.1 Blocks in $\mathcal{B}_j$ for $j = 0, 1, 2, 3$ when $d = 2, N = 3, L = 2$.

- (i) Given two blocks $B, B' \in \mathcal{B}_j$ that *do not touch*, $\zeta_j|_B$ and $\zeta_j|_{B'}$ are independent identically distributed Gaussian fields.
- (ii) Given any block $b \in \mathcal{B}_{j-1}$, the field $\zeta_j|_b$ is *approximately constant* in the sense that the gradient of the covariance obeys an upper bound that is smaller by a factor $L^{-(j-1)}$ than the upper bound for the covariance itself.

A *hierarchical* field is a Gaussian field on Λ_N with a decomposition $\varphi = \zeta_1 + \dots + \zeta_N$ in which the two properties (i) and (ii) above are replaced by the following stronger versions (i') and (ii').

Definition 4.1.2. A Gaussian field φ on Λ_N is *hierarchical* if there exist independent Gaussian fields $\zeta_1, \dots, \zeta_N$ on Λ_N , called *the fluctuation fields*, such that

$$\varphi \stackrel{D}{=} \zeta_1 + \dots + \zeta_N, \quad (4.1.1)$$

where the fields ζ_j obey:

- (i') Given two blocks $B, B' \in \mathcal{B}_j$ that are *not identical*, $\zeta_j|_B$ and $\zeta_j|_{B'}$ are independent identically distributed Gaussian fields.
- (ii') Given any block $b \in \mathcal{B}_{j-1}$, the field $\zeta_j|_b$ is *constant*: $\zeta_{j,x} = \zeta_{j,y}$ almost surely for all $x, y \in b$.

The replacement of (i–ii) by (i'–ii') is a major technical simplification for the study of the renormalisation group. Condition (ii') means that when x, y are in the same block, $\zeta_x - \zeta_y$ has zero variance and therefore the covariance of ζ is not positive definite; we have allowed for this in Definition 2.1.2. The condition (4.1.13) that appears below implies other linear combinations also have zero variance.

Exercise 4.1.3. The nesting of blocks can be represented as a rooted tree, in which the root is given by the unique block $\Lambda_N \in \mathcal{B}_N$, the blocks $B \in \mathcal{B}_j$ are the vertices at distance $N - j$ to the root, and the children of $B \in \mathcal{B}_j$ are the $b \in \mathcal{B}_{j-1}$ with

$b \subset B$. Represent the hierarchical field in terms of independent Gaussian variables associated to the edges of the tree. [\[Solution\]](#)

Remark 4.1.4. The *finite-range* decomposition of Corollary 3.4.1 is a representation of the GFF in which property (i) is as close to its hierarchical version (i') as possible. The price is that property (ii') needs to be weakened to (ii). There is an alternative decomposition of the GFF such that property (i) is replaced by dependence that decays exponentially with distance, and property (ii) holds. In this alternate decomposition, known as the *block spin* decomposition, ζ_j has the hierarchical features that it is a function of L^d independent Gaussian fields per block, subject to a zero-sum rule as in (4.1.3) below. The block spin decomposition was used, e.g., in [\[91, 95, 104, 108\]](#).

4.1.2 Construction of hierarchical GFF

The hierarchical GFF is defined in terms of the hierarchical Laplacian, which is itself defined in terms of certain projections. We start with the projections.

Let $d \geq 2$. Given a scale $j = 0, 1, \dots, N$ and $x \in \Lambda$, we write B_x for the unique j -block that contains x . Then we define the matrices of symmetric operators Q_j and P_j , acting on $\ell^2(\Lambda)$, by

$$Q_{j,xy} = \begin{cases} L^{-dj} & B_x = B_y \\ 0 & B_x \neq B_y \end{cases} \quad (j = 0, 1, \dots, N), \quad (4.1.2)$$

$$P_j = Q_{j-1} - Q_j \quad (j = 1, \dots, N). \quad (4.1.3)$$

Lemma 4.1.5. *The operators $P_1, \dots, P_N, Q_N$ are orthogonal projections whose ranges are disjoint and provide a direct sum decomposition of $\ell^2(\Lambda)$:*

$$P_j P_k = P_k P_j = \begin{cases} P_j & (j = k) \\ 0 & (j \neq k), \end{cases} \quad \sum_{j=1}^N P_j + Q_N = \text{Id}. \quad (4.1.4)$$

Proof. The second equation is an immediate consequence of the definition (4.1.2) of P_j , together with the fact that $Q_0 = \text{Id}$. For the other properties, we claim that

$$Q_j Q_k = Q_{j \vee k} = Q_k Q_j. \quad (4.1.5)$$

In particular, the case $j = k$ shows that Q_j is an orthogonal projection. To prove (4.1.5), it suffices to consider $j \leq k$. We use primes to denote blocks in the larger scale $\mathcal{B}_k$, and unprimed blocks are in $\mathcal{B}_j$. Then the x, y matrix element of the product is given by

$$\begin{aligned}
\sum_z Q_{j;xz} Q_{k;zy} &= L^{-d(j+k)} \sum_z \mathbb{1}_{B_x=B_z} \mathbb{1}_{B'_z=B'_y} = L^{-d(j+k)} \sum_{z \in B_x} \mathbb{1}_{B_x \subset B'_y} \\
&= L^{-dk} \mathbb{1}_{B_x \subset B'_y} = L^{-dk} \mathbb{1}_{B'_x=B'_y} = Q_{k;xy},
\end{aligned} \tag{4.1.6}$$

as claimed. Thus $\{Q_j\}_{j=0,\dots,N}$ is a sequence of commuting decreasing projections that starts with $Q_0 = \text{Id}$. By (4.1.5) it readily follows that $P_1, \dots, P_N, Q_N$ are orthogonal projections that obey (4.1.4). ■

The next exercise identifies the subspaces in the direct sum decomposition given in Lemma 4.1.5.

Exercise 4.1.6. For $j = 0, \dots, N$, let X_j denote the subspace of $\ell^2(\Lambda)$ consisting of vectors that are constant on blocks in $\mathcal{B}_j$, so $X_0 = \ell^2(\Lambda) \supset X_1 \supset \dots \supset X_N = \text{span}(1, \dots, 1)$. For $j = 0, \dots, N$, show that the range of the projection Q_j is X_j . For $j = 1, \dots, N$, show that the range of the projection P_j is the orthogonal complement of X_j in X_{j-1} , i.e., the set of vectors constant on $(j-1)$ -blocks whose restriction to any j -block has zero sum. [Solution]

Definition 4.1.7. The *hierarchical Laplacian* $\Delta_{H,N}$ is the operator on $\ell^2(\Lambda)$ given by

$$-\Delta_{H,N} = \sum_{j=1}^N L^{-2(j-1)} P_j. \tag{4.1.7}$$

The hierarchical Laplacian generates a certain hierarchical random walk; this point of view is developed in the next exercise (see also [40]). Its decay properties mirror those of the Laplacian on $\mathbb{Z}^d$, and this fact is established in Exercise 4.1.13.

Exercise 4.1.8. Let j_x be the smallest j such that 0 and x are in the same j -block; we call j_x the *coalescence scale* for the points 0, x . Show that

$$\Delta_{H,N;0x} = \begin{cases} -\frac{1-L^{-d}}{1-L^{-(d+2)}} (1-L^{-(d+2)N}) & (x=0) \\ \frac{L^2-1}{1-L^{-(d+2)}} L^{-(d+2)j_x} + \frac{1-L^{-d}}{1-L^{-(d+2)}} L^{-(d+2)N} & (x \neq 0). \end{cases} \tag{4.1.8}$$

In particular, $\Delta_{H,N;00} < 0$ and, for $x \neq 0$, $\Delta_{H,N;0x} > 0$. Show also that $\sum_{x \in \Lambda} \Delta_{H,N;0,x} = 0$. This implies that $\Delta_{H,N}$ is the infinitesimal generator (also called a Q -matrix [128]) of a continuous-time random walk. What steps does it take? [Solution]

Given $m^2 > 0$, we set

$$\gamma_j = \frac{L^{2(j-1)}}{1 + L^{2(j-1)} m^2} \tag{4.1.9}$$

and for $j = 1, \dots, N$ define matrices

$$C_{j;xy}(m^2) = \gamma_j P_{j;xy}, \quad C_{\hat{N};xy}(m^2) = \frac{1}{m^2} Q_{N;xy}. \tag{4.1.10}$$

It follows from Lemma 4.1.5 that

$$C_j C_k = 0 \quad (j \neq k). \quad (4.1.11)$$

Note that $C_{j;xy}(m^2)$ is actually well defined for all $m^2 \geq 0$, and is independent of N in the sense that the C_j defined in terms of any $N \geq j$ are naturally identified. In contrast, $C_{\hat{N};xy}(m^2)$ is not defined for $m^2 = 0$ and does depend on N . In fact, $C_{\hat{N};xy}(m^2) = m^{-2} L^{-dN}$ for all $x, y \in \Lambda$ because Λ is a single block at scale N .

The special role of $C_{\hat{N}}$ is analogous to the situation for the Euclidean torus decomposition $(-\Delta_\Lambda + m^2)^{-1} = \sum_{j=1}^{N-1} C_j + C_{N,N}$ of (3.4.4). There the term $C_{N,N}$ is special as it is the term that takes the finite-volume torus into account. Similarly, in the hierarchical setting we isolate the finite-volume effect by writing the decomposition in the form $\sum_{j=1}^N C_j + C_{\hat{N}}$, with $\zeta_{\hat{N}}$ the field that takes the finite volume into account.

Proposition 4.1.9. *For $m^2 > 0$, $x, y \in \Lambda_N$ and $j = 0, \dots, N-1$,*

$$C_{j+1;xx}(m^2) = \frac{1}{1 + m^2 L^{2j}} L^{-(d-2)j} (1 - L^{-d}), \quad (4.1.12)$$

$$\sum_x C_{j+1;0x} = 0, \quad (4.1.13)$$

$$C_{\hat{N};xy}(m^2) = L^{-dN} m^{-2}. \quad (4.1.14)$$

The matrix $C = (m^2 - \Delta_{H,N})^{-1}$ has the decomposition

$$C = C_1 + \dots + C_N + C_{\hat{N}}. \quad (4.1.15)$$

Let ζ_j be independent fields, Gaussian with covariance C_j . Then the field $\varphi = \zeta_1 + \dots + \zeta_N + \zeta_{\hat{N}}$ is a hierarchical field as in Definition 4.1.2.

Proof. The variance statement (4.1.12) is immediate by setting $x = y$ in the definition (4.1.3) of P_j . The identity (4.1.13) follows from (4.1.2)–(4.1.3), since

$$\sum_{x \in \Lambda} P_{j+1;0x} = \sum_{x \in \Lambda} Q_{j;0x} - \sum_{x \in \Lambda} Q_{j+1;0x} = L^{dj} L^{-dj} - L^{d(j+1)} L^{-d(j+1)} = 0. \quad (4.1.16)$$

Also, (4.1.14) follows from the definition (4.1.10) of $C_{\hat{N}}$.

The decomposition statement (4.1.15) and $C = (-\Delta_{H,N} + m^2)^{-1}$ follow from the independence of the fields in the decomposition $\varphi = \zeta_1 + \dots + \zeta_N + \zeta_{\hat{N}}$ and from (4.1.4) which together with (4.1.7) shows that $P_1, \dots, P_N, Q_N$ are spectral projections for $-\Delta_{H,N}$. In fact, let $f(t) = (t + m^2)^{-1}$ and $\lambda_j = L^{-2(j-1)}$. By the spectral calculus,

$$\begin{aligned}
(-\Delta_{H,N} + m^2)^{-1} &= f(-\Delta_{H,N}) = f\left(\sum_{j=1}^N \lambda_j P_j + 0 Q_N\right) \\
&= \sum_{j=1}^N f(\lambda_j) P_j + f(0) Q_N \\
&= \sum_{j=1}^N C_j(m^2) + C_N(m^2), \tag{4.1.17}
\end{aligned}$$

because $f(\lambda_j) = \gamma_j$ and $f(0) = m^{-2}$.

The independence required by Definition 4.1.2(i') holds by construction, and (ii') follows from the easily checked fact that $\text{Var}_j(\zeta_{j;x} - \zeta_{j;y}) = 0$ if x, y both lie in the same block $b \in \mathcal{B}_{j-1}$. This completes the proof. ■

Although we have the explicit formulas (4.1.10) and (4.1.7) for the covariances C_j and for $\Delta_{H,N}$, for our purposes these explicit formulas are not very important because almost everything in the following chapters uses only the properties listed in Proposition 4.1.9 and Definition 4.1.2. However, to be concrete, we call the particular random field ϕ defined by these explicit formulas the *hierarchical Gaussian free field (hGFF)*. The justification for this terminology is that (4.1.12) has the same scaling as its counterpart for the Gaussian free field, according to (3.3.7). Similarly, $\Delta_{H,N}$ has properties in common with the standard lattice Laplacian. Note that we use C for both the hierarchical and usual covariances. It should be clear from context which is intended.

Equation (4.1.13) holds both for block spins and for the hierarchical model, and this leads to simplifications in perturbation theory. However, it does not hold for the Euclidean model with finite-range decomposition, and perturbation theory is therefore more involved [21]. Not all authors include the $\Delta_{H,N}$ properties or property (4.1.13) when defining massless hierarchical fields.

4.1.3 Properties of hierarchical covariances

Exercise 4.1.10. Show that $\sum_{x \in \Lambda} C_{0x}(m^2) = m^{-2}$. (Cf. Exercise 1.5.2.) [Solution]

By (4.1.10), the hierarchical covariance is given, for $j < N$ and for x, y in the same $(j+1)$ -block, by

$$C_{j+1;xy}(m^2) = \begin{cases} L^{-(d-2)j}(1+m^2L^{2j})^{-1}(1-L^{-d}) & (b_x = b_y) \\ -L^{-(d-2)j}(1+m^2L^{2j})^{-1}L^{-d} & (b_x \neq b_y), \end{cases} \tag{4.1.18}$$

where b_x denotes the j -block containing x ; if x, y are not in the same $(j+1)$ -block then $C_{j+1;xy}(m^2) = 0$. We write the diagonal entry as

$$c_j = C_{j+1;00}(m^2) = L^{-(d-2)j}(1 + m^2 L^{2j})^{-1}(1 - L^{-d}), \quad (4.1.19)$$

and for $n \in \mathbb{N}$ define

$$c_j^{(n)} = \sum_{x \in \Lambda} (C_{j+1;0x}(m^2))^n. \quad (4.1.20)$$

The fact that C_{j+1} is positive definite and translation invariant implies that $c_j^{(1)} \geq 0$. For our specific choice of C_{j+1} , it follows from (4.1.13) (or directly from (4.1.18)) that $c_j^{(1)} = 0$.

Exercise 4.1.11. Use (4.1.18) to show that, for $j < N$,

$$c_j^{(2)} = L^{-(d-4)j}(1 + m^2 L^{2j})^{-2}(1 - L^{-d}), \quad (4.1.21)$$

$$c_j^{(3)} = L^{-(2d-6)j}(1 + m^2 L^{2j})^{-3}(1 - 3L^{-d} - 2L^{-2d}), \quad (4.1.22)$$

$$c_j^{(4)} = L^{-(3d-8)j}(1 + m^2 L^{2j})^{-4}(1 - 4L^{-d} + 6L^{-2d} - 3L^{-3d}). \quad (4.1.23)$$

[Solution]

Exercise 4.1.12. Recall the bubble diagram $B_{m^2} = \sum_{x \in \mathbb{Z}^d} ((-\Delta + m^2)_{0x}^{-1})^2$ defined in (1.5.23). The infinite-volume hierarchical bubble diagram is defined by

$$B_{m^2}^H = \lim_{N \rightarrow \infty} \sum_{x \in \Lambda_N} ((-\Delta_{H,N} + m^2)_{0x}^{-1})^2, \quad (4.1.24)$$

where $\Delta_{H,N}$ is the hierarchical Laplacian on Λ_N . Prove that, for $m^2 \geq 0$,

$$B_{m^2}^H = \sum_{j=0}^{\infty} c_j^{(2)}, \quad (4.1.25)$$

with $c_j^{(2)}$ given by (4.1.20). In particular, $B_{m^2}^H$ is finite in all dimensions for $m^2 > 0$, whereas B_0^H is finite if and only if $d > 4$. Prove that, as $m^2 \downarrow 0$,

$$B_{m^2}^H \sim \begin{cases} \text{const } m^{-(4-d)} & (d < 4) \\ \frac{1-L^{-d}}{\log L} \log m^{-1} & (d = 4). \end{cases} \quad (4.1.26)$$

[Solution]

The asymptotic behaviour for the hierarchical bubble in (4.1.26) is analogous to that of Exercise 1.5.4 for the bubble diagram of the GFF. Another correspondence between the hGFF and the GFF is that in the critical case $m^2 = 0$ in the infinite-volume limit, the covariance of the hGFF has the same large- $|x|$ decay as the GFF. This is shown in the following exercise.

Exercise 4.1.13. (i) Verify that

$$C(m^2) = \gamma_1 Q_0 + \sum_{j=1}^{N-1} (\gamma_{j+1} - \gamma_j) Q_j + (m^{-2} - \gamma_N) Q_N. \quad (4.1.27)$$

(ii) Using the result of part (i), prove that as $|x| \rightarrow \infty$ the hierarchical covariance obeys

$$\lim_{m^2 \downarrow 0} \lim_{N \rightarrow \infty} [C_{0x}(m^2) - C_{00}(m^2)] \begin{cases} \asymp -|x| & (d=1) \\ = -(1-L^{-2}) \log_L |x| + O(1) & (d=2) \end{cases} \quad (4.1.28)$$

and

$$\lim_{m^2 \downarrow 0} \lim_{N \rightarrow \infty} C_{0x}(m^2) \asymp |x|^{-(d-2)} \quad (d > 2). \quad (4.1.29)$$

[Solution]

On the other hand, the effect of the mass $m > 0$ is not as strong for the hierarchical covariance as it is for the Euclidean one. The Euclidean covariance with mass m decays exponentially with rate $\sim m$ as $m \downarrow 0$, while the hierarchical covariance decays only polynomially in $m|x|$. This results from the fact that $-\Delta_H$ is not local; its matrix elements decay only polynomially.

4.2 Hierarchical $|\varphi|^4$ model

Recall from Section 1.6.1 that the n -component $|\varphi|^4$ model on a set Λ is defined by the expectation

$$\langle F \rangle_{g,v,\Lambda} = \frac{1}{Z_{g,v,\Lambda}} \int_{\mathbb{R}^{n\Lambda}} F(\varphi) e^{-H(\varphi)} d\varphi \quad (4.2.1)$$

with

$$H(\varphi) = \frac{1}{2} \sum_{x \in \Lambda} \varphi_x \cdot (-\Delta_\beta \varphi)_x + \sum_{x \in \Lambda} \left(\frac{1}{4} g |\varphi_x|^4 + \frac{1}{2} v |\varphi_x|^2 \right). \quad (4.2.2)$$

Here $g > 0$, $v \in \mathbb{R}$, $d\varphi = \prod_{x \in \Lambda} d\varphi_x$ is the Lebesgue measure on $(\mathbb{R}^n)^\Lambda$, and β is a $\Lambda \times \Lambda$ symmetric matrix with non-negative entries. The GFF is the degenerate case $w(\varphi) = \frac{1}{2} m^2 |\varphi|^2$. The commonest short-range spin-spin interaction is the nearest-neighbour choice $\Delta_\beta = \Delta_\Lambda$.

Our topic now is the hierarchical $|\varphi|^4$ model, in which Δ_β is replaced by the hierarchical Laplacian Δ_H of Section 4.1. This choice significantly simplifies the analysis in the renormalisation group approach. According to Exercise 4.1.8, $-\Delta_H$ is ferromagnetic. Moreover, for $x \neq 0$, and in the simplifying case of the limit $N \rightarrow \infty$, $\Delta_{H;0x}$ is proportional to $L^{-(d+2)j_x}$ where j_x is the coalescence scale. Therefore $\Delta_{H;0x}$ is bounded above and below by multiples of $|x-y|^{-d-2}$. Thus, although the matrix Δ_H is long-range, it is almost short-range in the sense that its variance is only borderline divergent. Although it does not respect the symmetries of the Euclidean

lattice $\mathbb{Z}^d$, but rather those of a hierarchical group, it nevertheless shares essential features of the Euclidean nearest-neighbour model.

We denote expectation in the n -component hierarchical $|\varphi|^4$ model by

$$\langle F \rangle_{g,v,N} = Z_{g,v,N}^{-1} \int_{\mathbb{R}^{n\Lambda}} F(\varphi) e^{-\sum_{x \in \Lambda} (\frac{1}{2} \varphi_x \cdot (-\Delta_H \varphi)_x + \frac{1}{4} g |\varphi_x|^4 + \frac{1}{2} v |\varphi_x|^2)} d\varphi. \quad (4.2.3)$$

The finite-volume susceptibility is

$$\chi_N(g, v) = \sum_{x \in \Lambda} \langle \varphi_0^1 \varphi_x^1 \rangle_{g,v,N}, \quad (4.2.4)$$

and the susceptibility in infinite volume is

$$\chi(g, v) = \lim_{N \rightarrow \infty} \sum_{x \in \Lambda_N} \langle \varphi_0^1 \varphi_x^1 \rangle_{g,v,N}. \quad (4.2.5)$$

Existence of this limit is part of the statement of the following theorem. The theorem provides the hierarchical version of (1.6.12). Its proof occupies the rest of the book.

Theorem 4.2.1. *Let $d = 4$ and $n \geq 1$, let $L > 1$ be large, and let $g > 0$ be small. For the hierarchical $|\varphi|^4$ model, there exists $v_c = v_c(g, n) < 0$ such that, with $v = v_c + \varepsilon$ and as $\varepsilon \downarrow 0$,*

$$\chi(g, v) \sim A_{g,n} \frac{1}{\varepsilon} (\log \varepsilon^{-1})^{(n+2)/(n+8)}. \quad (4.2.6)$$

In particular, the limit defining $\chi(g, v)$ exists. Also, as $g \downarrow 0$,

$$A_{g,n} \sim \left(\frac{(1 - L^{-d})(n+8)g}{\log L} \right)^{\frac{n+2}{n+8}}, \quad v_c(g, n) \sim -(n+2)g(-\Delta_H)_{00}^{-1}. \quad (4.2.7)$$

The L -dependence present in (4.2.7) is a symptom of the fact that in our hierarchical model the definition of the model itself depends on L . This is in contrast to the Euclidean case, where the corresponding formulas for $A_{g,n}$ and v_c are independent of L in Theorem 1.6.1.

Hierarchical fields were introduced in 1969 by Dyson [75] for the study of the 1-dimensional Ising model with long-range spin-spin coupling with decay $r^{-\alpha}$ ($\alpha \in (1, 2)$). Three years later, the hierarchical model was defined independently by Baker [13]. In the context of the renormalisation group, the idea was taken up by Bleher and Sinai, who investigated both the Gaussian [35] and non-Gaussian regimes [36].

Since then, the hierarchical approximation has played an important role as a test case for the development of renormalisation group methods. The hierarchical 1-component φ^4 model is studied in [92, 152, 153] for $d = 4$, and in [112, 154] for $d = 3$. An analysis of the hierarchical 4-dimensional Ising model appears in [106]. The hierarchical version of the 4-dimensional weakly self-avoiding walk is analysed in [40, 49, 50]. The ε -expansion in the long-range (non-Gaussian) hierarchical setting is developed in [36, 64, 94].

Hierarchical models are remarkably parallel to Euclidean models, and our analysis is designed so that the Euclidean proofs closely follow the hierarchical proofs. An alternate approach to hierarchical models is explored in depth in [3]. In [3], continuum limits of hierarchical models are defined with p -adic numbers playing the role of $\mathbb{R}^d$, and spatially varying coupling constants are permitted. The search for parallels continues in [2] where hierarchical conformal invariance is studied.

4.3 GFF and $|\varphi|^4$ model

Now we make the connection between the n -component hierarchical $|\varphi|^4$ measure and an n -component Gaussian measure. The exponent

$$\frac{1}{2}\varphi \cdot (-\Delta_{H,N}\varphi) + \frac{1}{4}g|\varphi|^4 + \frac{1}{2}v|\varphi|^2 \quad (4.3.1)$$

in (4.2.3) has two quadratic terms, so it is tempting to use these two terms to define a Gaussian measure and write the $|\varphi|^4$ measure relative to this Gaussian measure. However, the corresponding Gaussian measure does not exist when v is negative, and we are interested in the critical value v_c which is negative. Also, the hierarchical Laplacian itself is not positive definite, so it is not possible to define a Gaussian measure using only the $\varphi(-\Delta_{H,N}\varphi)$ term, without restriction on the domain of $\Delta_{H,N}$.

Given a mass parameter $m^2 > 0$, we define $v_0 = v - m^2$ and

$$V_{g,v_0}(\varphi) = \frac{1}{4}g|\varphi|^4 + \frac{1}{2}v_0|\varphi|^2. \quad (4.3.2)$$

Leaving implicit the volume parameter N on the right-hand side, and writing $C = (-\Delta_{H,N} + m^2)^{-1}$, we have

$$\langle F \rangle_{g,v,N} = \frac{\mathbb{E}_C F e^{-\sum_{x \in \Lambda} V_{g,v_0}(\varphi_x)}}{\mathbb{E}_C e^{-V_{g,v_0}}}. \quad (4.3.3)$$

The finite-volume susceptibility corresponds to the choice $F(\varphi) = \sum_{x \in \Lambda} \varphi_0^1 \varphi_x^1$ on the left-hand side of (4.3.3). It can be studied using the Laplace transform, as in the next exercise. We define

$$Z_0(\varphi) = e^{-V_{g,v_0}(\varphi)} \quad (4.3.4)$$

and, for $f : \mathbb{R}^\Lambda \rightarrow \mathbb{R}^n$,

$$\Sigma_N(f) = \mathbb{E}_C(e^{(f,\varphi)} Z_0(\varphi)). \quad (4.3.5)$$

By Exercise 2.1.10,

$$\Sigma_N(f) = e^{\frac{1}{2}(f,Cf)} (\mathbb{E}_C \theta Z_0)(Cf). \quad (4.3.6)$$

Derivatives of functionals of fields, in the directions of test functions h_i , are defined by

$$D^n F(f; h_1, \dots, h_n) = \frac{d^n}{ds_1 \dots ds_n} F(f + s_1 h_1 + \dots + s_n h_n) \Big|_{s_1 = \dots = s_n = 0}. \quad (4.3.7)$$

Exercise 4.3.1. For $v_0 = v - m^2$,

$$\begin{aligned} \sum_{x \in \Lambda} \langle \varphi_0^\dagger \varphi_x^\dagger \rangle_{g, v, N} &= \frac{1}{|\Lambda|} \frac{D^2 \Sigma_N(0; \mathbb{1}, \mathbb{1})}{\Sigma_N(0)} \\ &= \frac{1}{m^2} + \frac{1}{m^4 |\Lambda|} \frac{D^2 Z_{\hat{N}}(0; \mathbb{1}, \mathbb{1})}{Z_{\hat{N}}(0)}, \end{aligned} \quad (4.3.8)$$

where $\mathbb{1}$ denotes the constant test function $\mathbb{1}_x = (1, 0, \dots, 0)$ for all $x \in \Lambda$, and where $Z_{\hat{N}} = \mathbb{E}_C \theta Z_0$ with Z_0 given by (4.3.4) and the convolution $\mathbb{E}_C \theta$ is given by Definition 2.1.5. Hint: use Exercise 4.1.10 for the second equality in (4.3.8). [Solution]

Using the renormalisation group method we will compute the *effective mass* $m^2 > 0$, as a function of $v > v_c$, with the property that the term involving $Z_{\hat{N}}$ on the right-hand side of (4.3.8) goes to zero as $N \rightarrow \infty$. By Exercise 4.1.10, this expresses the infinite-volume susceptibility of the interacting model at v as the susceptibility of the free model at m^{-2} .

Much of the literature on the triviality (Gaussian nature) of the 4-dimensional $|\varphi|^4$ model has focussed on the *renormalised coupling constant* g_{ren} , e.g., [10, 89]. This is defined in terms of the *truncated four-point function* $\bar{u}_4$, which for simplicity we discuss here for the 1-component model. In finite volume, let

$$\begin{aligned} \bar{u}_{4,N} &= \sum_{x,y,z \in \Lambda} \left(\langle \varphi_0 \varphi_x \varphi_y \varphi_z \rangle_N - \langle \varphi_0 \varphi_x \rangle_N \langle \varphi_y \varphi_z \rangle_N \right. \\ &\quad \left. - \langle \varphi_0 \varphi_y \rangle_N \langle \varphi_x \varphi_z \rangle_N - \langle \varphi_0 \varphi_z \rangle_N \langle \varphi_x \varphi_y \rangle_N \right) \\ &= \sum_{x,y,z \in \Lambda} \langle \varphi_0 \varphi_x \varphi_y \varphi_z \rangle_N - 3|\Lambda| \chi_N^2. \end{aligned} \quad (4.3.9)$$

Then we define $\bar{u}_4 = \lim_{N \rightarrow \infty} \bar{u}_{4,N}$ (assuming the limit exists), and set

$$g_{\text{ren}} = -\frac{1}{6} \frac{\bar{u}_4}{\xi^d \chi^2} \quad (4.3.10)$$

where ξ is the correlation length. The $\frac{1}{6}$ is simply a normalisation factor.

Exercise 4.3.2. (i) In the setup of Exercise 4.3.1 with $n = 1$, prove that

$$\bar{u}_{4,N} = \frac{1}{m^8 |\Lambda|} \left(\frac{D^4 Z_{\hat{N}}(0; \mathbb{1}, \mathbb{1}, \mathbb{1}, \mathbb{1})}{Z_{\hat{N}}(0)} - 3 \left(\frac{D^2 Z_{\hat{N}}(0; \mathbb{1}, \mathbb{1})}{Z_{\hat{N}}(0)} \right)^2 \right). \quad (4.3.11)$$

Here $\mathbb{1}_x = 1$ for all $x \in \Lambda$.

(ii) As discussed below Exercise 4.3.1, we will prove that in infinite volume the susceptibility is $\chi = m^{-2}$. As in Theorem 1.6.2, for $d = 4$ we expect the correlation length to have the same leading asymptotic behaviour as the square root of the

susceptibility. Thus, for $d = 4$, we define

$$\tilde{g}_{\text{ren}} = -\frac{1}{6} \frac{\bar{u}_4}{\chi^4} = -\frac{1}{6} m^8 \bar{u}_4. \quad (4.3.12)$$

If $Z_{\hat{N}}$ is replaced in (4.3.11) by $e^{-V_N(\Lambda)}$ with $V_N(\Lambda) = \sum_{x \in \Lambda} (\frac{1}{4} g_N \phi_x^4 + \frac{1}{2} v_N \phi_x^2 + u_N)$, prove that the right-hand side of (4.3.12) then becomes $g_\infty = \lim_{N \rightarrow \infty} g_N$ (assuming again that the limit exists). This explains the name “renormalised coupling constant.” [\[Solution\]](#)

Part II
**The renormalisation group: Perturbative
analysis**

Chapter 5

The renormalisation group map

The proof of Theorem 4.2.1 uses the renormalisation group method, and occupies the remainder of the book. An advantage of the hierarchical model is that the analysis can be reduced to individual blocks (recall Definition 4.1.1); this is not the case in the Euclidean setting. We explain this reduction in Section 5.1. The renormalisation group map is defined in Section 5.2. It involves the notion of flow of coupling constants (u_j, g_j, v_j) , as well as the flow of an infinite-dimensional non-perturbative coordinate K_j . The flow of coupling constants is given to leading order by perturbation theory, which is the subject of Section 5.3.

5.1 Reduction to block analysis

5.1.1 Progressive integration

Our starting point for the proof of Theorem 4.2.1 is a formula for the finite volume susceptibility $\chi_N(g, v)$ of (4.2.4). It can be rewritten, as in Exercise 4.3.1, as follows. Given (g_0, v_0) and $m^2 > 0$, we write $C = C(m^2) = (-\Delta_{H,N} + m^2)^{-1}$ and

$$Z_{\hat{N}} = \mathbb{E}_C \theta Z_0, \quad Z_0(\varphi) = e^{-\sum_{x \in \Lambda} (\frac{1}{4} g_0 |\varphi_x|^4 + \frac{1}{2} v_0 |\varphi_x|^2)}, \quad (5.1.1)$$

where the convolution $\mathbb{E}_C \theta$ is defined in Definition 2.1.5, and we emphasise that it here refers to an n -component Gaussian field as in Example 2.1.4. Then, for any $m^2 > 0$, and for $g_0 = g$ and $v_0 = v - m^2$,

$$\chi_N(g, v) = \frac{1}{m^2} + \frac{1}{m^4 |\Lambda|} \frac{D^2 Z_{\hat{N}}(0; \mathbb{1}, \mathbb{1})}{Z_{\hat{N}}(0)}, \quad (5.1.2)$$

where $\mathbb{1}$ denotes the constant test function $\mathbb{1}_x = (1, 0, \dots, 0)$ for all $x \in \Lambda$.

Thus, to compute the susceptibility, it suffices to understand $Z_{\hat{N}}$. The formula (5.1.2) requires that $v_0 = v - m^2$, but the right-hand side makes sense as a function of

three independent variables (m^2, g_0, v_0) , with $m^2 > 0$, $g_0 = g > 0$, $v_0 \in \mathbb{R}$. Although (5.1.2) no longer holds without the requirement that $v_0 = v - m^2$, it is nevertheless useful to analyse $Z_{\hat{N}}$ as a function of three independent variables for now, and to restrict v_0 later. We will do so.

The starting point for the renormalisation group is to evaluate $Z_{\hat{N}}$ as the last term in a sequence $Z_0, Z_1, \dots, Z_N, Z_{\hat{N}}$ generated by

$$Z_{j+1} = \mathbb{E}_{C_{j+1}} \theta Z_j \quad (j < N), \quad Z_{\hat{N}} = \mathbb{E}_{C_{\hat{N}}} \theta Z_N, \quad (5.1.3)$$

where $C = \sum_{j=1}^N C_j + C_{\hat{N}}$ is as in Proposition 4.1.9. It follows from the above recursion and Corollary 2.1.11 that

$$Z_{\hat{N}} = \mathbb{E}_{C_{\hat{N}}} \theta \circ \mathbb{E}_{C_N} \theta \circ \dots \circ \mathbb{E}_{C_1} \theta Z_0 = \mathbb{E}_{C(m^2)} \theta Z_0, \quad (5.1.4)$$

consistent with (5.1.1). The effect of finite volume is concentrated entirely in the last covariance $C_{\hat{N}}$.

The first equation of (5.1.3) can be rewritten as

$$Z_{j+1}(\varphi) = \mathbb{E}_{C_{j+1}} Z_j(\varphi + \zeta), \quad (5.1.5)$$

where the expectation on the right-hand side integrates with respect to ζ leaving φ fixed. By the definition of the hierarchical GFF in Definition 4.1.2,

- the restriction of $x \mapsto \zeta_x$ to a block $b \in \mathcal{B}_j$ is constant;
- the restriction of $x \mapsto \varphi_x$ to a block $B \in \mathcal{B}_{j+1}$ is constant.

The fluctuation field ζ is Gaussian with covariance C_{j+1} , while the block-spin field φ is Gaussian with covariance $C_{j+2} + \dots + C_N + C_{\hat{N}}$.

From now on, we often fix a *scale* j and omit it from the notation. We then write $+$ instead of $j+1$. In particular, we write C_+ for C_{j+1} , $\mathcal{B}$ for $\mathcal{B}_j$, and $\mathcal{B}_+$ for $\mathcal{B}_{j+1}$. We also abbreviate $\mathbb{E}_+ = \mathbb{E}_{C_{j+1}}$, and we typically use b to denote a block at scale j and B to denote a block at scale $j+1$ when blocks at both scales are being used.

5.1.2 Polynomials in the hierarchical field

We use the notation

$$\tau = \frac{1}{2}|\varphi|^2, \quad \tau^2 = \frac{1}{4}|\varphi|^4, \quad (5.1.6)$$

and write, for example, $\tau_x^2 = \frac{1}{4}|\varphi_x|^4$.

Definition 5.1.1. We set $\mathcal{V} = \mathbb{R}^2$, $\mathcal{U} = \mathbb{R}^3$ and write their elements as $V = (g, v) \in \mathcal{V}$ and $U = (u, V) \in \mathcal{U}$. We identify V and U with the polynomials $V = g\tau^2 + v\tau$ and $U = g\tau^2 + v\tau + u$. Given $V \in \mathcal{V}$ or $U \in \mathcal{U}$ and $X \subset \Lambda$, we set

$$V(X, \varphi) = \sum_{x \in X} (g\tau_x^2 + v\tau_x), \quad (5.1.7)$$

$$U(X, \varphi) = \sum_{x \in X} (g\tau_x^2 + v\tau_x + u). \quad (5.1.8)$$

Furthermore we set $U_x(\varphi) = g\tau_x^2 + v\tau_x + u$ and similarly for $V_x(\varphi)$.

Exercise 5.1.2. Show that, for an arbitrary covariance C , $\mathbb{E}_C \theta$ acts as a map $\mathcal{U} \rightarrow \mathcal{U}$ identified as polynomials in the field by setting $X = \{x\}$ in (5.1.8). (Recall Proposition 2.1.6). [Solution].

Recall that the set $\mathcal{B}_j$ of j -blocks is defined in Definition 4.1.1. We use multi-index notation: $\alpha = (\alpha_1, \dots, \alpha_n)$ is a vector of nonnegative integers, and we write $|\alpha| = \sum_{i=1}^n \alpha_i$, $\alpha! = \prod_{i=1}^n \alpha_i!$, and $\zeta^\alpha = \prod_{i=1}^n (\zeta^i)^{\alpha_i}$ for $\zeta \in \mathbb{R}^n$.

Lemma 5.1.3. If $U, U' \in \mathcal{U}$ and $B \in \mathcal{B}_+$ then there exist coefficients $p, q, r, s \in \mathbb{R}$, bilinear in U, U' , such that

$$\text{Cov}_+(\theta U_x, \theta U'(B)) = \sum_{x \in B} (p + q\tau_x + r\tau_x^2 + s\tau_x^3). \quad (5.1.9)$$

If $c_+^{(1)} = 0$ as in (4.1.13) then $s = 0$, and hence $\text{Cov}_+(\theta U_x, \theta U'(B))$ identifies with $(p, r, q) \in \mathcal{U}$ via (5.1.8) with $X = B$.

Proof. By Taylor's theorem, $U_x(\varphi + \zeta) = \sum_{|\alpha| \leq 4} \frac{1}{\alpha!} U^{(\alpha)}(\varphi) \zeta_x^\alpha$, so

$$\text{Cov}_+(\theta U_x, \theta U'(B)) = \sum_{|\alpha|, |\alpha'| \leq 4} \frac{1}{\alpha!} \frac{1}{\alpha'!} U^{(\alpha)} U'^{(\alpha')} \sum_{x' \in B'} \text{Cov}_+(\zeta_x^\alpha, \zeta_{x'}^{\alpha'}). \quad (5.1.10)$$

Terms with $|\alpha| = 0$ or $|\alpha'| = 0$ vanish since the covariance does. The same is true when $|\alpha| + |\alpha'|$ is odd, due to the $\zeta \mapsto -\zeta$ symmetry. When $|\alpha| = |\alpha'| = 1$, the covariance vanishes unless $\alpha = \alpha'$, and in this case the sum over x' is $c_+^{(1)}$. When $c_+^{(1)}$ is nonzero, the $O(n)$ -invariance of the covariance ensures that the resulting φ -dependence is of the form $|\varphi|^6$.

This leaves only terms where $|\alpha| + |\alpha'| \in \{4, 6, 8\}$. Such terms respectively produce contributions which are quartic, quadratic, and constant in φ . The fact that the covariance is $O(n)$ -invariant ensures that the quartic and quadratic terms are multiples of $|\varphi|^4$ and $|\varphi|^2$, and the proof is complete. ■

5.1.3 Functionals of the hierarchical field

Definition 5.1.4. For $B \in \mathcal{B}_j$, let $J(B)$ denote the set of constant maps from B to $\mathbb{R}^n$, and let $j_B : J(B) \rightarrow \mathbb{R}^n$ be the map that identifies the constant in the range of a map in $J(B)$, i.e.,

$$j_B(\varphi) = \varphi_x \quad (x \in B, \varphi \in J(B)). \quad (5.1.11)$$

Let $\mathcal{N}(B)$ be the vector space of functions that have the form $F \circ j_B : J(B) \rightarrow \mathbb{R}$, where $F : \mathbb{R}^n \rightarrow \mathbb{R}$ is a function with $p_{\mathcal{N}}$ continuous derivatives. In the proof of Theorem 4.2.1, we take $p_{\mathcal{N}} = \infty$, though the proof also works for any finite value $p_{\mathcal{N}} \geq 10$.

Definition 5.1.5. Let $\mathcal{F} = \mathcal{F}_j \subset \bigoplus_{B \in \mathcal{B}_j} \mathcal{N}(B)$ be the vector space of functions $F(B, \varphi)$ that obey the following properties for all $B \in \mathcal{B}_j$:

- *locality*: $F(B) \in \mathcal{N}(B)$,
- *spatial homogeneity*: $F(B) = F \circ j_B$ where $F : \mathbb{R}^n \rightarrow \mathbb{R}$ is the same for all blocks $B \in \mathcal{B}_j$,
- *$O(n)$ -invariance*: $F(B, \varphi) = F(B, T\varphi)$ for all $T \in O(n)$, where T acts on $J(B)$ by $(T\varphi)_x = T\varphi_x$ for $x \in B$.

The property *locality* is already included in the condition that $\mathcal{F}$ is a subspace of $\bigoplus_B \mathcal{N}(B)$, and is written for emphasis only. For $X \subset \mathcal{B}_j$, let $\mathcal{B}_j(X)$ denote the set of j -blocks comprising X . For $F_j \in \bigoplus_{B \in \mathcal{B}_j} \mathcal{N}(B)$, in particular for $F_j \in \mathcal{F}_j$, we define

$$F_j^X = \prod_{B \in \mathcal{B}_j(X)} F_j(B). \quad (5.1.12)$$

5.1.4 Global to local reduction

Let $V_0 = g_0 \tau_x^2 + v_0 \tau_x$. We define $F_0 \in \mathcal{F}_0$ by

$$F_0(\{x\}, \varphi) = e^{-V_0(\varphi_x)}. \quad (5.1.13)$$

By definition, Z_0 of (5.1.1) can be written in the notation (5.1.12) as

$$Z_0 = F_0^\Lambda \quad (5.1.14)$$

The product $F_0^\Lambda = \prod_{x \in \Lambda} F_0(\{x\})$ in (5.1.14) is the same as the product over 0-blocks $B \in \mathcal{B}_0$, because a 0-block B equals $\{x\}$ for some lattice point x . A principal feature of the hierarchical model is the stability of a product form for Z_j for every j , analogous to (5.1.14), as in the following lemma. In its statement, in accordance with (5.1.12) we write $F_k^B = \prod_{b \in \mathcal{B}_k(B)} F_k(b)$ for $B \in \mathcal{B}_{k+1}$, and $F_j^\Lambda = \prod_{B \in \mathcal{B}_j} F_j(B)$.

Lemma 5.1.6. *The sequence F_j defined inductively by*

$$F_{k+1}(B) = \mathbb{E}_{C_{k+1}} \theta F_k^B \quad (B \in \mathcal{B}_{k+1}), \quad (5.1.15)$$

with initial condition (5.1.13), defines a sequence $F_j \in \mathcal{F}_j$ when $F(B) = F(B, \varphi)$ is restricted to the domain $J(B)$. Moreover,

$$Z_j = F_j^\Lambda. \quad (5.1.16)$$

Proof. For $j = 0$, the claim (5.1.16) holds by (5.1.14) and $F_0 \in \mathcal{F}_0$ as remarked above. We apply induction, and assume that (5.1.16) holds for some j with $F_j \in \mathcal{F}_j$. In particular, for B in $\mathcal{B}_j$, $F_j(B)$ depends only on $\varphi|_B$ and this field is constant on B . Following the definition of $\mathbb{E}_{C_{j+1}}\theta$ we replace $F_j(B, \varphi)$ by $F_j(B, \varphi + \zeta)$ where ζ is Gaussian with covariance C_{j+1} and the expectation is over ζ . The covariance C_{j+1} is such that $\zeta|_B$ and $\zeta|_{B'}$ are independent for distinct blocks $B, B' \in \mathcal{B}_{j+1}$. Consequently, by the inductive hypothesis

$$\mathbb{E}_{C_{j+1}}\theta Z_j = \mathbb{E}_{C_{j+1}}\theta \prod_{B \in \mathcal{B}_{j+1}} F_j^B = \prod_{B \in \mathcal{B}_{j+1}} \mathbb{E}_{C_{j+1}}\theta F_j^B = F_{j+1}^\Lambda \quad (5.1.17)$$

as claimed. To prove that $F_{j+1} \in \mathcal{F}_{j+1}$ as in Definition 5.1.5 we use the inductive hypothesis $F_j \in \mathcal{F}_j$, the recursive definition (5.1.15) and that φ_{j+1} is constant on B . These immediately imply that F_{j+1} satisfies locality and homogeneity. By Exercise 2.1.13, $T \circ \mathbb{E}_{C_{j+1}}\theta = \mathbb{E}_{C_{j+1}}\theta \circ T$ for any $T \in O(n)$. This implies that F_{j+1} is $O(n)$ -invariant, which completes the proof that $F_{j+1} \in \mathcal{F}_{j+1}$, and completes the proof of the lemma. ■

According to (5.1.16) the sequence Z_j is determined by the sequence F_j . A key point is the simplifying feature that F_j is *local*, i.e., $F(B)$ depends only on φ_x for $x \in B$, while Z_j is *global*, i.e., it depends on φ_x for all $x \in \Lambda$.

In order to define the renormalisation group map, we make a conceptual shift in thinking about Lemma 5.1.6. Namely, we broaden our perspective, and no longer consider the input to the expectation $\mathbb{E}_{C_{j+1}}\theta$ as necessarily being determined by a specific sequence Z_j with initial condition Z_0 . Instead, we consider a generic $F \in \mathcal{F}_j$, define $Z = F^\Lambda$, and assume that Z is integrable. Then we consider $\mathbb{E}_+\theta = \mathbb{E}_{C_{j+1}}\theta$ as a map acting on this class of Z . The calculation in (5.1.17) shows that the map $F \mapsto F_+$ defined by $F_+(B) = \mathbb{E}_+\theta F^B$ is a lift of the map $Z \mapsto Z_+ = \mathbb{E}_+Z$. See Figure 5.1. As discussed above, this is a global to local reduction.

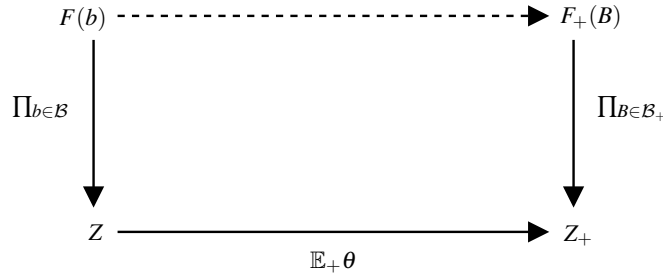


Fig. 5.1 The map $\mathbb{E}_+\theta : Z \mapsto Z_+$ is lifted to $F \mapsto F_+$.

5.2 The renormalisation group map

5.2.1 Local coordinates

To describe the map $F \mapsto F_+$, defined by $F_+(B) = \mathbb{E}_+ \theta F^B$ for integrable $F \in \mathcal{F}$, we introduce coordinates. Ideally, we would like to replace F by e^{-U} with $U \in \mathcal{U}$. This is not exactly possible, as we will need more degrees of freedom for a typical F than just three real parameters (u, g, v) . In particular, it is in general not the case that there exists $U_+ \in \mathcal{U}$ such that $\mathbb{E}_{C_+} \theta e^{-U(B)}$ will be equal to $e^{-U_+(B)}$. So instead, we make an approximate replacement of F by e^{-U} , and keep track of the error in this replacement.

In detail, given $U = (u, V) \in \mathcal{U}$, we define $I \in \mathcal{F}$ by

$$I(b) = e^{-V(b)}, \quad (5.2.1)$$

and write $F \in \mathcal{F}$ as

$$F(b) = e^{-u|b|}(I(b) + K(b)), \quad (5.2.2)$$

where K is defined so that (5.2.2) holds: $K(b) = e^{u|b|}F(b) - I(b)$. Then (5.2.2) represents F by *local coordinates*

$$(u, V, K) = (U, K), \quad (5.2.3)$$

with $u \in \mathbb{R}$, $V \in \mathbb{R}^2$, and $K \in \mathcal{F}$. We can turn this around: given coordinates (U, K) , the formula (5.2.2) defines F . If we define $Z = F^\Lambda$, then (U, K) also determines Z . Note that Z_0 of (5.1.14) is of this form, with $F = e^{-V}$, corresponding to $u = 0, K = 0$.

Given *any* $U_+ \in \mathcal{U}$, a simple algebraic manipulation shows that $F_+(B) = \mathbb{E}_{C_+} \theta F^B$ can be expressed in the same form

$$F_+(B) = \mathbb{E}_+ \theta F^B = e^{-u_+|B|}(I_+(B) + K_+(B)), \quad (5.2.4)$$

with $I_+ \in \mathcal{F}_+$ defined by $I_+(B) = e^{-V_+(B)}$ and with K_+ uniquely defined by

$$K_+(B) = e^{(u_+ - u)|B|} \mathbb{E}_+ \theta (I + K)^B - I_+(B). \quad (5.2.5)$$

It is straightforward to check that this is the solution that makes the diagram commutative in Figure 5.2. With $Z_+ = \mathbb{E}_+ \theta Z = \mathbb{E}_+ \theta F^\Lambda = F_+^\Lambda$ (we used (5.1.17) for the last equality), we obtain

$$Z_+ = e^{-u_+|\Lambda|}(I_+ + K_+)^{\Lambda} = e^{-u|\Lambda|} \mathbb{E}_+ \theta (I + K)^{\Lambda} = \mathbb{E}_+ \theta Z. \quad (5.2.6)$$

To be useful, we will need to make an intelligent choice of U_+ . Our choice is made in Section 5.2.4. It is designed in such a way that we will be able to prove that if $K = K_j$ is third order in the coefficients of U , then K_+ will be third order in the coefficients of U_+ *uniformly in the scale j* . The coordinate K is thus an error coordinate which gathers third order errors. Detailed second-order information is

retained in the polynomials U and U_+ , and this is the information that is primary in the computation of critical exponents.

We emphasise that the remainder coordinate K is not written in the exponent, i.e., we use the form $F(b) = e^{-u|b|}(e^{-V(b)} + K(b))$ instead of $e^{-u|b| - V(b) + K(b)}$. Since $K(b)$ contains contributions that are, e.g., degree-6 in the field ϕ and of uncontrolled sign, it is useful not to exponentiate them. Note that requiring that $K(b)$ be $O(|\phi|^6)$ as $\phi \rightarrow 0$ would be a natural condition to fix the choice of V and K ; however, we do not impose it and allow $K(b)$ to contain sufficiently small contributions of lower order in ϕ . This gives a somewhat more flexible representation whose generalisation is particularly useful in the Euclidean setting.

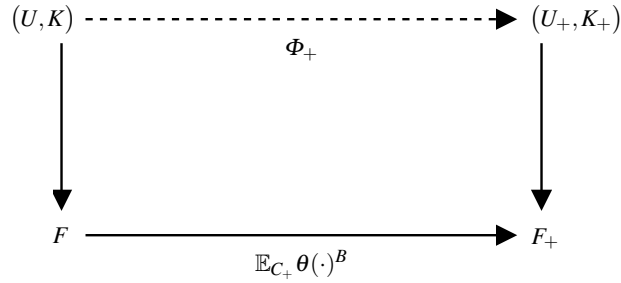


Fig. 5.2 The map $F \mapsto F_+$ is lifted to map $(U, K) \mapsto (U_+, K_+)$. The lift is not unique.

5.2.2 Localisation

A crucial idea for the renormalisation group method is that the coordinate K in (5.2.3) must contract under change in scale. A full discussion of this contraction involves the introduction of a norm to measure the size of K . We defer this to future chapters. In this section, we restrict attention to the definition of a map which extracts from a functional of the field, such as K , a local polynomial in the field which represents the parts of the functional which do *not* contract. This map is called *Loc*.

The monomials which comprise the range of *Loc* are those which do not contract under change of scale, in the following sense. For dimension $d = 4$, by (4.1.12) the approximate size (square root of the variance) of the fluctuation field $|\zeta_{j+1}|$ is L^{-j} . For $V = |\zeta_{j+1}|^p$, the size of $V(b)$ is approximately $L^{4j}L^{-pj} = L^{(4-p)j}$ because this is the number $|b| = L^{4j}$ of fields in b times the approximate size L^{-pj} of the monomial $|\zeta_{j+1;x}|^p$ at a point $x \in b$. Under this measure of size, the monomial grows exponentially with the scale if $p < 4$, it neither grows nor contracts if $p = 4$, and it contracts with the scale if $p > 4$. This motivates the following definition.

Definition 5.2.1. A homogeneous polynomial in ζ of degree p is *relevant* if $p < 4$, *marginal* if $p = 4$, and *irrelevant* if $p > 4$.

For a functional F of the field, we will use Taylor expansion to define $\text{Loc} F$ as the projection onto the relevant and marginal monomials of F . For this, we first develop the theory of Taylor expansion.

Recall that for a sufficiently smooth function $F : \mathbb{R}^n \rightarrow \mathbb{R}$ and a point $\varphi \in \mathbb{R}^n$, the p^{th} derivative $F^{(p)}(\varphi)$ of F at φ is the p -linear function of directions $\dot{\varphi}^p = (\dot{\varphi}_1, \dots, \dot{\varphi}_p) \in (\mathbb{R}^n)^p$ given by

$$F^{(p)}(\varphi; \dot{\varphi}^p) = \frac{\partial}{\partial t_1} \dots \frac{\partial}{\partial t_p} F(\varphi + \sum_{i=1, \dots, p} t_i \dot{\varphi}_i), \quad (5.2.7)$$

with the derivatives evaluated at $t_1 = \dots = t_p = 0$.

Definition 5.2.2. For smooth $F : \mathbb{R}^n \rightarrow \mathbb{R}$ and $k \geq 0$, we define $\text{Tay}_k F$ to be the k^{th} -order Taylor polynomial at 0, i.e., for $\varphi \in \mathbb{R}^n$,

$$\text{Tay}_k F(\varphi) = \sum_{p=0}^k \frac{1}{p!} F^{(p)}(0; \varphi^p), \quad \varphi^p = \varphi, \dots, \varphi. \quad (5.2.8)$$

For $b \in \mathcal{B}$ and $F(b) = F \circ j_b \in \mathcal{N}(b)$, and for a field φ that is constant on b , we define

$$\text{Tay}_k F(b) = (\text{Tay}_k F) \circ j_b. \quad (5.2.9)$$

We define the *localisation* operator Loc by

$$\text{Loc} F(b) = \text{Tay}_4 F(b). \quad (5.2.10)$$

By definition, Tay_k is a projection. More generally, $\text{Tay}_k \text{Tay}_l F(b) = \text{Tay}_{k \wedge l} F(b)$. We need $\text{Loc} = \text{Tay}_4$ only for $O(n)$ -invariant $F(b)$, and in this case it simplifies.

Lemma 5.2.3. *Let $b \in \mathcal{B}$ and suppose that $F(b) = F \circ j_b$ is in $\mathcal{N}(b)$ and is $O(n)$ -invariant. Then, for constant φ on b ,*

$$\text{Loc} F(b, \varphi) = F(0) + \frac{1}{2!} F^{(2)}(0; e_1^2) |\varphi|^2 + \frac{1}{4!} F^{(4)}(0; e_1^4) |\varphi|^4, \quad (5.2.11)$$

where $e_1 = (1, 0, \dots, 0)$ in $\mathbb{R}^n$ and $e_1^p = e_1, \dots, e_1$ in $\prod_{i=1, \dots, p} \mathbb{R}^n$. In particular, there is a unique element U_F of $\mathcal{U}$ such that $\text{Loc} F(b) = U_F(b)$, and we identify $\text{Loc} F$ with this element U_F .

Proof. By hypothesis, $F(T\varphi) = F(\varphi)$ for every $T \in O(n)$. With the choice $T = -I$, we see that $F^{(\alpha)}(0) = 0$ for all odd $|\alpha|$. With T chosen so that $T\varphi = (|\varphi|, 0, \dots, 0)$ (a rotation), we obtain (5.2.11). ■

Example 5.2.4. Consider 1-component fields φ defined on $\mathbb{R}^{\Lambda_N}$ which are constant on the block $b \in \mathcal{B}$.

(i) Let

$$F(b, \varphi) = \sum_{x \in b} \varphi_x^2. \quad (5.2.12)$$

This is an element of $\mathcal{N}(b)$ as in Definition 5.1.4. In particular, $F(b) = F \circ j_b$ with $F(u) = |b|u^2$, so $\text{Tay}_4 F(u) = |b|u^2$. Therefore $(\text{Tay}_4 F) \circ j_b(\varphi) = |b|(j_b(\varphi))^2$. Equivalently, $(\text{Tay}_4 F) \circ j_b(\varphi) = \sum_{x \in b} \varphi_x^2$. By Definition 5.2.2,

$$\text{Loc } F(b, \varphi) = \sum_{x \in b} \varphi_x^2 = F(b, \varphi). \quad (5.2.13)$$

Similarly, $\text{Loc } F(b, \varphi) = F(b, \varphi)$ if

$$F(b, \varphi) = \sum_{x \in b} \left(\frac{1}{4} g \varphi_x^4 + \frac{1}{2} v \varphi_x^2 + u \right). \quad (5.2.14)$$

(ii) Let

$$F(b, \varphi) = e^{\sum_{x \in b} v \varphi_x^2}. \quad (5.2.15)$$

Then $F(b) = F \circ j_b$ with $F(u) = e^{|b|v u^2}$, so $\text{Tay}_4 F(u) = 1 + |b|v u^2 + \frac{1}{2}|b|^2 v^2 u^4$. Therefore $(\text{Tay}_4 F) \circ j_b(\varphi) = 1 + |b|v(j_b(\varphi))^2 + \frac{1}{2}|b|^2 v^2(j_b(\varphi))^4$. Equivalently, $(\text{Tay}_4 F) \circ j_b(\varphi) = 1 + \sum_{x \in b} v \varphi_x^2 + \frac{1}{2}|b| \sum_{x \in b} v^2 \varphi_x^4$. By Definition 5.2.2,

$$\text{Loc } F(b, \varphi) = \sum_{x \in b} \left(\frac{1}{|b|} + v \varphi_x^2 + \frac{1}{2}|b|v^2 \varphi_x^4 \right). \quad (5.2.16)$$

In (5.2.13) and (5.2.16), the output of Loc has been written as a local polynomial in the field, summed over the block b . For the hierarchical model this could be seen as a redundant formulation, since the field is constant on b and hence, e.g., $\sum_{x \in b} \varphi_x^2 = |b|\varphi^2$. However, in the Euclidean model the field is no longer constant on blocks, and (5.2.13) and (5.2.16) have direct Euclidean counterparts. This illustrates the general theme that Euclidean formulas specialised to the case where fields are constant on blocks reduce to hierarchical formulas.

Ultimately, the proof that K_+ contracts relative to K requires an estimate on $1 - \text{Loc}$. A general version of this crucial estimate is given in Section 7.5, and its specific application occurs in Section 10.5.

5.2.3 Perturbative map

In this section, C_+ is any covariance with the property that the corresponding fields are constants on blocks in $b \in \mathcal{B}$. It will be taken to be either C_{j+1} for some $j < N$, or $C_{\hat{N}}$ when $j = N$. We sometimes write $\mathbb{E}_+$ in place of $\mathbb{E}_{C_+}$.

As discussed in Section 5.2.1, it is in general not the case that there exists $U_+ \in \mathcal{U}$ such that $\mathbb{E}_+ e^{-U(B)}$ will be equal to $e^{-U_+(B)}$. The *perturbative map* is a map $U \mapsto U_{\text{pt}}$ such that, in a sense to be made precise below, $\mathbb{E}_+ e^{-U(B)}$ is approximately equal to $e^{-U_{\text{pt}}(B)}$. The map is defined as follows.

Definition 5.2.5. Recall that $\mathbb{E}(\theta A; \theta B)$ is defined in (2.2.6); it is the same as the covariance $\text{Cov}(\theta A, \theta B)$. Given $U \in \mathcal{U}$, we define

$$U_{\text{pt}}(B) = \mathbb{E}_{C_+} \theta U(B) - \frac{1}{2} \text{Loc} \mathbb{E}_{C_+} (\theta U(B); \theta U(B)) \quad (B \in \mathcal{B}_{j+1}). \quad (5.2.17)$$

Exercise 5.1.2 shows that $\mathbb{E}_{C_+} \theta U(B)$ determines an element of $\mathcal{U}$, and the range of Loc is also $\mathcal{U}$, so $U_{\text{pt}}(B)$ determines an element of $\mathcal{U}$. We define the *perturbative map* $\Phi_{\text{pt}} : \mathcal{U} \rightarrow \mathcal{U}$ by setting $\Phi_{\text{pt}}(U)$ to be this element. Then (5.2.17) can also be written as

$$\Phi_{\text{pt}}(U; B) = \mathbb{E}_{C_+} \theta U(B) - \frac{1}{2} \text{Loc} \mathbb{E}_{C_+} (\theta U(B); \theta U(B)). \quad (5.2.18)$$

We also define

$$W_+(B) = \frac{1}{2} (1 - \text{Loc}) \mathbb{E}_{C_+} (\theta U(B); \theta U(B)). \quad (5.2.19)$$

By Lemma 5.1.3, if $c^{(1)} = 0$ then the definition of $U_{\text{pt}}(B)$ is not changed if Loc is removed from the right-hand side of (5.2.17). Also by Lemma 5.1.3, $W(B)$ is proportional to $\sum_{x \in B} \tau_x^3$, and is in fact zero if $c^{(1)} = 0$.

The polynomial $U_{\text{pt}} \in \mathcal{U}$ can be calculated explicitly, and the result of this calculation is given in Proposition 5.3.1. The coefficients of U_{pt} are explicit quadratic polynomials in the coefficients of U , and the coefficients of these quadratic polynomials are explicit functions of the covariance C_+ .

The sense in which the expectation $\mathbb{E}_+ \theta e^{-U}$ is approximately $e^{-U_{\text{pt}}(U)}$ is made precise by Lemma 5.2.6. Given U , we define

$$\delta U = \theta U - U_{\text{pt}}(U). \quad (5.2.20)$$

The following lemma illustrates what the definition of U_{pt} achieves. It shows that the difference between $\mathbb{E} e^{-\theta U}$ and $e^{-U_{\text{pt}}}$ is the sum of three terms. The term involving W is second order in U but is zero as long as $c^{(1)} = 0$, which does hold for all scales except the last scale by (4.1.13). The term involving $(\text{LocVar}(\theta U))^2$ is fourth-order in U , and there is a term that is formally third order in δU , defined in terms of

$$A_3(B) = \frac{1}{2!} \int_0^1 (-\delta U(B))^3 e^{-t\delta U(B)} (1-t)^2 dt. \quad (5.2.21)$$

In Section 10.3.2, we provide a careful analysis of the term involving A_3 .

Lemma 5.2.6. *For any polynomial $U \in \mathcal{U}$ such that the expectations exist, and for $B \in \mathcal{B}_+$,*

$$\mathbb{E}_+ e^{-\theta U(B)} = e^{-U_{\text{pt}}(B)} \left(1 + W_+(B) + \frac{1}{8} (\text{LocVar}(\theta U))^2 + \mathbb{E}_+ A_3(B) \right). \quad (5.2.22)$$

In particular, if the covariance satisfies the zero-sum condition $c^{(1)} = 0$, then

$$\mathbb{E}_+ e^{-\theta U(B)} = e^{-U_{\text{pt}}(B)} \left(1 + \frac{1}{8} (\text{Var}(\theta U(B)))^2 + \mathbb{E}_+ A_3(B) \right). \quad (5.2.23)$$

Proof. We drop the block B and subscript $+$ from the notation. Then we can rewrite the desired formula (5.2.22) as

$$\mathbb{E} e^{-\delta U} = 1 + W + \frac{1}{8} (\text{LocVar}(\theta U))^2 + \mathbb{E} A_3. \quad (5.2.24)$$

By the Taylor remainder formula,

$$e^{-\delta U} = 1 - \delta U + \frac{1}{2} (\delta U)^2 + A_3. \quad (5.2.25)$$

By Definition 5.2.5,

$$\mathbb{E} \delta U = \frac{1}{2} \text{LocVar}(\theta U). \quad (5.2.26)$$

Also, $\text{Var}(\delta U) = \text{Var}(\theta U)$ and $W = \frac{1}{2}(1 - \text{Loc})\text{Var}\theta U$, so

$$\begin{aligned} \mathbb{E} \left(-\delta U + \frac{1}{2} (\delta U)^2 \right) &= -\mathbb{E}(\delta U) + \frac{1}{2} \text{Var}(\delta U) + \frac{1}{2} (\mathbb{E}(\delta U))^2 \\ &= -\frac{1}{2} \text{LocVar}(\theta U) + \frac{1}{2} \text{Var}(\theta U) + \frac{1}{8} (\text{LocVar}(\theta U))^2 \\ &= W + \frac{1}{8} (\text{LocVar}(\theta U))^2. \end{aligned} \quad (5.2.27)$$

By (5.2.25), this leads to (5.2.24). Finally, (5.2.23) then follows immediately since when $c^{(1)} = 0$ we have $W = 0$ and $\text{LocVar}\theta U = \text{Var}\theta U$. This completes the proof. ■

A more naive idea would be to expand $e^{-\theta U}$ into a power series before computing the expectation $\mathbb{E}_+ e^{-\theta U}$, but such expansions can behave badly under the expectation, as illustrated by the following exercise.

Exercise 5.2.7. Observe that, for any $g \geq 0$,

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-gx^4} e^{-\frac{1}{2}x^2} dx \leq 1, \quad (5.2.28)$$

whereas, on the other hand, the series

$$\sum_{n=0}^{\infty} \frac{1}{n!} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (-gx^4)^n e^{-\frac{1}{2}x^2} dx \quad (5.2.29)$$

is not absolutely convergent for any $g \neq 0$. [Solution]

5.2.4 Definition of the renormalisation group map

We now have all the ingredients needed to define the renormalisation group map. We define the map Φ_{pt} with $C_+ = C_{j+1}$ as in Proposition 4.1.9. As before, the scale

j is fixed and omitted from the notation. We assume here that $j < N$. In particular, this means that $c_+^{(1)} = 0$ holds so that Loc can be dropped from (5.2.18) and $W_+ = 0$.

Definition 5.2.8. For $m^2 \geq 0$, the *renormalisation group map*

$$\Phi_+ : (V, K) \mapsto (U_+, K_+) = (u_+, V_+, K_+) \quad (5.2.30)$$

is defined by

$$U_+ = \Phi_{\text{pt}}(V - \text{Loc}(e^V K)), \quad (5.2.31)$$

$$K_+(B) = e^{u_+|B|} \mathbb{E}_+ \theta(I + K)^B - I_+(B), \quad (5.2.32)$$

where $I = e^{-V}$, and $I_+ = e^{-V_+}$. The domain of Φ_+ consists of those $(V, K) \in \mathcal{V} \times \mathcal{F}$ such that $\mathbb{E}_{C_+} \theta(I(V) + K)^B$ is defined. We write the components of Φ_+ as

$$\Phi_+ = (\Phi_+^U, \Phi_+^K) = (\Phi_+^u, \Phi_+^V, \Phi_+^K) = (\Phi_+^u, \Phi_+^{(0)}). \quad (5.2.33)$$

Note that in (5.2.30) the input polynomial is $V \in \mathcal{V}$ rather than $U = u + V \in \mathcal{U}$. The reason why it is sufficient to consider the case $u = 0$ is discussed in Remark 5.2.9. Note also that $\hat{U}(b) = V(b) - \text{Loc}(e^{V(b)} K(b))$ in (5.2.31) defines a b -independent element $\hat{U} \in \mathcal{U}$, due to the spatial homogeneity imposed on $K \in \mathcal{F}$ by Definition 5.1.5. The formula (5.2.32) for K_+ is identical to (5.2.5), with the specific choice (5.2.31) for U_+ , and with $u = 0$.

Remark 5.2.9. The domain of Φ_+ involves $V \in \mathcal{V}$ instead of $U \in \mathcal{U}$, i.e., for $u = 0$, while the output of Φ_+ has a u -component. This is because the dependence of the expectation on u is of a trivial nature. Let $U = (u, g, v) = (u, V)$. Then $\hat{U} = u + \hat{V}$ and $U_{\text{pt}}(U) = u + U_{\text{pt}}(V)$, and thus $U_+(U, K) = u + U_+(V, K)$ and $K_+(U, K) = e^{u|B|} K_+(V, K)$. The effect of nonzero u can thus be incorporated in this manner. We refer to the transition from $U = (u, g, v)$ to $U_+ = (u_+, g_+, v_+)$ as the *flow of coupling constants*.

Remark 5.2.10. We emphasise that, while the hierarchical model is originally defined only for $m^2 > 0$, the covariances $C_1, \dots, C_N$ (but not $C_{\hat{N}}$) are well-defined also for $m^2 = 0$. This allows us to define the renormalisation group map also for $m^2 = 0$, as in Definition 5.2.8. Furthermore, the maps Φ_j for Λ_N are the same for all $N \geq j$.

There are two aspects to our choice of U_+ , which is the basis for the definition of the renormalisation group map Φ_+ , with K_+ given by (5.2.5).

Nonperturbative aspect. In $e^{-V} + K = e^{-V}(1 + e^V K)$, the term $e^V K$ can contain relevant and marginal contributions, so we isolate these as $e^V K = \text{Loc } e^V K + (1 - \text{Loc})e^V K$. We wish to absorb $\text{Loc } e^V K$ into V , which is in the exponent, so we approximate $e^{-V}(1 + \text{Loc } e^V K)$ by $e^{-\hat{U}}$ with $\hat{U} = V - \text{Loc } e^V K$. This approximate transfer of the marginal and relevant terms turns out to be sufficient since we will impose a hypothesis that the remainder K is higher order and therefore does not significantly affect the evolution of V .

Perturbative aspect. The expectation $\mathbb{E}\theta e^{-\hat{U}}$ is approximately $e^{-U_{\text{pt}}(\hat{U})}$, in the sense that is made precise by Lemma 5.2.6. The occurrence of Φ_{pt} in (5.2.31) is for this reason.

5.2.5 The last renormalisation group step

The final renormalisation group step concerns the integration with covariance $C_{\hat{N}} = m^{-2}Q_N$. This step is special. There is no more reblocking because after the integration with covariance C_N only the one block $B = \Lambda_N$ remains. Also, $\hat{c}_N^{(1)} = m^{-2} \neq 0$ so $W_{\hat{N}}$ is not zero, where $W_{\hat{N}}$ is given by (5.2.19) to be

$$W_{\hat{N}}(B) = \frac{1}{2}(1 - \text{Loc})\mathbb{E}_{C_{\hat{N}}}(\theta V(B); \theta V(B)) \quad (B = \Lambda). \quad (5.2.34)$$

Definition 5.2.11. The *final renormalisation group map* $(V, K) \mapsto (U_{\hat{N}}, K_{\hat{N}})$ is the map from $\mathcal{V} \times \mathcal{F}$ to $\mathcal{U} \times \mathcal{F}$ defined by

$$U_{\hat{N}} = U_{\text{pt}}(V), \quad (5.2.35)$$

$$K_{\hat{N}}(B) = e^{-V_{\hat{N}}(B)} \left(\frac{1}{8} (\text{LocVar}(\theta U))^2 + \mathbb{E}_{C_{\hat{N}}} A_3(B) \right) + e^{u_{\hat{N}}|B|} \mathbb{E}_{C_{\hat{N}}} \theta K(B), \quad (5.2.36)$$

where $B = \Lambda_N$, $U_{\hat{N}} = u_{\hat{N}} + V_{\hat{N}}$ with $V_{\hat{N}} \in \mathcal{V}$, δU and A_3 are as in (5.2.20) and (5.2.21). We assume the expectations in (5.2.36) exist.

The formula for $U_{\hat{N}}$ in Definition 5.2.11 does not have the Loc term present in (5.2.31), because it is not necessary to remove expanding parts of K when there are no more renormalisation group steps to cause K to expand.

The following proposition shows that Definition 5.2.11 and $\hat{c}_N^{(1)} \neq 0$ lead to a revised version of the representation (5.2.6) where now $I_{\hat{N}}$ is given by $e^{-V_{\hat{N}}}(1 + W_{\hat{N}})$ rather than simply by $e^{-V_{\hat{N}}}$ as in all earlier renormalisation group steps.

Proposition 5.2.12. With $U_{\hat{N}}$ and $K_{\hat{N}}$ as in Definition 5.2.11, with $B = \Lambda_N$, and assuming that the expectations exist,

$$\mathbb{E}_{C_{\hat{N}}} \left(e^{-\theta V(B)} + \theta K(B) \right) = e^{-u_{\hat{N}}|B|} \left(e^{-V_{\hat{N}}(B)} (1 + W_{\hat{N}}(B)) + K_{\hat{N}}(B) \right). \quad (5.2.37)$$

Proof. We drop B from the notation. By Lemma 5.2.6, the left-hand side of (5.2.37) is equal to

$$e^{-U_{\hat{N}}} \left(1 + W_{\hat{N}} + \frac{1}{8} (\text{LocVar}(\theta U))^2 + \mathbb{E}_{C_{\hat{N}}} A_3 \right) + \mathbb{E}_{C_{\hat{N}}} \theta K. \quad (5.2.38)$$

After an algebraic reorganisation, this is seen to equal the right-hand side of (5.2.37). $\blacksquare$

Remark 5.2.13. As discussed further in Chapter A, for the Euclidean finite-range decomposition it is the case that $c^{(1)} \neq 0$ for all covariances. This creates a need for a term W in all renormalisation group steps, not just in the last step as we have here for the hierarchical model.

5.3 Perturbative flow of coupling constants: the map Φ_{pt}

In this section, we explicitly compute the map $\Phi_{\text{pt}}(V) = \Phi_+^U(V, 0)$, which by definition is the map $V \mapsto U_{\text{pt}}$ of (5.2.17). Note that Φ_{pt} depends only on V , and not on K . We allow nonzero u in this section, so that, as discussed in Remark 5.2.9, Φ_{pt} acts on $U = u + V \in \mathcal{U}$ rather than on $V \in \mathcal{V}$. As we show in Chapter 6, Φ_{pt} represents the second-order part of the map Φ_+ , whose remaining parts are third-order. We write the image of U under Φ_{pt} as $(u_{\text{pt}}, g_{\text{pt}}, v_{\text{pt}})$. Thus our goal is the calculation of $(u_{\text{pt}}, g_{\text{pt}}, v_{\text{pt}})$ as a function of (u, g, v) . This functional dependence of the former on the latter is referred to as the *perturbative flow of coupling constants*.

5.3.1 Statement of the perturbative flow

The perturbative flow of coupling constants is best expressed in terms of the rescaled variables:

$$\mu = L^{2j} v, \quad \mu_{\text{pt}} = L^{2(j+1)} v_{\text{pt}}, \quad E_{\text{pt}} = L^{d(j+1)} (u_{\text{pt}} - u). \quad (5.3.1)$$

We generally omit the scale index j , and regard variables with index pt as scale- $(j+1)$ quantities. The powers of L in (5.3.1) correspond to the scaling of the monomials on a block as discussed above Definition 5.2.1: $v\varphi^2$ scales like vL^{2j} , $u\varphi^0$ scales like uL^{dj} , and g is unscaled since φ^4 is marginal.

Proposition 5.3.1. *Let $d = 4$, $\gamma = (n+2)/(n+8)$, and suppose that $c^{(1)} = 0$. Then the map $U \mapsto U_{\text{pt}}$ of (5.2.17) can be written as*

$$g_{\text{pt}} = g - \beta g^2, \quad (5.3.2)$$

$$\mu_{\text{pt}} = L^2 (\mu(1 - \gamma\beta g) + \eta g - \xi g^2), \quad (5.3.3)$$

$$E_{\text{pt}} = L^d (\kappa_g g + \kappa_\mu \mu - \kappa_{g\mu} g\mu - \kappa_{gg} g^2 - \kappa_{\mu\mu} \mu^2), \quad (5.3.4)$$

where $\beta, \eta, \xi, \kappa_*$ are j -dependent constants defined in (5.3.10)–(5.3.12) below.

Ultimately, the coefficient γ in (5.3.3) will become the exponent of the logarithm in Theorem 4.2.1. To define the coefficients that appear in (5.3.2)–(5.3.4), we recall the definitions (4.1.19)–(4.1.20), namely

$$c_j = L^{-(d-2)j} (1 + m^2 L^{2j})^{-1} (1 - L^{-d}), \quad (5.3.5)$$

$$c_j^{(n)} = \sum_{x \in \Lambda} (C_{j+1;0,x}(m^2))^n. \quad (5.3.6)$$

We define the coefficients

$$\eta'_j = (n+2)c_j, \quad \beta'_j = (n+8)c_j^{(2)}, \quad \xi'_j = 2(n+2)c_j^{(3)} + (n+2)^2 c_j c_j^{(2)}, \quad (5.3.7)$$

$$\kappa'_{g,j} = \frac{1}{4}n(n+2)c_j^2, \quad \kappa'_{v,j} = \frac{1}{2}n c_j, \quad \kappa'_{gv,j} = \frac{1}{2}n(n+2)c_j c_j^{(2)}, \quad (5.3.8)$$

$$\kappa'_{gg,j} = \frac{1}{4}n(n+2) \left(c_j^{(4)} + (n+2)c_j^2 c_j^{(2)} \right), \quad \kappa'_{vv,j} = \frac{1}{4}n c_j^{(2)}. \quad (5.3.9)$$

The primes in the above definitions indicate that they refer to unscaled variables; these primes are dropped in rescaled versions. For $d = 4$, the rescaled versions are defined by

$$\eta_j = L^{2j} \eta'_j, \quad \beta_j = \beta'_j, \quad \xi_j = L^{2j} \xi'_j, \quad (5.3.10)$$

$$\kappa_{g,j} = L^{4j} \kappa'_{g,j}, \quad \kappa_{v,j} = L^{2j} \kappa'_{v,j}, \quad \kappa_{g\mu,j} = L^{2j} \kappa'_{gv,j}, \quad (5.3.11)$$

$$\kappa_{gg,j} = L^{4j} \kappa'_{gg,j}, \quad \kappa_{\mu\mu,j} = \kappa'_{vv,j}. \quad (5.3.12)$$

All the above coefficients depend on the mass m^2 occurring in the covariance.

The coefficient β_j is of particular importance. The use of the Greek letter β is not entirely consistent with the term “beta function” in physics, which in our context would represent the difference between the coupling constant at two successive scales. In our formulation, β represents the coefficient of g^2 in the beta function.

Remark 5.3.2. A term corresponding to κ'_{gv} was incorrectly omitted in [18, (3.27)–(3.28)]. Its inclusion does not affect the conclusions of [18].

Definition 5.3.3. For $m > 0$, let j_m be the greatest integer j such that $L^j m \leq 1$, and set $j_m = \infty$ if $m = 0$. We call j_m the *mass scale*.

The mass scale is the scale j at which the effect of the mass becomes important in estimates. For the mass-dependent factor in (4.1.18), for $L \geq 2$ we have

$$(1 + m^2 L^{2j})^{-1} \leq L^{-2(j-j_m)_+} \leq 4^{-(j-j_m)_+} = \vartheta_j^2 \leq \vartheta_j, \quad (5.3.13)$$

where the equality defines

$$\vartheta_j = 2^{-(j-j_m)_+}. \quad (5.3.14)$$

The advantage of ϑ_j over the stronger upper bound $L^{-2(j-j_m)_+}$ is that ϑ_j is independent of L . We often use ϑ_j as an adequate way to take into account decay above the mass scale.

Lemma 5.3.4. For $d = 4$,

$$\eta_j = \eta_0^0(1 + m^2 L^{2j})^{-1}, \quad \beta_j = \beta_0^0(1 + m^2 L^{2j})^{-2}, \quad (5.3.15)$$

$$\xi_j = \xi_0^0(1 + m^2 L^{2j})^{-3}, \quad \kappa_{*,j} = O((1 + m^2 L^{2j})^{-1}), \quad (5.3.16)$$

where $\beta_0^0 = \beta_0(m^2 = 0) = (n+8)(1-L^{-d})$, and analogously for ξ and η . In particular, each of $\eta_j, \beta_j, \xi_j, \kappa_{*,j}$ is bounded above by $O(\vartheta_j)$.

Proof. This follows from the definitions and Exercise 4.1.11. $\blacksquare$

An essential property is that $\beta_0^0 > 0$. Recall from Exercise 4.1.12 that the hierarchical bubble diagram $B_{m^2}^H$ is given by

$$B_{m^2}^H = \sum_{j=1}^{\infty} c_j^{(2)}. \quad (5.3.17)$$

Therefore,

$$\sum_{j=1}^{\infty} \beta_j = (n+8)B_{m^2}^H, \quad (5.3.18)$$

which is finite for $d = 4$ if and only if $m^2 > 0$, and diverges logarithmically as $m^2 \downarrow 0$ for $d = 4$.

5.3.2 Proof of the perturbative flow

The flow of coupling constants is stated in Proposition 5.3.1 in terms of rescaled variables, but for the proof we find it more convenient to work with the original variables. Also, although $c^{(1)}$ of (5.3.6) is equal to zero by (4.1.13), the final covariance $C_{\hat{N}}$ does not sum to zero. We allow for nonzero $c^{(1)}$ in the following proposition, so that it also handles the case of $C_{\hat{N}}$. For this, we introduce the two coefficients

$$s'_{\tau^2,j} = 4(g^2(n+2)c_j + g\nu)c_j^{(1)}, \quad (5.3.19)$$

$$s'_{\tau,j} = (g^2(n+2)^2c_j^2 + 2g\nu(n+2)c_j + \nu^2)c_j^{(1)}, \quad (5.3.20)$$

which each vanish when $c_j^{(1)} = 0$. Proposition 5.3.1 is an immediate consequence of Proposition 5.3.5.

Proposition 5.3.5. *For $U = u + g\tau^2 + \nu\tau$, the polynomial U_{pt} defined in (5.2.17) has the form $U_{\text{pt}} = g_{\text{pt}}\tau^2 + \nu_{\text{pt}}\tau + u_{\text{pt}}$ with*

$$g_{\text{pt}} = g - \beta'_j g^2 - s'_{\tau^2,j}, \quad (5.3.21)$$

$$\nu_{\text{pt}} = \nu(1 - \gamma\beta'_j g) + \eta'_j g - \xi'_j g^2 - s'_{\tau,j}, \quad (5.3.22)$$

$$u_{\text{pt}} = u + \kappa'_{g,j} g + \kappa'_{\nu,j} \nu - \kappa'_{g\nu,j} g\nu - \kappa'_{gg,j} g^2 - \kappa'_{\nu\nu,j} \nu^2, \quad (5.3.23)$$

with $\beta'_j, \eta'_j, \xi'_j, \kappa'_{*,j}$ defined in (5.3.7)–(5.3.9). Also,

$$W_+ = -4c^{(1)}g^2\tau^3. \quad (5.3.24)$$

In particular, if $c^{(1)} = 0$, then $W_+ = 0$, U_{pt} contains no term proportional to τ^3 , and hence $U_{\text{pt}} \in \mathcal{U}$.

Recall the definition of U_{pt} from (5.2.17) and recall Exercise 5.1.2. The following lemma computes the terms in U_{pt} that are linear in V .

Lemma 5.3.6. For $U = u + \frac{1}{4}g|\varphi|^4 + \frac{1}{2}v|\varphi|^2$,

$$\mathbb{E}_{C_{j+1}}\theta U = \frac{1}{4}g|\varphi|^4 + \frac{1}{2}(v + \eta'_jg)|\varphi|^2 + (u + \kappa'_{g,j}g + \kappa'_{v,j}v). \quad (5.3.25)$$

Proof. We write $C = C_{j+1}$, and sometimes also omit other labels j . Recall the formula $\mathbb{E}_C\theta U = e^{\frac{1}{2}\Delta_C}U$ from Proposition 2.1.6. Using this, we obtain

$$\mathbb{E}_C\theta U = U + \frac{1}{2}\Delta_C(\frac{1}{4}g|\varphi|^4 + \frac{1}{2}v|\varphi|^2) + \frac{1}{8}\Delta_C^2\frac{1}{4}g|\varphi|^4. \quad (5.3.26)$$

The η' term in (5.3.22) arises from the coefficient of $\frac{1}{2}|\varphi|^2$ in $\frac{1}{2}\Delta_C\frac{1}{4}|\varphi|^4$. By definition,

$$\Delta_C|\varphi|^4 = c_j \sum_{i=1}^n \frac{\partial^2}{\partial(\varphi^i)^2} (|\varphi|^2)^2. \quad (5.3.27)$$

Since

$$\frac{\partial^2}{\partial(\varphi^i)^2} (|\varphi|^2)^2 = 4 \frac{\partial}{\partial\varphi^i} (|\varphi|^2\varphi^i) = 8(\varphi^i)^2 + 4|\varphi|^2, \quad (5.3.28)$$

this coefficient is η'_j given by (5.3.7), as required. The constant terms are $\kappa'_v = \frac{1}{4}\Delta_C|\varphi|^2$ and $\kappa'_g = \frac{1}{32}\Delta_C^2|\varphi|^4$. We leave the verification of the formulas for κ'_v, κ'_g in (5.3.9) to Exercise 5.3.7. ■

Proof of Proposition 5.3.5. The definition of U_{pt} is given in (5.2.17). We again write $C = C_{j+1}$ and omit other labels j . The linear terms in (5.3.21)–(5.3.23) are given by Lemma 5.3.6. Let $x \in B \in \mathcal{B}_{j+1}$. For the quadratic terms, we must compute

$$\begin{aligned} \sum_{y \in B} \mathbb{E}_C(\theta V_x; \theta V_y) &= \sum_{y \in B} \left(\frac{1}{16}g^2 \mathbb{E}_C(\theta|\varphi_x|^4; \theta|\varphi_y|^4) + \frac{1}{4}g\frac{1}{2}v \mathbb{E}_C(\theta|\varphi_x|^2; \theta|\varphi_y|^4) \right. \\ &\quad \left. + \frac{1}{4}g\frac{1}{2}v \mathbb{E}_C(\theta|\varphi_x|^4; \theta|\varphi_y|^2) + \frac{1}{4}v^2 \mathbb{E}_C(\theta|\varphi_x|^2; \theta|\varphi_y|^2) \right). \end{aligned} \quad (5.3.29)$$

By Exercise 2.2.4, $\mathbb{E}_C(\theta P; \theta Q) = F_C(\mathbb{E}_C\theta P, \mathbb{E}_C\theta Q)$, and hence it follows from Lemma 5.3.6 that the summand in (5.3.29) is equal to

$$\begin{aligned}
& \frac{1}{16}g^2 F_C(|\varphi_x|^4 + 2\eta'|\varphi_x|^2; |\varphi_y|^4 + 2\eta'|\varphi_y|^2) \\
& + \frac{1}{4}g\frac{1}{2}\nu F_C(|\varphi_x|^2; |\varphi_y|^4 + 2\eta'|\varphi_y|^2) + \frac{1}{4}g\frac{1}{2}\nu F_C(|\varphi_x|^4 + 2\eta'|\varphi_x|^2; |\varphi_y|^2) \\
& + \frac{1}{4}\nu^2 F_C(|\varphi_x|^2; |\varphi_y|^2).
\end{aligned} \tag{5.3.30}$$

The above is equal to

$$\begin{aligned}
& \frac{1}{16}g^2 F_C(|\varphi_x|^4; |\varphi_y|^4) \\
& + \left(\frac{1}{16}g^2(2\eta')^2 + \frac{1}{4}g\nu 2\eta' + \frac{1}{4}\nu^2 \right) F_C(|\varphi_x|^2; |\varphi_y|^2) \\
& + \left(\frac{1}{16}g^2 2\eta' + \frac{1}{4}g\frac{1}{2}\nu \right) \left(F_C(|\varphi_x|^2; |\varphi_y|^4) + F_C(|\varphi_x|^4; |\varphi_y|^2) \right).
\end{aligned} \tag{5.3.31}$$

This can be evaluated using the formula from Exercise 2.2.4:

$$F_C(P_x; Q_y) = \sum_{p=1}^4 \frac{1}{p!} C_{x,y}^p \sum_{i_1, \dots, i_p=1}^n \frac{\partial^p P_x}{\partial \varphi_x^{i_1} \cdots \partial \varphi_x^{i_p}} \frac{\partial^p Q_y}{\partial \varphi_y^{i_1} \cdots \partial \varphi_y^{i_p}}. \tag{5.3.32}$$

In the following, we examine an important sample term, and leave most details for Exercise 5.3.7.

Consider the term $F_C(|\varphi|^4; |\varphi|^4)$. For $p = 2$, four of the eight fields are differentiated and this produces a $|\varphi|^4$ term. Calculation as in (5.3.28) gives

$$\sum_{i,j=1}^n \frac{\partial^2 |\varphi_x|^4}{\partial \varphi_x^i \partial \varphi_x^j} \frac{\partial^2 |\varphi_y|^4}{\partial \varphi_y^i \partial \varphi_y^j} = 16(n+8)|\varphi|^4, \tag{5.3.33}$$

where the subscript has been dropped on φ on the right-hand side to reflect the fact that the field is constant on B . This shows that the contribution due to $p = 2$ that arises from $-\frac{1}{2}\frac{1}{16}g^2 \sum_{y \in B} F_C(|\varphi_x|^4; |\varphi_y|^4)$ is

$$-\left(\frac{1}{2} \frac{1}{16} g^2 \frac{1}{2!} c_j^{(2)} 16(n+8) \right) |\varphi|^4 = -\beta_j g^2 \frac{1}{4} |\varphi|^4, \tag{5.3.34}$$

which is a term in (5.3.21). The $p = 1$ term gives rise to $-4c_j^{(1)}g^2\tau^3$ in U_{pt} . The $p = 1$ term from the third line of (5.3.31) gives rise to s'_{τ^2} . No other $|\varphi|^4$ terms can arise from (5.3.31), and the proof of (5.3.21) is complete. For $p = 3$, a contribution to ξ' results, and for $p = 4$, a contribution to κ'_{gg} results. We leave these, as well as the contributions due to $F_C(|\varphi|^2; |\varphi|^2)$ and $F_C(|\varphi|^2; |\varphi|^4)$, for Exercise 5.3.7. ■

Exercise 5.3.7. Verify the formulas given for the κ' coefficients in (5.3.9), and the omitted details in Proposition 5.3.5 for the coefficients in (5.3.7). [Solution]

Chapter 6

Flow equations and main result

In Section 6.1, we provide a detailed and elementary analysis of the perturbative flow of coupling constants, i.e., of the iteration of the recursion given by Proposition 5.3.1. We denote this flow by $(\bar{g}_j, \bar{\mu}_j)$. In particular, we construct a perturbative critical initial value $\bar{\mu}_0$ for which $\bar{\mu}_j$ approaches zero as $j \rightarrow \infty$.

In Section 6.2, we state extensions of the results of Section 6.1 to the nonperturbative setting, in which the recursion of Proposition 5.3.1 is corrected by higher order terms, and show that these extensions allows for a proof of the main result Theorem 4.2.1. The proof of the nonperturbative versions is given in Chapters 8–10.

6.1 Analysis of perturbative flow

In this section, we study the perturbative flow of coupling constants $\bar{U}$, defined as the solution to the recursion $\bar{U}_{j+1} = \Phi_+^U(\bar{U}_j, 0) = \Phi_{\text{pt}}(\bar{U}_j)$. The analysis of the susceptibility does not require the sequence u_j , so we do not study $\bar{u}_j$ here though its analysis is analogous. Moreover, since $\bar{u}_{j+1} - \bar{u}_j$ is a function of $\bar{V}_j$, $\bar{u}_j$ can be computed once $\bar{V}_j$ is known. Thus, we are concerned only with the $\bar{V}_j$ part of $\bar{U}_j = (\bar{u}_j, \bar{V}_j)$. We study the rescaled version $(\bar{g}_j, \bar{\mu}_j)$ of $\bar{V}_j = (\bar{g}_j, \bar{v}_j)$, with $\bar{\mu}_j = L^{2j} \bar{v}_j$.

According to (5.3.2)–(5.3.3),

$$\bar{g}_{j+1} = \bar{g}_j - \beta_j \bar{g}_j^2, \quad (6.1.1)$$

$$\bar{\mu}_{j+1} = L^2 (\bar{\mu}_j (1 - \gamma \beta_j \bar{g}_j) + \eta_j \bar{g}_j - \xi_j \bar{g}_j^2). \quad (6.1.2)$$

By Lemma 5.3.4, $\beta_j = \beta_0^0 (1 + m^2 L^{2j})^{-2}$. In particular, β_j is constant when $m^2 = 0$. The system of equations (6.1.1)–(6.1.2) is *triangular* since the first equation only depends on $\bar{g}$. Thus the equations can be solved successively; and they are so simple that we can calculate anything we want to know. Triangularity no longer holds when the effect of K is included, and the analysis of Chapter 8 is used to deal with this.

6.1.1 Flow of $\bar{g}$

The flow of the coupling constant g_j under the renormalisation group map is fundamental. This flow adds a higher-order error term to the perturbative sequence $\bar{g}_j$. The analysis of the flow is the same with or without the error term, so we include the error term from the outset here.

Thus we generalise (6.1.1) by adding an error term, and for the moment consider a general sequence of coefficients a_j for the quadratic term:

$$g_{j+1} = g_j - a_j g_j^2 + e_j, \quad (6.1.3)$$

where we assume

$$0 \leq a \leq a_j \leq A, \quad |e_j| \leq M_j g_j^3, \quad M_j \leq M. \quad (6.1.4)$$

The recursion for $\bar{g}_j$ is the case $M = 0$ and $a_j = \beta_j$, and all of our analysis in this section applies also when $M = 0$. The above recursion appears in many applications and has been studied by many authors, e.g., [38, Section 8.5] for the case $a_j = a$ for all j .

Exercise 6.1.1. Suppose that $0 < a \leq A < \infty$. Prove that if $g_0 > 0$ is sufficiently small (depending on a, A, M) then $0 < \frac{1}{2}g_j < g_{j+1} < g_j$ for all $j \geq 0$. It follows that the limit $\lim_{n \rightarrow \infty} g_j$ exists and is nonnegative. Prove that this limit is zero. [Solution]

Recall that the mass scale j_m is defined in Definition 5.3.3, and that

$$\vartheta_j = 2^{-(j-j_m)_+} \quad (6.1.5)$$

is defined in (5.3.14). In our context, $a_j = \beta_j$ is independent of j when $m^2 = 0$, and when $m^2 > 0$ it begins to decay exponentially after the mass scale. This decay, which is an important feature in our applications, violates the hypothesis $a > 0$ in Exercise 6.1.1 and requires attention. Its principle effect is that the flow of $g_j(m^2)$ resembles that of $g_j(0)$ for scales $j \leq j_m$, whereas the flow effectively stops at the mass scale so that $g_j(m^2)$ resembles $g_{j_m}(0)$ for scales $j > j_m$.

As we show in the next proposition, the solution of the recursion is essentially the sequence t_j defined by

$$A_j = \sum_{i=0}^{j-1} \beta_i, \quad t_j = \frac{g_0}{1 + g_0 A_j}. \quad (6.1.6)$$

In particular, when $m^2 = 0$,

$$A_j(0) = \beta_0^0 j, \quad t_j(0) = \frac{g_0}{1 + g_0 \beta_0^0 j}. \quad (6.1.7)$$

Exercise 6.1.2. For $m^2 > 0$,

$$A_j(m^2) = \beta_0^0(j \wedge j_m) + O(1), \quad (6.1.8)$$

$$t_j(m^2) \asymp t_{j \wedge j_m}(0) = \frac{g_0}{1 + g_0 \beta_0^0(j \wedge j_m)}, \quad (6.1.9)$$

$$\sum_{l=0}^j \vartheta_l t_l \leq O(|\log t_j|). \quad (6.1.10)$$

[Solution]

The following proposition gives the asymptotic behaviour of the solution to the recursion (6.1.3) when $a_j = \beta_j$ and $M_j = M \vartheta_j$. The leading behaviour is not affected by the error term e_j in the recursion, as long as $|e_j| \leq M \vartheta_j g_j^3$. In particular, g_j and $\bar{g}_j$ have the same asymptotic behaviour as $j \rightarrow \infty$.

Proposition 6.1.3. *Let $m^2 \geq 0$ and consider the recursion (6.1.3) with $a_j = \beta_j$ and $M_j = M \vartheta_j$. Let $g_0 > 0$ be sufficiently small.*

(i) As $j \rightarrow \infty$,

$$g_j = t_j + O(t_j^2 |\log t_j|), \quad (6.1.11)$$

with the constant in the error term uniform in $m^2 \geq 0$. Also, $g_j = O(g_0)$ and $g_{j+1} \in [\frac{1}{2}g_j, 2g_j]$.

(ii) For $m^2 = 0$, we have $g_j(0) \sim 1/(\beta_0^0 j) \rightarrow 0$ as $j \rightarrow \infty$. For $m^2 > 0$, the limit $g_\infty(m^2) = \lim_{j \rightarrow \infty} g_j(m^2) > 0$ exists and obeys $g_\infty(m^2) \sim 1/(\beta_0^0 j_m)$ as $m^2 \downarrow 0$.

(iii) Suppose that e_j is continuous in $m^2 \geq 0$. Then $g_\infty(m^2)$ is continuous in $m^2 \geq 0$ and the convergence of g_j to g_∞ is uniform on compact intervals of $m^2 > 0$.

Proof. (i) We assume by induction that $g_j \leq 2t_j$. The induction hypothesis holds for $j = 0$ since $g_0 = t_0$. The recursion gives

$$\frac{1}{g_{j+1}} = \frac{1}{g_j} \frac{1}{1 - a_j g_j + e_j/g_j} = \frac{1}{g_j} + a_j + O(a_j + M_j)g_j. \quad (6.1.12)$$

We solve by iteration to get

$$\frac{1}{g_{j+1}} = \frac{1}{g_0} + A_{j+1} + E_{j+1}, \quad (6.1.13)$$

with $|E_{j+1}| \leq \sum_{i=0}^j O(a_i + M_i)g_i$. By the induction hypothesis and (6.1.10), $|E_{j+1}| \leq \sum_{i=1}^j O(\vartheta_i t_i) \leq O(|\log t_j|)$. This gives

$$g_{j+1} = \frac{g_0}{1 + g_0 A_{j+1} + g_0 E_{j+1}} = t_{j+1}(1 + O(t_{j+1} E_{j+1})), \quad (6.1.14)$$

which in particular allows the induction to be advanced. It also proves the desired formula for g_j .

Finally, (6.1.9) implies that $t_j = O(g_0)$, and by $g_j \leq 2t_j$ this proves that $g_j = O(g_0)$. For the proof of $g_{j+1} \in [\frac{1}{2}g_j, 2g_j]$ see Exercise 6.1.1.

(ii) For $m^2 = 0$, (6.1.13) becomes

$$\frac{1}{g_{j+1}(0)} = \frac{1}{g_0} + \beta_0^0 j + O(\log j), \quad (6.1.15)$$

which proves that $g_j(0) \sim 1/(\beta_0^0 j)$. For $m^2 > 0$, (6.1.13) becomes instead

$$\frac{1}{g_{j+1}(m^2)} = \frac{1}{g_0} + \beta_0^0(j \wedge j_m) + O(1) + O(\log(j \wedge j_m)), \quad (6.1.16)$$

which proves that the limit $g_\infty(m^2)$ exists and is asymptotic to $1/(\beta_0^0 j_m)$ as $m^2 \downarrow 0$.

(iii) By definition, $\beta_j(m^2)$ is continuous in $m^2 > 0$, and $e_j(m^2)$ is continuous by hypothesis. On a compact subinterval of $m^2 \in (0, \infty)$, both β_j and e_j are uniformly bounded by exponentially decaying sequences. Consequently the sums A_{j+1} and E_{j+1} which appear in (6.1.13) converge uniformly to limits, and these limits are continuous by dominated convergence. This proves the uniform continuity on compact mass subintervals. The continuity at $m^2 = 0$ follows from the fact that the $j \rightarrow \infty$ limit of the right-hand side of (6.1.16) tends to infinity as $m^2 \downarrow 0$, and hence $\lim_{m^2 \downarrow 0} g_\infty(m^2) = 0 = g_\infty(0)$. This completes the proof. ■

The next exercise provides an extension of (6.1.10).

Exercise 6.1.4. Each of the sequences $g_j, \bar{g}_j, t_j$ obeys $g_{j+1} = g_j(1 + O(g_0))$, as well as the inequalities $\vartheta_j(m^2)g_j(m^2) \leq O(g_j(0))$ and

$$\sum_{l=j}^{\infty} \vartheta_l g_l^p \leq O(\vartheta_j g_j^{p-1}) \quad (p > 1), \quad (6.1.17)$$

$$\sum_{l=0}^j \vartheta_l g_l \leq O(|\log g_j|). \quad (6.1.18)$$

(The combination $\vartheta_j \bar{g}_j$ typically appears in our upper bounds.) [\[Solution\]](#)

Given $\tilde{m}^2 \geq 0$, we define the *mass domain*

$$\mathbb{I}_j(\tilde{m}^2) = \begin{cases} [0, L^{-2j}] & (\tilde{m}^2 = 0) \\ [\frac{1}{2}\tilde{m}^2, 2\tilde{m}^2] & (\tilde{m}^2 > 0). \end{cases} \quad (6.1.19)$$

The next exercise implies that any of the sequences $g_j, \bar{g}_j, t_j$ are comparable in value when evaluated at m^2 or $\tilde{m}^2$ if $m^2 \in \mathbb{I}_j(\tilde{m}^2)$.

Exercise 6.1.5. For $\tilde{m}^2 \geq 0$ and $m^2 \in \mathbb{I}_j(\tilde{m}^2)$, each of the sequences $g_j, \bar{g}_j, t_j$ obeys

$$g_j(m^2) = g_j(\tilde{m}^2) + O(g_j(\tilde{m}^2)^2). \quad (6.1.20)$$

[\[Solution\]](#)

6.1.2 Perturbative stable manifold

In this section, we obtain a simple 2-dimensional version of an infinite-dimensional counterpart in the next section. It is useful for illustrative purposes, though we do not use the 2-dimensional version later. We do however use the following lemma both for the two-dimensional and infinite-dimensional results.

Lemma 6.1.6. *Assume that the sequence g satisfies the recursion (6.1.3) with $a_j = \beta_j$ and $M_j = M\vartheta_j$, and with e_j continuous in $m^2 \geq 0$. For any fixed $\gamma \in \mathbb{R}$, let*

$$\Pi_{i,j} = \prod_{k=i}^j (1 - \gamma \beta_k g_k). \quad (6.1.21)$$

There exists $c_i = 1 + O(\vartheta_i \bar{g}_i)$, which is a continuous function of $m^2 \geq 0$, such that

$$\Pi_{i,j} = \left(\frac{g_{j+1}}{g_i} \right)^\gamma (c_i + O(\vartheta_j \bar{g}_j)). \quad (6.1.22)$$

Proof. By Proposition 6.1.3 the sequences g_j and $\bar{g}_j$ are comparable; we use $\bar{g}_j$ for error terms. Since $(1-x)^\gamma = (1-\gamma x)(1+O(x^2))$ as $x \rightarrow 0$, there exist $s_k = O(\vartheta_k^2 \bar{g}_k^2)$ such that

$$\Pi_{i,j} = \prod_{k=i}^j (1 - \beta_k g_k)^\gamma (1 + s_k). \quad (6.1.23)$$

By (6.1.3), and since $g_{j+1} \in [\frac{1}{2}g_j, 2g_j]$ by Proposition 6.1.3,

$$1 - \beta_k g_k = \frac{g_{k+1} - e_k}{g_k} = \frac{g_{k+1}}{g_k} (1 + O(\vartheta_k \bar{g}_k^2)). \quad (6.1.24)$$

Therefore, there exist $v_k = O(\vartheta_k \bar{g}_k^2)$ such that

$$\Pi_{i,j} = \prod_{k=i}^j \left(\frac{g_{k+1}}{g_k} \right)^\gamma (1 + v_k) = \left(\frac{g_{j+1}}{g_i} \right)^\gamma \prod_{k=i}^j (1 + v_k). \quad (6.1.25)$$

Since $\log(1+x) = O(x)$, the product obeys

$$\prod_{k=i}^j (1 + v_k) = \exp \left(\sum_{k=i}^j O(v_k) \right) = \exp \left(O(1) \sum_{k=i}^j \vartheta_k \bar{g}_k^2 \right). \quad (6.1.26)$$

By (6.1.17), the infinite product converges and we can define

$$c_i = \prod_{k=i}^{\infty} (1 + v_k) = 1 + O(\vartheta_i \bar{g}_i). \quad (6.1.27)$$

We then obtain the desired formula for $\Pi_{i,j}$ from

$$\prod_{k=i}^j (1 + v_k) = c_i \exp \left(- \sum_{k=j+1}^{\infty} \log(1 + v_k) \right) = c_i + O(\vartheta_j \bar{g}_{j+1}). \quad (6.1.28)$$

Finally, the continuity of c_i in m^2 follows from the uniform upper bound $v_k \leq O((\bar{g}_k(0))^2)$ by Exercise 6.1.4, the continuity of the β_k and $r_{g,k}$ and therefore of the g_k and t_k , and the dominated convergence theorem. This completes the proof. ■

The next proposition constructs an initial condition $\bar{\mu}_0^c$ for which the perturbative flow $(\bar{g}_j, \bar{\mu}_j)$ satisfies $\bar{\mu}_j \rightarrow 0$. For $m^2 = 0$, the set $(g, \bar{\mu}_0^c(g))$ plays the role of a stable manifold for the fixed point $(0, 0)$ of the dynamical system $(\bar{g}, \bar{\mu}) \mapsto (\bar{g}_+, \bar{\mu}_+)$. A schematic depiction of the stable manifold is given in Figure 6.1.

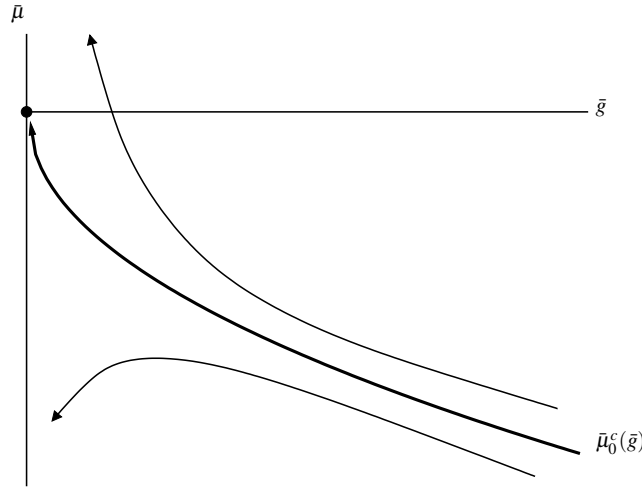


Fig. 6.1 Schematic depiction of the stable manifold for the perturbative flow $(\bar{g}_j, \bar{\mu}_j)$.

Proposition 6.1.7. *Given $m^2 \geq 0$, and given $g_0 > 0$ sufficiently small, there exists a unique $\bar{\mu}_0^c = \bar{\mu}_0^c(g_0, m^2)$ such that if $\bar{V}_0 = (g_0, \bar{\mu}_0^c)$ then the solution to the recursion (6.1.1)–(6.1.2) satisfies*

$$\bar{g}_j \rightarrow \bar{g}_\infty \geq 0, \quad \bar{\mu}_j \rightarrow 0. \quad (6.1.29)$$

More precisely, it obeys $\bar{\mu}_j = O(\vartheta_j \bar{g}_j)$, and $\bar{g}_\infty \sim c(\log m^{-2})^{-1}$ as $m^2 \downarrow 0$ (for some $c > 0$).

In particular, if $m^2 = 0$, then $\bar{g}_\infty = 0$ and $\bar{V}_j \rightarrow 0$. This is the famous observation of infrared *asymptotic freedom*, and inspires the prediction that scaling limits of the model near the critical point are described by the free field.

Proof of Proposition 6.1.7. Given an initial condition $\bar{g}_0$, a sequence $\bar{g}_j$ is determined by (6.1.1), and this sequence obeys the conclusions of Proposition 6.1.3. For the sequence $\bar{\mu}_j$, we rewrite the recursion (6.1.2) backwards as

$$\bar{\mu}_j = (1 - \gamma\beta_j\bar{g}_j)^{-1} (L^{-2}\bar{\mu}_{j+1} - \eta_j\bar{g}_j + \xi_j\bar{g}_j^2). \quad (6.1.30)$$

By iteration, with $\Pi_{i,j}$ given by Lemma 6.1.6, it follows that

$$\bar{\mu}_j = L^{-2(k+1-j)} \Pi_{j,k}^{-1} \bar{\mu}_{k+1} + \sum_{l=j}^k L^{-2(l-j)} \Pi_{j,l}^{-1} (-\eta_l\bar{g}_l + \xi_l\bar{g}_l^2). \quad (6.1.31)$$

Motivated by this, we define

$$\bar{\mu}_j = \sum_{l=j}^{\infty} L^{-2(l-j)} \Pi_{j,l}^{-1} (-\eta_l\bar{g}_l + \xi_l\bar{g}_l^2). \quad (6.1.32)$$

Since $\bar{g}_l = O(1)$ and, by Lemma 6.1.6, $\Pi_{j,l}^{-1}$ is slowly varying compared with $L^{-2(l-j)}$, the above sum converges, and $\bar{\mu}_j = O(\vartheta_j\bar{g}_j)$. It is easy to check that $\bar{\mu}_j$ given by (6.1.32) obeys the recursion (6.1.2), and that $\bar{\mu}_0$ is the unique initial value that leads to a zero limit for the sequence. ■

Given any initial condition $(\bar{g}_0, \bar{\mu}_0)$, the equations (6.1.1)–(6.1.2) can be solved by forward iteration. This defines sequences $\bar{g}_j, \bar{\mu}_j$ for arbitrary initial conditions, and the sequence $\bar{\mu}_j$ can be differentiated with respect to $\bar{\mu}_0$. This derivative is considered in the next proposition; its value is independent of the initial condition $\bar{\mu}_0$.

Proposition 6.1.8. *Given any small $\bar{g}_0 > 0$,*

$$\frac{\partial \bar{\mu}_j}{\partial \bar{\mu}_0} = L^{2j} \left(\frac{\bar{g}_j}{\bar{g}_0} \right)^{\gamma} (c + O(\vartheta_j\bar{g}_j)) \quad \text{with} \quad c = 1 + O(\bar{g}_0). \quad (6.1.33)$$

Consequently, there exists $c' > 0$ such that

$$\lim_{j \rightarrow \infty} L^{-2j} \frac{\partial \bar{\mu}_j}{\partial \bar{\mu}_0} \sim c' (\log m^{-2})^{\gamma} \quad \text{as } m^2 \downarrow 0. \quad (6.1.34)$$

Proof. By the chain rule, $\frac{\partial \bar{\mu}_j}{\partial \bar{\mu}_0} = \prod_{k=0}^{j-1} \frac{\partial \bar{\mu}_{k+1}}{\partial \bar{\mu}_k}$. We compute the factors in the product by differentiating the recursion relation (6.1.2) for $\bar{\mu}$. Since $\bar{g}_j$ is independent of $\bar{\mu}$, we obtain

$$\frac{\partial \bar{\mu}_j}{\partial \bar{\mu}_0} = \prod_{k=0}^{j-1} L^2 (1 - \gamma\beta_k\bar{g}_k). \quad (6.1.35)$$

Now we apply Lemma 6.1.6 for (6.1.33), and Proposition 6.1.3(ii) for (6.1.34). ■

6.2 Reduction of proof of Theorem 4.2.1

In this section, we prove Theorem 4.2.1 subject to Theorem 6.2.1 and Proposition 6.2.2. Theorem 6.2.1 is a non-perturbative versions of Propositions 6.1.7–6.1.8, with no uncontrolled remainder. Proposition 6.2.2 is a relatively minor result which incorporates the effect of the last renormalisation group step, corresponding to the Gaussian integration with covariance $C_{\hat{N}}$. Their proofs occupy the rest of the book.

Throughout this section, we fix

$$g_0 = g \quad (6.2.1)$$

and drop g from the notation when its role is insignificant. Our starting point is (5.1.2), which asserts that for $m^2 > 0$ and for

$$v_0 = v - m^2, \quad (6.2.2)$$

the susceptibility is given by

$$\chi_N(v) = \frac{1}{m^2} + \frac{1}{m^4|\Lambda|} \frac{D^2 Z_{\hat{N}}(0; \mathbb{1}, \mathbb{1})}{Z_{\hat{N}}(0)}, \quad (6.2.3)$$

with $Z_{\hat{N}} = \mathbb{E}_C \theta Z_0$ and $Z_0 = e^{-\sum_{x \in \Lambda} (g_0 \tau_x^2 + v_0 \tau_x)}$. As discussed below (5.1.2), we can regard the right-hand side as a function of two independent variables (m^2, v_0) , without enforcing (6.2.2), even though the equality in (6.2.3) is guaranteed only when (6.2.2) does hold. We define a function $\hat{\chi}_N(m^2, v_0)$ by the right-hand side of (6.2.3) with *independent* variables (m^2, v_0) . Thus $\hat{\chi}_N$ is a function of two variables (with dependence on g left implicit), and

$$\chi_N(v_0 + m^2) = \hat{\chi}_N(m^2, v_0). \quad (6.2.4)$$

To prove Theorem 4.2.1, the general strategy is to prove that for $m^2 \geq 0$ there is a *critical* initial value $v_0 = v_0(m^2)$ (depending also on g_0 but independent of the volume parameter N) such that starting from the initial condition $V_0 = g_0 \tau^2 + v_0 \tau$ and $K_0 = 0$ it is possible to iterate the renormalisation group map indefinitely. This iteration produces a sequence $(U_j, K_j) = (u_j, V_j, K_j)$ which represents Z_j via (5.2.6) as long as $j \leq N$. The sequence $(U_j, K_j) = (u_j, V_j, K_j)$ is *independent* of N for $j \leq N$, and thus in the limit $N \rightarrow \infty$ is a global renormalisation group trajectory. For finite N , (U_N, K_N) represents Z_N . Finally, there is the step of (5.1.3) which is the first and only step where a finite volume system deviates from the global trajectory. This step maps Z_N to $Z_{\hat{N}}$ with $Z_{\hat{N}}$ represented by $(U_{\hat{N}}, K_{\hat{N}}) = (u_{\hat{N}}, V_{\hat{N}}, K_{\hat{N}})$ from which $\hat{\chi}_N(m^2, v_0)$ is computed with (6.2.3). The critical initial value is intimately related to the critical point v_c . The global trajectory has the property that V_j and K_j both go to zero as $j \rightarrow \infty$, which is *infrared asymptotic freedom*. This property characterises $v_0(m^2)$ uniquely.

Given $m^2 \geq 0$, we can regard v_0 as a function $v_0(g_0)$ of the initial value $g_0 = g$. The construction of $v_0(g_0)$ corresponds schematically to the construction of the

stable manifold depicted in Figure 6.1 for the perturbative flow. However, the dynamical system here is more elaborate than the perturbative 2-dimensional dynamical system. Now it is instead infinite-dimensional due to the presence of the non-perturbative coordinate K_j , and it is also non-autonomous because K_j lies in different spaces $\mathcal{F}_j$ (which will be equipped with different norms) for different values of j . The dynamical system is nonhyperbolic, with expanding coordinate μ_j , contracting coordinate K_j , and with coordinate g_j which is neither contracting nor expanding. Its local phase diagram is shown schematically in Figure 6.2. The fixed point is $(g, \mu, K) = (0, 0, 0)$. Given small g_0 and K_0 , the flow of the dynamical system is towards $(0, 0, 0)$ when $v_0 = v_0(g_0, K_0)$ is chosen correctly (we focus on the case $K_0 = 0$ which is the only case we need). This choice defines the stable manifold, which has co-dimension 1 corresponding to the variable μ_j . If v_0 were chosen off the stable manifold, the flow of μ_j would explode exponentially taking the trajectory outside the domain of our RG map.

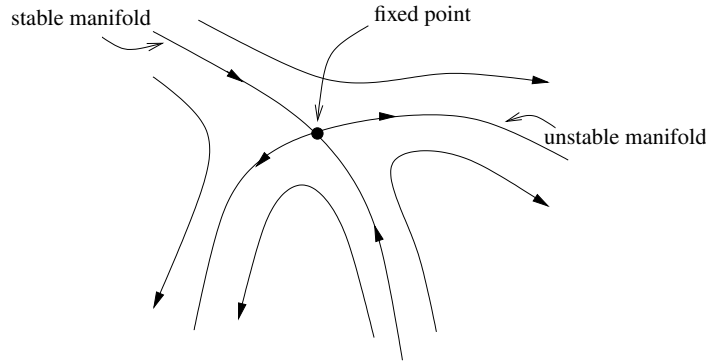


Fig. 6.2 Phase diagram for the dynamical system.

The sequence U_j is determined recursively from

$$U_{j+1}(U_j, K_j) = u_j + U_{j+1}(V_j, K_j) = u_j + \Phi_{j+1}^U(V_j, K_j), \quad (6.2.5)$$

with $U_{j+1}(V_j, K_j) = \Phi_{\text{pt}}(V_j - \text{Loc}(e^{V_j} K_j))$ as in (5.2.31). We have already analysed the map Φ_{pt} explicitly and in detail. It is defined by (5.2.18), and its explicit quadratic form is given in Proposition 5.3.1. Thus, to understand the sequence U_j , it suffices to analyse the sequence

$$R_{j+1}^U(V, K) = (r_{g,j}, r_{v,j}, r_{u,j}) \quad (6.2.6)$$

defined by

$$R_{j+1}^U(V_j, K_j) = \Phi_{j+1}^U(V_j, K_j) - \Phi_{\text{pt}}(V_j). \quad (6.2.7)$$

The following is a non-perturbative version of Propositions 6.1.7–6.1.8. Its proof is given in Section 8.4.

Theorem 6.2.1. *Fix L sufficiently large and $g_0 > 0$ sufficiently small.*

(i) *There exists a continuous function $v_0^c(m^2)$ of $m^2 \geq 0$ (depending on g_0) such that if $v_0 = v_0^c(m^2)$ then, for all $j \in \mathbb{N}$,*

$$r_{g,j} = O(\vartheta_j^3 g_j^3), \quad L^{2j} r_{v,j} = O(\vartheta_j^3 g_j^3), \quad L^{dj} r_{u,j} = O(\vartheta_j^3 g_j^3), \quad (6.2.8)$$

and

$$L^{2j} |v_j| = O(\vartheta_j g_j), \quad |K_j(0)| + L^{-2j} |D^2 K_j(0; \mathbb{1}, \mathbb{1})| = O(\vartheta_j^3 g_j^3). \quad (6.2.9)$$

(ii) *There exists $c = 1 + O(g_0)$ such that for $m^2 \geq 0$ and $j \in \mathbb{N}$, and with all derivatives evaluated at $(m^2, v_0^c(m^2))$,*

$$\frac{\partial \mu_j}{\partial v_0} = L^{2j} \left(\frac{g_j}{g_0} \right)^c (c + O(\vartheta_j g_j)), \quad \frac{\partial g_j}{\partial v_0} = O \left(g_j^2 \frac{\partial \mu_j}{\partial v_0} \right), \quad (6.2.10)$$

$$L^{-2j} \left| \frac{\partial}{\partial v_0} K_j(0) \right| + L^{-4j} \left| \frac{\partial}{\partial v_0} D^2 K_j(0; \mathbb{1}, \mathbb{1}) \right| = O \left(\vartheta_j^3 g_j^2 \left(\frac{g_j}{g_0} \right)^c \right). \quad (6.2.11)$$

From the first bound in (6.2.8) and Proposition 6.1.3, it follows that

$$\lim_{j \rightarrow \infty} g_j(m^2) = g_\infty(m^2) \quad (m^2 \geq 0), \quad (6.2.12)$$

where $g_\infty(0) = 0$, $g_\infty(m^2) > 0$ if $m^2 > 0$, and the limit is uniform on compact intervals of $m^2 > 0$. Also by Proposition 6.1.3,

$$g_\infty(m^2) \sim \frac{1}{\beta_0^0 \log_L m^{-2}} \quad (m^2 \downarrow 0). \quad (6.2.13)$$

The first N renormalisation group steps correspond to integration over the covariances $C_1 + \dots + C_N$. In finite volume, we are left with the final covariance $C_{\hat{N}}$. Unlike the other covariances, it has $\sum_y C_{\hat{N};xy} \neq 0$. The next proposition shows that its contribution is negligible. The proof, which requires only slight modifications to the analysis of a typical renormalisation group step, is given in Section 10.7.

Proposition 6.2.2. *Fix L sufficiently large and $g_0 > 0$ sufficiently small, and suppose that $m^2 L^{2N} \geq 1$ (i.e., that the last scale N is beyond the mass scale j_m). The expectations in Proposition 5.2.12 do exist at scale N for $(V, K) = (V_N, K_N)$ and (5.2.37) holds, i.e.,*

$$\mathbb{E}_{C_{\hat{N}}} \left(e^{-\theta V_N(\Lambda)} + K_N(\Lambda) \right) = e^{-u_{\hat{N}} |\Lambda|} \left(e^{-V_{\hat{N}}(\Lambda)} (1 + W_{\hat{N}}(\Lambda)) + K_{\hat{N}}(\Lambda) \right), \quad (6.2.14)$$

with $W_{\hat{N}} = -\frac{1}{2} c_{\hat{N}}^{(1)} g_{\hat{N}}^2 |\varphi|^6$. The estimates (6.2.9)–(6.2.11) hold with j replaced by $j = \hat{N}$ on the left-hand sides and $j = N$ on the right-hand sides, and $g_{\hat{N}} = g_N(1 + O(\vartheta_N g_N))$.

The following corollary shows that the susceptibility $\chi(v_0^c(m^2) + m^2)$ is simply the susceptibility of the hierarchical GFF with mass m (recall Exercise 4.1.10). Thus $v_0^c(m^2)$ has the property that

$$\chi(g, v_0^c(m^2) + m^2) = \chi(0, m^2). \quad (6.2.15)$$

In other words, m^2 represents the deviation from the critical value $v_0^c(m^2, g)$ such that the susceptibility of the interacting model is equal to the susceptibility of the noninteracting model with mass m^2 . Physicists refer to m^2 as an *effective* or *renormalised mass* and $v_0^c(m^2) + m^2$ as a *bare mass*.

Corollary 6.2.3. *Fix L sufficiently large and $g > 0$ sufficiently small, and let $m^2 > 0$ and $v_0 = v_0^c(m^2)$. The limit $\chi(v_0^c(m^2) + m^2) = \lim_{N \rightarrow \infty} \chi_N(v_0^c(m^2) + m^2)$ exists, uniformly on compact intervals of $m^2 > 0$, and*

$$\chi(v_0^c(m^2) + m^2) = \frac{1}{m^2}. \quad (6.2.16)$$

Proof. We will in fact prove the finite volume estimate

$$\chi_N(v_0^c(m^2) + m^2) = \frac{1}{m^2} \left(1 + \frac{O(\vartheta_N g_N)}{m^2 L^{2N}} \right), \quad (6.2.17)$$

which implies (6.2.16). By (6.2.3), to prove (6.2.17) it is sufficient to show that, for $v_0 = v_0^c(m^2)$,

$$\frac{1}{|\Lambda|} \frac{D^2 Z_{\hat{N}}(0; \mathbb{1}, \mathbb{1})}{Z_{\hat{N}}(0)} = O(L^{-2N} \vartheta_N g_N). \quad (6.2.18)$$

At scale N , Λ is a single block, so $Z_{\hat{N}} = e^{-u_{\hat{N}}|\Lambda|}(I_{\hat{N}} + K_{\hat{N}})$, where by definition $I_{\hat{N}} = e^{-V_{\hat{N}}(\Lambda)}(1 + W_{\hat{N}}(\Lambda))$. By Proposition 6.2.2, $W_{\hat{N}}$ is proportional to $g_N^2 |\varphi|^6$. Since $V_{\hat{N}}(\Lambda) = W_{\hat{N}}(\Lambda) = 0$ when $\varphi = 0$, we have $I_{\hat{N}} = 1$ when $\varphi = 0$. Also,

$$\frac{1}{|\Lambda|} D^2 I_{\hat{N}}(\varphi = 0; \mathbb{1}, \mathbb{1}) = -v_{\hat{N}}, \quad (6.2.19)$$

and hence

$$\frac{1}{|\Lambda|} \frac{D^2 Z_{\hat{N}}(0; \mathbb{1}, \mathbb{1})}{Z_{\hat{N}}(0)} = \frac{-v_{\hat{N}} + L^{-dN} D^2 K_{\hat{N}}(0; \mathbb{1}, \mathbb{1})}{1 + K_{\hat{N}}(0)} = O(L^{-2N} \vartheta_N g_N), \quad (6.2.20)$$

where the final estimate holds by Proposition 6.2.2 (with the final scale version of (6.2.9)). This proves (6.2.18). The convergence is uniform in compact intervals of $m^2 > 0$, due to the factor L^{-2N} . This completes the proof. ■

Corollary 6.2.4. *Fix L sufficiently large and $g > 0$ sufficiently small. There exists $B = B_{g,n} > 0$ such that*

$$\left. \frac{\partial \chi}{\partial v} \right|_{v=v_0^c(m^2)+m^2} \sim -B \frac{1}{m^4 (\log m^{-2})^\gamma} \quad \text{as } m^2 \downarrow 0. \quad (6.2.21)$$

Proof. By (6.2.4), the finite-volume version $\frac{\partial}{\partial v} \chi_N$ of the left-hand side of (6.2.21) is equal to $\frac{\partial}{\partial v_0} \hat{\chi}_N(m^2, v_0)$ evaluated at $v_0 = v_0^c(m^2)$. All v_0 derivatives in the proof are evaluated at this value, and we denote them by primes. We compute $\frac{\partial}{\partial v_0} \hat{\chi}_N(m^2, v_0)$ by differentiation of the right-hand side of (6.2.3), using $Z_{\hat{N}} = e^{-u_{\hat{N}}|\Lambda|} (I_{\hat{N}} + K_{\hat{N}})$ with $I_{\hat{N}}$ as in the proof of Corollary 6.2.3. This gives

$$\begin{aligned} \frac{\partial \chi_N}{\partial v} &= \frac{1}{m^4 L^{dN}} \frac{\partial}{\partial v_0} \frac{D^2 Z_{\hat{N}}(0; \mathbb{1}, \mathbb{1})}{Z_{\hat{N}}(0)} \\ &= \frac{1}{m^4 L^{dN}} \left(\frac{D^2 Z'_{\hat{N}}(0; \mathbb{1}, \mathbb{1})}{Z_{\hat{N}}(0)} - \frac{Z'_{\hat{N}}(0) D^2 Z_{\hat{N}}(0; \mathbb{1}, \mathbb{1})}{Z_{\hat{N}}(0)^2} \right). \end{aligned} \quad (6.2.22)$$

The factor $e^{-u_{\hat{N}}|\Lambda|}$ cancels in numerator and denominator of (6.2.3), and in particular need not be differentiated. We view $V_{\hat{N}}$ and $W_{\hat{N}}$ as functions of v_0 . As in (6.2.19),

$$I_{\hat{N}}(\varphi = 0) = 1, \quad \frac{1}{|\Lambda|} D^2 I_{\hat{N}}(\varphi = 0; \mathbb{1}, \mathbb{1}) = -v_{\hat{N}}, \quad (6.2.23)$$

and hence

$$I'_{\hat{N}}(\varphi = 0) = 0, \quad \frac{1}{|\Lambda|} D^2 I'_{\hat{N}}(\varphi = 0; \mathbb{1}, \mathbb{1}) = -v'_{\hat{N}}. \quad (6.2.24)$$

With some arguments omitted to simplify the notation, this leads to

$$\frac{\partial \chi_N}{\partial v} = \frac{1}{m^4} \left(\frac{-v'_{\hat{N}} + L^{-dN} D^2 K'_{\hat{N}}}{1 + K_{\hat{N}}} - \frac{K'_{\hat{N}}(-v_{\hat{N}} + L^{-dN} D^2 K_{\hat{N}})}{(1 + K_{\hat{N}})^2} \right). \quad (6.2.25)$$

By Proposition 6.2.2, as $N \rightarrow \infty$ the derivative $\frac{\partial \chi_N}{\partial v}$ has the same limit as $-m^{-4} v'_{\hat{N}}$, and the omitted terms go to zero uniformly on compact intervals in $m^2 > 0$ because $v_{\hat{N}}$ does. Therefore, by Proposition 6.2.2 with (6.2.10), and by (6.2.12),

$$\lim_{N \rightarrow \infty} \frac{\partial \chi_N}{\partial v} = -\frac{c}{m^4} \left(\frac{g_\infty}{g_0} \right)^\gamma. \quad (6.2.26)$$

The limit is again uniform on compact mass intervals, since the same is true of the limit in (6.2.12).

Since

$$\frac{\partial}{\partial v} \chi_N(v) = \frac{\partial}{\partial v_0} \hat{\chi}_N(m^2, v_0^c(m^2)), \quad (6.2.27)$$

and since v_0^c is a continuous function of m^2 , the limit $\lim_{N \rightarrow \infty} \frac{\partial}{\partial v} \chi_N(v)$ converges uniformly in compact intervals of v in the image of $m^2 + v_0^c(m^2, g_0)$ for $m^2 > 0$. Therefore the differentiation and the limit may be interchanged, so that

$$\frac{\partial}{\partial v} \chi(v) = \lim_{N \rightarrow \infty} \frac{\partial}{\partial v} \chi_N(v) = -\frac{c}{m^4} \left(\frac{g_\infty}{g_0} \right)^\gamma. \quad (6.2.28)$$

By (6.2.13), there is a positive constant $B_{g,n}$ such that

$$c \left(\frac{g_\infty}{g_0} \right)^\gamma \sim \frac{B_{g,n}}{(\log m^{-2})^\gamma} \quad (m^2 \downarrow 0). \quad (6.2.29)$$

This proves (6.2.21), and the proof is complete. $\blacksquare$

Finally, we need the next lemma which establishes that $v_0^c(m^2) + m^2$ is an increasing function of small m^2 .

Lemma 6.2.5. *Fix L sufficiently large and $g > 0$ sufficiently small. For $\delta > 0$ sufficiently small and for $m^2 \in [0, \delta]$, $v_0^c(m^2) + m^2$ is a continuous increasing function of m^2 .*

Proof. Set $v^*(m^2) = v_0^c(m^2) + m^2$. The continuity of v^* in $m^2 \in [0, \delta]$ is immediate from Theorem 6.2.1. For $m^2 > 0$, (6.2.16) and (6.2.21) imply

$$\chi(v^*(m^2)) = \frac{1}{m^2} < \infty, \quad (6.2.30)$$

$$\frac{\partial}{\partial v} \chi(v^*(m^2)) < 0. \quad (6.2.31)$$

We used the hypothesis $m^2 \in [0, \delta]$ with δ small to obtain (6.2.31). Let $I = \{v^*(m^2) : m^2 \in [0, \delta]\}$. By continuity of v^* in m^2 , and since continuous functions map an interval to an interval, I is an interval (which cannot be a single point due to (6.2.30)). Since $\chi(v)$ is decreasing in $v \in I$ for small m^2 by (6.2.31), and since the composition $\chi(v^*(m^2)) = \frac{1}{m^2}$ is decreasing in $m^2 > 0$, it follows that $v^*(m^2)$ is increasing in small m^2 . $\blacksquare$

Now we can complete the proof of Theorem 4.2.1, subject to Theorem 6.2.1 and Proposition 6.2.2, using Corollaries 6.2.3 and 6.2.4 and Lemma 6.2.5.

Proof of Theorem 4.2.1. Define

$$v_c = v_0^c(0). \quad (6.2.32)$$

By Lemma 6.2.5, the function $m^2 \mapsto v_0^c(m^2) + m^2$ is continuous and increasing as a function of $m^2 \in [0, \delta]$. It therefore has a continuous inverse. Its range is a closed interval of the form $[v_c, v_c + \varepsilon']$ for some $\varepsilon' > 0$. The inverse map associates to each $v = v_c + \varepsilon$, for $\varepsilon \in [0, \varepsilon']$, a unique m^2 . Using this relationship, we see from (6.2.21) and (6.2.16) that, as $m^2 \downarrow 0$ or equivalently $v \downarrow v_c$,

$$\frac{\partial}{\partial v} \chi(g, v) \sim -B \frac{1}{m^4 (\log m^{-2})^\gamma} \sim -B \chi(g, v)^2 (\log \chi(g, v))^{-\gamma}. \quad (6.2.33)$$

It is now an exercise in calculus to deduce that, as $\varepsilon \downarrow 0$,

$$\chi(g, v_c + \varepsilon) \sim \frac{1}{B} \frac{1}{\varepsilon} (\log \varepsilon^{-1})^\gamma. \quad (6.2.34)$$

This proves (4.2.6) with $A = B^{-1}$.

The constant B arises in (6.2.29), and by Theorem 6.2.1,

$$B = (1 + O(g))(\log L)^\gamma (g\beta_0^0)^{-\gamma} \quad (6.2.35)$$

with $\beta_0^0 = (n+8)(1-L^{-d})$ given by Lemma 5.3.4. This proves that $A = B^{-1} \sim (g\beta_0^0/(\log L)^\gamma)$, as claimed in (4.2.7).

It remains to prove the asymptotic formula for the critical point in (4.2.7). For the rest of the proof, we set $m^2 = 0$. To begin, we note that it follows from Proposition 5.3.1, (6.2.7) and (6.2.8) that g_j and μ_j are determined recursively from

$$g_{j+1} = g_j - \beta_j g_j^2 + O(\vartheta_j^3 g_j^3), \quad (6.2.36)$$

$$\mu_{j+1} = L^2 (\mu_j (1 - \gamma \beta_j g_j) + \eta_j g_j - \xi_j g_j^2) + O(\vartheta_j^3 g_j^3), \quad (6.2.37)$$

with initial condition $(g_0, \mu_0) = (g, v_c)$. Just as (6.1.32) defines a solution to the perturbative flow with zero final condition, backwards solution of (6.2.37) gives

$$\mu_0 = - \sum_{l=0}^{\infty} L^{-2l} \Pi_{0,l}^{-1} (\eta_l g_l + O(\vartheta_l \bar{g}_l^2)). \quad (6.2.38)$$

By (5.3.7), (5.3.10), and (4.1.19),

$$\eta_l = (n+2)L^{2l} C_{l+1;0,0}(0). \quad (6.2.39)$$

By Lemma 6.1.6, we obtain from (6.2.38)–(6.2.39) that

$$\mu_0 = -(n+2)(1 + O(g_0))g_0^\gamma \sum_{l=0}^{\infty} \left(C_{l+1;0,0} g_l^{1-\gamma} + O(L^{-2l} \bar{g}_l^{2-\gamma}) \right). \quad (6.2.40)$$

Since $C(0) = \sum_{l=0}^{\infty} C_{l+1;0,0}$, this gives

$$\begin{aligned} \mu_0 &= -(n+2)C(0)g_0(1 + O(g_0)) - (n+2)(1 + O(g_0))g_0^\gamma \sum_{l=0}^{\infty} C_{l+1;0,0}(g_l^{1-\gamma} - g_0^{1-\gamma}) \\ &\quad + g_0^\gamma \sum_{l=0}^{\infty} O(L^{-2l} \bar{g}_l^{2-\gamma}). \end{aligned} \quad (6.2.41)$$

We show that the last two terms are $O(g_0^2)$. This suffices, as it gives the desired result

$$\mu_0 = -(n+2)C(0)g_0 + O(g_0^2). \quad (6.2.42)$$

The last term in (6.2.41) is $O(g_0^2)$, since $g_l = O(g_0)$ by Proposition 6.1.3(i). For the more substantial sum in (6.2.41), by Taylor's theorem, and again using the fact $g_l = O(g_0)$, for any $\gamma < 1$ we have

$$g_l^{1-\gamma} - g_0^{1-\gamma} = (g_l - g_0)O(g_0^{-\gamma}). \quad (6.2.43)$$

By the recursion (6.2.36) for g_l and $C_{l+1;0,0} = O(L^{-2l})$, this gives

$$g_0^\gamma \sum_{l=0}^{\infty} C_{l+1;0,0} (g_l^{1-\gamma} - g_0^{1-\gamma}) = O(1) \sum_{l=0}^{\infty} L^{-2l} \sum_{k=0}^l \vartheta_k g_k^2 = O(g_0^2). \quad (6.2.44)$$

This completes the proof. ■

Exercise 6.2.6. Use (6.2.33) to prove (6.2.34). [\[Solution\]](#)

Part III

The renormalisation group: Nonperturbative analysis

Chapter 7

The T_z -seminorm

“...I went to the hotel, quite tired, and I went to sleep. I dreamed I was in a very long corridor, with no ceiling, and nothing in front of me, only two very long walls extremely high. Then I woke up and understood immediately that I was trapped inside a norm!!”¹

In order to analyse the renormalisation group map defined in Definition 5.2.8, we use certain seminorms. The seminorms are designed to measure the size of the non-perturbative coordinate $K_+(B)$ defined in (5.2.32), which is a function of the field φ and also of (V, K) . Since $K_+(B)$ is a function of fields that are constant on blocks B as in Definition 5.1.4, it is natural (and sufficient) to define the seminorm on functions of the constant value $\varphi \in \mathbb{R}^n$. We encode estimates of $K_+(B)$, and of its derivatives with respect to the three variables φ, V, K , in a single seminorm.

In this chapter, we define the seminorm that will be used for this purpose, the T_z -seminorm, and the T_φ -seminorm which is a special case. The T_z -seminorm is defined on functions of a variable z that lies in a product $\mathcal{Z} = \mathcal{Z}_1 \times \mathcal{Z}_2 \times \mathcal{Z}_3$ of three normed spaces $\mathcal{Z}_s$ ($s = 1, 2, 3$). The space $\mathcal{Z}_1 = \mathbb{R}^n$ is a space of values of field configurations φ . The space $\mathcal{Z}_2 = \mathcal{V}$ is the space of interactions V as in Definition 5.1.1 and $\mathcal{Z}_3 = \mathcal{N}(b)$ is the space of K as in Definition 5.1.4. These choices motivate this chapter, but the results are valid for arbitrary normed spaces $\mathcal{Z}_2, \mathcal{Z}_3$.

7.1 Definition of the T_z -seminorm

Let X be a normed vector space, and let X^p denote the Cartesian product of p copies of X . Given $\dot{x}_i \in X$, we write $\dot{x}^p = \dot{x}_1, \dots, \dot{x}_p$. A function $M : X^p \rightarrow \mathbb{R}$ is said to be p -linear if $M(\dot{x}^p)$ is linear in each of its p arguments $\dot{x}_1, \dots, \dot{x}_p$. The norm of M is defined by

¹ From an email from Benedetto Scoppola in 2005

$$\|M\|_X = \sup_{\dot{x}^p \in X(1)^p} |M(\dot{x}^p)|, \quad (7.1.1)$$

where $X(1)$ is the unit ball in X centred on the origin.

Given a function $F : X \rightarrow \mathbb{R}$, the Fréchet derivative $F^{(p)}(x)$ of order p , when it exists, is a *symmetric* p -linear function of p directions $\dot{x}^p = \dot{x}_1, \dots, \dot{x}_p$. It obeys, in particular,

$$F^{(p)}(x; \dot{x}^p) = \frac{\partial^p}{\partial t_1 \cdots \partial t_p} \bigg|_{t_1 = \cdots = t_p = 0} F(x + \sum t_i \dot{x}_i), \quad (7.1.2)$$

$$\left\| F(x + t\dot{x}) - \sum_{p' < p} \frac{t^{p'}}{p'!} F^{(p')}(x; \dot{x}, \dots, \dot{x}) \right\|_X = o(t^p), \quad (7.1.3)$$

where $o(t)$ is uniform for $\dot{x} \in X(1)^p$. For differential calculus of functions on Banach spaces see [65], and for the more general setting of normed vector spaces, see [5, Appendix D.2].

Example 7.1.1. Let M be a symmetric k -linear function on X , and let $F(x) = M(x, \dots, x)$, i.e., F is M evaluated on the diagonal sequence whose k -components are all equal to x . Then the p th derivative $F^{(p)}(x)$ at x is zero for $p > k$, and otherwise is the p -linear form $\dot{x}_1, \dots, \dot{x}_p \mapsto \frac{k!}{(k-p)!} M(\dot{x}_1, \dots, \dot{x}_p, x, \dots, x)$ (there are $k-p$ entries x). The combinatorial factor arises via the symmetry of M . By the definition (7.1.1),

$$\|F^{(p)}(x)\|_X \leq \frac{k!}{(k-p)!} \|M\|_X \|x\|^{k-p}. \quad (7.1.4)$$

Given normed spaces $\mathcal{Z}_s$ ($s = 1, 2, 3$), let $\mathcal{Z} = \mathcal{Z}_1 \times \mathcal{Z}_2 \times \mathcal{Z}_3$, and let $F : \mathcal{Z} \rightarrow \mathbb{R}$ be a function on $\mathcal{Z}$. Consider the Fréchet derivative of order p_1 with respect to z_1 , of order p_2 with respect to z_2 , and of order p_3 with respect to z_3 . Let $p = p_1, p_2, p_3$. Then the Fréchet derivative

$$F^{(p)}(z; \dot{z}_1^{p_1}; \dot{z}_2^{p_2}; \dot{z}_3^{p_3}) \quad (7.1.5)$$

of F at $z \in \mathcal{Z}$ is p_1 -linear in $\dot{z}_1^{p_1}$, p_2 -linear in $\dot{z}_2^{p_2}$, and p_3 -linear in $\dot{z}_3^{p_3}$, where $\dot{z}_s^{p_s} = \dot{z}_{s,1}, \dots, \dot{z}_{s,p_s} \in \mathcal{Z}_s^{p_s}$ for $s = 1, 2, 3$. The norm of this derivative is, by definition,

$$\|F^{(p)}(z)\|_{\mathcal{Z}} = \sup_{\dot{z}_s^{p_s} \in \mathcal{Z}_s(1)^{p_s}, s=1,2,3} |F^{(p)}(z; \dot{z}_1^{p_1}; \dot{z}_2^{p_2}; \dot{z}_3^{p_3})|. \quad (7.1.6)$$

We use this three-variable formalism in preference to uniting arguments using a larger normed space, in order to avoid testing differentiation in unwanted directions such as $\dot{\phi} + \dot{V}$.

Let $\mathfrak{h}$ be a positive number and let $|\cdot|$ be the Euclidean norm on $\mathbb{R}^n$. For the remainder of Chapter 7, we set

$$\mathcal{Z}_1 = \mathbb{R}_{\mathfrak{h}}^n = \mathbb{R}^n \text{ with norm } \mathfrak{h}^{-1}|\cdot|. \quad (7.1.7)$$

Let $\mathcal{Y} = \mathcal{Z}_2 \times \mathcal{Z}_3$ so that $\mathcal{Z} = \mathbb{R}_h^n \times \mathcal{Y}$. We write $z = (\varphi, y) \in \mathcal{Z}$. We use multi-index notation, in which we write $p! = \prod_s p_s!$, we write $p \leq p'$ to mean that $p_s \leq p'_s$ for each s , and we use multi-binomial coefficients defined by

$$\binom{p'}{p} = \frac{p'!}{p!(p'-p)!} = \prod_s \frac{p'_s!}{p_s!(p'_s - p_s)!}. \quad (7.1.8)$$

Definition 7.1.2. Let $p_{\mathcal{Z}} = (p_{\mathcal{Z}_1}, p_{\mathcal{Z}_2}, p_{\mathcal{Z}_3})$ where each $p_{\mathcal{Z}_s}$ is a non-negative integer or ∞ , and define $p_{\mathcal{N}} = p_{\mathcal{Z}_1}$ and $p_{\mathcal{Y}} = (p_{\mathcal{Z}_2}, p_{\mathcal{Z}_3})$. Given a function $F : \mathcal{Z} \rightarrow \mathbb{R}$ with norm-continuous Fréchet derivatives of orders up to $p_{\mathcal{Z}}$ we define the $T_z = T_z(h)$ -seminorm of F by

$$\|F\|_{T_z} = \sum_{p \leq p_{\mathcal{Z}}} \frac{1}{p!} \|F^{(p)}(z)\|_{\mathcal{Z}}. \quad (7.1.9)$$

The triangle inequality holds for $\|\cdot\|_{T_z}$ by definition. The T_{φ} -seminorm is defined by the same formula with $p_{\mathcal{Y}} = (0, 0)$, and is denoted by $\|F\|_{T_{\varphi}} = \|F\|_{T_{\varphi}(h)}$. The T_{φ} -seminorm does not examine derivatives with respect to y , and is defined on functions $F : \mathcal{Z}_1 \rightarrow \mathbb{R}$, or on functions $F : \mathcal{Z} \rightarrow \mathbb{R}$ with $z = (\varphi, y)$ where y is held fixed.

Later in this chapter and also in subsequent chapters, in a slight abuse of notation we apply the T_z -seminorm to elements of the space $\mathcal{N}(B)$ of Definition 5.1.4. An element $F(B) \in \mathcal{N}(B)$ determines a function $F : \mathbb{R}^n \rightarrow \mathbb{R}$ via the relation $F(B) = F \circ j_B$, and when we write $\|F(B)\|_{T_z}$ we mean $\|F\|_{T_z}$.

We only need the case $p_{\mathcal{Z}} = (\infty, \infty, \infty)$, but we include finite choices to emphasise that there is no need for analyticity in φ, V, K in this chapter. For $p_{\mathcal{Z}} = (0, 0, 0)$, the T_z -seminorm is simply the absolute value of $F(z) \in \mathbb{R}$. The name T_z refers to the Taylor expansion at z . Just as the Taylor expansion of the product of two functions is the product of the Taylor expansions, the seminorm of Definition 7.1.2 shares with the absolute value the following *product property*. A more general product property is proved in [54].

Lemma 7.1.3. For $F, G : \mathcal{Z} \rightarrow \mathbb{R}$ and $z \in \mathcal{Z}$,

$$\|FG\|_{T_z} \leq \|F\|_{T_z} \|G\|_{T_z} \quad (7.1.10)$$

Proof. It is a consequence of the definition of the norm and the product rule for differentiation that, for $p \leq p_{\mathcal{Z}}$,

$$\|(FG)^{(p)}(z)\|_{\mathcal{Z}} \leq \sum_{p' \leq p} \binom{p}{p'} \|F^{(p')}(z)\|_{\mathcal{Z}} \|G^{(p-p')}(z)\|_{\mathcal{Z}}, \quad (7.1.11)$$

where we have used the notation (7.1.8). Therefore,

$$\begin{aligned}
\|FG\|_{T_z} &\leq \sum_{p \leq p_Z} \frac{1}{p!} \sum_{p' \leq p} \binom{p}{p'} \|F^{(p')}(z)\|_{\mathcal{Z}} \|G^{(p-p')}(z)\|_{\mathcal{Z}} \\
&= \sum_{p' \leq p_Z} \frac{1}{p'!} \|F^{(p')}(z)\|_{\mathcal{Z}} \sum_{p: p' \leq p \leq p_Z} \frac{1}{(p-p')!} \|G^{(p-p')}(z)\|_{\mathcal{Z}} \\
&= \sum_{p' \leq p_Z} \frac{1}{p'!} \|F^{(p')}(z)\|_{\mathcal{Z}} \sum_{q \leq p_Z - p'} \frac{1}{q!} \|G^{(q)}(z)\|_{\mathcal{Z}} \\
&\leq \|F\|_{T_z} \|G\|_{T_z},
\end{aligned} \tag{7.1.12}$$

and the proof is complete. $\blacksquare$

The product property simplifies control of smoothness. For example,

$$\|e^F\|_{T_z} \leq e^{\|F\|_{T_z}}. \tag{7.1.13}$$

This follows by expanding the exponential in a Taylor expansion and applying the product property term by term.

Given $\mathfrak{h} > 0$, we define the function

$$P_{\mathfrak{h}}(t) = 1 + |t|/\mathfrak{h} \quad (t \in \mathbb{R}). \tag{7.1.14}$$

Exercise 7.1.4. Let M be a symmetric k -linear function on $\mathbb{R}_{\mathfrak{h}}^n$ that does not depend on the variables $y \in \mathcal{Y}$. Let $F(\varphi) = M(\varphi, \dots, \varphi)$ denote the result of evaluating M on the sequence $\varphi, \dots, \varphi$ with k components. Then

$$\|F\|_{T_z} \leq \|M\|_{\mathcal{Z}} P_{\mathfrak{h}}^k(\varphi). \tag{7.1.15}$$

Combine this with the product property to prove that, for a nonnegative integer p , $\|(\varphi \cdot \varphi)^p\|_{T_z} \leq (|\varphi| + \mathfrak{h})^{2p}$. Similarly, for a vector $\zeta \in \mathbb{R}^n$, $\|(\zeta \cdot \varphi)(\varphi \cdot \varphi)^p\|_{T_z} \leq |\zeta|(|\varphi| + \mathfrak{h})^{2p+1}$. [\[Solution\]](#)

Let $F : \mathcal{Z} \rightarrow \mathbb{R}$ and recall that $\mathcal{Z} = \mathbb{R}_{\mathfrak{h}}^n \times \mathcal{Y}$ with elements denoted $z = (\varphi, y)$. We define the norm

$$\|F\|_{T_{\infty, y}} = \sup_{\varphi \in \mathbb{R}^n} \|F\|_{T_z}, \tag{7.1.16}$$

where, in $z = (\varphi, y)$, y is held fixed. The T_z -seminorm and the T_{∞} -norm are monotone decreasing in the norms on $\mathbb{R}_{\mathfrak{h}}^n$ and $\mathcal{Y}$ and therefore monotone increasing in $\mathfrak{h}$. The product property for the T_z -seminorm immediately implies that the T_{∞} -norm also has the analogous product property. When $p_{\mathcal{Y}} = (0, 0)$ the norm (7.1.16) is equivalent to the $\mathcal{C}^{p_{\mathcal{N}}}$ norm, but is preferable for our purposes because it has the product property.

The following lemma provides an estimate which compares the norm of a polynomial in φ for two different values of the parameter $\mathfrak{h}$ for the norm on $\mathcal{Z}_1 = \mathbb{R}_{\mathfrak{h}}^n$ of (7.1.7), with the norm on $\mathcal{Y}$ unchanged.

Lemma 7.1.5. For a function $F : \mathcal{Z} \rightarrow \mathbb{R}$, which is polynomial of degree $k \leq p_{\mathcal{Z}_1}$ in φ , and for $\mathfrak{h}, \mathfrak{h}' > 0$,

$$\|F\|_{T_{0,y}(\mathfrak{h},\lambda)} \leq \left(\frac{\mathfrak{h}}{\mathfrak{h}'} \vee 1\right)^k \|F\|_{T_{0,y}(\mathfrak{h}',\lambda)}. \quad (7.1.17)$$

Proof. A unit norm direction in $\mathcal{Z}_1(\mathfrak{h})$ is a direction in $\mathcal{Z}_1(\mathfrak{h}')$ with norm $\frac{\mathfrak{h}}{\mathfrak{h}'}$, whereas, for $s = 2, 3$, norms of directions $\dot{y}$ in $\mathcal{Y}_s$ are the same in $\mathcal{Z}_s(\mathfrak{h})$ and $\mathcal{Z}_s(\mathfrak{h}')$. Consequently, $\|F^{(p)}((0,y))\|_{\mathcal{Z}(\mathfrak{h})} = (\frac{\mathfrak{h}}{\mathfrak{h}'})^{p_1} \|F^{(p)}((0,y))\|_{\mathcal{Z}(\mathfrak{h}')}$. Therefore, with $\alpha = \frac{\mathfrak{h}}{\mathfrak{h}'} \vee 1$,

$$\begin{aligned} \|F\|_{T_{0,y}(\mathfrak{h})} &= \sum_{p \leq (k, p_{\mathcal{Y}})} \frac{1}{p!} \|F^{(p)}((0,y))\|_{\mathcal{Z}(\mathfrak{h})} \\ &\leq \alpha^k \sum_{p \leq (k, p_{\mathcal{Y}})} \frac{1}{p!} \|F^{(p)}((0,y))\|_{\mathcal{Z}(\mathfrak{h}')} = \alpha^k \|F\|_{T_{0,y}(\mathfrak{h}')}, \end{aligned} \quad (7.1.18)$$

and the proof is complete. $\blacksquare$

7.2 Control of derivatives

The following two lemmas indicate how the $T_{\mathcal{Z}}$ -seminorm provides estimates on derivatives.

In the statement of the next lemma, for $F : \mathcal{Z} \rightarrow \mathbb{R}$, we define $F^{(p)}(y)$ to be the function $\varphi \mapsto F^{(p)}(\varphi, y)$ with y held fixed.

Lemma 7.2.1. *For $F : \mathcal{Z} \rightarrow \mathbb{R}$, for $p \leq p_{\mathcal{Y}}$, and for directions $\dot{y}^q = (\dot{z}_2^{p_2}, \dot{z}_3^{p_3})$ which have unit norm in $\mathcal{Y} = \mathcal{Z}_2 \times \mathcal{Z}_3$, for any $(\varphi, y) \in \mathcal{Z}$,*

$$\|F^{(0, p_2, p_3)}(y; \dot{y}^q)\|_{T_{\varphi}} \leq p_2! p_3! \|F\|_{T_{\varphi, y}}. \quad (7.2.1)$$

Proof. By Definition 7.1.2, the $T_{\varphi, y}$ -seminorm obeys the inequality

$$\sum_{p_1} \frac{1}{p_1!} \|F^{(p_1, q)}(z)\|_{\mathcal{Z}} \leq q! \|F\|_{T_{\varphi, y}}, \quad (7.2.2)$$

where $q = (p_2, p_3)$ and $q! = p_2! p_3!$. This implies that, for any fixed unit directions $\dot{y}^q$,

$$\sum_{p_1} \frac{1}{p_1!} \sup_{\dot{\varphi}^p \in \mathcal{Z}_1(1)^{p_1}} \left| F^{(p_1, q)}(z; \dot{\varphi}^{p_1}, \dot{y}^q) \right| \leq q! \|F\|_{T_{\varphi, y}}, \quad (7.2.3)$$

and this is (7.2.1) by the definition of the T_{φ} norm and of $F^{(p_1, q)}(y)$. $\blacksquare$

Lemma 7.2.2. *Let $F : \mathcal{Z} \rightarrow \mathbb{R}$ be polynomial in φ of degree $k \leq p_{\mathcal{N}}$. Then for $r \leq k$ and for directions $\dot{\varphi}^r$ which have unit norm in $\mathbb{R}^n$,*

$$\|F^{(r,0,0)}(\dot{\phi}^r)\|_{T_{\phi,y}} \leq 2^k \frac{r!}{\mathfrak{h}^r} \|F\|_{T_{\phi,y}}. \quad (7.2.4)$$

Proof. By Definition 7.1.2 and the hypotheses,

$$\|F^{(r,0,0)}(\dot{\phi}^r)\|_{T_{\phi,y}} \leq \sum_{p_1 \leq k-r} \frac{1}{p_1!} \sum_q \frac{1}{q!} \|F^{(p_1+r,q)}(z)\|_{\mathcal{Z}} \mathfrak{h}^{-r}. \quad (7.2.5)$$

The $\mathfrak{h}^{-r}$ factor in the right-hand side occurs because the $\mathcal{Z}$ norm on the right-hand side is defined in (7.1.6) as a supremum over directions with unit norm in $\mathbb{R}_\mathfrak{h}^n$ whereas in the left-hand side we are testing the derivative on directions with unit norm in $\mathbb{R}^n$. We shift the index by writing $p'_1 = p_1 + r$, and use $\frac{1}{(p'_1-r)!} = \frac{r!}{p'_1!} \binom{p'_1}{r}$ followed by $\binom{p'_1}{r} \leq \sum_{r \leq p'_1} \binom{p'_1}{r} = 2^{p'_1} \leq 2^k$, and obtain

$$\begin{aligned} \|F^{(r,0,0)}(\dot{\phi}^r)\|_{T_{\phi,y}} &\leq \sum_{p'_1=r}^k \frac{1}{(p'_1-r)!} \sum_q \frac{1}{q!} \|F^{(p'_1,q)}(z)\|_{\mathcal{Z}} \mathfrak{h}^{-r} \\ &\leq r! 2^k \sum_{p'_1,q} \frac{1}{p'_1! q!} \|F^{(p'_1,q)}(z)\|_{\mathcal{Z}} \mathfrak{h}^{-r}. \end{aligned} \quad (7.2.6)$$

The right-hand side is $r! 2^k \mathfrak{h}^{-r} \|F^{(r,q)}\|_{T_{\phi,y}}$, as desired. $\blacksquare$

7.3 Expectation and the T_z -seminorm

In (5.1.5), we encounter an expectation $(\mathbb{E}_+ \theta Z)(\varphi) = \mathbb{E}_+ Z(\varphi + \zeta)$, where the integration is with respect to ζ with φ held fixed. Similarly, in the definition (5.2.32) of $K_+(B)$ we encounter $(\mathbb{E}_+ \theta F^B)(\varphi) = \mathbb{E}_+ (\prod_{b \in B} F(b; \varphi + \zeta))$. The field ζ is constant on blocks $b \in \mathcal{B}$, while φ is constant on blocks $B \in \mathcal{B}_+$. In this section, we show in a general context how such convolution integrals can be estimated using the T_z -seminorm.

Given a block B , an n -component field φ which is constant on B , an n -component field ζ which is constant on blocks $b \in \mathcal{B}(B)$, and given $F(\cdot, \zeta) \in \mathcal{N}(b)$ with ζ regarded as fixed, we define $F_\zeta \in \mathcal{N}(b)$ by $F_\zeta(\varphi) = F(\varphi, \zeta)$. Similarly, we define $\theta_\zeta F \in \mathcal{N}(b)$ by $(\theta_\zeta F)(\varphi) = F(\varphi + \zeta)$. Although F is a function of $(\varphi, y) \in \mathcal{Z}$, we do not exhibit the dependence of F on y in our notation. We can take the $T_{\phi,y}$ -seminorm of F_ζ , obtaining $\|F_\zeta\|_{T_{\phi,y}}$ which depends on the variable ζ that is held fixed. Also, with φ fixed, we can integrate $F_\zeta(\varphi)$ with respect to ζ . These last two facts are relevant for the interpretation of (7.3.2) in the following proposition.

Proposition 7.3.1. *For $b \in \mathcal{B}$ and $F \in \mathcal{N}(b)$,*

$$\|\theta_\zeta F\|_{T_{\phi,y}} = \|F\|_{T_{\phi+\zeta,y}}. \quad (7.3.1)$$

For $B \in \mathcal{B}_+$, for ζ a field which is constant on blocks $b \in \mathcal{B}(b)$, and for $F = F(\varphi, \zeta)$ with $F(\cdot, \zeta) \in \mathcal{N}(B)$,

$$\|\mathbb{E}_C F_\zeta\|_{T_{\varphi,y}} \leq \mathbb{E}_C \|F_\zeta\|_{T_{\varphi,y}}. \quad (7.3.2)$$

For a family $F(b) \in \mathcal{N}(b)$, where b ranges over $\mathcal{B}(B)$ with $B \in \mathcal{B}_+$, and for $F^B = \prod_{b \in B} F(b)$ as in (5.1.12),

$$\|\mathbb{E}_C \theta F^B\|_{T_{\varphi,y}} \leq \mathbb{E}_C \left(\prod_{b \in \mathcal{B}(B)} \|F(b)\|_{T_{\varphi+\zeta|_b,y}} \right). \quad (7.3.3)$$

Proof. The identity (7.3.1) follows immediately from the definition of $\theta_\zeta F$, by commuting derivatives with the translation $\varphi \mapsto \varphi + \zeta$. The inequality (7.3.2) is obtained by commuting derivatives past the expectation,

$$\left| \frac{\partial^p}{\partial \varphi^p} \mathbb{E}_C F(\varphi, \zeta) \right| \leq \mathbb{E}_C \left| \frac{\partial^p F_\zeta(\varphi)}{\partial \varphi^p} \right| \quad (7.3.4)$$

and then the desired result follows from the Definition 7.1.2 of the $T_{\varphi,y}$ -seminorm. For the inequality (7.3.3), we first use (7.3.2), then that θ is a homomorphism, then the product property of the T_z -seminorm, and finally (7.3.1), to obtain

$$\begin{aligned} \|\mathbb{E} \theta_\zeta F^B\|_{T_{\varphi,y}} &\leq \mathbb{E} \|\theta_\zeta F^B\|_{T_{\varphi,y}} = \mathbb{E} \|(\theta_\zeta F)^B\|_{T_{\varphi,y}} \\ &\leq \mathbb{E} \left(\|\theta_\zeta F\|_{T_{\varphi,y}}^B \right) = \mathbb{E} \left(\prod_{b \in \mathcal{B}(B)} \|F(b)\|_{T_{\varphi+\zeta|_b,y}} \right), \end{aligned} \quad (7.3.5)$$

as required. ■

7.4 Exponentials and the T_z -seminorm

As a consequence of the product property, the T_z -norm interacts well with the exponential function. The following lemma, which is based on [54, Proposition 3.8], is an extension of (7.1.13). It improves on (7.1.13) when $F(z) < 0$.

Lemma 7.4.1. For $F : \mathcal{Z} \rightarrow \mathbb{R}$,

$$\|e^F\|_{T_z} \leq e^{F(z) + (\|F\|_{T_z} - |F(z)|)}. \quad (7.4.1)$$

Lemma 7.4.1 is an immediate consequence of the following proposition, which holds in any unital algebra $\mathcal{A}$ with seminorm obeying the product property $\|FG\| \leq \|F\| \|G\|$ for all $F, G \in \mathcal{A}$. To deduce (7.4.1) from Proposition 7.4.2, we simply take r to be the value $F(z)$ (not the function) and use the fact that $\|F - F(z)\|_{T_z} = \|F\|_{T_z} - |F(z)|$ by definition of the T_z -norm.

Proposition 7.4.2. *Let $\mathcal{A}$ be a unital algebra with seminorm obeying the product property. For any $F \in \mathcal{A}$ and $r \in \mathbb{R}$,*

$$\|e^F\| \leq e^{r+\|F-r\|}. \quad (7.4.2)$$

Proof. It suffices to show that

$$\|e^F\| \leq \liminf_{n \rightarrow \infty} \|1 + F/n\|^n, \quad (7.4.3)$$

since, for any $r \in \mathbb{R}$ and $n \geq |r|$,

$$\|1 + F/n\| = \|1 + r/n + (F - r)/n\| \leq 1 + r/n + \|F - r\|/n, \quad (7.4.4)$$

and hence, by (7.4.3) and the fact that $e^x = \lim_{n \rightarrow \infty} (1 + x/n)^n$,

$$\|e^F\| \leq \liminf_{n \rightarrow \infty} (1 + r/n + \|F - r\|/n)^n = e^{r+\|F-r\|}. \quad (7.4.5)$$

To prove (7.4.3), it suffices to restrict to the case $n > \|F\|$, so that $(1 + F/n)^{-1}$ is well-defined by its power series. We first use the product property to obtain

$$\begin{aligned} \|e^F\| &= \left\| (e^{F/n})^n (1 + F/n)^{-n} (1 + F/n)^n \right\| \\ &\leq \left\| e^{F/n} (1 + F/n)^{-1} \right\|^n \left\| (1 + F/n) \right\|^n. \end{aligned} \quad (7.4.6)$$

Let $R_n = e^{F/n} - 1 - F/n$. By expanding the exponential, we find that $\|R_n\| = O(n^{-2})$. Therefore,

$$\|e^{F/n} (1 + F/n)^{-1}\| = \|(1 + F/n + R_n)(1 + F/n)^{-1}\| = 1 + O(n^{-2}). \quad (7.4.7)$$

Since $(1 + O(n^{-2}))^n \rightarrow 1$, (7.4.3) follows after taking the $\liminf$ in (7.4.6). $\blacksquare$

7.5 Taylor's theorem and the T_z -seminorm

As in Definition 5.2.2, we write $\text{Tay}_k F$ for the degree- k Taylor polynomial of $F : \mathcal{Z} \rightarrow \mathbb{R}$ in $\varphi \in \mathbb{R}_{\mathfrak{h}}^n$, with $y \in \mathcal{Y}$ held fixed, i.e.,

$$\text{Tay}_k F(\varphi, y) = \sum_{r \leq k} \frac{1}{r!} F^{(r,0,0)}(0, y; \varphi^r). \quad (7.5.1)$$

The following lemma relates the seminorms of $\text{Tay}_k F$ and F . Given $\mathfrak{h} > 0$, we write $P_{\mathfrak{h}}(t) = 1 + |t|/\mathfrak{h}$ for $t \in \mathbb{R}$, as in (7.1.14).

Lemma 7.5.1. *For $F : \mathcal{Z} \rightarrow \mathbb{R}$ and $k \leq p_{\mathcal{N}}$,*

$$\|\text{Tay}_k F\|_{T_{\varphi,y}} \leq \|F\|_{T_{0,y}} P_{\mathfrak{h}}^k(\varphi). \quad (7.5.2)$$

In particular,

$$\|\text{Tay}_k F\|_{T_{0,y}} \leq \|F\|_{T_{0,y}}. \quad (7.5.3)$$

Proof. According to (7.5.1), $\text{Tay}_k F$ is a sum of terms $M_r(z) = F^{(r,0,0)}(0,y;\varphi^r)$ with $r \leq k$, where all the components of the sequence φ^r are equal to φ . We therefore begin with an estimate for the T_z -seminorm of M_r obtained by generalising Example 7.1.1 to include y -dependence. If $p_1 > r$ then $M_r^{(p)} = 0$. For $p = (p_1, q)$ with $p_1 \leq r$ and $q = (p_2, p_3)$, and for unit norm directions $\dot{z}^p$,

$$\begin{aligned} |M_r^{(p)}(z; \dot{z}^p)| &= \frac{r!}{(r-p_1)!} \left| F^{(r,q)}(0,y; \dot{\varphi}^{p_1}, \varphi^{r-p_1}; \dot{y}^q) \right| \\ &\leq \frac{r!}{(r-p_1)!} \|F^{(r,q)}(0,y)\|_{\mathcal{Z}} \|\varphi\|_{\mathbb{R}_{\mathfrak{h}}^n}^{r-p_1}. \end{aligned} \quad (7.5.4)$$

We take the supremum over $\dot{z}^p$ and obtain

$$\|M_r^{(p)}(\varphi, y)\|_{\mathcal{Z}} \leq \frac{r!}{(r-p_1)!} \|F^{(r,q)}(0,y)\|_{\mathcal{Z}} \|\varphi\|_{\mathbb{R}_{\mathfrak{h}}^n}^{r-p_1}. \quad (7.5.5)$$

By dividing by $p! = p_1!q!$ and summing over p with $p_1 \leq r$, and by Definition 7.1.2 of the T_z -seminorm, this gives

$$\begin{aligned} \|M_r\|_{T_z} &\leq \sum_{p_1, q} \binom{r}{p_1} \frac{1}{q!} \|F^{(r,q)}(0,y)\|_{\mathcal{Z}} \|\varphi\|_{\mathbb{R}_{\mathfrak{h}}^n}^{r-p_1} \\ &= \sum_q \frac{1}{q!} \|F^{(r,q)}(0,y)\|_{\mathcal{Z}} P_{\mathfrak{h}}^r(\varphi), \end{aligned} \quad (7.5.6)$$

where we evaluated the sum over $p_1 \leq r$ by the binomial theorem, obtaining $(1 + \|\varphi\|_{\mathbb{R}_{\mathfrak{h}}^n})^r$ which equals $P_{\mathfrak{h}}(\varphi)$ by (7.1.7) and (7.1.14). We replace P^r by P^k , which is larger because $r \leq k$, and insert the resulting bound into the definition (7.5.1) of $\text{Tay}_k F$, to obtain

$$\|\text{Tay}_k F\|_{T_{\varphi,y}} \leq \sum_{r \leq k} \frac{1}{r!} \|M_r\|_{T_{\varphi,y}} \leq \|F\|_{T_{0,y}} P_{\mathfrak{h}}^k(\varphi). \quad (7.5.7)$$

This completes the proof. ■

Exercise 7.5.2. Suppose that $F(z)$ is a polynomial in φ of degree $k \leq p_{\mathcal{N}}$, with coefficients that are functions of y . Then

$$\|F\|_{T_{\varphi,y}} \leq \|F\|_{T_{0,y}} P_{\mathfrak{h}}^k(\varphi). \quad (7.5.8)$$

[Solution]

Lemma 7.5.1 shows that the Taylor polynomial of F is effectively bounded in norm by the norm of F . The following lemma shows how the Taylor remainder $(1 - \text{Tay}_k)F$ can be bounded in terms of the norm of F . In the remainder estimate, an important feature is that the norm of $(1 - \text{Tay}_k)F$ is computed for the field $\varphi \in \mathbb{R}_{\mathfrak{h}_+}^n$, whereas in the norm of F the field lies in $\mathbb{R}_{\mathfrak{h}}^n$. In our applications, the change from $\mathfrak{h}$ to $\mathfrak{h}_+$ corresponds to a change in scale, with small ratio $\mathfrak{h}_+/\mathfrak{h}$. The small factor $(\frac{\mathfrak{h}_+}{\mathfrak{h}})^{k+1}$ present in the upper bound of (7.5.9) is ultimately what leads to the crucial contraction estimate for the renormalisation group map; see Proposition 10.5.1. In the lemma, we make explicit the $\mathfrak{h}$ dependence of the norm on $\mathcal{Z}$ by writing $T_z(\mathfrak{h})$ and $\|\cdot\|_{\mathcal{Z}(\mathfrak{h})}$.

Lemma 7.5.3. *For $k < p_{\mathcal{N}}$, $\mathfrak{h}_+ \leq \mathfrak{h}$, and $F : \mathcal{Z} \rightarrow \mathbb{R}$,*

$$\|(1 - \text{Tay}_k)F\|_{T_z(\mathfrak{h}_+)} \leq 2 \left(\frac{\mathfrak{h}_+}{\mathfrak{h}}\right)^{k+1} P_{\mathfrak{h}_+}^{k+1}(\varphi) \sup_{0 \leq t \leq 1} \|F\|_{T_{z_t}(\mathfrak{h})}, \quad (7.5.9)$$

with $z = (\varphi, y)$ and $z_t = (t\varphi, y)$.

Proof. We write $R = (1 - \text{Tay}_k)F$ and $\bar{F} = \sup_{0 \leq t \leq 1} \|F\|_{T_{z_t}(\mathfrak{h})}$, so that our goal becomes

$$\|R\|_{T_z(\mathfrak{h}_+)} \leq 2 \left(\frac{\mathfrak{h}_+}{\mathfrak{h}}\right)^{k+1} P_{\mathfrak{h}_+}^{k+1}(\varphi) \bar{F}. \quad (7.5.10)$$

By definition, with $q = (p_2, p_3)$ and $q! = p_2! p_3!$,

$$\|R\|_{T_z(\mathfrak{h}_+)} = \sum_{p=0}^k \sum_{q \leq p_{\mathcal{Y}}} \frac{1}{p!} \frac{1}{q!} \|R^{(p,q)}(\varphi)\|_{\mathcal{Z}(\mathfrak{h}_+)} + \sum_{p=k+1}^{p_{\mathcal{N}}} \sum_{q \leq p_{\mathcal{Y}}} \frac{1}{p!} \frac{1}{q!} \|F^{(p,q)}(\varphi)\|_{\mathcal{Z}(\mathfrak{h}_+)}, \quad (7.5.11)$$

where the replacement of R by F in the second sum is justified by the fact that p φ -derivatives of $\text{Tay}_k F$ vanish when $p > k$. We estimate the two sums on the right-hand side of (7.5.11) separately.

For the first sum, we fix $p \leq k$ and use the fact that the first $k - p$ φ -derivatives of R , evaluated at zero field, are equal to zero. Let $f(t) = R^{(p,q)}(z_t; \dot{\varphi}^p; \dot{y}^q)$, where $\dot{\varphi}^p; \dot{y}^q$ are $\mathcal{Z}(\mathfrak{h}_+)$ unit norm directions of differentiation in $\mathbb{R}_{\mathfrak{h}_+}^n \times \mathcal{Y}$. The Taylor expansion of $f(t)$ at $t = 1$ to order $k - p$ about $t = 0$ vanishes, so by the integral form of the Taylor remainder, and again replacing R by F as in the second sum of (7.5.11), we obtain

$$R^{(p,q)}(z; \dot{\varphi}^p; \dot{y}^q) = \int_0^1 dt \frac{(1-t)^{k-p}}{(k-p)!} F^{(k+1,q)}(z_t; \dot{\varphi}^p, \varphi^{k+1-p}; \dot{y}^q), \quad (7.5.12)$$

where $\varphi^{k+1-p} = \varphi, \dots, \varphi \in (\mathbb{R}^n)^{k+1-p}$. We take the supremum over the directions and apply the definition (7.1.6) of the $\mathcal{Z}$ -norm of derivatives. This yields

$$\|R^{(p,q)}(z)\|_{\mathcal{Z}(\mathfrak{h}_+)} \leq \int_0^1 dt \frac{(1-t)^{k-p}}{(k-p)!} \|F^{(k+1,q)}(z_t)\|_{\mathcal{Z}(\mathfrak{h})} \left(\frac{\mathfrak{h}_+}{\mathfrak{h}}\right)^p \left(\frac{|\varphi|}{\mathfrak{h}}\right)^{k+1-p}. \quad (7.5.13)$$

Since

$$\sum_{q \leq p_{\mathcal{Y}}} \frac{1}{q!} \|F^{(k+1,q)}(z_t)\|_{\mathcal{Z}(\mathfrak{h})} \leq (k+1)! \|F\|_{T_{\mathfrak{z}}(\mathfrak{h})} \leq (k+1)! \bar{F}, \quad (7.5.14)$$

this gives

$$\begin{aligned} \sum_{p=0}^k \sum_{q \leq p_{\mathcal{Y}}} \frac{1}{p!} \frac{1}{q!} \|R^{(p,q)}(\varphi)\|_{\mathcal{Z}(\mathfrak{h}_+)} &\leq \frac{\bar{F}}{\mathfrak{h}^{k+1}} \sum_{p=0}^k \binom{k+1}{p} \mathfrak{h}_+^p |\varphi|^{k+1-p} \\ &\leq \left(\frac{\mathfrak{h}_+}{\mathfrak{h}}\right)^{k+1} P_{\mathfrak{h}_+}^{k+1}(\varphi) \bar{F}. \end{aligned} \quad (7.5.15)$$

In the last step, we extended the sum to $p \leq k+1$, applied the binomial theorem, and used the definition of $P_{\mathfrak{h}_+}$ from (7.1.14).

For the second sum in (7.5.11), we observe that the definition 7.1.6 of the $\mathcal{Z}$ -norm implies that $\|F^{(p,q)}(\varphi)\|_{\mathcal{Z}(\mathfrak{h}_+)} = \left(\frac{\mathfrak{h}_+}{\mathfrak{h}}\right)^p \|F^{(p,q)}(\varphi)\|_{\mathcal{Z}(\mathfrak{h})}$. Since $\frac{\mathfrak{h}_+}{\mathfrak{h}} \leq 1$, we obtain

$$\begin{aligned} \sum_{p=k+1}^{p_{\mathcal{N}}} \sum_{q \leq p_{\mathcal{Y}}} \frac{1}{p!} \frac{1}{q!} \|F^{(p,q)}(\varphi)\|_{\mathcal{Z}(\mathfrak{h}_+)} &= \sum_{p=k+1}^{p_{\mathcal{N}}} \sum_{q \leq p_{\mathcal{Y}}} \frac{1}{p!} \frac{1}{q!} \|F^{(p,q)}(\varphi)\|_{\mathcal{Z}(\mathfrak{h})} \left(\frac{\mathfrak{h}_+}{\mathfrak{h}}\right)^p \\ &\leq \left(\frac{\mathfrak{h}_+}{\mathfrak{h}}\right)^{k+1} \|F\|_{T_{\mathfrak{z}}(\mathfrak{h})}. \end{aligned} \quad (7.5.16)$$

Since $P_{\mathfrak{h}_+}(\varphi) \geq 1$ and $\|F\|_{T_{\mathfrak{z}}(\mathfrak{h})} \leq \bar{F}$, the above estimate, together with (7.5.11) and (7.5.15), completes the proof of (7.5.10). $\blacksquare$

In subsequent chapters, we will make use of two choices of $\mathfrak{h}$, namely ℓ and h with $\ell \leq h$. The following corollary shows that the $T_{0,y}(\ell)$ - and $T_{\infty,y}(h)$ -seminorms together also control the $T_{\varphi,y}(\ell)$ -seminorm.

Corollary 7.5.4. *For $k < p_{\mathcal{N}}$ and $0 < \mathfrak{h} \leq h$,*

$$\|F\|_{T_{\varphi,y}(\mathfrak{h})} \leq P_{\mathfrak{h}}^{k+1}(\varphi) \left(\|F\|_{T_{0,y}(\mathfrak{h})} + 2 \left(\frac{\mathfrak{h}}{h}\right)^{k+1} \|F\|_{T_{\infty,y}(h)} \right) \quad (7.5.17)$$

Proof. Let $k < p_{\mathcal{N}}$. With Tay_k defined by (7.5.1), we write

$$F = \text{Tay}_k F + (1 - \text{Tay}_k) F. \quad (7.5.18)$$

Then (7.5.17) follows from Lemmas 7.5.1 and 7.5.3 with $\mathfrak{h} = h$ and $\mathfrak{h}_+ = \mathfrak{h}$. $\blacksquare$

7.6 Polynomial estimates

In this section, we obtain estimates on the covariance of two polynomials in the field φ , and on $\delta U = \theta U - U_{\text{pt}}(U)$. Here $\theta U(B)$ is defined by

$$\theta_\zeta U(B) = \sum_{x \in B} U(\varphi_x + \zeta_x) = \sum_{x \in B} U(\varphi + \zeta_x), \quad (7.6.1)$$

where φ is constant on $B \in \mathcal{B}_+$ and ζ is constant on smaller blocks $b \in \mathcal{B}$.

Given a covariance C_+ , we write

$$\mathfrak{c}_+^2 = \max_{x,y \in A} |C_{+,x,y}| = C_{+,x,x}. \quad (7.6.2)$$

The second equality follows from the fact that since $C_{+,x,y}$ is positive-semidefinite, $C_{+,x,y}^2$ is bounded by $C_{+,x,x}C_{+,y,y}$, and $C_{+,x,x} = C_{+,y,y}$ for our covariances.

Lemma 7.6.1. *There exists $c > 0$ such that for $\mathfrak{h} \geq \mathfrak{c}_+ > 0$, for $U = u + V$ and $U' = u' + V'$ polynomials of degree 4 with constant parts u, u' , and for $x \in B \in \mathcal{B}_+$,*

$$\begin{aligned} & \|\text{LocCov}_+(\theta U_x, \theta U'(B))\|_{T_{\varphi,y}(\mathfrak{h}, \lambda)} \\ & \leq c \left(\frac{\mathfrak{c}_+}{\mathfrak{h}} \right)^4 \|V_x\|_{T_{0,y}(\mathfrak{h}, \lambda)} \|V'(B)\|_{T_{0,y}(\mathfrak{h}, \lambda)} P_{\mathfrak{h}}^4(\varphi). \end{aligned} \quad (7.6.3)$$

Proof. Without loss of generality, we can and do assume that $u = u' = 0$ since constants do not contribute to the covariance. Recall the multi-index notation from above Lemma 5.1.3. Let $S = \{(\alpha, \alpha') : |\alpha| \leq 4, |\alpha'| \leq 4, |\alpha| + |\alpha'| \in \{4, 6, 8\}\}$. As in the proof of Lemma 5.1.3,

$$\text{LocCov}_+(\theta V_x, \theta V'(B)) = \sum_{(\alpha, \alpha') \in S} \frac{1}{\alpha! \alpha'!} V^{(\alpha)} V'^{(\alpha')} \sum_{x' \in B} \text{Cov}_+(\zeta_x^\alpha, \zeta_{x'}^{\alpha'}). \quad (7.6.4)$$

If U is a polynomial in φ of degree at most 4, and if $|\alpha| = p \leq 4$, then it follows from Exercise 7.5.2 and Lemma 7.2.2 that

$$\|U^{(\alpha)}\|_{T_{\varphi,y}} \leq \|U^{(\alpha)}\|_{T_{0,y}} P_{\mathfrak{h}}^{4-p}(\varphi) \leq O(\mathfrak{h}^{-p}) \|U\|_{T_{0,y}} P_{\mathfrak{h}}^{4-p}(\varphi). \quad (7.6.5)$$

Therefore, with $p = |\alpha|$ and $p' = |\alpha'|$,

$$\begin{aligned} & \|\text{LocCov}_+(\theta V_x, \theta V'(B))\|_{T_{\varphi,y}} \\ & \leq O(1) \|V\|_{T_{0,y}} \|V'\|_{T_{0,y}} \sum_{(\alpha, \alpha') \in S} \mathfrak{h}^{-p-p'} P_{\mathfrak{h}}^{8-p-p'}(\varphi) \sum_{x' \in B} \left| \text{Cov}_+(\zeta_x^\alpha, \zeta_{x'}^{\alpha'}) \right|. \end{aligned} \quad (7.6.6)$$

It follows from Exercise 2.1.7 and the definition of $\mathfrak{c}_+$ that the covariance is bounded by $O(\mathfrak{c}_+^{p+p'})$. After inserting this bound, there is no dependence on x' so the sum over x' becomes a factor $|B|$ which together with $\|V'\|_{T_{0,y}}$ equals $\|V'(B)\|_{T_{0,y}}$ because φ is constant on B . Also, $P_{\mathfrak{h}}^{8-p-p'}(\varphi) \leq P_{\mathfrak{h}}^4(\varphi)$, and since $\mathfrak{h} \geq \mathfrak{c}_+$, the proof of (7.6.3) is complete since $\sum_{(\alpha, \alpha') \in S} (\mathfrak{c}_+/\mathfrak{h})^{p+p'} \leq O(\mathfrak{c}_+/\mathfrak{h})^4$. This completes the proof. ■

Exercise 7.6.2. By adapting the proof of Lemma 7.6.1, and with the same hypotheses, show that

$$\|\mathbb{E}_{C_+}(\theta U(B) - U(B))\|_{T_{\varphi,y}(\mathfrak{h},\lambda)} \leq c \left(\frac{\epsilon_{\pm}}{\mathfrak{h}}\right)^2 \|V(B)\|_{T_{0,y}(\mathfrak{h},\lambda)} P_{\mathfrak{h}}^2(\varphi). \quad (7.6.7)$$

[Solution]

The next lemma and proposition include as hypothesis $\|V(B)\|_{T_{0,y}(\mathfrak{h},\lambda)} \leq 1$. The upper bound 1 serves merely to avoid introduction of a new constant, and any finite upper bound would serve the same purpose. In both the lemmas and Proposition, U can be replaced by V in the statement and proof. For the lemma this is because if $U = u + V$ then $U_{\text{pt}}(U) = u + U_{\text{pt}}(V)$ so u cancels in the left-hand side. Likewise, for $\delta U = \theta U - U_{\text{pt}}(U)$ as in (5.2.20), we have $\delta U = \theta U - U_{\text{pt}}(U) = \theta V - U_{\text{pt}}(V) = \delta V$.

Lemma 7.6.3. *There exists $c > 0$ such that, for $\mathfrak{h} \geq \epsilon_+ > 0$, and for all $U = u + V$ with $\|V(B)\|_{T_{0,y}(\mathfrak{h},\lambda)} \leq 1$,*

$$\|U_{\text{pt}}(B) - U(B)\|_{T_{\varphi,y}(\mathfrak{h},\lambda)} \leq c \left(\frac{\epsilon_{\pm}}{\mathfrak{h}}\right) \|V(B)\|_{T_{0,y}(\mathfrak{h},\lambda)} P_{\mathfrak{h}}^4(\varphi). \quad (7.6.8)$$

Proof. As discussed above, we can replace U by V . By Definition 5.2.5,

$$U_{\text{pt}}(B) - V(B) = \mathbb{E}_{C_{j+1}}(\theta V(B) - V(B)) + \frac{1}{2} \text{LocVar}_+(\theta V(B)). \quad (7.6.9)$$

The desired inequality then follows from Exercise 7.6.2 and Lemma 7.6.1, together with $P_{\mathfrak{h}}^2 \leq P_{\mathfrak{h}}^4$ and $\frac{\epsilon_{\pm}}{\mathfrak{h}} \leq 1$. ■

Proposition 7.6.4. *There exists $c > 0$ such that, for $\mathfrak{h} \geq \epsilon_+ > 0$, and for all $m \geq 1$ and all $U = u + V$ with $\|V(B)\|_{T_{0,y}(\mathfrak{h},\lambda)} \leq 1$,*

$$\|\delta U(B)_{\zeta}\|_{T_{\varphi,y}(\mathfrak{h},\lambda)}^m \leq c^m O\left(\frac{\epsilon_{\pm}}{\mathfrak{h}}\right)^m \|V(B)\|_{T_{0,y}(\mathfrak{h},\lambda)}^m P_{\mathfrak{h}}^{4m}(\varphi) \frac{1}{|B|} \sum_{x \in B} P_{\epsilon_+}^{4m}(\zeta_x). \quad (7.6.10)$$

Proof. As discussed above, we can replace U by V . Also, it suffices to prove the case $m = 1$, since this case implies the general case by the product property of the norm and Jensen's inequality in the form $|B|^{-1} \sum_{x \in B} |a_x|^m \leq |B|^{-1} \sum_{x \in B} |a_x|$. Let $m = 1$.

By the triangle inequality,

$$\|\delta V(B)_{\zeta}\|_{T_{\varphi,y}} = \|\theta_{\zeta} V(B) - V(B)\|_{T_{\varphi,y}} + \|V(B) - U_{\text{pt}}(B)\|_{T_{\varphi,y}}. \quad (7.6.11)$$

The second term obeys the desired estimate, by Lemma 7.6.3.

For the first term, it suffices to prove that

$$\left\| \frac{d}{dt} \theta_t \zeta V_x \right\|_{T_{\varphi,y}} \leq O\left(\frac{\epsilon_{\pm}}{\mathfrak{h}}\right) \|V\|_{T_0} P_{\mathfrak{h}}^3(\varphi) P_{\epsilon}^4(\zeta_x), \quad (7.6.12)$$

since integration over $t \in [0, 1]$ and summing over $x \in B$ then leads to the desired estimate. For $j = 1, \dots, n$, let e_j denote the multi-index which has 1 in j^{th} position and 0 elsewhere. We apply the chain rule, Proposition 7.3.1, and (7.6.5) to obtain

$$\begin{aligned} \left\| \frac{d}{dt} \theta_t \zeta V_x \right\|_{T_{\varphi, y}} &\leq \sum_{j=1}^n \left\| \theta_t \zeta V_x^{(e_j)} \right\|_{T_{\varphi, y}} |\zeta_x^j| = \sum_{j=1}^n \left\| V_x^{(e_j)} \right\|_{T_{\varphi+t\zeta, y}} |\zeta_x^j| \\ &\leq O(\mathfrak{h}^{-1}) \|V_x\|_{T_{0, y}} P_{\mathfrak{h}}^3(\varphi + t\zeta_x) |\zeta_x|. \end{aligned} \quad (7.6.13)$$

For $t \in [0, 1]$, $P_{\mathfrak{h}}(\varphi + t\zeta) \leq P_{\mathfrak{h}}(\varphi)P_{\mathfrak{h}}(\zeta) \leq P_{\mathfrak{h}}(\varphi)P_{\mathfrak{c}_+}(\zeta)$, where we used $\mathfrak{h} \geq \mathfrak{c}_+$ in the final inequality. Also, $|\zeta| \leq \mathfrak{c}_+ P_{\mathfrak{c}_+}(\zeta)$. This gives (7.6.12) and completes the proof. $\blacksquare$

Chapter 8

Global flow: Proof of Theorem 4.2.1

The main theorem proved in this book is Theorem 4.2.1, which provides the asymptotic behaviour of the susceptibility of the 4-dimensional hierarchical model. Chapter 6 proves Theorem 4.2.1 subject to Theorem 6.2.1 and Proposition 6.2.2. In this chapter, we state the two main theorems concerning the renormalisation group, namely Theorems 8.2.4–8.2.5, and use these theorems to prove Theorem 6.2.1. This then proves Theorem 4.2.1 subject to Theorems 8.2.4–8.2.5 and Proposition 6.2.2. The proof of Theorem 8.2.5 is given in Chapter 9 and the proofs of Theorem 8.2.4 and Proposition 6.2.2 are given in Chapter 10.

We begin in Section 8.1 with a discussion of fields and domains for the renormalisation group coordinate V . Our choice of norms for the coordinate K is introduced in Section 8.2.2. The main theorems about the renormalisation group map, Theorems 8.2.4–8.2.5, are stated in Section 8.2.3. In Section 8.3, we apply these main theorems to construct the critical point and a global renormalisation group flow started from the critical point. Finally, in Section 8.4, we apply the main theorems to prove Theorem 6.2.1.

8.1 Fluctuation and block-spin fields

8.1.1 Hierarchical field

For the analysis of the renormalisation group map Φ_{j+1} defined in Definition 5.2.8, the scale $0 \leq j < N$ is fixed, and we often drop the subscript j and replace the subscript $j+1$ by $+$. Thus we write $(V, K) \mapsto (U_+, K_+)$ when discussing the map Φ_+ . All results are uniform in the scale j . We write B for an arbitrary *fixed* block in $\mathcal{B}_+(\Lambda)$, whereas blocks in $\mathcal{B}(B)$ are denoted by b .

We recall the decomposition of the covariance $C = C_1 + \cdots + C_{N-1} + C_{N,N}$ from Proposition 4.1.9. For the last step, we further divide $C_{N,N} = C_N + C_{\hat{N}}$. Given j , only the covariances C_{j+1} and $C_{j+2} + \cdots + C_{N,N}$ are of importance.

By definition of the hierarchical GFF,

- the restriction of $x \mapsto \zeta_x$ to a block $b \in \mathcal{B}$ is constant;
- the restriction of $x \mapsto \varphi_x$ to a block $B \in \mathcal{B}_+$ is constant.

When attention is on fields ζ_x with x restricted to a specified scale j block then we often omit x and write ζ instead. For the same reason we write φ instead of φ_x when $x \in B$.

The analysis of Φ_+ relies on perturbation theory and Taylor approximation in powers of the field φ about $\varphi = 0$. These are only good approximations when fields are small. Large fields are handled by non-perturbative estimates which show that large fields are unlikely. Implementing this apparently simple idea leads to notorious complications in rigorous renormalisation group analysis that are collectively known as the *large-field problem*. The subsequent chapters provide a way to solve the large-field problem in the hierarchical setting, where the difficulties are fewer than in the Euclidean setting.

Two mechanisms suppress large fields, one for the fluctuation field and one for the block-spin field.

- The fluctuation field suppression comes from the low probability that a Gaussian field is much larger than its standard deviation.
- The block-spin suppression comes from the factor $e^{-g\tau^2}$ in e^{-V} . This is more subtle because it is a non-Gaussian effect.

8.1.2 Fluctuation field

By (4.1.12), for $j+1 \leq N$, the variance of the fluctuation field $\zeta_x = \zeta_{j+1,x}$ at any point x is

$$C_{j+1;x,x}(m^2) = (1 + m^2 L^{2j})^{-1} L^{-(d-2)j} (1 - L^{-d}). \quad (8.1.1)$$

Given $m \geq 0$, the mass scale j_m is defined in Definition 5.3.3. As in (5.3.14), the exponential decay beyond the mass scale due to the factor $(1 + m^2 L^{2j})^{-1}$ is encoded by the larger (L -independent) sequence ϑ_j^2 , with

$$\vartheta_j = 2^{-(j-j_m)_+}. \quad (8.1.2)$$

We fix an L -dependent constant

$$\ell_0 = L^{1+d/2}, \quad (8.1.3)$$

and define the *fluctuation-field scale*

$$\ell = \ell_j = \ell_0 L^{-j(d-2)/2}. \quad (8.1.4)$$

Then $C_{+,x,x}$ is bounded by $\vartheta^2 \ell_0^{-2} \ell^2$. Therefore, with $\mathfrak{c}_+$ given by (7.6.2), a typical fluctuation field has size on the order of

$$\mathfrak{c}_+ = C_{+,x,x}^{1/2} \leq \vartheta \ell_0^{-1} \ell \leq \vartheta_+ \ell_+, \quad (8.1.5)$$

where we used (8.1.3) and $L \geq 2$ for the last inequality.

We use $\mathfrak{c}_+$ to control the covariance when it is important to know its decay as a function of the mass. The parameter ℓ_j is an upper bound for the covariance which is independent of the mass and which we use in the definition of norms.

8.1.3 Block-spin field

For the block-spin field, we fix strictly positive parameters $\tilde{g} = \tilde{g}_j$ and $\tilde{g}_+ = \tilde{g}_{j+1}$ obeying

$$\tilde{g}_+ \in [\tfrac{1}{2}\tilde{g}, 2\tilde{g}], \quad (8.1.6)$$

and also fix a small constant $k_0 > 0$ whose value is determined in Proposition 10.2.1. Then we define the *large-field scale*

$$h = h_j = k_0(L^{dj}\tilde{g}_j)^{-1/4}. \quad (8.1.7)$$

The definition of h_j is arranged so that if $g \asymp \tilde{g}_j$ and $|\varphi_x| \asymp h_j$, then $\sum_{x \in b} g |\varphi_x|^4 \asymp k_0^4$ is positive uniformly in all parameters j, L, g . In other words, the exponential decay due to $e^{-\frac{1}{4}g \sum_{x \in b} |\varphi_x|^4}$ becomes significant once $|\varphi|$ exceeds the large-field scale h_j , provided that the coupling constant g is close to its reference value $\tilde{g}$. The latter condition is encoded by the *stability domain* for the coupling constants, defined by

$$\mathcal{D}_j^{\text{st}} = \left\{ (g, \mathbf{v}, u) : k_0 \tilde{g}_j < g < k_0^{-1} \tilde{g}_j, |\mathbf{v}| < \tilde{g}_j h_j^2, |u| < \tilde{g}_j h_j^4 \right\}. \quad (8.1.8)$$

Indeed, the domain $\mathcal{D}^{\text{st}}$ is defined to make the following estimate work.

Exercise 8.1.1. Show that if $U \in \mathcal{D}^{\text{st}}$ then

$$U(\varphi) \geq \tfrac{1}{8} k_0 \tilde{g} |\varphi|^4 - \tfrac{3}{2} k_0^3 L^{-dj}, \quad (8.1.9)$$

and hence, if k_0 is chosen small enough that $e^{\frac{3}{2}k_0^3} \leq 2$, then

$$e^{-U(\varphi)} \leq \left(e^{\frac{3}{2}k_0^3} e^{-\frac{1}{8}k_0^5 |\varphi/h_j|^4} \right)^{L^{-dj}} \leq \left(2 e^{-\frac{1}{8}k_0^5 |\varphi/h_j|^4} \right)^{L^{-dj}}. \quad (8.1.10)$$

[Solution]

8.2 Main estimate on renormalisation group map

8.2.1 Domain for V

In addition to the stability domain, which ensures the stability estimate (8.1.10), we define a smaller domain $\mathcal{D}$ which puts constraints on the coupling constants that we use to ensure that the non-perturbative flow remains close to the perturbative flow defined by the map Φ_{pt} of (5.2.18). Thus we estimate $\Phi_+(V, K)$ for V in the domain in $\mathcal{V}$ defined by

$$\mathcal{D}_j = \{(g, v) : 2k_0 \tilde{g}_j < g < (2k_0)^{-1} \tilde{g}_j, |v| < (2k_0)^{-1} \tilde{g}_j L^{-(d-2)j}\}. \quad (8.2.1)$$

The following lemma shows that $\mathcal{D}$ is contained in both $\mathcal{D}^{\text{st}}$ and $\mathcal{D}_+^{\text{st}}$.

Lemma 8.2.1. *For $d = 4$, and for each scale j ,*

$$\mathcal{D}_j \subset \mathcal{D}_j^{\text{st}} \cap \mathcal{D}_{j+1}^{\text{st}}. \quad (8.2.2)$$

Proof. To see that $\mathcal{D} \subset \mathcal{D}^{\text{st}}$, we examine the three coupling constants one by one. The inclusion for g is immediate from the definition of the domains. For v , the $\mathcal{D}$ condition is $|v| < (2k_0)^{-1} \tilde{g} L^{-2j}$. For $\tilde{g}$ small depending on k_0 this implies $|v| < k_0^{-1/2} \tilde{g}^{1/2} L^{-2j}$ which is the desired $\mathcal{D}^{\text{st}}$ condition. For u , since $u = 0$ for elements of $\mathcal{D}_j$, there is nothing to check. The inclusion $\mathcal{D} \subset \mathcal{D}_+^{\text{st}}$ follows similarly, using (8.1.6). For v we must take $\tilde{g}$ small depending on L . ■

8.2.2 Norms

Our estimates on the renormalisation group map are expressed in terms of certain norms. These norms are constructed from the $T_\varphi(\mathfrak{h})$ -seminorm of Definition 7.1.2. At present, we do not use any auxiliary space $\mathcal{Y}$; that will become advantageous in Chapters 9–10. To obtain estimates that are useful for both the fluctuation-field scale ℓ and the large-field scale h , we use the two choices $\mathfrak{h} = \ell$ and $\mathfrak{h} = h$. We fix the parameter $p_{\mathcal{N}}$, which guarantees sufficient smoothness in φ in Definition 7.1.2, with

$$p_{\mathcal{N}} \geq 10, \quad \text{where } p_{\mathcal{N}} = \infty \text{ is permitted.} \quad (8.2.3)$$

The choice $p_{\mathcal{N}} = \infty$ provides analyticity, whereas the choice of finite $p_{\mathcal{N}}$ shows that analyticity is not required for the method to apply. For the small field scale $\mathfrak{h} = \ell$, we usually set φ equal to 0 for all estimates, i.e., we use the $T_0(\ell)$ -seminorm. The following exercise, in particular (8.2.5), shows that the $T_0(\ell)$ norm of the polynomial $V(b)$ is $O_L(\max\{g, |\mu|\})$.

Exercise 8.2.2. For $U = \frac{1}{4}g|\varphi|^4 + \frac{1}{2}v|\varphi|^2 + u$ and $\mathfrak{h} > 0$ prove that

$$\|U_x\|_{T_0(\mathfrak{h})} = \frac{1}{4}|g|\mathfrak{h}^4 + \frac{1}{2}|\mathbf{v}|\mathfrak{h}^2 + |u|. \quad (8.2.4)$$

Therefore, by (8.1.4), for $b \in \mathcal{B}$,

$$\|V(b)\|_{T_0(\ell)} = \frac{1}{4}\ell_0^4|g|L^{-(d-4)j} + \frac{1}{2}\ell_0^2|\mu|. \quad (8.2.5)$$

In particular, by (8.2.1) and (8.1.7), for $V \in \mathcal{D}$,

$$\|V(b)\|_{T_0(\ell)} \leq \frac{\ell_0^4}{2k_0}\tilde{g}, \quad \|V(b)\|_{T_0(h)} \leq \frac{k_0^3}{2}. \quad (8.2.6)$$

Hint: find the norm of each monomial separately. [\[Solution\]](#)

On the other hand, estimates in the large field scale $\mathfrak{h} = h$ will be uniform in φ , i.e., we use the $T_\infty(h)$ -norm of (7.1.16).

The input bounds on K we require for Φ_+ are:

$$\|K(b)\|_{T_0(\ell)} \leq O(\vartheta^3\tilde{g}^3), \quad (8.2.7)$$

$$\|K(b)\|_{T_\infty(h)} \leq O(\vartheta^3\tilde{g}^{3/4}). \quad (8.2.8)$$

A hint of the choice of powers of $\tilde{g}$ in the above two right-hand sides can be gleaned from the intuition that K captures higher-order corrections to second-order perturbation theory, and is dominated by contributions containing a third power of $\delta V = \theta V - U_{\text{pt}}(V)$. According to Proposition 7.6.4, the $T_\varphi(\mathfrak{h})$ -seminorm of $(\delta V(b))^3$ has an upper bound that includes a factor $[(\mathfrak{c}_+/\mathfrak{h})\|V(b)\|_{T_0(\mathfrak{h})}]^3$. By (8.2.6), and by the fact that $\mathfrak{c}_+ \leq \ell$ by (8.1.5), this factor is order $\tilde{g}^3$ for $\mathfrak{h} = \ell$ and is order $\tilde{g}^{3/4}$ for $\mathfrak{h} = h$.

Recall the definition of the vector space $\mathcal{F}$ in Definition 5.1.5. We create a norm on $\mathcal{F}$ that combines (8.2.7)–(8.2.8) into a single estimate, namely (for any $b \in \mathcal{B}$)

$$\|K\|_{\mathcal{W}} = \|K(b)\|_{T_0(\ell)} + \tilde{g}^{9/4}\|K(b)\|_{T_\infty(h)}. \quad (8.2.9)$$

Then the statement that $\|K\|_{\mathcal{W}} \leq M\vartheta^3\tilde{g}^3$ implies the two estimates $\|K(b)\|_{T_0(\ell)} \leq M\vartheta^3\tilde{g}^3$ and $\|K(b)\|_{T_\infty(h)} \leq M\vartheta^3\tilde{g}^{3/4}$, as in (8.2.7)–(8.2.8).

The $\mathcal{W}$ -norm does not obey the product property, whereas the T_φ -seminorms do. For this reason, our procedure is to first obtain estimates for $T_0(\ell)$ and $T_\varphi(h)$, and to then combine them into an estimate for the $\mathcal{W}$ -norm. The next lemma shows that the $\mathcal{W}$ -norm also controls the $T_\varphi(\ell)$ -norm for nonzero φ .

Lemma 8.2.3. *For $K \in \mathcal{F}$ and $b \in \mathcal{B}$,*

$$\|K(b)\|_{T_\varphi(\ell)} \leq P_\ell^{10}(\varphi)\|K\|_{\mathcal{W}}. \quad (8.2.10)$$

Proof. The inequality is an immediate consequence of Corollary 7.5.4 with $k = 9 < p_{\mathcal{N}}$ by (8.2.3), since $(2\frac{\ell}{h})^{10}$ is $o(\tilde{g}^{9/4})$. $\blacksquare$

8.2.3 Main result

The renormalisation group map Φ_+ depends on the mass $m^2 \geq 0$. Our estimates for Φ_+ would be most easily stated for a fixed m^2 . However, to prove the continuity of the critical point as a function of the mass as in Theorem 6.2.1, we regard Φ_+ as a function jointly in (V, K, m^2) . Because Φ_+ has strong dependence on m^2 , this requires care to obtain estimates that are uniform in (V, K, m^2) . To achieve this, we fix $\tilde{m}^2 \geq 0$ and employ the mass domain $\mathbb{I}_j(\tilde{m}^2)$ defined in (6.1.19), and regard the renormalisation group map as a function of $m^2 \in \mathbb{I}_+(\tilde{m}^2)$. Then m^2 is essentially fixed to $\tilde{m}^2$, but can still be varied. We also define the sequence

$$\tilde{\vartheta}_j = 2^{-(j-j\tilde{m})_+}. \quad (8.2.11)$$

By our assumption that $m^2 \geq \frac{1}{2}\tilde{m}^2$, we have $\vartheta_j \leq 2\tilde{\vartheta}_j$.

Given $C_{\text{RG}} > 0$, for $\tilde{g} > 0$ and $\tilde{m} \geq 0$, the domain of Φ_+ is defined to be the set of (V, K) in

$$\mathbb{D} = \mathcal{D} \times \{K \in \mathcal{F} : \|K\|_{\mathcal{W}} < C_{\text{RG}} \tilde{\vartheta}^3 \tilde{g}^3\}, \quad (8.2.12)$$

where the norm of $V \in \mathcal{D}$ is $\|V(b)\|_{T_0(\ell)}$. We will write $\mathcal{V}(\ell)$ to denote the vector space $\mathcal{V}$ with norm $\|V(b)\|_{T_0(\ell)}$. Note that the norm on $\mathcal{W}$ and the domain $\mathbb{D}$ are defined in terms of both $\tilde{g} > 0$ and $\tilde{m}^2 \geq 0$ (through $\tilde{\vartheta}$). We sometimes emphasise this dependence by writing

$$\mathbb{D} = \mathbb{D}(\tilde{m}^2, \tilde{g}^2). \quad (8.2.13)$$

We always assume that $\tilde{g}$ and $\tilde{g}_+$ have bounded ratios as in (8.1.6).

The following theorem is proved in Chapter 10. Besides providing estimates on Φ_+^K , the theorem also specifies the constant C_{RG} occurring in $\mathbb{D}$. The first case in (8.2.16) shows that $\|K\|_{\mathcal{W}} \leq C_{\text{RG}} \tilde{\vartheta}^3 \tilde{g}^3$ implies that $\|K_+\|_{\mathcal{W}_+} \leq C_{\text{RG}} \tilde{\vartheta}_+^3 \tilde{g}_+^3$. This shows that K does not expand as the scale is advanced. The proof of this crucial fact is based on the third case in (8.2.16), which shows that the K -derivative of the map taking K to K_+ can be made as small as desired by a choice of sufficiently large L , so the map Φ_+^K is contractive.

To formulate bounds on derivatives, we consider maps $F : \mathcal{V}(\ell) \times \mathcal{W} \rightarrow \mathcal{X}$ taking values in a normed space $\mathcal{X}$, where here $\mathcal{X}$ is $\mathcal{U}_+(\ell_+)$ or $\mathcal{W}_+$. For $y = (V, K) \in \mathcal{V}(\ell) \times \mathcal{W}$, the derivative $D^{p_2} D^{p_3} F(V, K)$ at (V, K) is a multilinear map $\mathcal{V}(\ell)^{p_2} \times \mathcal{W}^{p_3} \rightarrow \mathcal{X}$. We write

$$\|D_V^{p_2} D_K^{p_3} F(V, K)\|_{\mathcal{V}(\ell) \times \mathcal{W} \rightarrow \mathcal{X}} \quad (8.2.14)$$

for the norm of this multilinear map. In the next theorem, and for other statements that are uniform in all (V, K) considered, we typically omit the argument (V, K) from the notation.

Theorem 8.2.4. *Let $\tilde{m}^2 \geq 0$, let L be sufficiently large, let $\tilde{g}$ be sufficiently small (depending on L), and let $p, q \in \mathbb{N}_0$. Let $0 \leq j < N$. There exist L -dependent $C_{\text{RG}}, M_{p,q} > 0$ and $\kappa = O(L^{-2})$ such that the map*

$$\Phi_+^K : \mathbb{D} \times \mathbb{I}_+ \rightarrow \mathcal{W}_+ \quad (8.2.15)$$

satisfies the estimates

$$\|D_V^p D_K^q \Phi_+^K\|_{\mathcal{V}(\ell) \times \mathcal{W} \rightarrow \mathcal{W}_+} \leq \begin{cases} C_{\text{RG}} \tilde{\vartheta}_+^3 \tilde{g}_+^3 & (p=0, q=0) \\ M_{p,0} \tilde{\vartheta}_+^3 \tilde{g}_+^{3-p} & (p>0, q=0) \\ \kappa & (p=0, q=1) \\ M_{p,q} \tilde{g}_+^{-p-\frac{9}{4}(q-1)} & (p \geq 0, q \geq 1). \end{cases} \quad (8.2.16)$$

In addition, Φ_+^K and all Fréchet derivatives $D_V^p D_K^q \Phi_+^K$ are jointly continuous in all arguments $V, K, \dot{V}, \dot{K}$, as well as in $m^2 \in \mathbb{I}_+$.

The map $\Phi_{\text{pt}}^U(V)$ is the same as $\Phi_+^U(V, 0)$, and it has been analysed explicitly in Section 5.3. Thus, to complete the understanding of the map Φ_+^U , we recall from (6.2.7) the definition

$$R_+^U(V, K) = \Phi_+^U(V, K) - \Phi_+^U(V, 0). \quad (8.2.17)$$

By definition, $R_+^U(V, K)$ is an element of $\mathcal{U}$. Similar to the notation $\mathcal{V}(\ell)$ we introduced for the vector space $\mathcal{V}$ with norm $\|V(b)\|_{\mathcal{T}_0(\ell)}$, we write $\mathcal{U}_+(\ell_+)$ for the vector space $\mathcal{U}$ with norm $\|U(B)\|_{\mathcal{T}_0(\ell_+)}$ for $B \in \mathcal{B}_+$. The following theorem is proved in Chapter 9.

Theorem 8.2.5. *Let $\tilde{m}^2 \geq 0$, let $\tilde{g}$ be sufficiently small (depending on L), and let $p, q \in \mathbb{N}_0$. Let $0 \leq j < N$. There exists an L -dependent constant $M_{p,q} > 0$ such that the map*

$$R_+^U : \mathbb{D} \times \mathbb{I}_+ \rightarrow \mathcal{U}_+(\ell_+) \quad (8.2.18)$$

satisfies the estimates

$$\|D_V^p D_K^q R_+^U\|_{\mathcal{V}(\ell) \times \mathcal{W} \rightarrow \mathcal{U}_+(\ell_+)} \leq \begin{cases} M_{p,0} \tilde{\vartheta}_+^3 \tilde{g}_+^3 & (p \geq 0, q=0) \\ M_{p,q} & (p \geq 0, q=1, 2) \\ 0 & (p \geq 0, q \geq 3). \end{cases} \quad (8.2.19)$$

In addition, R_+^U and all Fréchet derivatives $D_V^p D_K^q R_+^U$ are jointly continuous in all arguments $V, K, \dot{V}, \dot{K}$, as well as in $m^2 \in \mathbb{I}_+$.

As in (6.2.6), we write the components of R_+^U as $(r_{g,j}, r_{v,j}, r_{u,j})$. By (8.2.17), these components give the remainder terms of the renormalisation group flow relative to the perturbative flow of Proposition 5.3.1. By combining (8.2.5) at scale $j+1$ and (8.2.19), we see that

$$r_{g,j} = O_L(\vartheta_j^3 g_j^3), \quad L^{2j} r_{v,j} = O_L(\vartheta_j^3 g_j^3). \quad (8.2.20)$$

8.3 Construction of critical point

In this section, for $m^2 \geq 0$ we construct a critical value $v_0^c(m^2)$ such that the renormalisation group flow exists for all scales, and prove that $v_0^c(m^2)$ is continuous in m^2 . To do so, we apply Theorems 8.2.4–8.2.5.

The u -component of Φ_+ does not play any role and we therefore write $\Phi_+ = (\Phi_+^V, \Phi_+^K)$. Given $m^2 \geq 0$ and an integer $k \geq 0$, the sequence $(V_j, K_j)_{j \leq k}$ is a *flow* (up to scale k) of the renormalisation group map Φ if, for all $j < k$, (V_j, K_j) is in the domain $\mathbb{D}_j(m^2, g_0)$ and

$$(V_{j+1}, K_{j+1}) = \Phi_{j+1}(V_j, K_j; m^2). \quad (8.3.1)$$

To apply Theorems 8.2.4–8.2.5, we need to fix the sequence $\tilde{g}$. Given $\tilde{m}^2 \geq 0$, we define $\tilde{g}_j = \tilde{g}_j(\tilde{m}^2)$ with $\tilde{g}_j$ given by (6.1.1). This choice obeys the condition (8.1.6) by Proposition 6.1.3(i). Thus we now have three similar sequences: g , $\bar{g}$, and $\tilde{g}$. The sequence g is the coupling constant in the true renormalisation group flow and is given in terms of a complicated equation involving the nonperturbative coordinate K_j and also μ_j . On the other hand, the sequence $\bar{g}$ is explicit in terms of the parameters (g_0, m^2) and a simple quadratic recursion. The reason for introducing $\tilde{g}$ in addition to $\bar{g}$ is that $\tilde{g}$ does not depend on m^2 . However, for $m^2 \in \mathbb{I}_+(\tilde{m}^2)$, the sequences $\bar{g}$ and $\tilde{g}$ are comparable, and under the condition that V is in the domain $\mathcal{D}$, all three of $g, \bar{g}, \tilde{g}$ are comparable. In particular, we will use without further comment that for V in $\mathcal{D}$, error estimates $O(\tilde{\vartheta}_j^3 \tilde{g}_j^3)$ as in the statement of Theorem 8.2.5 are equivalent to $O(\vartheta_j^3 g_j^3)$, and similarly with other powers.

Given $c_0 \in (0, (2k_0)^{-1})$, we define the intervals

$$G_j = G_j(m^2) = [-\tfrac{1}{2}\tilde{g}_j(m^2), 2\tilde{g}_j(m^2)], \quad (8.3.2)$$

$$J_j = J_j(m^2; c_0) = [-c_0\vartheta_j(m^2)\tilde{g}_j(m^2), c_0\vartheta_j(m^2)\tilde{g}_j(m^2)]. \quad (8.3.3)$$

By the definition of $\mathcal{D}_j$ in (8.2.1), we have $G_j \times L^{-2j}J_j \subset \mathcal{D}_j$. The set $G_j \times J_j$ is a domain for (g_j, μ_j) as opposed to (g_j, v_j) , hence the factor L^{-2j} .

For the statement of the next proposition, we fix $c_\eta > 0$ such that the coefficient η of (5.3.10) obeys $|\eta_j| \leq c_\eta \vartheta_j$. By (4.1.19), we can choose $c_\eta = n + 2$. For the following, we assume that k_0 is small enough to ensure that $12c_\eta \leq (2k_0)^{-1}$, i.e.,

$$k_0 \leq \frac{1}{24(n+2)}. \quad (8.3.4)$$

This insures that the choice $c_0 = 12c_\eta$ obeys the requirement $c_0 \in (0, (2k_0)^{-1})$; this choice occurs in the proof of Proposition 8.3.3. As always, we also assume that k_0 is small enough to satisfy the restriction imposed by Proposition 10.2.1.

The next proposition characterises the critical value v_0^c . The essence of the proof of its part (i) is known as the *Bleher–Sinai argument* [35].

Proposition 8.3.1. *Let $K_0 = 0$ and fix $g_0 > 0$ sufficiently small. Let $m^2 \geq 0$.*

- (i) There exists $v_0^c(m^2)$ such that the flow (V_j, K_j) of Φ with initial condition g_0 and $v_0 = v_0^c(m^2)$ exists for all $j \in \mathbb{N}$ and is such that $(g_j, \mu_j) \in G_j(m^2) \times J_j(m^2; 4c_\eta)$ and $\|K_j\|_{\mathcal{W}_j} \leq C_{\text{RG}} \tilde{g}_j^3 g_j^3$ for all j . In fact, $g_j = \tilde{g}_j + O(\tilde{g}_j^2 |\log \tilde{g}_j|)$.
- (ii) Let $c_0 \in [4c_\eta, (2k_0)^{-1})$. The value of $v_0^c(m^2)$ is unique in the sense that if a global flow exists started from some v_0 and if this flow obeys $\mu_j \in J_j(m^2; c_0)$ for all j , then $v_0 = v_0^c(m^2)$.

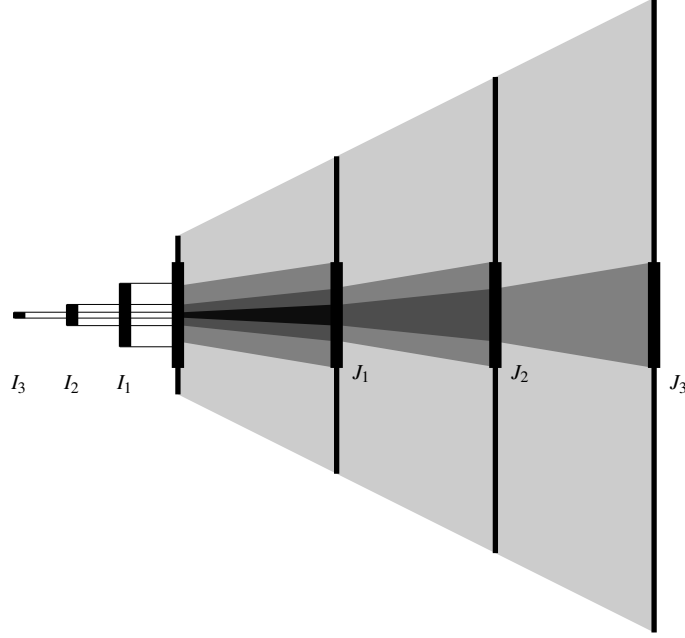


Fig. 8.1 Illustration of the Bleher–Sinai argument. The intervals J_j are indicated by vertical bars, where the scale of the figure is chosen such that the intervals J_j have the same length. Since the map $\mu_j \mapsto \mu_{j+1}$ expands, the condition that the image of μ_0 is contained in J_j after j iterations of the map, for all j , determines μ_0 uniquely.

Proof. (i) Let $K_0 = 0$, and fix $g_0 > 0$, $m^2 \geq 0$, and $\tilde{g} = \tilde{g}(m^2)$. Throughout the proof, we fix any $c_0 \in [4c_\eta, (2k_0)^{-1})$ and set $J_j = J_j(m^2; c_0)$, and we drop m^2 from the notation.

We apply induction in k . The induction hypothesis is that there is a closed interval I_k such that for $\mu_0 \in I_k$, the flow $(V_j, K_j)_{j \leq k}$ exists, $\mu_k \in J_k$, and every $\mu_k \in J_k$ has some preimage $\mu_0 \in I_k$. For $k = 0$, the inductive hypothesis clearly holds. The intervals are illustrated in Figure 8.1.

To advance the induction: the inductive hypothesis implies that $(V_k, K_k) \in \mathbb{D}_k$. Then Theorem 8.2.4 implies that K_{k+1} is defined and satisfies $\|K_{k+1}\|_{\mathcal{W}_{k+1}} \leq$

$C_{RG} \tilde{\vartheta}_{k+1}^3 \tilde{g}_{k+1}^3$. By Theorem 8.2.5, in particular (8.2.20), V_{k+1} is given by Proposition 5.3.1 with corrections from R^U which are $O(\tilde{\vartheta}^3 \tilde{g}_j^3)$. Therefore the flow exists up to scale $k+1$ and

$$g_{j+1} = g_j - \beta_j g_j^2 + O(\vartheta_j^3 g_j^3), \quad (8.3.5)$$

$$\mu_{j+1} = L^2(\mu_j + e_j). \quad (8.3.6)$$

Using Proposition 6.1.3(i) and (8.3.5), followed by (6.1.20), we obtain that $g_j = \tilde{g}_j(1 + O(\tilde{g}_j |\log \tilde{g}_j|)) = \tilde{g}_j(1 + O(\tilde{g}_j |\log \tilde{g}_j|))$. In particular, this shows that $g_j \in G_j$. By (5.3.3), the term e_j in (8.3.6) satisfies

$$|e_j| \leq |\eta_j| g_j + O(\vartheta_j g_j^2) \leq c_\eta \vartheta_j g_j + O(\vartheta_j g_j^2) \leq 2c_\eta \vartheta_j \tilde{g}_j. \quad (8.3.7)$$

By (8.3.6) and (8.3.7), $\mu_{k+1} \geq L^2(\mu_k - 2c_\eta \vartheta_k \tilde{g}_k) \geq L^{\frac{1}{2}} \mu_k$ if $\mu_k = c_0 \vartheta_k \tilde{g}_k$ is the largest point in J_k , and $\mu_{k+1} \leq L^2(\mu_k + 2c_\eta \vartheta_k \tilde{g}_k) \leq L^{\frac{1}{2}} \mu_k$ if $\mu_k = -c_0 \vartheta_k \tilde{g}_k$ is the smallest point in J_k . Together with the continuity of the map $\mu_0 \mapsto \mu_{k+1}$, it follows that the set of μ_{k+1} 's produced from $\mu_0 \in I_k$ includes the interval $\frac{1}{2} L^2 J_k$, which strictly includes J_{k+1} for large L^2 . Thus we can define a new interval $I_{k+1} \subset I_k$ as the inverse image of J_{k+1} under the map $\mu_0 \mapsto \mu_{k+1}$, and it has the required properties. This advances the induction. By construction, $\mu_j \in J_j$.

Finally, since either ϑ_j or $\tilde{g}_j$ decreases to 0, the sequences $\vartheta_j \tilde{g}_j$ tends to 0 for any $m^2 \geq 0$, and the intersection $\cap_{j \geq 1} I_j$ must consist of a single point. We choose $v_0^c(m^2)$ to be that point.

(ii) The proof is a corollary of the proof of part (i). We fix $m^2 \geq 0$ and drop it from the notation.

As shown in the proof of Proposition 8.3.1, the existence of a global flow obeying $\mu_j \in J_j(c_0)$ characterises $v_0 = v_0^c$ uniquely, although in principle it could be the case that v_0 depends on c_0 , i.e., $v_0 = v_0^c(c_0)$. To see that it does not depend on c_0 , note that the flow started from $v_0(4c_\eta)$ is such that $\mu_j \in J_j(4c_\eta)$ for all j . Since $J_j(4c_\eta) \subset J_j(c_0)$ when $c_0 \geq 4c_\eta$, the flow started from $v_0^c(4c_\eta)$ is such that $\mu_j \in J_j(c_0)$ as well. Since the proof of Proposition 8.3.1 shows that only $v_0^c(c_0)$ has this last property, we conclude that $v_0^c(c_0) = v_0^c(4c_\eta)$, and the proof is complete. ■

Let $v_0^c(m^2)$ be the initial condition uniquely defined by Proposition 8.3.1. To prepare for a proof of the continuity of $v_0^c(m^2)$ in m^2 , we first prove the following lemma. Let $j^*(g_0, v_0, m^2)$ be the largest integer j^* such that the flow $(V_j, K_j)_{j \leq j^*}$ of Φ with initial condition $V_0 = (g_0, v_0)$, $K_0 = 0$ exists. Recall that the term “flow” includes the condition $(V_j, K_j) \in \mathbb{D}_j(m^2, g_0)$ for all $j \leq j^*$.

Lemma 8.3.2. *Let (V_j, K_j) be the flow with initial condition $V_0 = (g_0, v_0)$ and $K_0 = 0$. Given $(\tilde{g}_0, \tilde{v}_0, \tilde{m}^2)$ and $j < j^*(\tilde{g}_0, \tilde{v}_0, \tilde{m}^2)$, the map $(g_0, v_0, m^2) \mapsto V_j$ is continuous in a neighbourhood of $(\tilde{g}_0, \tilde{v}_0, \tilde{m}^2)$.*

Proof. By Theorems 8.2.4–8.2.5, the map Φ_{j+1} is jointly continuous in (V, K, m^2) as a map $\mathbb{D}_j(\tilde{m}^2, \tilde{g}_j) \times \mathbb{I}_j(\tilde{m}^2) \subset \mathcal{V} \times \mathcal{W}_j(\tilde{g}_j) \times \mathbb{I}_j(\tilde{m}^2) \rightarrow \mathcal{V} \times \mathcal{W}_{j+1}(\tilde{g}_{j+1})$. The claim then follows from the continuity of the projection $(V, K, m^2) \mapsto V$. ■

Proposition 8.3.3. *The function $m^2 \mapsto v_0^c(m^2)$ is continuous in $m^2 \geq 0$, including right-continuity at $m^2 = 0$. For the corresponding flow, the function $m^2 \mapsto V_j$ is continuous in $m^2 \geq 0$, for each $j \in \mathbb{N}$.*

Proof. Limit points of the set $\{v_0^c(m^2) : m^2 \geq 0\}$ exist because the set is bounded. Suppose that $m^2 \rightarrow \tilde{m}^2$, and let $\tilde{v}_0$ be any limit point of $v_0^c(m^2)$ as $m^2 \rightarrow \tilde{m}^2$. It suffices to show $\tilde{v}_0 = v_0^c(\tilde{m}^2)$. For this, consider the flow $(\tilde{V}_j, \tilde{K}_j)$ with mass $\tilde{m}$ and initial condition $(g_0, \tilde{v}_0)$. By Proposition 8.3.1(ii), the continuity of v_0^c would follow from the condition that $\tilde{V}_j \in \mathcal{D}_j(\tilde{m}^2)$ and $\tilde{\mu}_j \in J_j(\tilde{m}^2; 12c_\eta)$ for all $j \in \mathbb{N}$. Then the continuity of V_j would follow from the continuity of V_j at $\tilde{m}^2$.

To verify the above condition, we use the fact that for any given k , the endpoints of $J_k(m^2)$ and those of the intervals defining the domain $\mathcal{D}_k(m^2)$ can jump at most by a multiplicative factor 3 when m^2 is varied. More precisely, we write $\vartheta^+(\tilde{m}^2) = \limsup_{m^2 \rightarrow \tilde{m}^2} \vartheta(m^2)$ and $\vartheta^-(\tilde{m}^2) = \liminf_{m^2 \rightarrow \tilde{m}^2} \vartheta(m^2)$. Then, for every k , $\frac{1}{2}\vartheta_k^-(\tilde{m}^2) \leq \vartheta_k(\tilde{m}^2) \leq 2\vartheta_k^+(\tilde{m}^2)$. Since the endpoints of J_k and $\mathcal{D}_k$ are defined in terms of ϑ_k , and of $\tilde{g}_k$ which jumps at most by a factor $1 + O(g_0)$, we conclude that the endpoints of J_k and $\mathcal{D}_k$ can jump at most by a factor $2 + O(g_0) < 3$.

By Lemma 8.3.2, (g_k, μ_k) is continuous in (v_0, m^2) in a neighbourhood of $(\tilde{v}_0, \tilde{m}^2)$. Since $m^2 \rightarrow \tilde{m}^2$ and $v_0^c(m^2) \rightarrow \tilde{v}_0$, therefore $g_j(v_0^c(m^2), m^2) \rightarrow g_j(\tilde{v}_0, \tilde{m}^2)$ and the sequence μ_j with mass m is continuous as $m^2 \rightarrow \tilde{m}^2$. Since $V_k(m^2) \in G_k(m^2) \times J_k(m^2; 4c_\eta)$ for any $m^2 \geq 0$, and using the above bound on the jumps of the endpoints of the intervals in $\mathcal{D}_k$, we see that $V_k \in \mathcal{D}_k(\tilde{m}^2)$ for all $k \leq j$, when $m^2 \rightarrow \tilde{m}^2$. Moreover, by Proposition 8.3.1, $\mu_j(\mu_0(m^2), m^2) \in J_j(m^2; 4c_\eta)$ for any m^2 , and thus $\mu_j(\tilde{\mu}_0, \tilde{m}^2) \in J_j(\tilde{m}^2; 12c_\eta)$. As discussed above, this completes the proof. ■

8.4 Proof of Theorem 6.2.1

We now restate and prove Theorem 6.2.1, and thereby complete the proof of Theorem 4.2.1 subject to Theorems 8.2.4–8.2.5 and Proposition 6.2.2.

Theorem 8.4.1. *Fix L sufficiently large and $g_0 > 0$ sufficiently small.*

(i) *There exists a continuous function $v_0^c(m^2)$ of $m^2 \geq 0$ (depending on g_0) such that if $v_0 = v_0^c(m^2)$ then, for all $j \in \mathbb{N}$,*

$$r_{g,j} = O(\vartheta_j^3 g_j^3), \quad L^{2j} r_{v,j} = O(\vartheta_j^3 g_j^3), \quad L^{dj} r_{u,j} = O(\vartheta_j^3 g_j^3), \quad (8.4.1)$$

$$L^{2j} |v_j| = O(\vartheta_j g_j), \quad |K_j(0)| + L^{-2j} |D^2 K_j(0; \mathbb{1}, \mathbb{1})| = O(\vartheta_j^3 g_j^3). \quad (8.4.2)$$

(ii) There exists $c = 1 + O(g_0)$ such that for $m^2 \geq 0$ and $j \in \mathbb{N}$, with all derivatives evaluated at $(m^2, v_0^c(m^2))$,

$$\frac{\partial \mu_j}{\partial v_0} = L^{2j} \left(\frac{g_j}{g_0} \right)^\gamma (c + O(\vartheta_j g_j)), \quad \frac{\partial g_j}{\partial v_0} = O \left(g_j^2 \frac{\partial \mu_j}{\partial v_0} \right), \quad (8.4.3)$$

$$L^{-2j} \left| \frac{\partial}{\partial v_0} K_j(0) \right| + L^{-4j} \left| \frac{\partial}{\partial v_0} D^2 K_j(0; \mathbb{1}, \mathbb{1}) \right| = O \left(\vartheta_j^3 g_j^2 \left(\frac{g_j}{g_0} \right)^\gamma \right). \quad (8.4.4)$$

In the following, we fix $g_0 > 0$ small and drop it from the notation and discussion. Also, the dependence of $\mathcal{W}_j$ on (g_0, m^2) is left implicit. We use primes to denote derivatives with respect to $v_0 = \mu_0$. Let $V_j = V_j(m^2)$ be the infinite sequence given by Proposition 8.3.1 with initial condition $(g_0, v_0^c(m^2))$. Let $K_j = K_j(m^2)$ be the corresponding sequence given by Proposition 8.3.1. Let (V'_j, K'_j) denote the sequence of derivatives along this solution, with respect to the initial condition μ_0 , and with the derivative K'_j taken in the space $\mathcal{W}_j$. The following lemma isolates a continuity property of V'_j .

Lemma 8.4.2. *The function $m^2 \mapsto V'_j(m^2)$ is continuous in $m^2 \geq 0$.*

Proof. By definition, $V'_j(m^2)$ is the derivative of V_j with respect to the initial condition v_0 , evaluated at $(g_0, v_0^c(m^2), m^2)$. By Theorem 8.2.4–8.2.5 and the chain rule, V'_j is continuous in (v_0, m^2) . By Proposition 8.3.3, the function $v_0^c(m^2)$ is continuous, so it follows that $V'(m^2)$ is continuous in m^2 . ■

Proof of Theorem 8.4.1. (i) The function $v_0^c(m^2)$ is given by Proposition 8.3.1. Proposition 8.3.1 also implies that $(V_j, K_j) \in \mathbb{D}_j(m^2)$, since $G_j \times J_j \subset \mathcal{D}_j$ as noted below (8.3.3). Theorem 8.2.5 (in particular (8.2.20)) yields (8.4.1), and $V_j \in \mathcal{D}_j$ immediately gives the bound on v_j in (8.4.2).

For the bound on K_j , we use the fact that the $\mathcal{W}$ -norm dominates the T_0 -seminorm by (8.2.9). Since the T_0 -seminorm dominates the absolute value and $(V_j, K_j) \in \mathbb{D}_j(m^2)$, we have in particular that $|K_j(0)| \leq \|K_j\|_{\mathcal{W}_j} \leq O(\vartheta_j^3 g_j^3)$. For the φ -derivative $D^2 K_j$ in (8.4.2), we use the fact that by definition of the T_φ -seminorm, the derivative in the direction of a test function f obeys

$$|D^2 F(0; f, f)| \leq 2 \|F\|_{T_{0,j}} \|f\|_{\Phi_j}^2. \quad (8.4.5)$$

The norm of the constant test function $\mathbb{1} \in \Phi_j$ is

$$\|\mathbb{1}\|_{\Phi_j} = \ell_j^{-1} \sup_x |\mathbb{1}_x| = \ell_j^{-1} = O(L^{j(d-2)/2}). \quad (8.4.6)$$

Therefore,

$$L^{-2j} |D^2 K_j(0; \mathbb{1}, \mathbb{1})| \leq 2 \|K_j\|_{T_{0,j}} \leq O(\vartheta_j^3 g_j^2). \quad (8.4.7)$$

This completes the proof of (8.4.1)–(8.4.2).

(ii) We prove that there exists a continuous function $c : [0, \infty) \rightarrow \mathbb{R}$, which satisfies $c(m^2) = 1 + O(g_0)$, such that for all $j \in \mathbb{N}_0$:

$$\mu'_j = L^{2j} \left(\frac{g_j}{g_0} \right)^\gamma (c(m^2) + O(\vartheta_j g_j)), \quad g'_j = O(\mu'_j g_j^2), \quad (8.4.8)$$

and

$$\|K'_j\|_{\mathcal{W}_j} = O(\vartheta_j^3 \mu'_j g_j^2). \quad (8.4.9)$$

Then (8.4.3) follows from (8.4.8). Also, as in the proof of part (i), (8.4.9) implies that

$$|K'_j(0)| + L^{-2j} |D^2 K'_j(0; \mathbb{1}, \mathbb{1})| \leq O(\vartheta_j^3 \mu'_j g_j^2), \quad (8.4.10)$$

and then (8.4.4) follows from (8.4.3). It remains to prove (8.4.8)–(8.4.9).

Recall the definition of $\Pi_{i,j}$ from Lemma 6.1.6. We define $\Pi_j^* = \Pi_j(m^2)$ by

$$\Pi_j^* = L^{2j} \Pi_{0,j-1} = L^{2j} \prod_{l=0}^{j-1} (1 - \gamma \beta_l g_l). \quad (8.4.11)$$

By Lemma 6.1.6, and by the continuity of V_j in m^2 provided by Proposition 8.3.3, there is a continuous function $\Gamma_\infty(m^2) = O(g_0)$ such that

$$\Pi_j^* = L^{2j} \left(\frac{g_j}{g_0} \right)^\gamma (1 + \Gamma_\infty(m^2) + O(\vartheta_j g_j)). \quad (8.4.12)$$

We also define $\Sigma_j = \Sigma_j(m^2)$ by

$$\mu'_j = \Pi_j^* (1 + \Sigma_j) \quad (j \geq 0), \quad \Sigma_{-1} = 0. \quad (8.4.13)$$

We will use induction on j , where $j < N$. The inductive assumption is that there exist $M_1 \gg M_2 \gg 1$ such that for $k \leq j$,

$$|\Sigma_k - \Sigma_{k-1}| \leq O(M_1 + M_2) \vartheta_k g_k^2, \quad |g'_k| \leq M_1 \vartheta_k^3 \Pi_k^* g_k^2, \quad (8.4.14)$$

$$\|K'_k\|_{\mathcal{W}_k} \leq M_2 \vartheta_k^3 \Pi_k^* g_k^2. \quad (8.4.15)$$

Since $(g'_0, \mu'_0, K'_0) = (0, 1, 0)$, $\Sigma_0 = 0$ and so the inductive assumption (8.4.14)–(8.4.15) is true for $j = 0$. By summing the first inequality in (8.4.14)–(8.4.15) over $k \leq j$ and by applying (8.4.12)–(8.4.13) we conclude that, if $L \gg 1$ and if g_0 is sufficiently small, then

$$|\mu'_j| \leq 2\Pi_j^*, \quad \vartheta_j^3 \Pi_j^* g_j^2 \leq \frac{1}{2} \vartheta_{j+1}^3 \Pi_{j+1}^* g_{j+1}^2, \quad (8.4.16)$$

where we used (6.1.17) for the first inequality.

As a first step in advancing the induction, we differentiate

$$(V_{j+1}, K_{j+1}) = (\Phi_{\text{pt}}^V(V_j) + R_{j+1}^V(V_j, K_j), \Phi_{j+1}^K(V_j, K_j)), \quad (8.4.17)$$

where R_{j+1}^V denotes the V -component of R_{j+1}^U . By the chain rule, with $F = R_{j+1}^V$ or $F = \Phi_{j+1}^K$,

$$F'(V_j, K_j) = D_VF(V_j, K_j)V_j' + D_KF(V_j, K_j)K_j'. \quad (8.4.18)$$

The g_j' estimate in (8.4.14) and the μ_j' estimate in (8.4.16) bound the coefficients of V_j' . By combining this bound on the coefficients with (8.2.5) we obtain $\|V_j'(b)\|_{T_0(\ell_j)} = O_L(M_1 + 2)\Pi_j^*$. We also have the estimate for K_j' in (8.4.15). By applying Theorems 8.2.4–8.2.5, we obtain

$$\|D_VF(V_j, K_j)V_j'\| \leq O(\vartheta_j^3 g_j^2)(M_1 g_j^2 + 2)\Pi_j^* \leq O(\vartheta_j^3 \Pi_j^* g_j^2), \quad (8.4.19)$$

$$\|D_K R_{j+1}^V(V_j, K_j)K_j'\| \leq O(M_2)\vartheta_j^3 \Pi_j^* g_j^2, \quad (8.4.20)$$

$$\|D_K K_{j+1}(V_j, K_j)K_j'\| \leq M_2 \vartheta_j^3 \Pi_j^* g_j^2. \quad (8.4.21)$$

with the norms dictated by Theorems 8.2.4–8.2.5 on the left-hand sides. For example, if $F = R_{j+1}^V$ (which is in $\mathcal{U}_{j+1}(\ell_{j+1})$) the norm is $\|R_{j+1}^V(B)\|_{T_0(\ell_{j+1})}$, and if $F = \Phi_{j+1}^K$ the norm is $\mathcal{W}_{j+1}$. This implies, for $M_2 \gg 1$,

$$\|(R_{j+1}^V)'(V_j, K_j)\| \leq O(M_2)\vartheta_j^3 \Pi_j^* g_j^2, \quad \|K_{j+1}'(V_j, K_j)\| \leq 2M_2 \vartheta_j^3 \Pi_j^* g_j^2. \quad (8.4.22)$$

With the second inequality of (8.4.16), this advances the induction for K' .

For μ' , the induction is advanced using the recursion (8.4.17) with (5.3.3), (8.4.22), together with the estimates

$$\eta_j g_j', \xi_j (g_j^2)' = O(M_1 \vartheta_j^3 \Pi_j^* g_j^2) \quad (8.4.23)$$

which follow from (8.4.14). We obtain

$$\begin{aligned} \mu_{j+1}' &= L^2 \mu_j' (1 - \gamma \beta_j g_j) + O((M_1 + M_2)\vartheta_j^3 \Pi_j^* g_j^2) \\ &= \Pi_{j+1}^* (1 + \Sigma_j) + O((M_1 + M_2)\vartheta_{j+1}^3 \Pi_{j+1}^* g_{j+1}^2). \end{aligned} \quad (8.4.24)$$

This advances the induction for μ' , namely the first estimate of (8.4.14).

The advancement of the induction for g is similar, as follows. We use the recursion relation (8.4.17) with (5.3.2) and (8.4.22), and choose $M_1 \gg M_2$ to obtain

$$\begin{aligned} |g_{j+1}'| &\leq (M_1(1 + O(g_j)) + O(M_2))\vartheta_j^3 \Pi_j^* g_j^2 \\ &\leq 2M_1 \vartheta_j^3 \Pi_j^* g_j^2 \leq M_1 \vartheta_{j+1}^3 \Pi_{j+1}^* g_{j+1}^2. \end{aligned} \quad (8.4.25)$$

This advances the induction for g' .

By Lemma 8.4.2, V_j' is continuous in m^2 . Since Π_j^* is continuous, it follows that $\Sigma_j(m^2)$ is continuous in m^2 , for each $j \in \mathbb{N}_0$. Since $\sum_{j=1}^\infty \vartheta_j g_j^2 = O(g_0)$ by (6.1.17), it follows from (8.4.14) that the limit $\Sigma_\infty = \lim_{j \rightarrow \infty} \Sigma_j = \sum_{j=1}^\infty (\Sigma_j - \Sigma_{j-1})$ exists with $\Sigma_\infty = O(g_0)$. Continuity of Σ_∞ in m^2 follows from the dominated convergence theorem, with Proposition 6.1.7. Also, again by (6.1.17),

$$\Sigma_\infty - \Sigma_j = O\left(\sum_{k=j+1}^{\infty} \vartheta_k g_j^2\right) = O(\vartheta_j g_j). \quad (8.4.26)$$

From (8.4.12)–(8.4.13) and (8.4.26), we obtain the equation for μ'_j in (8.4.8), with $c(m^2) = (1 + \Sigma_\infty(m^2))(1 + \Gamma_\infty(m^2))$. This $c(m^2)$ is indeed continuous, since Σ_∞ and Γ_∞ are. With this, (8.4.14)–(8.4.15) implies the last inequality in (8.4.8) and (8.4.9), and the proof is complete. ■

Chapter 9

Nonperturbative contribution to Φ_+^U : Proof of Theorem 8.2.5

In this chapter, we prove the estimates of Theorem 8.2.5, which for convenience we restate below as Theorem 9.1.1. The continuity statement of Theorem 8.2.5 is deferred to Section 10.6.

The proof of Theorem 9.1.1 makes use of certain norm estimates on polynomials in the field, which are developed in Section 9.3. A more comprehensive set of estimates on polynomials is needed for Chapter 10, and we present these estimates also in Section 9.3.

9.1 The polynomial R_+^U

The non-perturbative contribution R_+^U to Φ_+^U is defined in (6.2.7) as

$$R_+^U(V, K) = \Phi_+^U(V, K) - \Phi_+^U(V, 0). \quad (9.1.1)$$

By definition, $R_+^U(V, K)$ is an element of $\mathcal{U}$. To any element of $\mathcal{U}$, we may associate an element of $\mathcal{N}(B)$. We denote this element by $R_+^U(B)$ and then omit the argument (V, K) .

The map Φ_+^U is defined in Definition 5.2.8 as

$$\Phi_+^U(V, K) = \Phi_{\text{pt}}(V - Q), \quad (9.1.2)$$

with $Q \in \mathcal{U}$ equal to the b -independent polynomial defined by

$$Q(b) = \text{Loc}(e^{V(b)} K(b)) \quad (9.1.3)$$

for $b \in \mathcal{B}$. The map Φ_{pt} is given in (5.2.18) as

$$\Phi_{\text{pt}}(U; B) = \mathbb{E}_+ \theta U(B) - \frac{1}{2} \mathbb{E}_+ (\theta U(B); \theta U(B)). \quad (9.1.4)$$

Recall that, by definition of the renormalisation group map in Definition 5.2.8, the expectation $\mathbb{E}_+$ here is with respect to a covariance obeying the zero-sum condition $c^{(1)} = 0$. This allows us to drop the operator Loc from the definition of Φ_{pt} in (9.1.4), as remarked under (5.2.18).

After simplification, the above formulas lead to

$$R_+^U(B) = -\mathbb{E}_{C_+} \theta Q(B) + \text{Cov}_+(\theta(V(B) - \tfrac{1}{2}Q(B)), \theta Q(B)). \quad (9.1.5)$$

According to Exercise 5.1.2 and Lemma 5.1.3, the right-hand side is indeed an element of $\mathcal{U}$. By definition, Q is linear in K . Thus R_+^U is a quadratic function of K . The explicit form of (9.1.5) makes the analysis of R_+^U relatively easy.

In this chapter, we prove the estimates of Theorem 8.2.5, which we restate here as follows. Although the domain of R_+^U in (9.1.6) is stated in terms of $\mathbb{D}$, in which K is measured with the $\mathcal{W}$ -norm, in the proof we actually only use the weaker hypothesis that $\|K(b)\|_{T_0(\ell)} \leq C_{\text{RG}} \vartheta^3 \tilde{g}^3$.

Theorem 9.1.1. *Let $\tilde{m}^2 \geq 0$, let $\tilde{g}$ be sufficiently small (depending on L), and let $p, q \in \mathbb{N}_0$. Let $0 \leq j < N$. There exists an L -dependent constant $M_{p,q} > 0$ such that the map*

$$\mathbb{R}_+^U : \mathbb{D} \times \mathbb{I}_+ \rightarrow \mathcal{U}_+ \quad (9.1.6)$$

satisfies the estimates

$$\|D_V^p D_K^q R_+^U\|_{\mathcal{V}(\ell) \times \mathcal{W} \rightarrow \mathcal{U}_+(\ell_+)} \leq \begin{cases} M_{p,0} \tilde{\vartheta}_+^3 \tilde{g}_+^3 & (p \geq 0, q = 0) \\ M_{p,q} & (p \geq 0, q = 1, 2) \\ 0 & (p \geq 0, q \geq 3). \end{cases} \quad (9.1.7)$$

9.2 The standard and extended norms

9.2.1 Utility of the extended norm

Theorems 8.2.4–8.2.5 (and so Theorem 9.1.1) are expressed in terms of the *standard* norms. For V , this is the norm $\|V(b)\|_{T_0(\ell)}$ on the space $\mathcal{V}(\ell)$. For K , these are the seminorms $T_\infty(h)$ and $T_0(\ell)$ and their combination as the $\mathcal{W}$ -norm, on the space $\mathcal{F}$. Functions of (V, K) such as Φ_+ are functions on $\mathcal{V} \times \mathcal{F}$. The standard norm is good for estimates that hold for fixed (V, K) . The dependence of a function $F(V, K)$ on (V, K) is controlled by derivatives with respect to (V, K) in these norms. For a given function F , these can in principle be computed using the usual rules of calculus. In practice, for complicated functions F such as Φ_+ , this can become unwieldy.

To handle derivatives systematically, we use the *extended* norm which encodes not only point-wise dependence of a function $F(V, K)$ on (V, K) but also their derivatives. Thus the extended norm of $F(V, K)$ is not a norm for (V, K) fixed, but requires an infinitesimal neighbourhood of some reference point (V, K) . It is in fact an instance of the T_z -norm, with $\mathcal{Y}$ chosen to be a subspace of $\mathcal{V} \times \mathcal{F}$.

We denote the coordinate maps for (V, K) using stars, i.e., $V^* : \mathcal{Y} \rightarrow \mathcal{V}$ and $K^* : \mathcal{Y} \rightarrow \mathcal{F}$ are defined by

$$V^*(V, K) = V, \quad K^*(V, K) = K. \quad (9.2.1)$$

For the coordinate maps, by Definition 7.1.2 with $y = (V, K)$, the extended and standard norms are related by

$$\|V^*(b)\|_{T_{\phi, y}} = \|V^*(b)\|_{T_{\phi, V, K}} = \|V(b)\|_{T_{\phi}} + \sup_{(\dot{V}, \dot{K})} \frac{\|\dot{V}(b)\|_{T_{\phi}}}{\|(\dot{V}, \dot{K})\|_{\mathcal{Y}}}, \quad (9.2.2)$$

$$\|K^*(b)\|_{T_{\phi, y}} = \|K^*(b)\|_{T_{\phi, V, K}} = \|K(b)\|_{T_{\phi}} + \sup_{(\dot{V}, \dot{K})} \frac{\|\dot{K}(b)\|_{T_{\phi}}}{\|(\dot{V}, \dot{K})\|_{\mathcal{Y}}}, \quad (9.2.3)$$

where b is an arbitrary block in $\mathcal{B}_j(B)$. In particular, the standard norm can be recovered from the extended norm with the limiting choice $\|y\|_{\mathcal{Y}} = \|(V, K)\|_{\mathcal{Y}} = \infty$. The standard norm of V or K only differs from the extended norm of V^* or K^* at (V, K) by an additive constant. Thus, for the coordinate maps, the additional information encoded by the extended norm is trivial. However, for maps that are non-linear in (V, K) , the extended norm is a significant help because it can often be bounded in the same way as the standard norm, yet a bound of the extended norm of a function of (V, K) yields also bounds on the derivatives of this function by Lemma 7.2.1 (see also the special case Lemma 9.2.1 below).

9.2.2 Choice of the space $\mathcal{Y}$

We now specify the space $\mathcal{Y}$ used to define the extended norm. We set $p_{\mathcal{Y}} = \infty$ and fix nonnegative parameters $\lambda = (\lambda_V, \lambda_K)$, which for the moment are arbitrary. Recall that $\mathcal{V} = \mathbb{R}^2$ is defined in Definition 5.1.1. We define $\mathcal{Y} \subset \mathcal{V} \times \mathcal{F}$ to be the space of $y = (V, K)$ with finite norm

$$\|y\|_{\mathcal{Y}} = \max \left\{ \frac{\|V(b)\|_{T_0(\ell)}}{\lambda_V}, \frac{\|K\|_{\mathcal{W}}}{\lambda_K} \right\}, \quad (9.2.4)$$

where b is an arbitrary block in $\mathcal{B}(B)$. We also define

$$X = \mathbb{R}_{\mathfrak{h}}^n, \quad \mathcal{Z} = X \times \mathcal{Y} \subset \mathbb{R}_{\mathfrak{h}}^n \times (\mathcal{V} \times \mathcal{F}). \quad (9.2.5)$$

The case $\lambda_V = \lambda_K = 0$, which superficially appears to prescribe division by zero in (9.2.4), is equivalent to taking $p_{\mathcal{Y}} = 0$ in (7.1.9), or equivalently to a norm that does not measure the size of derivatives with respect to (V, K) ; in this case the norm on $\mathcal{Y}$ is not used and there is no division by zero.

In the same spirit as in the extension of the T_{ϕ} -seminorms to the $T_{\phi, y}$ -seminorms, we extend the definition (8.2.9) of the $\mathcal{W}$ -norm to incorporate the parameter λ , by

defining the extended $\mathcal{W}_y(\lambda)$ norm of a function $F(V, K)$ to be

$$\|F\|_{\mathcal{W}_y(\lambda)} = \|F(b)\|_{T_{0,y}(\ell, \lambda)} + \tilde{g}_+^{9/4} \|F(b)\|_{T_{\infty,y}(h, \lambda)}. \quad (9.2.6)$$

The following lemma is a special case of Lemma 7.2.1.

Lemma 9.2.1. *The following hold for any $y = (V, K) \in \mathcal{V} \times \mathcal{W}$.*

(i) *For $F : \mathcal{V}(\ell) \times \mathcal{W} \rightarrow \mathcal{U}_+(\ell_+)$,*

$$\|D_V^{p_2} D_K^{p_3} F(V, K)\|_{\mathcal{V}(\ell) \times \mathcal{W} \rightarrow \mathcal{U}_+(\ell_+)} \leq \frac{p_2! p_3!}{\lambda_V^{p_2} \lambda_K^{p_3}} \|F\|_{T_{0,y}(\ell_+, \lambda)}. \quad (9.2.7)$$

(ii) *For $F : \mathcal{V}(\ell) \times \mathcal{W} \rightarrow \mathcal{W}_+$,*

$$\|D_V^{p_2} D_K^{p_3} F(V, K)\|_{\mathcal{V}(\ell) \times \mathcal{W} \rightarrow \mathcal{W}_+} \leq \frac{p_2! p_3!}{\lambda_V^{p_2} \lambda_K^{p_3}} \|F\|_{\mathcal{W}_{y,+}(\lambda)}. \quad (9.2.8)$$

Proof. In (9.2.4), the norm $\|y\|_{\mathcal{Y}}$ of $y = (V, K)$ is defined such that the unit ball $\|y\|_{\mathcal{Y}} \leq 1$ corresponds to $\|V(b)\|_{T_0(\ell)} \leq \lambda_V$ and $\|K\|_{\mathcal{W}} \leq \lambda_K$. The norm of a multilinear map defined on $\mathcal{V}(\ell) \times \mathcal{W}$ is, however, defined with respect to unit directions of (V, K) . By Lemma 7.2.1, therefore

$$\begin{aligned} \|D_{z_2}^{p_2} D_{z_3}^{p_3} F(V, K)\|_{\mathcal{V}(\ell) \times \mathcal{W} \rightarrow T_\varphi(\mathfrak{h})} &= \frac{1}{\lambda_V^{p_2} \lambda_K^{p_3}} \|D_{z_2}^{p_2} D_{z_3}^{p_3} F(V, K)\|_{\mathcal{Y} \rightarrow T_\varphi(\mathfrak{h})} \\ &\leq \frac{p_2! p_3!}{\lambda_V^{p_2} \lambda_K^{p_3}} \|F\|_{T_{\varphi,y}(\mathfrak{h}, \lambda)}. \end{aligned} \quad (9.2.9)$$

In the above inequality we set (i) $\mathfrak{h} = \ell_+$ and $\varphi = 0$, and (ii) $\mathfrak{h} = h_+$. The result is the two estimates:

$$\|D_{z_2}^{p_2} D_{z_3}^{p_3} F(V, K)\|_{T_0(\ell) \times \mathcal{W} \rightarrow T_0(\ell_+)} \leq \frac{p_2! p_3!}{\lambda_V^{p_2} \lambda_K^{p_3}} \|F\|_{T_{0,y}(\ell_+, \lambda)}, \quad (9.2.10)$$

$$\|D_{z_2}^{p_2} D_{z_3}^{p_3} F(V, K)\|_{T_0(\ell) \times \mathcal{W} \rightarrow T_\varphi(h_+)} \leq \frac{p_2! p_3!}{\lambda_V^{p_2} \lambda_K^{p_3}} \|F\|_{T_{\varphi,y}(h_+, \lambda)}. \quad (9.2.11)$$

The first estimate is (9.2.7), since the $T_0(\ell_+)$ -seminorm on a block is the same as the $\mathcal{U}_+(\ell_+)$ norm by definition of the latter. In the second estimate we replace T_φ by T_∞ by taking the supremum over φ . By the definitions (8.2.9) and (9.2.6) of the $\mathcal{W}$ - and $\mathcal{W}(\lambda)$ -norms, the desired result (9.2.8) is an immediate consequence of multiplying the second estimate by $\tilde{g}_+^{9/4}$ and adding the two estimates. ■

9.3 Norms of polynomials

In this section, we develop a comprehensive set of norm estimates on polynomials in the field. For the proof of Theorem 9.1.1, we only need the $\mathfrak{h} = \ell$ case of the bound on V given in (9.3.4), and the bound on Q given in (9.3.20). The remaining results in this section prepare the ground for our analysis of K in Chapter 10. Recall that the domains $\mathcal{D}$ and $\mathcal{D}^{\text{st}}$ are defined by (8.2.1) and (8.1.8).

We first observe that it follows from Exercise 8.2.2 that the norm of $U = g\tau^2 + \nu\tau + u$ on a block $b \in \mathcal{B}$ is given by

$$\|U(b)\|_{T_0(\mathfrak{h})} = L^{jd} \left(\frac{1}{4}g\mathfrak{h}^4 + \frac{1}{2}|\nu|\mathfrak{h}^2 + |u| \right). \quad (9.3.1)$$

By the definitions of ℓ and h in (8.1.4) and (8.1.7),

$$L^{jd}\tilde{g}h^4 = k_0^4, \quad \frac{\ell}{h} = \frac{\ell_0}{k_0}\tilde{g}^{1/4}L^{-j(d-4)/4}, \quad h^2 = k_0^2\tilde{g}^{-1/2}L^{-jd/2}. \quad (9.3.2)$$

The constant k_0 is small and independent of L , whereas ℓ_0 is large and is equal to $L^{1+d/2}$ by (8.1.3). By (9.3.1) and the first equality of (9.3.2), for $V = g\tau^2 + \nu\tau$ we have

$$\|V(b)\|_{T_0(\mathfrak{h})} = \frac{1}{4}k_0^4\frac{g}{\tilde{g}}\left(\frac{\mathfrak{h}}{h}\right)^4 + \frac{1}{2}L^{jd}|\nu|h^2\left(\frac{\mathfrak{h}}{h}\right)^2. \quad (9.3.3)$$

The equalities of (9.3.2) are useful for application of (9.3.3). The following lemma uses (9.3.3) to obtain estimates. For our application to dimension $d = 4$ the factors $L^{-(d-4)j}$ in the following lemma equal one.

Lemma 9.3.1. *Let $d \geq 4$, $U = g\tau^2 + \nu\tau + u = V + u$, and $b \in \mathcal{B}$.*

(i) *If $V \in \mathcal{D}$ then*

$$\|V(b)\|_{T_0(\mathfrak{h})} \leq \begin{cases} O_L(\tilde{g})L^{-(d-4)j} & (\mathfrak{h} = \ell) \\ k_0^3 & (\mathfrak{h} = h). \end{cases} \quad (9.3.4)$$

(ii) *If $V \in \mathcal{D}^{\text{st}}$ then*

$$\|V(b)\|_{T_0(\mathfrak{h})} \leq \begin{cases} O_L(\tilde{g}^{1/2})L^{-(d-4)j/2} & (\mathfrak{h} = \ell) \\ k_0^3 & (\mathfrak{h} = h). \end{cases} \quad (9.3.5)$$

Proof. We use (9.3.2) without comment in the proof.

Suppose first that $V \in \mathcal{D}$. By the definition of $\mathcal{D}$ in (8.2.1), $g < (2k_0)^{-1}\tilde{g}$ and $|\nu| < (2k_0)^{-1}\tilde{g}L^{-(d-2)j}$, so

$$\|V(b)\|_{T_0(\mathfrak{h})} \leq \frac{1}{8}k_0^3\left(\frac{\mathfrak{h}}{h}\right)^4 + \frac{1}{4}\tilde{g}^{1/2}k_0L^{-j(d-4)/2}\left(\frac{\mathfrak{h}}{h}\right)^2. \quad (9.3.6)$$

For $\mathfrak{h} = \ell$, the right-hand side is $\left(\frac{1}{8}\frac{\ell_0^4}{k_0} + \frac{1}{4}\frac{\ell_0^2}{k_0}\right)\tilde{g}L^{-j(d-4)}$. Since $\ell_0 \geq 1$, this is less than $k_0^{-1}\ell_0^4\tilde{g}L^{-j(d-4)}$, and this proves the case $\mathfrak{h} = \ell$ of (9.3.4). For $\mathfrak{h} = h$ and $\tilde{g} \leq k_0^4$, the right-hand side of (9.3.6) is at most k_0^3 , as desired for the case $\mathfrak{h} = h$ of (9.3.4).

Suppose now that $V \in \mathcal{D}^{\text{st}}$. By (8.1.8), we have $g \leq k_0^{-1}\tilde{g}$ and $|v| \leq \tilde{g}h^2$. With (9.3.3), we obtain

$$\|V(b)\|_{T_0(\mathfrak{h})} \leq \frac{1}{4}k_0^3\left(\frac{\mathfrak{h}}{h}\right)^4 + \frac{1}{2}k_0^4\left(\frac{\mathfrak{h}}{h}\right)^2. \quad (9.3.7)$$

For $\mathfrak{h} = \ell$ we use $k_0^3 \leq \ell_0^2$ (since $k_0 \leq 1 \leq L$) to obtain

$$\|V(b)\|_{T_0(\ell)} \leq \frac{1}{4}\frac{\ell_0^4}{k_0}\tilde{g}L^{-j(d-4)} + \frac{1}{2}k_0^2\ell_0^2\tilde{g}^{1/2}L^{-j(d-4)/2} \leq \frac{\ell_0^4}{k_0}\tilde{g}^{1/2}L^{-j(d-4)/2}, \quad (9.3.8)$$

as desired for the case $\mathfrak{h} = \ell$ of (9.3.5). Finally, for $\mathfrak{h} = h$, the right-hand side of (9.3.7) is $\frac{1}{4}k_0^3 + \frac{1}{2}k_0^4 \leq k_0^3$, which proves the case $\mathfrak{h} = h$ of (9.3.5). ■

Next, we obtain bounds in the extended norm. By (9.2.2)–(9.2.3),

$$\|V^*(b)\|_{T_{0,y}(\mathfrak{h},\lambda)} = \|V(b)\|_{T_0(\mathfrak{h})} + \lambda_V \sup_{\dot{V} \in \mathcal{V}} \frac{\|\dot{V}(b)\|_{T_0(\mathfrak{h})}}{\|\dot{V}(b)\|_{T_0(\ell)}}, \quad (9.3.9)$$

$$\|K^*(b)\|_{T_{\varphi,y}(\mathfrak{h},\lambda)} = \|K(b)\|_{T_{\varphi}(\mathfrak{h})} + \lambda_K \sup_{\dot{K} \in \mathcal{F}} \frac{\|\dot{K}(b)\|_{T_{\varphi}(\mathfrak{h})}}{\|\dot{K}\|_{\mathcal{W}}}. \quad (9.3.10)$$

In particular, this shows that the norms of U^* and K^* are monotone increasing in $\mathfrak{h}$. Furthermore, for $V \in \mathcal{V}$,

$$\|V^*(b)\|_{T_{0,y}(\ell,\lambda)} = \|V(b)\|_{T_0(\ell)} + \lambda_V, \quad (9.3.11)$$

$$\|K^*(b)\|_{T_{0,y}(\ell,\lambda)} \leq \|K(b)\|_{T_0(\ell)} + \lambda_K, \quad (9.3.12)$$

$$\|K^*(b)\|_{T_{\varphi,y}(h,\lambda)} \leq \|K(b)\|_{T_{\varphi}(h)} + \lambda_K \tilde{g}^{-9/4}. \quad (9.3.13)$$

The bound (9.3.11) is obtained by setting $\mathfrak{h} = \ell$ in (9.3.9). Likewise, (9.3.12)–(9.3.13) follow from (9.3.10) and the definition of the $\mathcal{W}$ -norm in (8.2.9).

The above estimates do not yet include a bound on $\|V^*(b)\|_{T_{0,y}(h,\lambda)}$. The following lemma fills this gap. For the proof of Proposition 10.2.1, it will be important that the coefficient $\frac{3}{8}$ on the first right-hand side of (9.3.14) is smaller than $\frac{1}{2}$.

Lemma 9.3.2. *Let $d = 4$. Let $(g, v, 0) \in \mathcal{D}^{\text{st}}$, $\lambda_V \leq \tilde{g}$, $b \in \mathcal{B}$, and let L be sufficiently large. Then*

$$\|(g\tau^2)^*(b)\|_{T_{0,y}(h,\lambda)} \leq \frac{3}{8}\frac{g}{\tilde{g}}k_0^4, \quad \|(v\tau)^*(b)\|_{T_{0,y}(h,\lambda)} \leq k_0^4. \quad (9.3.14)$$

In particular, the extended norm of $V = g\tau^2 + v\tau$ obeys

$$\|V^*(b)\|_{T_{0,y}(h,\lambda)} \leq \frac{11}{8}k_0^3. \quad (9.3.15)$$

Proof. As in (9.3.9),

$$\|(g\tau^2)^*(b)\|_{T_{0,y}(h,\lambda)} = \|g\tau^2(b)\|_{T_0(h)} + \lambda_V \sup_{\dot{V} \in \mathcal{V}: \dot{v}=0} \frac{\|\dot{V}(b)\|_{T_0(h)}}{\|\dot{V}(b)\|_{T_0(\ell)}}. \quad (9.3.16)$$

By (9.3.3) with $v = 0$ and $\mathfrak{h} = h$, the first term on the right-hand side is $\frac{1}{4}k_0^4 g \tilde{g}^{-1}$. Similarly, by comparing (9.3.3) with $\mathfrak{h} = h$ and with $\mathfrak{h} = \ell$, the ratio in the second term is $h^4 \ell^{-4} = \tilde{g}^{-1} k_0^4 \ell_0^{-4}$. Therefore,

$$\|(g\tau^2)^*(b)\|_{T_{0,y}(h,\lambda)} = \left(\frac{1}{4}\frac{g}{\tilde{g}} + \frac{\lambda_V}{\tilde{g}\ell_0^4}\right)k_0^4. \quad (9.3.17)$$

By hypothesis, $\frac{\lambda_V}{\tilde{g}\ell_0^4} \leq \ell_0^{-4}$, and by the definition of ℓ_0 in (8.1.3) we have $\ell_0 \rightarrow \infty$ as $L \rightarrow \infty$. Hence, since $k_0 < \frac{g}{\tilde{g}}$ by definition of $\mathcal{D}^{\text{st}}$ in (8.1.8), if L is sufficiently large then $\ell_0^{-4} \leq \frac{1}{8}k_0 < \frac{1}{8}\frac{g}{\tilde{g}}$. This proves the first inequality of (9.3.14). Similarly, from (9.3.3) and (9.3.2),

$$\|(v\tau)^*(b)\|_{T_{0,y}(h,\lambda)} = \left(\frac{1}{2}\frac{|v|L^{dj/2}}{\tilde{g}^{1/2}} + \frac{\lambda_V}{\tilde{g}^{1/2}\ell_0^2}\right)k_0^2 \leq \left(\frac{1}{2}k_0^2 + \frac{\tilde{g}^{1/2}}{\ell_0^2}\right)k_0^2 \leq k_0^4. \quad (9.3.18)$$

This proves the second inequality of (9.3.14). Finally, (9.3.15) follows from the triangle inequality and the bound $g/\tilde{g} \leq k_0^{-1}$ (due to (8.1.8)). ■

Finally, we obtain estimates for $Q(b)$ of (9.1.3). We view Q as a function of (V, K) , so we use the extended norm.

Lemma 9.3.3. *For all $(V, K) \in \mathcal{V} \times \mathcal{F}$, and for all $\mathfrak{h} > 0$,*

$$\|Q(b)\|_{T_{\varphi,y}(\mathfrak{h},\lambda)} \leq e^{\|V^*(b)\|_{T_{0,y}(\mathfrak{h},\lambda)}} \|K^*\|_{T_{0,y}(\mathfrak{h},\lambda)} P_{\mathfrak{h}}^4(\varphi). \quad (9.3.19)$$

In particular,

$$\|Q(b)\|_{T_{0,y}(\ell,\lambda)} \leq e^{\|V(b)\|_{T_0(\ell)} + \lambda_V} (\|K\|_{T_0(\ell)} + \lambda_K). \quad (9.3.20)$$

Suppose now that $\lambda_V \leq \tilde{g}$, $\lambda_K \leq \tilde{g}$, $\mathfrak{h}_+ \geq \ell$, $V \in \mathcal{D}^{\text{st}}$, and $\|K(b)\|_{T_0(\ell)} \leq \tilde{g}$. Then, for L sufficiently large,

$$\|Q(B)\|_{T_{\varphi,y}(\mathfrak{h}_+,\lambda)} \leq O(\ell_0^{-4}) \left(\frac{\mathfrak{h}_+}{h_+}\right)^4 P_{\mathfrak{h}_+}^4(\varphi). \quad (9.3.21)$$

Proof. Recall that $\text{Loc} = \text{Tay}_4$. By the definition of Q in (9.1.3) and the bound on Tay_4 in (7.5.3), for $\mathfrak{h} > 0$,

$$\|Q(b)\|_{T_{0,y}(\mathfrak{h},\lambda)} \leq \|e^{V^*(b)} K^*(b)\|_{T_{0,y}(\mathfrak{h},\lambda)} \leq e^{\|V^*(b)\|_{T_{0,y}(\mathfrak{h},\lambda)}} \|K^*\|_{T_{0,y}(\mathfrak{h},\lambda)}, \quad (9.3.22)$$

which proves (9.3.19). By setting $\mathfrak{h} = \ell$ and inserting (9.3.11)–(9.3.12),

$$\|Q(b)\|_{T_{0,y}(\ell,\lambda)} \leq e^{\|V(b)\|_{T_{0,y}(\ell,\lambda)} + \lambda_V} (\|K\|_{T_{0,y}(\ell,\lambda)} + \lambda_K), \quad (9.3.23)$$

which proves (9.3.20).

Next we prove (9.3.21). By (7.5.8), it suffices to prove the result just for $\varphi = 0$. By the hypothesis $\mathfrak{h}_+ \geq \ell$ and Lemma 7.1.5 with $\mathfrak{h}' = \ell$ and $\mathfrak{h} = \mathfrak{h}_+$,

$$\|Q(b)\|_{T_{0,y}(\mathfrak{h}_+,\lambda)} \leq \left(\frac{\mathfrak{h}_+}{\ell}\right)^4 \|Q(b)\|_{T_{0,y}(\ell,\lambda)}. \quad (9.3.24)$$

We insert (9.3.20) in the right-hand side and use (9.3.5), which implies $e^{\|V(b)\|_{T_{0,y}(\ell)}} \leq 2$ (for $\tilde{g}$ small). Using also the hypotheses, we obtain

$$\|Q(b)\|_{T_{0,y}(\mathfrak{h}_+,\lambda)} \leq \left(\frac{\mathfrak{h}_+}{\ell}\right)^4 2e^{\lambda_V} (\tilde{g} + \lambda_K) \leq 5\tilde{g} \left(\frac{\mathfrak{h}_+}{\ell}\right)^4. \quad (9.3.25)$$

By (8.1.4) and (9.3.2), $\frac{\mathfrak{h}_+}{\ell} = L^{-1} \frac{\mathfrak{h}_+}{\ell_+} = L^{-1} \frac{\mathfrak{h}_+}{h_+} \frac{h_+}{\ell_+} = L^{-1} \frac{\mathfrak{h}_+}{h_+} \frac{k_0}{\ell_0} \tilde{g}_+^{-1/4}$. Therefore

$$\|Q(b)\|_{T_{0,y}(\mathfrak{h}_+,\lambda)} \leq O(k_0^4) L^{-4} \ell_0^{-4} \left(\frac{\mathfrak{h}_+}{h_+}\right)^4, \quad (9.3.26)$$

where we used (8.1.6). Since $\|Q(B)\|_{T_{0,y}(\mathfrak{h}_+,\lambda)} = L^d \|Q(b)\|_{T_{0,y}(\mathfrak{h}_+,\lambda)}$, this implies (9.3.21) for $\varphi = 0$ as desired. The proof is complete. $\blacksquare$

Next we bound $\hat{V} = V - Q$.

Lemma 9.3.4. *Let $\lambda_V \leq \tilde{g}$ and $\lambda_K \leq \tilde{g}$. Let $V \in \mathcal{D}$ and $\|K(b)\|_{T_0(\ell)} \leq \tilde{g}$. Then, for L sufficiently large,*

$$\|\hat{V}(B)\|_{T_{0,y}(\ell_+,\lambda)} \leq O_L(\tilde{g}_+), \quad (9.3.27)$$

$$\|\hat{V}(B)\|_{T_{0,y}(h_+,\lambda)} \leq 1. \quad (9.3.28)$$

The same estimates hold when $\hat{V}$ is replaced by V^* .

Proof. Since we are using the extended norm, we write $\hat{V} = V^* - Q$. We will bound V^* and Q individually. The proof for V^* instead of $\hat{V}$ is obtained by forgetting Q . We will use (8.1.6) without comment to replace $O(\tilde{g})$ by $O(\tilde{g}_+)$.

We begin with (9.3.27). The $T_{0,y}(\ell_+,\lambda)$ norm of $V^*(B)$ increases when ℓ_+ is replaced by ℓ . Furthermore, the $T_{0,y}(\ell,\lambda)$ norm of $V^*(B)$ is L^d times the same norm of $V^*(b)$, which, by (9.3.11) and (9.3.4), is $O_L(\tilde{g})$. Therefore

$$\|\hat{V}(B)\|_{T_{0,y}(\ell_+,\lambda)} = O_L(\tilde{g}_+). \quad (9.3.29)$$

By (9.3.21) with $\mathfrak{h}_+ = \ell_+$ and (9.3.2) to show that $\left(\frac{\ell_+}{h_+}\right)^4 = O_L(\tilde{g}_+)$ we find that $\|Q(B)\|_{T_{0,y}(\ell_+,\lambda)}$ is $O_L(\tilde{g}_+)$. Therefore, by the triangle inequality,

$$\|\hat{V}(B)\|_{T_{0,y}(\ell_+,\lambda)} \leq \|V^*(B)\|_{T_{0,y}(\ell_+,\lambda)} + \|Q(B)\|_{T_{0,y}(\ell_+,\lambda)} \leq O_L(\tilde{g}_+) \quad (9.3.30)$$

as desired.

For (9.3.28), by the triangle inequality and (9.3.9) followed by (9.3.3) and (9.3.2), as in (9.3.17)–(9.3.18) we obtain

$$\begin{aligned} \|V^*(B)\|_{T_{0,y}(h_+, \lambda)} &\leq \left(\frac{1}{4} |B| \frac{g_+}{\tilde{g}_+} \tilde{g}_+ h_+^4 + \lambda_V \frac{|B| h_+^4}{|b| \ell^4} \right) + \left(\frac{1}{2} |v| |B| h_+^2 + \lambda_V \frac{|B| h_+^2}{|b| \ell^2} \right) \\ &= \left(\frac{1}{4} \frac{g_+}{\tilde{g}_+} + \frac{\lambda_V}{\ell_0^4} \right) k_0^4 + \left(\frac{1}{2} \frac{|v| L^{d(j+1)/2}}{\tilde{g}_+^{1/2}} + \frac{\tilde{g}_+^{1/2}}{\ell_0^2} \right) k_0^2. \end{aligned} \quad (9.3.31)$$

By inserting the definition (8.2.1) of $V \in \mathcal{D}$, (8.1.6), and using (9.3.21) with $\mathfrak{h}_+ = h_+$ for Q , we obtain

$$\|V^*(B)\|_{T_{0,y}(h_+, \lambda)} + \|Q(B)\|_{T_{0,y}(h_+, \lambda)} \leq \frac{1}{4} k_0^3 + O_L(\tilde{g}_+^{1/2}) + O(\ell_0^{-4}) \leq 1 \quad (9.3.32)$$

because $\tilde{g}_+$ and ℓ_0^{-4} are small depending on L and $k_0 \leq 1$. This proves (9.3.28). ■

The following lemma is a consequence of the previous two.

Lemma 9.3.5. *Let $\lambda_V \leq \tilde{g}_+$ and $\lambda_K \leq \tilde{g}_+$. Let $V \in \mathcal{D}$ and $\|K(b)\|_{T_0(\ell)} \leq \tilde{g}_+$. Then, for L sufficiently large,*

$$\|U_+(B) - V^*(B)\|_{T_{\varphi,y}(h_+, \lambda)} \leq O(\ell_0^{-4}) P_{h_+}^4(\varphi). \quad (9.3.33)$$

Proof. By (7.5.8), it suffices to consider the case $\varphi = 0$. Recall the definition $U_+ - V = U_{\text{pt}}(\hat{V}) - V$ with $\hat{V} = V - Q$. By the triangle inequality,

$$\|U_+(B) - V^*(B)\|_{T_{0,y}(h_+, \lambda)} \leq \|U_{\text{pt}}(\hat{V}, B) - \hat{V}(B)\|_{T_{0,y}(h_+, \lambda)} + \|Q\|_{T_{0,y}(h_+, \lambda)}. \quad (9.3.34)$$

We use (9.3.28) and Lemma 7.6.3 for the first term, and (9.3.21) for the second term. With the bound on $\mathfrak{c}_+$ of (8.1.5), this leads to

$$\|U_+(B) - V^*(B)\|_{T_{0,y}(h_+, \lambda)} \leq O\left(\frac{\mathfrak{c}_+}{h_+}\right) + O(\ell_0^{-4}) = O\left(\tilde{g}_+^{1/4} + \ell_0^{-4}\right), \quad (9.3.35)$$

and the proof is complete since we can choose $\tilde{g}_+$ small depending on L . ■

9.4 Proof of Theorem 9.1.1

The following lemma is the basis for the proof of Theorem 9.1.1.

Lemma 9.4.1. *Let $\mathcal{Y} = \mathcal{V} \times \mathcal{F}$ have the norm (9.2.4), with arbitrary λ . For $(V, K) \in \mathcal{V} \times \mathcal{F}$, let $r_1 = \|V(b)\|_{T_0(\ell)} + \lambda_V$ and $r_2 = \|K\|_{T_0(\ell)} + \lambda_K$, and assume $r_2 \leq r_1$. Then*

$$\|R_+^U(B)\|_{T_{0,y}(\ell_+, \lambda)} = O(L^{2d}) (e^{2r_1} r_1 r_2). \quad (9.4.1)$$

Proof. By (9.3.20), $\ell_+ \leq \ell$ and monotonicity in $\mathfrak{h}$,

$$\|Q(B)\|_{T_{0,y}(\ell_+,\lambda)} \leq L^d e^{r_1} r_2, \quad (9.4.2)$$

where L^d is the number of blocks b in B . By Exercise 7.6.2, and since $\mathfrak{c}_+ \leq \ell_+$ by (8.1.5), this implies

$$\|\mathbb{E}_{C_+} \theta Q(B)\|_{T_{0,y}(\ell_+,\lambda)} \leq \|Q(B)\|_{T_{0,y}(\ell_+,\lambda)} \left(1 + c \left(\frac{\mathfrak{c}_+}{\ell_+}\right)^2\right) \leq O(L^d)(e^{r_1} r_2). \quad (9.4.3)$$

By Lemma 7.6.1, by $\mathfrak{c}_+ \leq \ell_+$, and by the hypothesis $r_2 \leq r_1$ and (9.4.2),

$$\begin{aligned} & \|\text{Cov}_+(\theta(V(B) - \tfrac{1}{2}Q(B)), \theta Q(B))\|_{T_{0,y}(\ell_+,\lambda)} \\ & \leq c|B|^2 \|V_x - \tfrac{1}{2}Q_x\|_{T_{0,y}(\ell_+,\lambda)} \|Q_x\|_{T_{0,y}(\ell_+,\lambda)} \leq O(L^{2d})(e^{2r_1} r_1 r_2). \end{aligned} \quad (9.4.4)$$

The desired result then follows by inserting (9.4.3)–(9.4.4) into (9.1.5). $\blacksquare$

Proof of Theorem 9.1.1. Since R_+^U is quadratic in K , the case $q \geq 3$ of (9.1.7) is immediate and we need only consider the cases $q = 0, 1, 2$. Let $p \geq 0$ and $q = 0, 1, 2$.

We will apply (9.2.7) and (9.4.1). Let $r_1 = \|V(b)\|_{T_0(\ell)} + \lambda_V$ and $r_2 = \|K\|_{T_0(\ell)} + \lambda_K$. By (9.3.4), and the hypothesis on K , we have $r_1 = O_L(\tilde{g}) + \lambda_V$ and $r_2 = O(\vartheta^{3/2} \tilde{g}^3) + \lambda_K$. We choose $\lambda_V = 1$. Then the hypothesis $r_2 \leq r_1$ of Lemma 9.4.1 applies as long as $\lambda_K \leq 1$, which we also assume. We apply Lemma 9.4.1 and (9.2.7) and obtain

$$\begin{aligned} \|D_V^p D_K^q R_+^U(B)\|_{\mathcal{V}(\ell) \times \mathcal{W} \rightarrow \mathcal{U}_+(\ell_+)} & \leq \lambda_K^{-q} p! q! \|R_+^U(B)\|_{T_{0,y}(\mathfrak{h}_+,\lambda)} \\ & \leq \lambda_K^{-q} p! q! O_L(\vartheta^3 \tilde{g}^3 + \lambda_K). \end{aligned} \quad (9.4.5)$$

We obtain the case $q = 0$ of (9.1.7) by setting $\lambda_K = 0$, and the cases $q = 1, 2$ by setting $\lambda_K = 1$. This completes the proof. $\blacksquare$

The polynomial R_+^U has the relatively simple explicit formula (9.1.5), so it is also possible to compute and estimate the derivatives directly using only the $T_0(\ell)$ -seminorm, and without introduction of the extended norm. This is the subject of the next exercise. Direct computation and estimation of derivatives of K_+ is less straightforward, and the profit from using the extended norm is larger.

Exercise 9.4.2. Compute the derivatives $D_V^p D_K^q R_+^U(V, K; \dot{V}^p, \dot{K}^q)$ explicitly, and use the result to estimate the norms in (9.1.7) directly using only the $T_0(\ell)$ -seminorms. [\[Solution\]](#)

Chapter 10

Bounds on Φ_+^K : Proof of Theorem 8.2.4

In this chapter, we prove Theorem 8.2.4 and, as a byproduct, also Proposition 6.2.2. We also prove the continuity assertion in Theorem 8.2.5. This then completes the proof of Theorem 4.2.1.

10.1 Main result

Our main goal is to prove the estimates of Theorem 8.2.4, which we restate here as follows. In Section 10.6, we verify the continuity assertions of Theorem 8.2.4, and also of Theorem 8.2.5.

Theorem 10.1.1. *Let $\tilde{m}^2 \geq 0$, let L be sufficiently large, let $\tilde{g}$ be sufficiently small (depending on L), and let $p, q \in \mathbb{N}_0$. Let $0 \leq j < N$. There exist L -dependent $C_{\text{RG}}, M_{p,q} > 0$ and $\kappa = O(L^{-2})$ such that the map*

$$\Phi_+^K : \mathbb{D} \times \mathbb{I}_+ \rightarrow \mathcal{W}_+ \quad (10.1.1)$$

satisfies the estimates

$$\|D_V^p D_K^q \Phi_+^K\|_{\mathcal{V}(\ell) \times \mathcal{W} \rightarrow \mathcal{W}_+} \leq \begin{cases} C_{\text{RG}} \tilde{\partial}_+^3 \tilde{g}_+^3 & (p=0, q=0) \\ M_{p,0} \tilde{\partial}_+^3 \tilde{g}_+^{3-p} & (p>0, q=0) \\ \kappa & (p=0, q=1) \\ M_{p,q} \tilde{g}_+^{-p-\frac{9}{4}(q-1)} & (p \geq 0, q \geq 1), \end{cases} \quad (10.1.2)$$

The proof is based on a decomposition of $\Phi_+(V, K)$ as a sum of two contributions, which are constructed as follows. Let $K_+ = \Phi_+^K(V, K)$. With Q defined in (9.1.3), we let

$$\hat{V} = V - Q, \quad \hat{K} = e^{-V} - e^{-\hat{V}} + K, \quad (10.1.3)$$

so that

$$e^{-V} + K = e^{-\hat{V}} + \hat{K}. \quad (10.1.4)$$

By (5.2.32),

$$\begin{aligned} K_+(B) &= e^{u_+|B|} \left(\mathbb{E}_+ \theta \left(e^{-\hat{V}} + \hat{K} \right)^B - e^{-U_+(B)} \right) \\ &= e^{u_+|B|} \left(\sum_{X \subset \mathcal{B}(B)} \mathbb{E}_+ \theta \left(e^{-\hat{V}(B \setminus X)} \hat{K}^X \right) - e^{-U_+(B)} \right). \end{aligned} \quad (10.1.5)$$

We isolate the term with $X = \emptyset$ and thus write $K_+(B)$ as

$$K_+ = S_0 + S_1, \quad (10.1.6)$$

with

$$S_0 = e^{u_+|B|} \left(\mathbb{E}_+ \theta e^{-\hat{V}(B)} - e^{-U_+(B)} \right), \quad (10.1.7)$$

$$S_1 = e^{u_+|B|} \sum_{\substack{X \subset \mathcal{B}(B) \\ |X| \geq 1}} \mathbb{E}_+ \theta \left(e^{-\hat{V}(B \setminus X)} \hat{K}^X \right). \quad (10.1.8)$$

The region X on the right-hand side of (10.1.8) is illustrated in Figure 10.1.

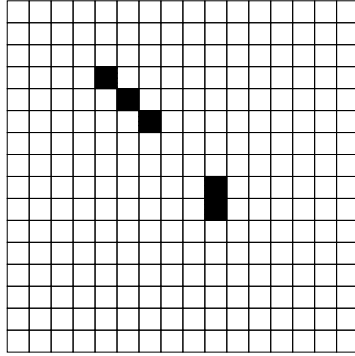


Fig. 10.1 Each block B is divided into L^d blocks b at the previous scale. The small black blocks represent the region X in (10.1.8).

Separate mechanisms are invoked to estimate S_0 and S_1 . The term S_0 will be shown to be third order in $\tilde{g}_+$ due to the fact that $U_+ = \Phi_{\text{pt}}(V - Q)$ (recall (5.2.31)) has been defined in such a manner to achieve this. Indeed, Lemma 5.2.6 implies (here $W_+ = 0$) that

$$\mathbb{E}_+ e^{-\theta \hat{V}(B)} - e^{-U_+(B)} = e^{-U_+(B)} \left(\frac{1}{8} (\text{LocVar}(\theta \hat{V}))^2 + \mathbb{E}_+ A_3(B) \right), \quad (10.1.9)$$

where A_3 (given by (5.2.21) with U replaced by $\hat{V}$) is third order in $\hat{V} - U_+$. The variance term is fourth order in $\hat{V}$. Using these facts, S_0 will be shown to the third order in $\tilde{g}_+$.

The contribution of S_1 to K_+ is small because it contains a factor $\hat{K}(b)$ for at least one small block $b \in B$. If $\hat{K}$ were roughly the same size as K , this would provide a good factor of order g^3 . However, there are L^d small blocks $b \in B$, so this good factor must be multiplied by L^d . Theorem 10.1.1 asserts that the K -derivative of the map $(V, K) \mapsto K_+$ is less than 1, and the naive argument just laid out cannot prove this, it is spoiled by the L^d . Instead, we make use of the crucial fact that the map $K \rightarrow \hat{K}$ on a single small block b has derivative of order L^{-d-2} because the terms $e^{-V} - e^{-\hat{V}}$ in (10.1.3) effectively cancel the relevant and marginal parts of K , leaving behind irrelevant parts that scale down. The good scaling factor L^{-d-2} and the bad entropic factor L^d combine to give L^{-2} . We can choose L large enough so that L^{-2} cancels any dimension-dependent (but L -independent) combinatorial factors that arise in the estimates. Consequently, S_1 remains small enough to prove Theorem 10.1.1.

The details for S_0 and S_1 are presented in the rest of the chapter. An important special case is obtained by setting $K = 0$ in (10.1.7) to yield $S_{0,0}$ defined by

$$S_{0,0} = e^{\mu_{\text{pt}}|B|} \left(\mathbb{E}_+ \theta e^{-V(B)} - e^{-U_{\text{pt}}(B)} \right). \quad (10.1.10)$$

We refer to $S_{0,0}$ as the *perturbative contribution* to K_+ , since it is the contribution when $K = 0$. As discussed below (5.2.3), at the initial scale 0 we have $K_0 = 0$, so for the first application of the renormalisation group map, from scale 0 to 1, $K_1 = \Phi_1^K(V_0, K_0)$ is equal to the perturbative contribution. This can be considered the genesis of K . At subsequent scales, the previous K creates an additional contribution to K_+ . The following proposition gives an estimate on $S_{0,0}$.

Proposition 10.1.2. *For L sufficiently large and $\tilde{g}$ sufficiently small, there is a constant $C_{\text{pt}} = C_{\text{pt}}(L)$ such that for all $V \in \mathcal{D}$ and $B \in \mathcal{B}_+$,*

$$\|S_{0,0}\|_{\mathcal{W}_+} \leq C_{\text{pt}} \vartheta_+^3 \tilde{g}_+^3. \quad (10.1.11)$$

The subscript in the constant C_{pt} in (10.1.11) stands for “perturbation theory.” The constant C_{RG} which appears in the domain $\mathbb{D}$ and in the first estimate of Theorem 10.1.1 is defined by

$$C_{\text{RG}} = C_{\text{RG}}(L) = 2C_{\text{pt}}. \quad (10.1.12)$$

We prove Proposition 10.1.2 as a special case of the following proposition, in which we use the extended norm (see Section 9.2). The definition of the extended norm requires specification of the parameters λ_V and λ_K . Throughout this chapter, we always require

$$\lambda_V \leq \tilde{g}. \quad (10.1.13)$$

For λ_K , we require either

$$\lambda_K \leq \tilde{g} \quad \text{or} \quad \lambda_K \leq \tilde{g}^{9/4}, \quad (10.1.14)$$

with the choice depending on the estimate being proved. For derivatives, we obtain the best estimates by taking the two parameters to be as large as possible. Smaller choices are also useful and permitted, including $\lambda_V = \lambda_K = 0$, which gives the best estimates on the functions themselves.

Proposition 10.1.3 is the main ingredient in the proof of the derivative estimates in (10.1.2), but on its own is not sufficient to prove the cases $(p, q) = (0, 0)$ and $(p, q) = (0, 1)$ with constants C_{RG} and κ in their upper bounds. Those cases are given separate treatment. Note that the constant $\bar{C}$ in (10.1.15) is not the same as C_{RG} in Proposition 10.1.3. The case $(p, q) = (0, 0)$ in (10.1.2) is proved using the crucial contraction, i.e., the case $(p, q) = (0, 1)$ of (10.1.2), and is discussed in detail later in this section.

Proposition 10.1.3. *Let L be sufficiently large and let $\tilde{g}$ be sufficiently small depending on L . Let C_{RG} be given by (10.1.12). Let $\lambda_V \leq \tilde{g}$ and $\lambda_K \leq \tilde{g}^{9/4}$. There is a constant $\bar{C} = \bar{C}(L)$ such that if $(V, K, m^2) \in \mathbb{D} \times \mathbb{I}_+$ then (recall $y = (V, K)$)*

$$\|K_+\|_{\mathcal{W}_{y,+}(\lambda)} \leq \bar{C}(\vartheta_+^3 \tilde{g}_+^3 + \lambda_K). \quad (10.1.15)$$

The next two lemmas give estimates on S_0 and S_1 . The lemmas are proved in Sections 10.3 and 10.4, respectively.

Lemma 10.1.4. *Let $V \in \mathcal{D}$ and $\|K(b)\|_{T_0(\ell)} \leq \tilde{g}$. If $\lambda_V \leq \tilde{g}$ and $\lambda_K \leq \tilde{g}$ then*

$$\|S_0\|_{\mathcal{W}_{y,+}(\lambda)} \leq O_L(\vartheta_+^3 \tilde{g}_+^3). \quad (10.1.16)$$

Lemma 10.1.5. *Let $(V, K) \in \mathbb{D}$. If $\lambda_V \leq \tilde{g}$ and $\lambda_K \leq \tilde{g}^{9/4}$ then*

$$\|S_1\|_{\mathcal{W}_{y,+}(\lambda)} \leq O_L(\vartheta_+^3 \tilde{g}_+^3 + \lambda_K). \quad (10.1.17)$$

Proof of Proposition 10.1.2. We take $K = 0$ in (10.1.6) so that $K_+ = S_{0,0}$ with $S_{0,0}$ given by (10.1.10). Taking $\lambda_V = \lambda_K = 0$ in Lemma 10.1.4, we get the estimate $\|S_{0,0}(V)\|_{T_\varphi(\mathfrak{h}_+)} \leq \bar{C}' \tilde{e}_+^3$. The constant $\bar{C}'$ does not depend on C_{RG} , because C_{RG} serves only to provide a limitation on the size of K , and we have set $K = 0$. Thus we obtain (10.1.11) with $C_{\text{pt}} = \bar{C}'$. ■

Proof of Proposition 10.1.3. The bound (10.1.15) is an immediate consequence of Lemmas 10.1.4–10.1.5, together with the decomposition $K_+ = S_0 + S_1$ of (10.1.6). ■

Proof of (10.1.2) except cases $(p, q) = (0, 0)$ and $(p, q) = (0, 1)$. We fix $(V, K, m^2) \in \mathbb{D} \times \mathbb{I}_+$. From (10.1.15) and Lemma 9.2.1 we have

$$\|D_V^p D_K^q K_+\|_{\mathcal{V}(\ell) \times \mathcal{W} \rightarrow \mathcal{W}_+} \leq \frac{p!q!}{\lambda_V^p \lambda_K^q} O_L(\vartheta_+^3 \tilde{g}_+^3 + \lambda_K). \quad (10.1.18)$$

For the case $p \geq 1$ with $q = 0$, we set $\lambda_K = 0$ and take equality for λ_V in (10.1.13), to get the desired result. For $(p, q) = (0, 0)$, the choice $\lambda_V = 0$ could be used, but this gives an upper bound with constant $2\tilde{C}$ rather than the required C_{RG} ; this case is discussed in the next proof.

For the case $p \geq 0$ with $q \geq 1$, we take equality for λ_V in (10.1.13) and set $\lambda_K = \tilde{g}^{9/4}$. This gives the desired result except for $(p, q) = (0, 1)$, where here we see an upper bound $2\tilde{C}(\vartheta^3 \tilde{g}^{3/4} + 1)$ rather than $\kappa = O(L^{-2})$; the bound with κ is the crucial contraction whose proof is given in Section 10.5. This completes the proof. ■

Proof of case $(p, q) = (0, 0)$ in (10.1.2). This proof uses the $\mathcal{W}$ -norm of (8.2.9), and not the extended $\mathcal{W}(\lambda)$ -norm of (9.2.6).

Let $f(t) = \Phi_+^K(V, tK)$. We apply Taylor's Theorem to f , with integral form of the remainder, and obtain

$$\Phi_+^K(V, K) = \Phi_+^K(V, 0) + D_K \Phi_+^K(V, 0; K) + R_+^K(V, K) \quad (10.1.19)$$

with

$$R_+^K(V, K) = \int_0^1 (1-t) \frac{d^2}{dt^2} \Phi_+(V, tK) dt. \quad (10.1.20)$$

By the triangle inequality,

$$\|\Phi_+^K(V, K)\|_{\mathcal{W}_+} = \|\Phi_+^K(V, 0)\|_{\mathcal{W}_+} + \|D_K \Phi_+^K(V, 0; K)\|_{\mathcal{W}_+} + \|R_+^K(V, K)\|_{\mathcal{W}_+}. \quad (10.1.21)$$

By Proposition 10.1.2,

$$\|\Phi_+^K(V, 0)\|_{\mathcal{W}_+} \leq 2C_{\text{pt}} \vartheta_+^3 \tilde{g}_+^3. \quad (10.1.22)$$

By the crucial contraction with $\kappa \leq c_\kappa L^{-2}$, (the case $(p, q) = (0, 1)$ of (10.1.2) proved in Section 10.5), and by the assumption that $\|K\|_{\mathcal{W}} \leq C_{\text{RG}} \vartheta^3 \tilde{g}^3$, we have

$$\|D_K \Phi_+^K(V, 0; K)\|_{\mathcal{W}_+} \leq \kappa C_{\text{RG}} \vartheta^3 \tilde{g}^3 \leq c_\kappa L^{-2} 2C_{\text{pt}} \vartheta^3 \tilde{g}^3. \quad (10.1.23)$$

Since $\vartheta \leq 2\vartheta_+$ and $\tilde{g} \leq 2\tilde{g}_+$, we can choose L so that $L^2 \geq 2 \cdot 2 \cdot 2^6 c_\kappa$ to conclude that

$$\|D_K \Phi_+^K(V, 0; K)\|_{\mathcal{W}_+} \leq \frac{1}{2} C_{\text{pt}} \vartheta_+^3 \tilde{g}_+^3. \quad (10.1.24)$$

By the case $(p, q) = (0, 2)$ of (10.1.2), for $\tilde{g}$ chosen sufficiently small depending on L to ensure that $M_{0,2}(2C_{\text{pt}}2^6)^2 \tilde{g}^{3/4} \leq \frac{1}{2} C_{\text{pt}}$, we also have

$$\|R_+^K(V, K)\|_{\mathcal{W}_+} \leq M_{0,2} \tilde{g}_+^{-9/4} (C_{\text{RG}} \vartheta^3 \tilde{g}^3)^2 \leq \frac{1}{2} C_{\text{pt}} \vartheta_+^3 \tilde{g}_+^3. \quad (10.1.25)$$

This gives the desired result, with $C_{\text{RG}} = 2C_{\text{pt}}$ as in (10.1.12). ■

It remains now to prove Lemmas 10.1.4–10.1.5, as well as the case $(p, q) = (0, 1)$ of (10.1.2). We do this in the remainder of the chapter.

10.2 Stability

This section is concerned with a collection of estimates which together go by the name of *stability* estimates. The domain $\mathcal{D}$ for $V = g\tau^2 + v\tau$ permits negative values of the coupling constant v , as it must in order to approach the critical value, which is itself negative. Thus, V can have a double well shape for $n = 1$, and a Mexican-hat shape for $n > 1$, so in $e^{-V(b)}$ there is a growing exponential factor $e^{-v\tau(b)}$ which must be compensated, or *stabilised*, by the decaying factor $e^{-g\tau^2(b)}$. Moreover, it is not only the value of $e^{-V(b)}$ itself that must be controlled, but also its derivatives with respect to the field. For this, we use the $T_\varphi(h)$ -seminorm with $h = k_0\tilde{g}^{-1/4}L^{-dj/4}$ given by (8.1.7).

Recall the definition of the domain $\mathcal{D}^{\text{st}}$ in (8.1.8). The definition guarantees that for $V = (g, v, 0) \in \mathcal{D}^{\text{st}}$ the stability estimate (8.1.10) holds. This is an estimate for $e^{-V(b)}$ pointwise in φ . In this section, we extend this estimate to an estimate for $T_{\varphi, y}$ -norms and also consider more general expressions than $e^{-V(b)}$. The stability domain is useful because, although it is not the case that $\mathcal{D}$ is contained in $\mathcal{D}_+$, according to Lemma 8.2.1 we do have $\mathcal{D} \subset \mathcal{D}^{\text{st}} \cap \mathcal{D}_+^{\text{st}}$. Therefore a hypothesis that $V \in \mathcal{D}$ ensures stability at both scales. This fact is used, e.g., in the proof of Lemma 10.2.3.

The following proposition is fundamental. It contains a hypothesis on the constant k_0 that appears in the definition (8.1.7) of the large-field scale h_j and in the definition (8.1.8) of the stability domain $\mathcal{D}^{\text{st}}$. Henceforth, we fix k_0 so that the conclusions of Proposition 10.2.1 hold; we also require that $k_0 \leq \frac{1}{24(n+2)}$ as in (8.3.4). The statement of the proposition involves the constant c^{st} defined by

$$c^{\text{st}} = \frac{1}{128}k_0^5. \quad (10.2.1)$$

By Exercise 8.1.1, k_0^5 is the best possible order in c^{st} , because the $T_{\varphi, y}$ norm dominates the absolute value.

Proposition 10.2.1. *For $k_0 > 0$ sufficiently small, $V = (g, v, 0) \in \mathcal{D}^{\text{st}}$, $\mathfrak{h} \leq h$, $t \geq 0$, and $\lambda_V \leq \tilde{g}$,*

$$\|e^{-tV^*(b)}\|_{T_{\varphi, y}(\mathfrak{h}, \lambda)} \leq 2^{t/8} e^{-8tc^{\text{st}}|\frac{\varphi}{h}|^4}. \quad (10.2.2)$$

Proof. The $T_{\varphi, y}(\mathfrak{h}, \lambda)$ -seminorm is monotone in $\mathfrak{h}$, so it suffices to consider $\mathfrak{h} = h$. By the product property and Lemma 7.4.1,

$$\|e^{-tV^*(b)}\|_{T_{\varphi, y}} \leq e^{-2tg\tau^2(b)} e^{t\|(g\tau^2)^*(b)\|_{T_{\varphi, y}}} e^{t\|(v\tau)^*(b)\|_{T_{\varphi, y}}}. \quad (10.2.3)$$

We write the total exponent on the right-hand side of (10.2.3) as tX , and set $s = |\varphi|/h$. Then it suffices to show that

$$X \leq \frac{1}{8} \log 2 - 8c^{\text{st}} s^4. \quad (10.2.4)$$

Recall that $\tau = \frac{1}{2}|\varphi|^2$. We estimate the $T_{\varphi,y}$ -norms in X by $T_{0,y}$ -norms using Exercise 7.5.2, and then apply Lemma 9.3.2 to bound the $T_{0,y}$ -norms. The result is

$$\begin{aligned} X &\leq -\frac{2}{4} g s^4 h^4 L^{dj} + \frac{3}{8} \frac{g}{g} k_0^4 (1+s)^4 + k_0^4 (1+s)^2 \\ &= -\frac{1}{2} \frac{g}{g} k_0^4 s^4 + \frac{3}{8} \frac{g}{g} k_0^4 (1+s)^4 + k_0^4 (1+s)^2, \end{aligned} \quad (10.2.5)$$

where we used the definition (8.1.7) of h in the first term. We split $-\frac{1}{2}s^4$ in the first term into $-\frac{1}{16}s^4$ and $-\frac{7}{16}s^4$ and obtain

$$\begin{aligned} X &\leq -\frac{1}{16} k_0^4 \frac{g}{g} s^4 + k_0^4 \frac{g}{g} \left(-\frac{7}{16} s^4 + \frac{3}{8} (1+s)^4 + \frac{g}{g} (1+s)^2 \right) \\ &\leq -\frac{1}{16} k_0^5 s^4 + k_0^3 \max \left(-\frac{7}{16} s^4 + \frac{3}{8} (1+s)^4 + k_0^{-1} (1+s)^2 \right), \end{aligned} \quad (10.2.6)$$

where we used the bounds on g given by $\mathcal{D}^{\text{st}}$ in (8.1.8). The maximum is positive and is $O(k_0^{-2})$ as $k_0 \downarrow 0$ so $X \leq -\frac{1}{16} k_0^5 s^4 + O(k_0)$, which is the same as $X \leq O(k_0) - 8c^{\text{st}} s^4$ by the definition (10.2.1) of c^{st} . Therefore there exists sufficiently small k_0 such that (10.2.4) holds. The proof is complete. ■

The next lemma gives an estimate for an extended T_φ -norm of $e^{tQ(B)}$. Since φ is a constant field, we have $Q(b) = L^{-d} Q(B)$, so the choice $t = L^{-d}$ gives a bound on the norm of $e^{Q(b)}$. The situation is similar in subsequent lemmas.

Lemma 10.2.2. *Let $\lambda_V \leq \tilde{g}$ and $\lambda_K \leq \tilde{g}$. Let $\mathfrak{h}_+ \leq h_+$ and $t \geq 0$. Let $V \in \mathcal{D}^{\text{st}}$ and $\|K(b)\|_{T_0(\ell)} \leq \tilde{g}$. Then, for L sufficiently large,*

$$\|e^{tQ(B)}\|_{T_{\varphi,y}(\mathfrak{h}_+,\lambda)} \leq 2^{t/8} e^{\frac{1}{8} t c^{\text{st}} |\frac{\varphi}{h_+}|^4}. \quad (10.2.7)$$

Proof. It is sufficient to set $\mathfrak{h} = h_+$. By the product property

$$\|e^{tQ(B)}\|_{T_{\varphi,y}(h_+,\lambda)} \leq e^{t\|Q(B)\|_{T_{\varphi,y}(h_+,\lambda)}}. \quad (10.2.8)$$

By (9.3.21) and the inequality $P_{h_+}^4(\varphi) \leq 2^4(1 + |\frac{\varphi}{h_+}|^4)$, the norm in the exponent on the right-hand side is bounded above by $O(\ell_0^{-4})(1 + |\frac{\varphi}{h_+}|^4)$. Since $\ell_0 \rightarrow \infty$ as $L \rightarrow \infty$ by the definition of ℓ_0 in (8.1.3), the prefactor on the right-hand side of (9.3.21) can be made as small as we wish, and the desired result follows. ■

For the statement of the remaining results we single out the following hypotheses

$$\lambda_V \leq \tilde{g}, \quad \lambda_K \leq \tilde{g}, \quad \|K(b)\|_{T_0(\ell)} \leq \tilde{g}. \quad (10.2.9)$$

Lemma 10.2.3. *Let $\mathfrak{h}_+ \leq h_+$, $t \geq 0$, $0 \leq s \leq 1$, $V \in \mathcal{D}$, and assume (10.2.9). For L sufficiently large,*

$$\|e^{-t(V^*-sQ)(B)}\|_{T_{\varphi,y}(\mathfrak{h}_+,\lambda)} \leq 2^{t/4} e^{-4tc^{\text{st}}|\frac{\varphi}{h_+}|^4}. \quad (10.2.10)$$

Proof. Since $V \in \mathcal{D}$, by Lemma 8.2.1, we have $V \in \mathcal{D}_+^{\text{st}}$ and we can apply Proposition 10.2.1 at the next scale. The claim follows by multiplying the estimates of Proposition 10.2.1 at the next scale and Lemma 10.2.2. ■

Lemma 10.2.4. *Let $\mathfrak{h}_+ \leq h_+$, $t \geq 0$, $V \in \mathcal{D}$, and assume (10.2.9). For L sufficiently large,*

$$\|e^{-tU_+(B)}\|_{T_{\varphi,y}(\mathfrak{h}_+,\lambda)} \leq 2^{t/2} e^{-2tc^{\text{st}}|\frac{\varphi}{h_+}|^4}. \quad (10.2.11)$$

The same estimate holds with U_+ replaced by $U_+ - u_+$ on the left-hand side. Furthermore, $e^{t\|u_+(B)\|_{T_{\varphi,y}(\mathfrak{h}_+,\lambda)}} \leq 2^{t/2}$.

Proof. We drop the subscript $T_{\varphi,y}(\mathfrak{h}_+,\lambda)$ from the seminorm in this proof, and by we monotonicity assume that $\mathfrak{h}_+ = h_+$. We write $U_+ = [U_+ - V^*] + V^*$ and apply the product property to conclude that

$$\|e^{-tU_+(B)}\| \leq e^{t\|U_+(B)-V^*(B)\|} \|e^{-tV^*(B)}\|. \quad (10.2.12)$$

We estimate the first factor on the right-hand side using (9.3.33), and the second factor using Proposition 10.2.1 at the next scale. This gives

$$\|e^{-tU_+(B)}\| \leq e^{tO(\ell_0^{-4})P_{h_+}^4(\varphi)} 2^{t/8} e^{-8tc^{\text{st}}|\frac{\varphi}{h_+}|^4}. \quad (10.2.13)$$

To complete the proof, we use $P_{h_+}^4(\varphi) \leq 2^4(1 + |\frac{\varphi}{h_+}|^4)$ and take L large, using $\ell_0 = L^{1+d/2}$ by (8.1.3).

Since $\|U_+(B) - |B|u_+ - V^*(B)\|_{T_{\varphi,y}(\mathfrak{h}_+,\lambda)} \leq \|U_+(B) - V^*(B)\|_{T_{\varphi,y}(\mathfrak{h}_+,\lambda)}$, the same argument shows that bound holds with U_+ replaced by $U_+ - u_+$. Similarly, for the bound involving u_+ , we use that $\|u_+\|_{T_{\varphi,y}(\mathfrak{h}_+,\lambda)} = \|u_+\|_{T_{0,y}(\mathfrak{h}_+,\lambda)} \leq \|U_+ - V\|_{T_{0,y}(\mathfrak{h}_+,\lambda)} \leq O(\ell_0^{-4})$, which implies the claimed bound on $e^{t\|u_+(B)\|_{T_{\varphi,y}(\mathfrak{h}_+,\lambda)}}$. ■

In the following proposition, the fluctuation field ζ is as usual constant on small blocks b , and we write its value on b as ζ_b .

Proposition 10.2.5. *Let $\mathfrak{h}_+ \leq h_+$, $t \in [0, 1]$, $V \in \mathcal{D}$, and assume (10.2.9). For L sufficiently large and c^{st} the constant of (10.2.1),*

$$\|(e^{-U_+(B)-t\delta\hat{V}(B)})_\zeta\|_{T_{\varphi,y}(\mathfrak{h}_+,\lambda)} \leq 2e^{-c^{\text{st}}|\varphi/h_+|^4} + 2e^{-c^{\text{st}}L^{-d}\sum_{b \in \mathcal{B}(B)}|\frac{\varphi+\zeta_b}{h_+}|^4}, \quad (10.2.14)$$

where $\hat{V} = V - Q$ and $\delta\hat{V} = \theta\hat{V} - U_+$. The same estimate holds with U_+ replaced by $U_+ - u_+$ on the left-hand side.

Proof. By monotonicity, we may assume that $\mathfrak{h}_+ = h_+$. It follows from the definition of θ_ζ that

$$(e^{-(1-t)U_+(B)-t\theta\hat{V}(B)})_\zeta = e^{-(1-t)U_+(B)} \prod_{b \in \mathcal{B}(B)} (e^{-t\theta\hat{V}(b)})_\zeta. \quad (10.2.15)$$

By the product property and Lemma 10.2.4,

$$\begin{aligned} & \| (e^{-(1-t)U_+(B)-t\theta\hat{V}(B)})_\zeta \|_{T_{\Phi,Y}(h_+,\lambda)} \\ & \leq \left(2^{1/2} e^{-2c^{\text{st}} |\frac{\Phi}{h_+}|^4} \right)^{(1-t)} \prod_{b \in \mathcal{B}(B)} \| (e^{-t\theta\hat{V}(b)})_\zeta \|_{T_{\Phi,Y}(h_+,\lambda)}. \end{aligned} \quad (10.2.16)$$

By Lemma 10.2.3, with $t = L^{-d}$, we estimate each factor under the product over blocks and multiply the resulting estimate to obtain

$$\begin{aligned} & \| (e^{-(1-t)U_+(B)-t\theta\hat{V}(B)})_\zeta \|_{T_{\Phi,Y}(h_+,\lambda)} \\ & \leq \left(2^{1/2} e^{-2c^{\text{st}} |\frac{\Phi}{h_+}|^4} \right)^{(1-t)} \left(2^{1/4} e^{-4c^{\text{st}} L^{-d} \sum_{b \in \mathcal{B}(B)} |\frac{\Phi + \zeta_b}{h_+}|^4} \right)^t. \end{aligned} \quad (10.2.17)$$

Then we apply the arithmetic mean inequality $a^{1-t}b^t \leq (1-t)a + tb \leq a + b$ to the right-hand side, to obtain the desired inequality. ■

10.3 Bound on S_0 : proof of Lemma 10.1.4

In this section, we prove Lemma 10.1.4. We begin with an estimate for Gaussian integrals that is useful in the proof of Lemma 10.1.4 and is also useful later.

10.3.1 Estimation of Gaussian moments

We exploit the fact that values of ζ significantly larger than ℓ_+ are unlikely, via the existence of high moments implied by the following lemma for powers of P_{ℓ_+} convolved with a quartic exponential factor. Recall from (7.1.14) that $P_{\mathfrak{h}}(t) = 1 + |t|/\mathfrak{h}$.

Lemma 10.3.1. *For $q \geq 0$ there exists $c_2 > 0$ (depending on q) such that for all $\mathfrak{h}_+ \geq \ell_+$ and $0 \leq c_1 \leq \frac{1}{8}L^4$,*

$$\mathbb{E}_+(P_{\ell_+}^q(\zeta_b) e^{-c_1 |\frac{\Phi + \zeta_b}{h_+}|^4}) \leq c_2 e^{-\frac{c_1}{2} |\frac{\Phi}{h_+}|^2}. \quad (10.3.1)$$

Proof. Throughout the proof, we write $\zeta = \zeta_b$. For $t, u, v \in \mathbb{R}^n$, we use the inequalities $|t|^2 \geq |t| - \frac{1}{4}$, $|u+v| \geq ||u| - |v||$, and $2|u||v| \leq \frac{1}{2}|u|^2 + 2|v|^2$, to conclude that

$$\begin{aligned} |u+v|^4 &\geq (|u| - |v|)^2 - \frac{1}{4} \\ &= |u|^2 + |v|^2 - 2|u||v| - \frac{1}{4} \geq \frac{1}{2}|u|^2 - |v|^2 - \frac{1}{4}. \end{aligned} \quad (10.3.2)$$

This gives

$$e^{-c_1 |\frac{\varphi+\zeta}{h_+}|^4} \leq e^{\frac{c_1}{4} - \frac{c_1}{2} |\frac{\varphi}{h_+}|^2} e^{c_1 |\frac{\zeta}{h_+}|^2}. \quad (10.3.3)$$

Since $P_{\ell_+}^q(\zeta) = O_q(e^{c_1 |\frac{\zeta}{\ell_+}|^2})$ and $h_+ \geq \ell_+$,

$$\mathbb{E}_+(P_{\ell_+}^q(\zeta) e^{-c_1 |\frac{\varphi+\zeta}{h_+}|^4}) \leq e^{\frac{c_1}{4} - \frac{c_1}{2} |\frac{\varphi}{h_+}|^2} O_q(\mathbb{E}_+ e^{2c_1 |\frac{\zeta}{\ell_+}|^2}). \quad (10.3.4)$$

To complete the proof it suffices to show that $\mathbb{E}_+ e^{2c_1 |\frac{\zeta}{\ell_+}|^2} \leq 2^{n/2}$. As in (7.6.2), we write $c_+^2 = C_{+,x,x}$. The coefficient $\frac{2c_1}{\ell_+^2}$ of $|\zeta|^2$ is bounded by

$$\frac{2c_1}{\ell_+^2} = \frac{2}{8c_+^2} \frac{8c_1 c_+^2}{\ell_+^2} \leq \frac{2}{8c_+^2} \frac{8c_1 L^2}{\ell_0^2} \leq \frac{2}{8c_+^2}, \quad (10.3.5)$$

where we used (8.1.5) followed by $\ell_0^2 = L^6$ from (8.1.3) and the c_1 hypothesis. Denote by X the first component of $c_+^{-1} \zeta \in \mathbb{R}^n$. Then X is a standard normal variable, so

$$\mathbb{E}_+(e^{\frac{2}{8} X^2}) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} z^2} e^{\frac{2}{8} z^2} dz = \sqrt{2}. \quad (10.3.6)$$

The components of ζ are independent, and hence, by (10.3.5), $\mathbb{E}_+ e^{2c_1 \ell_+^{-2} |\zeta|^2} \leq \mathbb{E}(e^{\frac{2}{8} c_+^{-2} |\zeta|^2}) = 2^{n/2}$ as desired. $\blacksquare$

10.3.2 Bound on S_0

We begin with preparation and explanation and then prove Lemma 10.1.4. We recall from the definition of S_0 in (10.1.7) that

$$S_0 = e^{u_+ |B|} \left(\mathbb{E}_+ \theta e^{-\hat{V}(B)} - e^{-U_+(B)} \right). \quad (10.3.7)$$

By Lemma 5.2.6 applied with $\hat{V}$ in place of V , and using $U_+ = U_{\text{pt}}(\hat{V})$,

$$\mathbb{E} e^{-\theta \hat{V}(B)} = e^{-U_+(B)} \left(1 + \frac{1}{8} (\text{Var}_+ \theta \hat{V}(B))^2 + \mathbb{E} A_3(B) \right), \quad (10.3.8)$$

where, with $\delta\hat{V} = \theta\hat{V}(B) - U_+(B)$,

$$A_3(B) = -\frac{1}{2!}(\delta\hat{V})^3 \int_0^1 e^{-t\delta\hat{V}} (1-t)^2 dt. \quad (10.3.9)$$

Thus

$$S_0 = e^{-U_+(B)+u_+|B|} \left(\frac{1}{8} (\text{Var}_+ \theta\hat{V}(B))^2 + \mathbb{E}A_3(B) \right). \quad (10.3.10)$$

Note that there is a cancellation in the exponent on the right-hand side, namely

$$U_+ - u_+ = g_+ \tau^2 + v_+ \tau. \quad (10.3.11)$$

In the proof of Lemma 10.1.4, the ratio $\mathfrak{c}_+/\mathfrak{h}_+$ occurs when estimating expectations in which some fields have been replaced by their typical values under the fluctuation-field expectation, which is $\mathfrak{c}_+$, rather than giving them size $\mathfrak{h}_+$ through the norm. Recall the definitions $\ell_j = \ell_0 L^{-j}$ and $h_j = k_0 \tilde{g}_j^{-1/4} L^{-j}$ from (8.1.4) and (8.1.7), and recall the inequality $\mathfrak{c}_+ \leq \vartheta_j L^{-j}$ from (8.1.5). These imply that

$$\frac{\mathfrak{c}_+}{\mathfrak{h}_+} \leq \begin{cases} \vartheta & (\mathfrak{h}_+ = \ell_+) \\ \vartheta L k_0^{-1} \tilde{g}_+^{1/4} & (\mathfrak{h}_+ = h_+). \end{cases} \quad (10.3.12)$$

This ratio sometimes occurs multiplied by $\|V(B)\|_{T_{0,y}(\mathfrak{h}_+)}$; this factor is the size of the sum of V over a block B if the field has size $\mathfrak{h}_+$. We define

$$\bar{\varepsilon} = \bar{\varepsilon}(\mathfrak{h}) = \begin{cases} \tilde{\vartheta} \tilde{g} & (\mathfrak{h} = \ell) \\ \tilde{\vartheta} \tilde{g}^{1/4} & (\mathfrak{h} = h). \end{cases} \quad (10.3.13)$$

By Lemma 9.3.4, if $V \in \mathcal{D}$ and $\mathfrak{h}_+ \in \{\ell_+, h_+\}$, then

$$\frac{\mathfrak{c}_+}{\mathfrak{h}_+} \|V^*(B)\|_{T_{0,y}(\mathfrak{h}_+, \lambda)} \leq O_L(\bar{\varepsilon}) \quad (10.3.14)$$

$$\frac{\mathfrak{c}_+}{\mathfrak{h}_+} \|\hat{V}(B)\|_{T_{0,y}(\mathfrak{h}_+, \lambda)} \leq O_L(\bar{\varepsilon}), \quad (10.3.15)$$

where we also assume $\|K(b)\|_{T_0(\ell)} \leq \tilde{g}$ for (10.3.15).

To prove Lemma 10.1.4 it suffices to show that, for $\mathfrak{h}_+ \in \{\ell_+, h_+\}$,

$$\|S_0\|_{T_{\varphi,y}(\mathfrak{h}_+, \lambda)} \leq O_L(\bar{\varepsilon}_+^3(\mathfrak{h})) P_{\mathfrak{h}_+}^{12}(\varphi) e^{-c_L |\varphi/h_+|^2}. \quad (10.3.16)$$

The desired inequality (10.1.16) then follows immediately from (10.3.16) and the definition of the $\mathcal{W}(\lambda)$ -norm in (9.2.6).

Note that h_+ appears in the right-hand side of (10.3.16), regardless of the choice of $\mathfrak{h}_+ \leq h_+$. The inequality (10.3.16) reveals why we need $\mathfrak{h}_+ = h_+$, and why it is not enough to use $\mathfrak{h}_+ = \ell_+$. Indeed, uniformly in $\tilde{g} > 0$ small, the supremum over φ of $P_{\mathfrak{h}_+}^p(\varphi) e^{-c|\varphi/h_+|^2}$ is bounded if $\mathfrak{h}_+ = h_+$, but diverges as $\tilde{g} \rightarrow 0$ if $\mathfrak{h}_+ = \ell_+$. We

control this large field problem with our choice $\mathfrak{h}_+ = h_+$. For $\mathfrak{h}_+ = \ell_+$ we still use the fact that trivially $P_{\mathfrak{h}_+}^p(\varphi)e^{-c|\varphi/h_+|^2} \leq 1$ when $\varphi = 0$.

The following proof relies heavily on our specific choice of the polynomial U_{pt} .

Proof of Lemma 10.1.4. As noted above, it suffices to prove (10.3.16). For this, by (10.3.10) and Lemma 7.5.1, it is enough to prove that there are constants c_L, C_L such that, for $\tilde{g}$ sufficiently small, L sufficiently large, $V \in \mathcal{D}$, $B \in \mathcal{B}_+$, and $\mathfrak{h}_+ \in \{\ell_+, h_+\}$,

$$\|e^{-(U_+ - u_+)(B)}\|_{T_{\varphi, y}(\mathfrak{h}_+, \lambda)} \leq 2e^{-c|\varphi/h_+|^4}, \quad (10.3.17)$$

$$\|\text{Var}_+(\theta\hat{V}(B))\|_{T_{\varphi, y}(\mathfrak{h}_+, \lambda)} \leq C_L \left(\frac{\varepsilon_+}{\mathfrak{h}_+}\right)^2 \bar{\varepsilon}^2 P_{\mathfrak{h}_+}^4(\varphi), \quad (10.3.18)$$

$$\|e^{-(U_+ - u_+)(B)} \mathbb{E}_+ A_3\|_{T_{\varphi, y}(\mathfrak{h}_+, \lambda)} \leq C_L \bar{\varepsilon}^3 P_{\mathfrak{h}_+}^{12}(\varphi) e^{-c_L|\varphi/h_+|^2}. \quad (10.3.19)$$

The following proof of these estimates shows that they also hold with V instead of $\hat{V}$ and U_{pt} instead of U_+ ; in fact, this replacement simplifies the proof.

The inequality (10.3.17) is proved in Lemma 10.2.4. For (10.3.18), Lemma 7.6.1 gives

$$\|\text{Var}_+(\theta\hat{V}(B))\|_{T_{\varphi, y}(\mathfrak{h}_+, \lambda)} \leq c \left(\frac{\varepsilon_+}{\mathfrak{h}_+}\right)^4 |B|^2 \|\hat{V}\|_{T_{0, y}(\mathfrak{h}_+, \lambda)}^2 P_{\mathfrak{h}_+}^4(\varphi), \quad (10.3.20)$$

where we recall that $\text{Var}_+(\theta\hat{V}(B))$ is a degree 4 monomial since $j < N$ and so Loc in (7.6.3) is the identity. Now (10.3.18) follows from (10.3.15).

It remains to prove (10.3.19). We write $\delta\hat{V}$ in place of $\delta\hat{V}(B)$. Starting with the definition (10.3.9) of A_3 , we use $\int_0^1 (1-t)^2 dt = \frac{1}{3}$, followed by (7.3.2), to obtain

$$\begin{aligned} & \|e^{-(U_+ - u_+)(B)} \mathbb{E}_+ A_3\|_{T_{\varphi, y}(\mathfrak{h}_+, \lambda)} \\ & \leq \frac{1}{3!} \sup_{t \in [0, 1]} \|\mathbb{E}_+ \left(\delta\hat{V}^3 e^{-(U_+ - u_+)(B) - t\delta\hat{V}} \right)\|_{T_{\varphi, y}(\mathfrak{h}_+, \lambda)} \\ & \leq \frac{1}{3!} \sup_{t \in [0, 1]} \mathbb{E}_+ \left(\|(\delta\hat{V})_\zeta\|_{T_{\varphi, y}(\mathfrak{h}_+, \lambda)}^3 \|(e^{-(U_+ - u_+)(B) - t\delta\hat{V}})_\zeta\|_{T_{\varphi, y}(\mathfrak{h}_+, \lambda)} \right). \end{aligned} \quad (10.3.21)$$

Since $\|\hat{V}(B)\|_{T_{0, y}(\mathfrak{h}_+, \lambda)} \leq 1$ by Lemma 9.3.4, we can apply Proposition 7.6.4 (with $m = 3$). In the bound of Lemma 9.3.4, the normalised sum over x can be replaced by a normalised sum over b since ζ is constant on small blocks. Together with (10.3.14), we conclude that there is a constant c such that

$$\|\delta\hat{V}_\zeta\|_{T_{\varphi, y}(\mathfrak{h}_+, \lambda)}^3 \leq c \bar{\varepsilon}^3 P_{\mathfrak{h}_+}^{12}(\varphi) \frac{1}{L^d} \sum_{b \in \mathcal{B}(B)} P_{\ell_+}^{12}(\zeta_b). \quad (10.3.22)$$

By (10.2.14),

$$\begin{aligned} \|(e^{-(U_+ - u_+)(B) - t\delta\hat{V}})_\zeta\|_{T_{\varphi, y}(\mathfrak{h}_+, \lambda)} & \leq 2e^{-c^{\text{st}}|\varphi/h_+|^4} + 2e^{-c^{\text{st}}L^{-d} \sum_{b \in \mathcal{B}(B)} |\frac{\varphi + \zeta_b}{h_+}|^4} \\ & \leq 2e^{-c^{\text{st}}|\varphi/h_+|^4} + 2e^{-c^{\text{st}}L^{-d} |\frac{\varphi + \zeta_b}{h_+}|^4}, \end{aligned} \quad (10.3.23)$$

where in the second line b is an arbitrary block in $\mathcal{B}(B)$.

By combining (10.3.21)–(10.3.23), $\|e^{-(U_+ - u_+)(B)} \mathbb{E}_+ A_3\|_{T_{\varphi, y}(\mathfrak{h}_+, \lambda)}$ is bounded by

$$O(\bar{\varepsilon}^3) P_{\mathfrak{h}}^{12}(\varphi) \frac{1}{L^d} \sum_{b \in \mathcal{B}(B)} \left(e^{-c^{\text{st}} |\varphi/h_+|^4} \mathbb{E}_+ \left(P_{\ell_+}^{12}(\zeta_b) \right) + \mathbb{E}_+ \left(P_{\ell_+}^{12}(\zeta_b) e^{-c^{\text{st}} L^{-d} \left| \frac{\varphi + \zeta_b}{h_+} \right|^4} \right) \right). \quad (10.3.24)$$

By (10.3.1) with $c_1 = 0$ the first expectation is at most c_2 ; by (10.3.1) with $c_1 = c^{\text{st}} L^{-d}$ we also bound the second expectation. After these bounds there is no longer any b dependence and the normalised sum drops out. Therefore

$$\|e^{-(U_+ - u_+)(B)} \mathbb{E}_+ A_3\|_{T_{\varphi, y}(\mathfrak{h}_+, \lambda)} \leq O(\bar{\varepsilon}^3) P_{\mathfrak{h}}^{12}(\varphi) \left(e^{-c^{\text{st}} |\varphi/h_+|^4} + e^{-\frac{c^{\text{st}} L^{-d}}{2} |\varphi/h_+|^2} \right). \quad (10.3.25)$$

This implies (10.3.19) and completes the proof. $\blacksquare$

10.4 Bound on S_1 : proof of Lemma 10.1.5

In this section, we prove Lemma 10.1.5. Recall from (10.1.8) that

$$S_1 = e^{u_+ |B|} \sum_{\substack{X \subset \mathcal{B}(B) \\ |X| \geq 1}} \mathbb{E}_+ \theta \left(e^{-\hat{V}(B \setminus X)} \hat{K}^X \right). \quad (10.4.1)$$

Let $(V, K) \in \mathbb{D}$, and recall the hypotheses that $\lambda_V \leq \tilde{g}$ and $\lambda_K \leq \tilde{g}^{9/4}$. We define $\lambda_K(\mathfrak{h}) \leq 1$ by

$$\lambda_K(\mathfrak{h}) = \begin{cases} \lambda_K & (\mathfrak{h} = \ell) \\ \lambda_K \tilde{g}^{-9/4} & (\mathfrak{h} = h). \end{cases} \quad (10.4.2)$$

Recall from (9.2.6) that the extended $\mathcal{W}$ -norm is defined to be $\|K\|_{\mathcal{W}_y(\lambda)} = \|K(b)\|_{T_{0, y}(\ell, \lambda)} + \tilde{g}^{9/4} \|K(b)\|_{T_{\infty, y}(h, \lambda)}$, where $\|K(b)\|_{T_{\infty}(h)} = \sup_{\varphi} \|K(b)\|_{T_{\varphi}(h)}$. To prove Lemma 10.1.5, it suffices to prove that, for $(\varphi, \mathfrak{h}_+)$ equal to either (∞, h_+) or $(0, \ell_+)$,

$$\|S_1\|_{T_{\varphi, y}(\mathfrak{h}_+, \lambda)} \leq O_L(\bar{\varepsilon}_+^3(\mathfrak{h}) + \lambda_K(\mathfrak{h})), \quad (10.4.3)$$

since these two bounds combine to give the desired estimate $\|S_1\|_{\mathcal{W}_+(\lambda)} \leq O_L(\vartheta_+^3 \tilde{g}_+^3)$. The inequality (10.4.3) is an immediate consequence of (10.4.1), the triangle inequality, and the following lemma because there are at most 2^{L^d} terms in the sum in (10.4.1).

Lemma 10.4.1. *For $(\varphi, \mathfrak{h}_+)$ equal to either (∞, h_+) or $(0, \ell_+)$,*

$$\|e^{u_+|B|}\mathbb{E}_+\theta(e^{-\hat{V}(B\setminus X)}\hat{K}^X)\|_{T_{\varphi,y}(\mathfrak{h}_+,\lambda)} \leq O_L(\bar{\varepsilon}_+^3(\mathfrak{h}) + \lambda_K(\mathfrak{h}))^{|X|}. \quad (10.4.4)$$

To prove Lemma 10.4.1, we first develop general estimates relating the norm of an expectation to the expectation of the norm, as well as estimates on $\hat{K}$. Note that it follows exactly as in the proof of (8.2.10) that, for $F \in \mathcal{F}$,

$$\|F\|_{T_{\varphi,y}(\ell,\lambda)} \leq P_\ell^{10}(\varphi)\|F\|_{\mathcal{W}(\lambda)}. \quad (10.4.5)$$

Lemma 10.4.2. *For a family $F(b) \in \mathcal{N}(b)$ where $b \in \mathcal{B}(B)$,*

$$\|\mathbb{E}_+\theta F^B\|_{T_{0,y}(\ell_+,\lambda)} \leq O_L(1) \prod_{b \in \mathcal{B}(B)} \|F(b)\|_{\mathcal{W}_+(\lambda)}, \quad (10.4.6)$$

$$\|\mathbb{E}_+\theta F^B\|_{T_{\infty,y}(h_+,\lambda)} \leq \prod_{b \in \mathcal{B}(B)} \|F(b)\|_{T_{\infty,y}(h_+,\lambda)}. \quad (10.4.7)$$

Proof. By (7.3.3),

$$\|\mathbb{E}_+\theta F^B\|_{T_{\varphi,y}(\mathfrak{h}_+,\lambda)} \leq \mathbb{E}_+ \left(\prod_{b \in \mathcal{B}(B)} \|F(b)\|_{T_{\varphi+\zeta_b,y}(\mathfrak{h}_+,\lambda)} \right). \quad (10.4.8)$$

This immediately implies (10.4.7). With (10.4.5), it also gives

$$\|\mathbb{E}_+\theta F^B\|_{T_{0,y}(\ell_+,\lambda)} \leq O(1) \left(\prod_{b \in \mathcal{B}(B)} \|F(b)\|_{\mathcal{W}_+(\lambda)} \right) \mathbb{E}_+ \left(\prod_{b \in \mathcal{B}(B)} P_{\ell_+}^{10}(\zeta_b) \right). \quad (10.4.9)$$

By Hölder's inequality,

$$\mathbb{E}_+ \left(\prod_{b \in \mathcal{B}(B)} P_{\ell_+}^{10}(\zeta_b) \right) \leq \mathbb{E}_+ P_{\ell_+}^{10L^d}(\zeta_b). \quad (10.4.10)$$

The expectation on the right-hand side is bounded by an L -dependent constant by Lemma 10.3.1. This completes the proof. $\blacksquare$

Lemma 10.4.3. *For $\lambda_K \geq 0$,*

$$\|\hat{K}(b)\|_{T_{\infty,y}(h,\lambda)} \leq O_L(\|K^*(b)\|_{T_{\infty,y}(h,\lambda)}), \quad (10.4.11)$$

$$\|\hat{K}(b)\|_{\mathcal{W}_y(\lambda)} \leq O_L(\|K^*(b)\|_{\mathcal{W}(\lambda)}). \quad (10.4.12)$$

In particular, for $(V, K) \in \mathbb{D}$,

$$\|\hat{K}(b)\|_{T_{\infty,y}(h,\lambda)} \leq O_L(\vartheta^3 \tilde{g}^3 + \lambda_K) \tilde{g}^{-9/4}, \quad (10.4.13)$$

$$\|\hat{K}(b)\|_{\mathcal{W}_y(\lambda)} \leq O_L(\vartheta^3 \tilde{g}^3 + \lambda_K). \quad (10.4.14)$$

Proof. We drop the block b from the notation. By the definition of $\hat{K}$ in (10.1.3),

$$\hat{K} = K + e^{-V} - e^{-V+Q} = K - \int_0^1 Q e^{-V+sQ} ds. \quad (10.4.15)$$

This implies that

$$\|\hat{K}\|_{T_{\varphi,y}(\mathfrak{h},\lambda)} \leq \|K^*\|_{T_{\varphi,y}(\mathfrak{h},\lambda)} + \|Q\|_{T_{\varphi,y}(\mathfrak{h},\lambda)} \sup_{s \in [0,1]} \|e^{-V+sQ}\|_{T_{\varphi,y}(\mathfrak{h},\lambda)}. \quad (10.4.16)$$

By (9.3.4), (9.3.11), (9.3.15) and (9.3.19)

$$\|Q\|_{T_{\varphi,y}(\mathfrak{h},\lambda)} \leq 2P_{\mathfrak{h}}^4(\varphi) \|K^*\|_{T_{0,y}(\mathfrak{h},\lambda)}. \quad (10.4.17)$$

We apply Lemma 10.2.3 to bound the exponential, after making use of the comment above Lemma 10.2.2 which permits it to be applied on a small block by choosing $t = L^{-d}$. Consequently, the product $P_{\mathfrak{h}}^4(\varphi) \|e^{-V+sQ}\|_{T_{\varphi,y}(\mathfrak{h},\lambda)}$ is bounded by $O(1)$ if $\varphi = 0$ and $\mathfrak{h} = \ell$, and uniformly in φ by $O(t^{-1}) = O(L^d)$ if $\mathfrak{h} = h$. Therefore,

$$\|\hat{K}\|_{T_{\infty,y}(h,\lambda)} \leq O(L^d) \|K^*\|_{T_{\infty,y}(h,\lambda)}, \quad \|\hat{K}\|_{T_{0,y}(\ell,\lambda)} \leq O(1) \|K^*\|_{T_{0,y}(\ell,\lambda)}. \quad (10.4.18)$$

This proves (10.4.11) and (10.4.12).

By (9.3.12), (9.3.13) and the definition (8.2.12) of $\mathbb{D}$,

$$\|K^*\|_{T_{0,y}(\ell,\lambda)} \leq C_{\text{RG}} \mathfrak{g}^3 \tilde{\mathfrak{g}}^3 + \lambda_K, \quad (10.4.19)$$

$$\|K^*\|_{T_{\infty,y}(h,\lambda)} \leq C_{\text{RG}} \mathfrak{g}^3 \tilde{\mathfrak{g}}^{3/4} + \lambda_K \tilde{\mathfrak{g}}^{-9/4}. \quad (10.4.20)$$

This implies (10.4.13) and (10.4.14) and completes the proof. $\blacksquare$

Proof of Lemma 10.4.1. Let $J = \mathbb{E}_+ \theta(e^{-\hat{V}(B \setminus X)} \hat{K}^X)$. By the product property followed by Lemma 10.2.4 with $t = 1$,

$$\|e^{u_+|B|} J\|_{T_{\varphi,y}(\mathfrak{h}_+,\lambda)} \leq \|e^{u_+|B|}\|_{T_{\varphi,y}(\mathfrak{h}_+,\lambda)} \|J\|_{T_{\varphi,y}(\mathfrak{h}_+,\lambda)} \leq 2^{1/2} \|J\|_{T_{\varphi,y}(\mathfrak{h}_+,\lambda)}. \quad (10.4.21)$$

Therefore we are reduced to proving that $\|J\|_{T_{\varphi,y}(\mathfrak{h}_+,\lambda)} = O_L(\bar{\mathfrak{E}}_+^3(\mathfrak{h}) + \lambda_K(\mathfrak{h}))^{|X|}$ for the two cases $(\varphi, \mathfrak{h}_+) = (\infty, h_+)$ and $(\varphi, \mathfrak{h}_+) = (0, \ell_+)$. We write $J = \mathbb{E}_+ \theta F^B$ with F defined by

$$F(b) = \begin{cases} \hat{K}(b) & (b \in X), \\ e^{-\hat{V}(b)} & (b \in B \setminus X). \end{cases} \quad (10.4.22)$$

Suppose first that $(\varphi, \mathfrak{h}_+) = (\infty, h_+)$. By (10.4.7),

$$\|J\|_{T_{\infty,y}(h_+,\lambda)} \leq \prod_{b \in \mathcal{B}(B)} \|F(b)\|_{T_{\infty,y}(h_+,\lambda)}. \quad (10.4.23)$$

For $b \in \mathcal{B}(X)$ we bound $F(b)$ using (10.4.13); for $b \in \mathcal{B}(B \setminus X)$ we bound $F(b)$ using Lemma 10.2.3 with $t = L^{-d}$ and $s = 1$. The result is

$$\|F(b)\|_{T_{\infty,y}(h_+,\lambda)} \leq \begin{cases} O_L(\vartheta^3 \tilde{g}^3 + \lambda_K) \tilde{g}^{-9/4} & (b \in X) \\ 2^{L^{-d}/4} & (b \in B \setminus X). \end{cases} \quad (10.4.24)$$

When $(\vartheta^3 \tilde{g}^3 + \lambda_K) \tilde{g}^{-9/4}$ is rewritten in terms of $\lambda_K(\mathfrak{h})$ defined in (10.4.2) and $\bar{\varepsilon}$ defined in (10.3.13), it becomes $\bar{\varepsilon}_+^3(\mathfrak{h}) + \lambda_K(\mathfrak{h})$. Therefore, from (10.4.23), we have $\|J\|_{T_{\infty,y}(h_+,\lambda)} \leq O_L(\bar{\varepsilon}_+^3(\mathfrak{h}) + \lambda_K(\mathfrak{h}))^{|X|}$ as desired.

Suppose now that $(\varphi, \mathfrak{h}_+) = (0, \ell_+)$. By (10.4.6),

$$\|J\|_{T_{0,y}(\ell_+,\lambda)} \leq O_L(1) \prod_{b \in \mathcal{B}(B)} \|F(b)\|_{\mathcal{W}_+(\lambda)}. \quad (10.4.25)$$

For $b \in \mathcal{B}(X)$ we bound $F(b)$ using (10.4.14); for $b \in \mathcal{B}(B \setminus X)$ we bound $F(b) = e^{-\hat{V}(b)}$ using Lemma 10.2.3 with $t = L^{-d}$ and $s = 1$. In more detail, Lemma 10.2.3 bounds the $T_{\infty,y}(h_+,\lambda)$ norm of $e^{-\hat{V}(b)}$ and this is one of the two terms in the definition (9.2.6) of the $\mathcal{W}_+(\lambda)$ -norm. However, the $T_{\varphi,y}(h_+,\lambda)$ -seminorm becomes the $T_{0,y}(h_+,\lambda)$ -seminorm by setting $\varphi = 0$, and the $T_{0,y}(h_+,\lambda)$ -seminorm is larger than the $T_{0,y}(\ell_+,\lambda)$ -seminorm because $h_+ \geq \ell_+$. Therefore Lemma 10.2.3 also bounds the other term in the $\mathcal{W}_+(\lambda)$ -norm. Thus we have

$$\|F(b)\|_{\mathcal{W}_+(\lambda)} \leq \begin{cases} O_L(\vartheta^3 \tilde{g}^3 + \lambda_K) \tilde{g}^{-9/4} & (b \in X) \\ 2^{L^{-d}/4} (1 + \tilde{g}^{9/4}) & (b \in B \setminus X). \end{cases} \quad (10.4.26)$$

By (10.4.6), this implies $\|J\|_{\mathcal{W}_+(\lambda)} \leq O_L(\bar{\varepsilon}_+^3(\mathfrak{h}) + \lambda_K(\mathfrak{h}))^{|X|}$ as desired. $\blacksquare$

10.5 Crucial contraction

Throughout this section, we work with the $T_\varphi(\mathfrak{h})$ -seminorm. In fact, the analysis presented here also applies for the $T_{\varphi,y}(\mathfrak{h}, \lambda)$ -seminorm, but we do not require the more detailed information that it encodes.

The crucial contraction is the $(p, q) = (0, 1)$ case of (10.1.2), which asserts that if $(V, K) \in \mathbb{D}$ then $\|D_K \Phi_+^K\| \leq \kappa$ with $\kappa = O(L^{-2})$. This estimate is the key fact used to prove that K does not grow from one scale to the next as long as (V, K) remains in the renormalisation group domain $\mathbb{D}$. It relies heavily on our specific choice in (5.2.31) of the polynomial $U_+ = \Phi_{\text{pt}}(V - \text{Loc}(e^V K))$ as part of the definition of the renormalisation group map. This choice transfers the growing contributions from K into V where they are dominated by terms that are quadratic in the coupling constants.

Proposition 10.5.1. *Let L be sufficiently large, and let $\tilde{g}$ be sufficiently small depending on L . For $(V, K) \in \mathbb{D}$, the Fréchet derivative of Φ_+^K as a map from $\mathcal{W} \rightarrow \mathcal{W}_+$ at $K = 0$ obeys*

$$\|D_K \Phi_+^K(V, 0)\|_{\mathcal{W} \rightarrow \mathcal{W}_+} \leq \kappa \quad (10.5.1)$$

with $\kappa = O(L^{-2}) < 1$.

As in (10.1.6), we write $K_+ = \Phi_+(V, K)$ and

$$K_+(B) = S_0 + S_1. \quad (10.5.2)$$

The next lemma shows that the K -derivative of S_0 is negligible.

Lemma 10.5.2. *Under the hypotheses of Proposition 10.5.1,*

$$\|D_K S_0\|_{\mathcal{V} \times \mathcal{W} \rightarrow \mathcal{W}_+} \leq O(\vartheta_+^3 \tilde{g}_+^2) \leq O(L^{-2}). \quad (10.5.3)$$

Proof. We apply Lemma 10.1.4 with $\lambda_V = \lambda_K = \tilde{g}$, and obtain

$$\|S_0\|_{\mathcal{W}_+(\lambda)} \leq O_L(\vartheta_+^3 \tilde{g}_+^3). \quad (10.5.4)$$

The desired result then follows immediately from Lemma 9.2.1. $\blacksquare$

Thus the main work in proving the crucial contraction rests with estimation of the K -derivative of S_1 . By the definition of S_1 in (10.1.8),

$$S_1 = e^{u_+|B|} \sum_{b \in B} \mathbb{E}_+ \theta(e^{-\hat{V}(B \setminus b)} \hat{K}(b)) + e^{u_+|B|} \sum_{\substack{X \subset \mathcal{B}(B) \\ |X| \geq 2}} \mathbb{E}_+ \theta(e^{-\hat{V}(B \setminus X)} \hat{K}^X). \quad (10.5.5)$$

For the first term, we write $Q(b) = \text{Loc}(e^{V(b)} K(b))$ as in (9.1.3), and use the definition of $\hat{K}$ in (10.1.3) to obtain

$$\hat{K} = e^{-V} (1 - e^Q + e^V K) = e^{-V} (1 - \text{Loc})(e^V K) + A, \quad (10.5.6)$$

with

$$A(b) = e^{-V(b)} (1 + Q(b) - e^{Q(b)}). \quad (10.5.7)$$

This gives

$$\sum_{b \in B} \mathbb{E}_+ \theta(e^{-\hat{V}(B \setminus b)} \hat{K}(b)) = \mathbb{E}_+ \theta T K(b) + \sum_{b \in B} \mathbb{E}_+ \theta(e^{-\hat{V}(B \setminus b)} A(b)), \quad (10.5.8)$$

with

$$T K = \sum_{b \in \mathcal{B}(B)} \left(e^{-V(B)} (1 - \text{Loc})(e^{V(b)} K(b)) \right). \quad (10.5.9)$$

We write $e^{u_+|B|} = e^{u_{\text{pt}}|B|} + \delta$ with $\delta = e^{u_+|B|} - e^{u_{\text{pt}}|B|}$. Then the above leads to

$$\begin{aligned} S_1 &= (e^{u_{\text{pt}}|B|} + \delta) \mathbb{E}_+ \theta T K + e^{u_+|B|} \sum_{b \in B} \mathbb{E}_+ \theta(e^{-\hat{V}(B \setminus b)} A(b)) \\ &\quad + e^{u_+|B|} \sum_{\substack{X \subset \mathcal{B}(B) \\ |X| \geq 2}} \mathbb{E}_+ \theta(e^{-\hat{V}(B \setminus X)} \hat{K}^X). \end{aligned} \quad (10.5.10)$$

We will show in the proof of Proposition 10.5.1 that the linear term $e^{u_{\text{pt}}|B|}\mathbb{E}_+\theta TK$ on the right-hand side is the Fréchet derivative of S_1 , and that the other terms are error terms.

Before doing so, in Lemma 10.5.3 we obtain an estimate for the norm of the linear operator T . In $TK(b)$ there are a dangerous number $|\mathcal{B}(B)| = L^4$ of terms in the sum over b . Thus, naively, the operator norm of T is not obviously small. On the other hand, the operator $1 - \text{Loc}$ has an important contractive property. According to Definition 5.2.2, $\text{Loc} = \text{Tay}_4$. The contractive property of $1 - \text{Loc}$ is given by Lemma 7.5.3, which asserts that if $F : \mathbb{R}^n \rightarrow \mathbb{R}$ is $O(n)$ -invariant and if $\mathfrak{h}_+ \leq \mathfrak{h}$, then

$$\|(1 - \text{Loc})F\|_{T_\varphi(\mathfrak{h}_+)} \leq 2 \left(\frac{\mathfrak{h}_+}{\mathfrak{h}} \right)^6 P_{\mathfrak{h}_+}^6(\varphi) \sup_{0 \leq t \leq 1} \|F\|_{T_{t\varphi}(\mathfrak{h})}. \quad (10.5.11)$$

The hypothesis that F is $O(n)$ -invariant has been used here to replace $\text{Loc} = \text{Tay}_4$ by Tay_5 , which is possible since $F^{(5)}(0) = 0$. (Since we have made the choice $p_{\mathcal{N}} = \infty$, the hypothesis concerning $p_{\mathcal{N}}$ in Lemma 7.5.3 is certainly satisfied.) We use (10.5.11) in Lemma 10.5.3 to obtain a factor $(\mathfrak{h}_+/\mathfrak{h})^6 = O(L^{-6})$, which more than compensates for the entropic factor L^4 , resulting in an estimate of order L^{-2} for the norm of T .

Lemma 10.5.3. *Let L be sufficiently large, and let $\tilde{g}$ be sufficiently small depending on L . For $V \in \mathcal{D}$ and $\dot{K} \in \mathcal{F}$,*

$$\|T\dot{K}\|_{\mathcal{W}_+} \leq O(L^{-2})\|\dot{K}\|_{\mathcal{W}}. \quad (10.5.12)$$

Proof. It suffices to prove that

$$\|T\dot{K}\|_{T_\infty(h_+)} \leq O(L^{-2})\|\dot{K}\|_{T_\infty(h)}, \quad (10.5.13)$$

$$\|T\dot{K}\|_{T_0(\ell_+)} \leq O(L^{-2})\|\dot{K}\|_{T_0(\ell)}. \quad (10.5.14)$$

By the definition of Loc , and by the $O(n)$ symmetry of V and K , the Taylor expansion of $(1 - \text{Loc})(e^{V(b)}\dot{K}(b))$ starts at order 6. Therefore, the same is true for $e^{-V(b)}(1 - \text{Loc})(e^{V(b)}\dot{K}(b))$. Thus $1 - \text{Loc}$ acts on it as the identity, and

$$\begin{aligned} e^{-V(B)}(1 - \text{Loc})(e^{V(b)}\dot{K}(b)) &= e^{-V(B \setminus b)}e^{-V(b)}(1 - \text{Loc})(e^{V(b)}\dot{K}(b)) \\ &= e^{-V(B \setminus b)}(1 - \text{Loc})e^{-V(b)}(1 - \text{Loc})(e^{V(b)}\dot{K}(b)). \end{aligned} \quad (10.5.15)$$

We insert this equality into the definition (10.5.9) of $T\dot{K}$ and write the result as $T\dot{K}(b) = T_1 + T_2$, where

$$T_1\dot{K} = \sum_{b \in \mathcal{B}(B)} e^{-V(B \setminus b)}(1 - \text{Loc})\dot{K}(b), \quad (10.5.16)$$

$$T_2\dot{K} = - \sum_{b \in \mathcal{B}(B)} e^{-V(B \setminus b)}(1 - \text{Loc})(e^{-V(b)}\dot{Q}(b)), \quad (10.5.17)$$

with $\dot{Q}(b) = \text{Loc}(e^{V(b)}\dot{K}(b))$. The T_1 term comes from the 1 and the T_2 term from the $-\text{Loc}$ in the inner $1 - \text{Loc}$ on the right-hand side of (10.5.15).

Since $\mathfrak{h}_+/\mathfrak{h} = O(L^{-1})$ for both $\mathfrak{h} = \ell$ and $\mathfrak{h} = h$, it follows from (10.5.11) that

$$\|T_1\dot{K}\|_{T_\varphi(\mathfrak{h}_+)} \leq L^d \|e^{-V(B\backslash b)}\|_{T_\varphi(\mathfrak{h}_+)} O(L^{-6}) P_{\mathfrak{h}_+}^6(\varphi) \sup_{0 \leq t \leq 1} \|\dot{K}(b)\|_{T_{t\varphi}(\mathfrak{h})}. \quad (10.5.18)$$

For $\mathfrak{h}_+ = \ell_+$ and $\varphi = 0$, this simplifies to

$$\|T_1\dot{K}\|_{T_0(\ell_+)} \leq O(L^{-2}) \|\dot{K}(b)\|_{T_0(\ell)}, \quad (10.5.19)$$

since the norm of $e^{-V(B\backslash b)}$ is $O(1)$ by Proposition 10.2.1 at the next scale (with $t = 1 - |b|/|B| = 1 - L^{-d}$). For $\mathfrak{h}_+ = h_+$, we again apply Proposition 10.2.1 to conclude that

$$\|e^{-V(B\backslash b)}\|_{T_\varphi(h_+)} P_{h_+}^6(\varphi) = O(1) \quad (10.5.20)$$

uniformly in φ . Therefore,

$$\|T_1\dot{K}\|_{T_\infty(h_+)} \leq O(L^{-2}) \|\dot{K}(b)\|_{T_\infty(h)}. \quad (10.5.21)$$

For T_2 the method is the same once we have verified that $e^{-V(b)}\dot{Q}(b)$ is bounded in $T_\infty(h)$ and in $T_0(\ell)$, which we now do. By (9.3.20) with $\lambda_V = \lambda_K = 0$ and (9.3.4),

$$\|e^{-V(b)}\dot{Q}(b)\|_{T_\varphi(\mathfrak{h})} \leq 2 \|e^{-V(b)}\|_{T_\varphi(\mathfrak{h})} P_{\mathfrak{h}}^4(\varphi) \|\dot{K}(b)\|_{T_0(\mathfrak{h})}. \quad (10.5.22)$$

By Proposition 10.2.1, $\|e^{-V(b)}\|_{T_\varphi(\mathfrak{h})} P_{\mathfrak{h}}^4(\varphi)$ is bounded by a constant uniformly in φ for $\mathfrak{h} = h$, and when $\varphi = 0$ for $\mathfrak{h} = \ell$. Now the bounds

$$\|T_2\dot{K}\|_{T_0(\ell_+)} = O(L^{-2}) \|\dot{K}\|_{T_0(\ell)}, \quad (10.5.23)$$

$$\|T_2\dot{K}\|_{T_\infty(h_+)} = O(L^{-2}) \|\dot{K}\|_{T_\infty(h)} \quad (10.5.24)$$

follow as in the analysis of T_1 . This completes the proof. $\blacksquare$

Proof of Proposition 10.5.1. We use the decomposition $\Phi_+^K = S_0 + S_1$. By (10.5.3), the Fréchet derivative of S_0 obeys

$$\|D_K S_0(V, 0)\|_{T_0(\ell) \times \mathcal{W} \rightarrow \mathcal{W}_+} = O(\tilde{g}^2) = O(L^{-2}). \quad (10.5.25)$$

Thus, it suffices to identify $e^{\mu_{\text{pt}}|B|}\mathbb{E}_+\theta T$ as the Fréchet derivative of S_1 and to prove that it is bounded in norm by $O(L^{-2})$.

We begin with the bound. By Lemma 10.2.4, $e^{\mu_{\text{pt}}|B|} \leq 2$. By Lemma 10.5.3, together with (10.4.6)–(10.4.7),

$$\|\mathbb{E}_+\theta T\dot{K}\|_{T_0(\ell_+)} \leq O(1) \|T\dot{K}\|_{\mathcal{W}_+} \leq O(L^{-2}) \|\dot{K}\|_{\mathcal{W}}, \quad (10.5.26)$$

$$\|\mathbb{E}_+\theta T\dot{K}\|_{T_\infty(h_+)} \leq \|T\dot{K}\|_{T_\infty(h_+)} \leq O(L^{-2}) \|\dot{K}\|_{T_\infty(h)}. \quad (10.5.27)$$

In particular, we have the desired bound

$$\|E_+ \theta T \dot{K}\|_{\mathcal{W}_+} \leq O(L^{-2}) \|\dot{K}\|_{\mathcal{W}}. \quad (10.5.28)$$

It remains to identify $e^{u_{\text{pt}}|B|} \mathbb{E}_+ \theta T$ as the Fréchet derivative of S_1 . For this, it suffices to prove that, for $(V, K) \in \mathbb{D}$,

$$\|S_1(V, K) - e^{u_{\text{pt}}|B|} \mathbb{E}_+ \theta T K\|_{T_0(\ell_+)} = O_L(\|K\|_{\mathcal{W}}^2), \quad (10.5.29)$$

$$\|S_1(V, K) - e^{u_{\text{pt}}|B|} \mathbb{E}_+ \theta T K\|_{T_\infty(h_+)} = O_L(\|K\|_{T_\infty(h)} \|K\|_{\mathcal{W}}). \quad (10.5.30)$$

To prove (10.5.29)–(10.5.30), we will show that the three terms on the right-hand side of the formula (10.5.10) for S_1 involving δ , A , and $|X| \geq 2$ are bounded by the right-hand sides of (10.5.29)–(10.5.30).

The δ term is $\delta \mathbb{E}_+ \theta T K$, with

$$\delta = e^{-u_+|B|} - e^{-u_{\text{pt}}|B|} = e^{-u_{\text{pt}}|B|} O(|u_+ - u_{\text{pt}}||B|). \quad (10.5.31)$$

By Lemma 10.2.4, the factor $e^{-u_{\text{pt}}|B|}$ is bounded by 2, and the factor $|u_+ - u_{\text{pt}}||B|$ is bounded by $O_L(\|K\|_{T_0(\ell)})$, by Lemma 9.4.1. With (10.5.26)–(10.5.27), this shows that the δ term obeys the required estimate.

The term involving A is $e^{u_+|B|} \sum_{b \in B} \mathbb{E}_+ \theta(e^{-\hat{V}(B \setminus b)} A(b))$, with $A(b) = e^{-V(b)}(1 + Q(b) - e^{Q(b)})$. By Taylor's formula,

$$1 + Q(b) - e^{Q(b)} = - \int_0^1 (1-s) Q(b)^2 e^{sQ(b)} ds. \quad (10.5.32)$$

This gives

$$\|A(b)\|_{T_\varphi(\mathfrak{h}_+)} \leq \sup_{s \in [0,1]} \|Q(b)\|_{T_\varphi(\mathfrak{h}_+)}^2 \|e^{-V(b)+sQ(b)}\|_{T_\varphi(\mathfrak{h}_+)}. \quad (10.5.33)$$

By (9.3.19) with $\lambda = 0$ and (9.3.4) to bound Q , and Lemma 10.2.3 with $t = L^{-d}$ to bound the exponential term,

$$\|A\|_{T_\varphi(\mathfrak{h}_+)} \leq 2 \|K(b)\|_{T_0(\mathfrak{h}_+)}^2 P_{\mathfrak{h}_+}^8(\varphi) e^{-c|\varphi/h_+|^4} \leq O_L(\|K(b)\|_{T_0(\mathfrak{h}_+)}^2) \quad (10.5.34)$$

if $\mathfrak{h}_+ = h_+$ or $\varphi = 0$. Also, $\|e^{-\hat{V}(B \setminus b)}\|_{T_\varphi(\mathfrak{h}_+)} \leq O(1)$ by Lemma 10.2.3, and $e^{u_+|B|} \leq 2$ by Lemma 10.2.4. Finally, we apply (10.4.6)–(10.4.7) to estimate the expectation.

The remaining term in (10.5.10) is

$$e^{u_+|B|} \sum_{\substack{X \subset \mathcal{B}(B) \\ |X| \geq 2}} \mathbb{E}_+ \theta(e^{-\hat{V}(B \setminus X)} \hat{K}^X). \quad (10.5.35)$$

In the proof of Lemma 10.4.1, an estimate is given for $T_0(\ell_+)$ - and $T_\infty(h_+)$ -seminorms of the terms in the above sum. These estimates show that the norm of the

sum is dominated by the terms with $|X| = 2$, and these are respectively $O(\|K\|_{\mathcal{W}}^2)$ and $O(\|K\|_{T_\infty(h)}^2)$. This completes the proof. $\blacksquare$

10.6 Continuity in the mass

In this section, we prove the continuity assertions of Theorems 8.2.4–8.2.5, which we restate as the following proposition. With m^2 fixed, the continuity in (V, K) follows from the differentiability in (V, K) , so our main attention is on continuity in the mass parameter m^2 .

In the proposition, the Fréchet derivatives $D_V^p D_K^q R_+^U$ and $D_V^p D_K^q \Phi_+^K$ are multilinear maps defined on directions $\dot{V} \in (T_0(\ell))^p$, $\dot{K} \in \mathcal{W}^q$ and taking values in $T_0(\ell_+)$ for R_+^U and in $\mathcal{W}_+$ for Φ_+^K .

Proposition 10.6.1. *Let $\tilde{m}^2 \geq 0$, let $\tilde{g}$ be sufficiently small (depending on L), and let $p, q \in \mathbb{N}_0$. Let $0 \leq j < N$, and let $p, q \geq 0$. For Φ_+^K , we also assume that L is sufficiently large. The maps $R_+^U : \mathbb{D} \times \mathbb{I}_+ \rightarrow \mathcal{U}_+$ and $\Phi_+^K : \mathbb{D} \times \mathbb{I}_+ \rightarrow \mathcal{W}_+$ and their Fréchet derivatives $D_V^p D_K^q R_+^U$ and $D_V^p D_K^q \Phi_+^K$ are jointly continuous in all arguments $V, K, \dot{V}, \dot{K}$, as well as in $m^2 \in \mathbb{I}_+$.*

The proof of Proposition 10.6.1 uses the following lemma. We use the extended norm in the proof as it controls the Fréchet derivatives as in Lemma 9.2.1.

Lemma 10.6.2. *Let $\tilde{m}^2 \geq 0$. Let $B \in \mathcal{B}_+$ and suppose that $F : \mathbb{D} \times \mathbb{I}_+ \rightarrow \mathcal{N}(B)$ obeys $\|F\|_{T_{\varphi,y}(\mathfrak{h},\lambda)} \leq c_F P_{\mathfrak{h}}^k(\varphi)$ for some $c_F, k \geq 0$. There exists a function $\eta(m^2)$, with $\eta(m^2) \rightarrow 0$ as $m^2 \rightarrow \tilde{m}^2$, such that*

$$\|\mathbb{E}_{C_+(m^2)} \theta F - \mathbb{E}_{C_+(\tilde{m}^2)} \theta F\|_{T_{\varphi,y}(\mathfrak{h},\lambda)} \leq \eta(m^2) c_F P_{\mathfrak{h}}^k(\varphi). \quad (10.6.1)$$

Proof. Let $C = C_+(m^2)$ and $\tilde{C} = C_+(\tilde{m}^2)$. Note that C and $\tilde{C}$ differ only in the multiplicative factor γ_j in (4.1.10). Since $j < N$ this factor γ_j is a continuous function of m^2 , including at $m^2 = 0$. According to the interpretation of Gaussian integration with respect to a positive semi-definite matrix given in (2.1.3), there is a positive definite matrix C' and a subspace Z of $\mathbb{R}^{n_A}$ such that

$$\mathbb{E}_C \theta F(\varphi) = \int_Z F(\varphi + \zeta) p_{C'}(\zeta) d\zeta \quad (10.6.2)$$

with

$$p_{C'}(\zeta) = \det(2\pi C')^{-1/2} e^{-\frac{1}{2}(\zeta, (C')^{-1} \zeta)}. \quad (10.6.3)$$

Let $\tilde{C}'$ be the positive definite matrix that similarly represents $\mathbb{E}_{\tilde{C}}$.

Since $P_{\mathfrak{h}}(\varphi + \zeta) \leq P_{\mathfrak{h}}(\varphi) P_{\mathfrak{h}}(\zeta)$, our assumption on F implies that

$$\|\mathbb{E}_C \theta F - \mathbb{E}_{\tilde{C}} \theta F\|_{T_{\varphi,y}(\mathfrak{h},\lambda)} \leq c_F P_{\mathfrak{h}}^k(\varphi) \int_Z |p_{C'}(\zeta) - p_{\tilde{C}'}(\zeta)| P_{\mathfrak{h}}^k(\zeta) d\zeta. \quad (10.6.4)$$

We define $\eta(m^2)$ to be the integral in the above right-hand side. It goes to zero as $m^2 \rightarrow \tilde{m}^2$ by dominated convergence, since γ_j is continuous. This completes the proof. $\blacksquare$

Proof of Proposition 10.6.1. We write $K_+ = \Phi_+^K(V, K)$. By Theorems 8.2.4–8.2.5, R_+ and K_+ and derivatives are smooth in (V, K) , uniformly in $m^2 \in \mathbb{I}_+$ and in unit directions $\hat{V}, \hat{K}$. To show the desired joint continuity in $(V, K, \hat{V}, \hat{K}, m^2)$, it therefore suffices to show that R_+ , K_+ and their derivatives are continuous in $m^2 \in \mathbb{I}_+$ uniformly in $(V, K) \in \mathbb{D}$. To do so, we will show that R_+ and K_+ are continuous in $m^2 \in \mathbb{I}_+$, uniformly in $y = (V, K)$, where we use the $T_{0,y}(\ell_+, \lambda)$ -norm for R_+ and the $\mathcal{W}_+(\lambda)$ -norm for K_+ . We require that λ satisfy (10.1.13)–(10.1.14). The continuity of the derivatives then follows from Lemma 9.2.1. (Note that although the inverse powers of λ in the bounds of Lemma 9.2.1 may appear dangerous, they do not create trouble because we are merely proving continuity and make no claim on the modulus of continuity.)

We begin with R_+ . By definition,

$$R_+ = \Phi_+^U(V, K) - \Phi_+^U(V, 0) = \Phi_{\text{pt}}(\hat{V}) - \Phi_{\text{pt}}(V), \quad (10.6.5)$$

where $\hat{V} = V - \text{Loc}(e^V K)$, and, as in (5.2.18),

$$\Phi_{\text{pt}}(V; B) = \mathbb{E}_{C_+} \theta V(B) - \frac{1}{2} \mathbb{E}_{C_+} (\theta V(B); \theta V(B)). \quad (10.6.6)$$

The Loc in (5.2.18) plays no role here since, with the hypothesis $j < N$, we have the $c^{(1)} = 0$ hypothesis of Proposition 5.3.5, so Loc is omitted in (10.6.6). Thus to prove the continuity in m^2 of R_+ , it suffices to prove the continuity of $\Phi_{\text{pt}}(\hat{V})$ and of $\Phi_{\text{pt}}(V)$. These are entirely analogous and we therefore only consider $\Phi_{\text{pt}}(\hat{V})$. By Lemma 9.3.4 and (7.5.8),

$$\|\hat{V}(B)\|_{T_{\varphi,y}(\mathfrak{h}_+, \lambda)} \leq P_{\mathfrak{h}}^4(\varphi). \quad (10.6.7)$$

The product property of the norm then implies that the norm of $\hat{V}(B)^2$ is bounded above by $P_{\mathfrak{h}}^8(\varphi)$. By (10.6.6), the continuity of $\Phi_{\text{pt}}(\hat{V})$ in m^2 (in $T_{0,y}(\ell_+, \lambda)$ -norm) then follows from Lemma 10.6.2.

Next, we prove the continuity of K_+ in m^2 , with $\mathcal{W}_+(\lambda)$ -norm. By definition,

$$K_+(B) = e^{u_+|B|} \left(\mathbb{E}_{C_+(m^2)} \theta (e^{-V} + K)^B - e^{-U_+(B)} \right), \quad (10.6.8)$$

where $U_+ = \Phi_+^U(V, K)$. We consider the two cases in the definition of the $\mathcal{W}_+(\lambda)$ -norm separately. That is, we consider the $T_{\infty,y}(h_+, \lambda)$ norm and the $T_{0,y}(\ell_+, \lambda)$ -seminorm. Since both norms satisfy the product property, it suffices to prove the continuity of $e^{u_+|B|}$, $e^{-U_+(B)}$ and of $\mathbb{E}_{C_+(m^2)} \theta (e^{-V} + K)^B$ separately, in both norms.

We first show that $e^{-U_+(B)}$ is continuous; the continuity of $e^{u_+|B|}$ is analogous and we do not enter into its details. We write $U_+ = U_+(B, m^2)$ and $\tilde{U}_+ = U_+(B, \tilde{m}^2)$.

By the Fundamental Theorem of Calculus,

$$e^{-U_+} - e^{-\tilde{U}_+} = \int_0^1 e^{-tU_+ - (1-t)\tilde{U}_+} (U_+ - \tilde{U}_+) dt. \quad (10.6.9)$$

We apply the product property of the norm, and use (10.2.11) to bound the norms of the exponential factors. This gives

$$\begin{aligned} \|e^{-U_+} - e^{-\tilde{U}_+}\|_{T_{\varphi,y}(\mathfrak{h}_+, \lambda)} &\leq \|U_+ - \tilde{U}_+\|_{T_{\varphi,y}(\mathfrak{h}_+, \lambda)} 2^{1/2} e^{-2c^{\text{st}}|\varphi_x/h_+|^4} \\ &\leq \|U_+ - \tilde{U}_+\|_{T_{0,y}(\mathfrak{h}_+, \lambda)} P_{\mathfrak{h}_+}^4(\varphi) 2^{1/2} e^{-2c^{\text{st}}|\varphi_x/h_+|^4}. \end{aligned} \quad (10.6.10)$$

For the $T_{0,y}(\ell_+, \lambda)$ -norm, the φ -dependent factors on the right-hand side are absent, and the continuity then follows from the fact shown earlier in the proof that $U_+ = \Phi_{\text{pt}}(\hat{V})$ is continuous in m^2 when considered as a map into $T_{0,y}(\ell_+, \lambda)$. For the $T_{\infty,y}(h_+, \lambda)$ -norm, we have a uniform bound on the product of the exponential and polynomial factors in the last line, and the norm on the right-hand side goes to zero as $m^2 \rightarrow \tilde{m}^2$ as a consequence of the $\mathfrak{h}_+ = \ell_+$ case and Lemma 7.1.5.

Finally, we prove the continuity of $\mathbb{E}_{C_+(m^2)} \theta(e^{-V} + K)^B$ in m^2 . Let $G = e^{-V} + K$ and $F = G^B$. By Lemma 10.6.2, it suffices to prove that there are constants c_F, k such that, for $h_+ = \ell_+$ and $\mathfrak{h}_+ = h_+$,

$$\|F\|_{T_{\varphi,y}(\mathfrak{h}_+, \lambda)} \leq c_F P_{\mathfrak{h}_+}^k(\varphi). \quad (10.6.11)$$

By the product property, (10.6.11) will follow once we prove that there are constants c_G, m such that

$$\|G\|_{T_{\varphi,y}(\mathfrak{h}_+, \lambda)} \leq c_G P_{\mathfrak{h}_+}^m(\varphi). \quad (10.6.12)$$

By (9.3.12), (9.3.13), (10.2.10), and the assumption $(V, K) \in \mathbb{D}$,

$$\|e^{-V^*(B)}\|_{T_{\infty,y}(\mathfrak{h}_+, \lambda)} \leq 2, \quad \|K^*\|_{T_{\infty,y}(h_+, \lambda)} \leq 2, \quad \|K^*\|_{T_{\varphi,y}(\ell_+, \lambda)} \leq 4P_\ell(\varphi)^{10}, \quad (10.6.13)$$

where for the last inequality we also used (7.5.17). Therefore, by the triangle inequality,

$$\|G\|_{T_{\infty,y}(h_+, \lambda)} \leq 4, \quad \|G\|_{T_{\varphi,y}(\ell_+, \lambda)} \leq 8P_\ell(\varphi)^{10}. \quad (10.6.14)$$

This gives (10.6.12) and completes the proof. $\blacksquare$

10.7 Last renormalisation group step: Proof of Proposition 6.2.2

In this section, we prove Proposition 6.2.2, which accounts for the last renormalisation group step. This last step is given by the map defined in Definition 5.2.11. It

does not change scale, and it does not extract the growing contributions from K as this is unnecessary because the map is not iterated. Until the last step, we have relied on the vanishing of $c^{(1)}$, but the last covariance does not satisfy $c_{\hat{N}}^{(1)} = 0$. The last step therefore involves the additional perturbative contribution $W_{\hat{N}} = -\frac{1}{2}c_{\hat{N}}^{(1)}g_{\hat{N}}^2|\phi|^6$ (recall (5.3.24)).

At scale N there is only one block $B = \Lambda \in \mathcal{B}_N(\Lambda)$, and

$$C_{\hat{N}} = m^{-2}Q_N \quad (10.7.1)$$

with Q_N defined by (4.1.2). Then, by definition, $c_{\hat{N}}^{(1)} = \sum_{x \in B} Q_{N;0x} = m^{-2}$, and by (7.6.2) $\mathfrak{c}_+ = m^{-1}L^{-dN/2}$.

According to Definition 5.2.11, the final renormalisation group map $(V, K) \mapsto (U_{\hat{N}}, K_{\hat{N}})$ is defined by

$$U_{\hat{N}} = U_{\text{pt}}(V) = g_{\hat{N}}\tau^2 + v_{\hat{N}}\tau + u_{\hat{N}}, \quad (10.7.2)$$

$$K_{\hat{N}}(B) = e^{-V_{\hat{N}}(B)} \left(\frac{1}{8} \left(\text{LocVar}_{C_{\hat{N}}}(\theta V) \right)^2 + \mathbb{E}_{C_{\hat{N}}} A_3(B) \right) + e^{u_{\hat{N}}|B|} \mathbb{E}_{C_{\hat{N}}} \theta K(B). \quad (10.7.3)$$

Here, as in (5.2.20) and (5.2.21),

$$\delta V = \theta V - U_{\text{pt}}(V), \quad (10.7.4)$$

$$A_3(B) = \frac{1}{2!} \int_0^1 (-\delta V(B))^3 e^{-t\delta V(B)} (1-t)^2 dt. \quad (10.7.5)$$

By Proposition 5.2.12, provided that the expectations on the right-hand side are well-defined,

$$\mathbb{E}_{C_{\hat{N}}} \left(e^{-\theta V_{\hat{N}}(B)} + \theta K_{\hat{N}}(B) \right) = e^{-u_{\hat{N}}|B|} \left(e^{-V_{\hat{N}}(B)} (1 + W_{\hat{N}}(B)) + K_{\hat{N}}(B) \right). \quad (10.7.6)$$

The following proposition is a restatement of Proposition 6.2.2.

Proposition 10.7.1. *Fix L sufficiently large and $g_0 > 0$ sufficiently small, and suppose that $m^2 L^{2N} \geq 1$. Let $(V_N, K_N) \in \mathbb{D}_N$. Derivatives with respect to v_0 are evaluated at $(m^2, v_0^c(m^2))$.*

(i) *The perturbative part of the last map obeys*

$$g_{\hat{N}} = g_N(1 + O(\vartheta_N g_N)), \quad L^{2N}|v_{\hat{N}}| = O(\vartheta_N g_N), \quad W_{\hat{N}} = -\frac{1}{2}c_{\hat{N}}^{(1)}g_{\hat{N}}^2|\phi|^6, \quad (10.7.7)$$

$$\frac{\partial v_{\hat{N}}}{\partial v_0} = \left(\frac{g_N}{g_0} \right)^{\gamma} (c + O(\vartheta_N g_N)), \quad \frac{\partial g_{\hat{N}}}{\partial v_0} = O \left(L^{2N} g_N^2 \left(\frac{g_N}{g_0} \right)^{\gamma} \right), \quad (10.7.8)$$

with $c = 1 + O(g_0)$ from Theorem 6.2.1.

(ii) *At scale N , the expectations on the right-hand side of (10.7.6) exist, and*

$$|K_{\hat{N}}(0)| + L^{-2N} |D^2 K_{\hat{N}}(0; \mathbb{1}, \mathbb{1})| = O(\vartheta_N^3 g_N^3), \quad (10.7.9)$$

$$L^{-2N} \left| \frac{\partial}{\partial v_0} K_{\hat{N}}(0) \right| + L^{-4N} \left| \frac{\partial}{\partial v_0} D^2 K_{\hat{N}}(0; \mathbb{1}, \mathbb{1}) \right| = O \left(\vartheta_N^3 g_N^2 \left(\frac{g_N}{g_0} \right)^\gamma \right). \quad (10.7.10)$$

Proof. We again use the parameter $\bar{\varepsilon}_N$ defined in (10.3.13), which obeys

$$\bar{\varepsilon}_N = \bar{\varepsilon}_N(\mathfrak{h}) \asymp \begin{cases} \vartheta_N g_N & (\mathfrak{h} = \ell) \\ \vartheta_N g_N^{1/4} & (\mathfrak{h} = h). \end{cases} \quad (10.7.11)$$

(i) The formula for $W_{\hat{N}}$ follows from (5.3.24).

By (10.7.2) and (5.2.18),

$$U_{\hat{N}}(B) = \mathbb{E}_{C_{\hat{N}}} \theta V_N(B) - \frac{1}{2} \text{LocVar}_{C_{\hat{N}}}(\theta V_N(B)). \quad (10.7.12)$$

Fix $\lambda_V \leq g_N$. By Lemma 9.3.4, $\|V_N(B)\|_{T_{0,y}(\mathfrak{h}_N, \lambda)}$ is at most 1 for $\mathfrak{h}_N = h_N$ and is at most $O(g_N)$ for $\mathfrak{h}_N = \ell_N$. Since $N > j_m$ by assumption, we have $(mL^N)^{-1} \leq \vartheta_N$. Therefore, by Lemma 7.6.3 and $\mathfrak{c}_+ = m^{-1}L^{-dN/2}$, there exists $c > 0$ such that

$$\begin{aligned} \|U_{\hat{N}}(B) - V_N(B)\|_{T_{\varphi,y}(\mathfrak{h}, \lambda)} &\leq c \frac{m^{-1}L^{-dN/2}}{\mathfrak{h}_N} \|V_N(B)\|_{T_{0,y}(\mathfrak{h}, \lambda)} P_{\mathfrak{h}}^4(\varphi) \\ &\leq O(\bar{\varepsilon}_N) P_{\mathfrak{h}}^4(\varphi). \end{aligned} \quad (10.7.13)$$

With $\mathfrak{h}_N = \ell_N$ and $\varphi = 0$, this implies in particular that $\mu_{\hat{N}} = \mu_N + O(\vartheta_N g_N)$, from which the estimate on $v_{\hat{N}}$ in (10.7.7) holds because v_N obeys that estimate. For the bound $g_{\hat{N}} = g_N(1 + O(\vartheta_N g_N))$, we observe that g_N is the only contribution to $g_{\hat{N}}$ from $\mathbb{E}_{C_{\hat{N}}} \theta V_N(B)$, so the difference is contained in the covariance term in (10.7.12), and this term obeys the quadratic upper bound (7.6.3). This then gives $g_{\hat{N}} - g_N = O(\vartheta_N g_N^2)$, and the proof of (10.7.7) is complete.

The proof of (10.7.8) follows as in (8.4.24)–(8.4.25) with j replaced by N and $j+1$ replaced by $\hat{N}$; in fact it is easier here because there is no dependence on K_N for $g_{\hat{N}}, v_{\hat{N}}$.

(ii) Fix $\lambda_V \leq g_N$ and let $\lambda_K(\mathfrak{h})$ be given by (10.4.2) with $\lambda_K \leq g_N^{9/4}$. It suffices to prove that for $(\varphi, \mathfrak{h}_N)$ equal to either (∞, h_N) or $(0, \ell_N)$,

$$\|K_{\hat{N}}(B)\|_{T_{\varphi,y}(\mathfrak{h}_N, \lambda)} \leq O(\bar{\varepsilon}_N^3 + \lambda_K(\mathfrak{h})), \quad \|K'_{\hat{N}}(B)\|_{T_{\varphi}(\mathfrak{h}_N)} \leq O(\vartheta_N^3 g_N^2 \mu'_N), \quad (10.7.14)$$

where $K' = \frac{\partial}{\partial v_0} K_{\hat{N}}$. Indeed, this is more than is needed, the case $(\varphi, \mathfrak{h}) = (0, \ell_N)$ suffices as it implies (10.7.9)–(10.7.10) because $\|\mathbb{1}\|_{\Phi_N(\ell_N)} = O(L^N)$ by (8.4.6) (recall (8.4.7)).

According to (10.7.3),

$$K_{\hat{N}}(B) = e^{-V_{\hat{N}}(B)} \left(\frac{1}{8} \left(\text{LocVar}_{C_{\hat{N}}}(\theta V_N) \right)^2 + \mathbb{E}_{C_{\hat{N}}} A_3(B) \right) + e^{u_{\hat{N}}|B|} \mathbb{E}_{C_{\hat{N}}} \theta K_N(B). \quad (10.7.15)$$

To estimate the terms in (10.7.15), we use the bounds, valid for $(\varphi, \mathfrak{h})$ equal to either (∞, h_N) or $(0, \ell_N)$,

$$e^{\|U_{\hat{N}}(B)\|_{T_{\varphi, y}(\mathfrak{h}_N, \lambda)}} \leq 2, \quad (10.7.16)$$

$$\|\text{LocVar}_{C_{\hat{N}}}(\theta V_N(B))\|_{T_{\varphi, y}(\mathfrak{h}_N, \lambda)} \leq O(\tilde{\varepsilon}_N^2) P_{\mathfrak{h}_N}^4(\varphi) \quad (10.7.17)$$

$$\|\mathbb{E}_{C_{\hat{N}}} A_3(B)\|_{T_{\varphi, y}(\mathfrak{h}_N, \lambda)} \leq O(\tilde{\varepsilon}_N^3), \quad (10.7.18)$$

$$\|\mathbb{E}_{C_{\hat{N}}} \theta K_N(B)\|_{T_{\varphi, y}(\mathfrak{h}_N, \lambda)} \leq O(\tilde{\varepsilon}_N^3 + \lambda_K(\mathfrak{h})). \quad (10.7.19)$$

The first three inequalities follow as in the proof of Lemma 10.1.4 in Section 10.3.2 (we also use $\text{Loc} = \text{Tay}_4$ and Lemma 7.5.1 for (10.7.17)), and we omit the details. The inequality (10.7.19) follows as in Lemma 10.4.2 (though here the product over blocks has only one block), together with the norm estimates on K given by (9.3.12)–(9.3.13) and our assumption that K_N lies in the domain.

Finally, to prove the bound (10.7.14) on $K'_{\hat{N}}$ we first use the chain rule to obtain

$$K'_{\hat{N}}(V_N, K_N) = D_V K_{\hat{N}}(V_N, K_N) V'_N + D_K K_{\hat{N}}(V_N, K_N) K'_N. \quad (10.7.20)$$

By Lemma 9.2.1 with $\lambda_V = g_N$ and $\lambda_K = g_N^{9/4}$, this gives

$$\begin{aligned} \|K'_{\hat{N}}\|_{T_{\varphi}(\mathfrak{h}_N)} &\leq (\lambda_V^{-1} \|V'_N\|_{T_0(\ell_N)} + \lambda_K^{-1} \|K'_N\|_{\mathcal{W}_N}) \|K_{\hat{N}}\|_{\mathcal{W}_N} \\ &\leq (g_N^{-1} \|V'_N\|_{T_0(\ell_N)} + g_N^{-9/4} \|K'_N\|_{\mathcal{W}_N}) O(\vartheta_N^3 g_N^3). \end{aligned} \quad (10.7.21)$$

The norms of the derivatives on the right-hand side are respectively bounded in (6.2.10) and (8.4.9), and using these bounds we obtain

$$\|K'_{\hat{N}}\|_{T_{\varphi}(\mathfrak{h}_N)} \leq (g_N^{-1} \mu'_N + g_N^{-9/4} \mu'_N g_N^2) O(\vartheta_N^3 g_N^3) = O(\vartheta_N^3 g_N^2 \mu'_N). \quad (10.7.22)$$

This completes the proof. ■

Remark 10.7.2. We emphasise that the first inequality of (10.7.14) proves a stronger result than is required for (10.7.9). Indeed, for the standard norm it implies that, for $(\varphi, \mathfrak{h})$ equal to either (∞, h_N) or $(0, \ell_N)$,

$$\|K_{\hat{N}}(B)\|_{T_{\varphi}(\mathfrak{h}_N)} \leq O(\tilde{\varepsilon}_N^3). \quad (10.7.23)$$

In particular, (10.7.9) is a consequence of this $T_0(\ell_N)$ estimate.

Part IV

Self-avoiding walk and supersymmetry

Chapter 11

Self-avoiding walk and supersymmetry

A strength of the renormalisation group method presented in this book is that it applies with little modification to models which incorporate fermion fields. This allows, in particular, for a rigorous analysis of a version of the continuous-time weakly self-avoiding walk (also known as the *lattice Edwards model*). The continuous-time weakly self-avoiding walk is predicted to lie in the same universality class as the standard self-avoiding walk. In this chapter, whose results are not used elsewhere in the book, we give an introduction to the continuous-time weakly self-avoiding walk and its representation as a supersymmetric spin system.

We begin in Section 11.1 with a brief discussion of the critical behaviour of the standard self-avoiding walk model, and then introduce the continuous-time weakly self-avoiding walk. Spin systems have been studied for many decades via their random walk representations, and in Section 11.2 we prove the BFS–Dynkin isomorphism theorem that implements this representation. In Section 11.3, we prove that a certain supersymmetric spin system has a representation in terms of the continuous-time weakly self-avoiding walk. In contrast to the usual application of the results of Section 11.2, in which the random walk representation is used to study the spin system, the results of Section 11.3 have been used in reverse. Namely, starting with the continuous-time weakly self-avoiding walk, we use the supersymmetric representation to convert the walk problem to a spin problem. Then the renormalisation group method in this book can be applied to analyse the spin system and thereby yield results about the weakly self-avoiding walk. Finally, in Section 11.4 we expand on the concept of supersymmetry.

11.1 Critical behaviour of self-avoiding walk

Our study of the continuous-time weakly self-avoiding walk is motivated by the standard model of self-avoiding walk, which is a model of discrete-time strictly self-avoiding walk. In Section 11.1.1, we provide some background on the self-avoiding walk. In Section 11.1.2, we discuss continuous-time random walk on $\mathbb{Z}^d$, and then

in Section 11.1.3 we define the continuous-time weakly self-avoiding walk and give examples of results that have been obtained using the renormalisation group method discussed in this book.

11.1.1 Self-avoiding walk

The self-avoiding walk on $\mathbb{Z}^d$ is a well-known and notoriously difficult mathematical model of linear polymer molecules. Further background and details can be found in [121].

Definition 11.1.1. An n -step *self-avoiding walk* is a sequence $\omega : \{0, 1, \dots, n\} \rightarrow \mathbb{Z}^d$ with: $\omega(0) = 0$, $\omega(n) = x$, $|\omega(i+1) - \omega(i)| = 1$, and $\omega(i) \neq \omega(j)$ for all $i \neq j$. We write $\mathcal{S}_n(x)$ for the set of n -step self-avoiding walks on $\mathbb{Z}^d$ from 0 to x , and write $\mathcal{S}_n = \cup_{x \in \mathbb{Z}^d} \mathcal{S}_n(x)$ for the set of n -step self-avoiding walk starting from the origin. We denote the cardinalities of these sets by $c_n(x) = |\mathcal{S}_n(x)|$ and $c_n = |\mathcal{S}_n| = \sum_{x \in \mathbb{Z}^d} c_n(x)$.

We define a probability measure on $\mathcal{S}_n$ by declaring all walks in $\mathcal{S}_n$ to be equally likely, and we write E_n for expectation with respect to uniform measure on $\mathcal{S}_n$. Then each walk has probability c_n^{-1} .

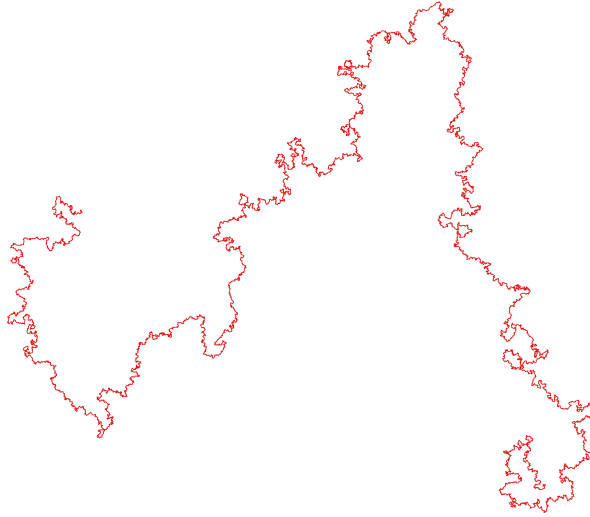


Fig. 11.1 A random 10^8 -step self-avoiding walk on $\mathbb{Z}^2$. Figure by Nathan Clisby (used with permission).

It is easy to see that $c_{m+n} \leq c_m c_n$. From this, it follows from Fekete's subadditivity lemma (see, e.g., [121, Lemma 1.2.2]) that

$$\mu = \mu(d) = \lim_{n \rightarrow \infty} c_n^{1/n} = \inf_{n \geq 1} c_n^{1/n}. \quad (11.1.1)$$

In particular, the limit exists, and $c_n \geq \mu^n$ for all $n \geq 1$. Thus, roughly speaking, c_n grows exponentially with growth rate μ . Crude bounds on the *connective constant* μ are given by the following exercise.

Exercise 11.1.2. For $d \geq 1$, show that $\mu \in [d, 2d - 1]$. [Solution]

The *two-point function* is defined by $G_{0x}(z) = \sum_{n=0}^{\infty} c_n(x) z^n$. Its radius of convergence is $z_c = \mu^{-1}$ for all x [121, Corollary 3.2.6], and z_c plays the role of a critical point for a spin system. It is predicted that there are universal critical exponents γ, ν, η such that

$$c_n \sim A \mu^n n^{\gamma-1}, \quad \mathbb{E}_n |\omega(n)|^2 \sim D n^{2\nu}, \quad G_{0x}(z_c) \sim C |x|^{-(d-2+\eta)}, \quad (11.1.2)$$

with γ, ν, η related by Fisher's relation $\gamma = (2 - \eta)\nu$. Since $|\omega(n)| \leq n$ by definition, it is always the case that $\mathbb{E}_n |\omega(n)|^2 \leq n^2$, so $\nu \leq 1$. Also, (11.1.1) implies that $c_n \geq \mu^n$ for all n , so $\gamma \geq 1$. For simple random walk, without the self-avoidance constraint, the number of n -step walks is $(2d)^n$, the mean-square displacement is equal to n , and the critical two-point function is the lattice Green function which has decay $|x|^{-(d-2)}$ (for $d > 2$). Thus the exponents for simple random walk are $\gamma = 1$, $\nu = \frac{1}{2}$, and $\eta = 0$.

The *susceptibility* and *correlation length* are defined by

$$\chi(z) = \sum_{n=0}^{\infty} c_n z^n = \sum_{x \in \mathbb{Z}^d} G_{0x}(z), \quad \frac{1}{\xi(z)} = - \lim_{n \rightarrow \infty} \frac{1}{n} \log G_{0, ne_1}(z), \quad (11.1.3)$$

and it is predicted that

$$\chi(z) \sim \frac{A'}{(z_c - z)^{\gamma}}, \quad \xi(z) \sim \frac{D'}{(z_c - z)^{\nu}} \quad \text{as } z \uparrow z_c. \quad (11.1.4)$$

The fact that γ appears both for c_n and its generating function $\chi(z)$ is an (in general conjectural) Abelian/Tauberian relation. The fact that the same exponent ν appears both for the mean-square displacement and the correlation length is an example of the general belief that a single critical exponent governs all natural critical length scales. For dimension $d = 4$, logarithmic corrections to simple random walk scaling are predicted [62, 73] (but not for the critical two-point function):

$$c_n \sim A \mu^n (\log n)^{1/4}, \quad \chi(z) \sim \frac{A' |\log(z_c - z)|^{1/4}}{z_c - z}, \quad (11.1.5)$$

$$\mathbb{E}_n |\omega(n)|^2 \sim D n (\log n)^{1/4}, \quad G_{0x}(z_c) \sim C |x|^{-2}. \quad (11.1.6)$$

For $d \leq 4$, very little has been proved. For the end-to-end distance, the best results are the following.

Theorem 11.1.3. [[120] (lower bound), [71] (upper bound)] For all $d \geq 2$,

$$\frac{1}{6}n^{4/3d} \leq \mathbb{E}_n |\omega(n)|^2 \leq o(n^2). \quad (11.1.7)$$

Theorem 11.1.3 can be paraphrased as $\frac{2}{3d} \leq \nu \leq 1^-$. It remains an open problem in dimensions 2, 3, 4 even to prove that $\mathbb{E}_n |\omega(n)|^2 \geq cn$ (i.e., that $\nu \geq \frac{1}{2}$), or that $\mathbb{E}_n |\omega(n)|^2 \leq O(n^{2-\varepsilon})$ for some $\varepsilon > 0$ (i.e., that $\nu < 1$). This lack of proof is in spite of the fact that it seems obvious that self-avoiding walk must move away from the origin at least as rapidly as simple random walk, yet should not move away from the origin with constant speed.

For dimensions $d \geq 5$, the lace expansion has been used to provide a thorough understanding of the critical behaviour. Some principal results are summarised in the following theorem.

Theorem 11.1.4. [105, 107]. *For $d \geq 5$, there are positive constants A, D, C (depending on d) such that*

$$c_n \sim A\mu^n, \quad \mathbb{E}_n |\omega(n)|^2 \sim Dn, \quad G_{0x}(z_c) \sim C|x|^{-(d-2)},$$

and $(\frac{1}{\sqrt{Dn}}\omega(\lfloor nt \rfloor))_{t \geq 0}$ converges in distribution to Brownian motion $(B_t)_{t \geq 0}$.

The above theorem shows that self-avoiding walk behaves like simple random walk when the dimension is above 4, in the sense that both models have $\gamma = 1$, $\nu = \frac{1}{2}$ and $\eta = 0$, and in both cases the scaling limit is Brownian motion. Some indication of the special role of $d = 4$ is provided by Exercise 1.5.6, which shows that the expected number of intersections of two independent simple random walks is finite if and only if $d > 4$. This suggests that elimination of self intersections may not play a big role in the global behaviour when $d > 4$. The proof of Theorem 11.1.4 relies heavily on the fact that the bubble diagram (see Section 1.5.3) is finite in dimensions $d \geq 5$, and indeed that it is not very large for $d = 5$.

For $d = 2$, there is a complete set of predictions: $\gamma = \frac{43}{32}$, $\nu = \frac{3}{4}$, $\eta = \frac{5}{24}$, and that the scaling limit is the Schramm–Loewner Evolution $\text{SLE}_{8/3}$ [115, 127], but none of this has been rigorously proved. For $d = 3$ there are good numerical results, e.g., $\nu = 0.58759700(40)$ [63].

11.1.2 Continuous-time random walk

The definition of the continuous-time weakly self-avoiding walk is based on a continuous-time random walk. We provide background on the latter here. For simplicity, we first consider the case of random walk on a finite set Λ , which may be but need not be a subset of $\mathbb{Z}^d$.

A continuous-time random walk X on Λ can be defined via specification of an *infinitesimal generator*, also called a Q -matrix [128], namely a $\Lambda \times \Lambda$ matrix (Q_{xy}) with the properties that $Q_{xx} < 0$, $Q_{xy} \geq 0$ for $x \neq y$, and $\sum_{y \in \Lambda} Q_{xy} = 0$. Such a random walk takes independent steps from x at rate $-Q_{xx}$, and jumps to y with probability $-\frac{Q_{xy}}{Q_{xx}}$. The statement that steps from x occur at rate $-Q_{xx}$ means that when the

random walk is in state x , it waits a random time σ before taking its next step, where σ has exponential distribution of rate $-Q_{xx}$ (i.e., with mean $-\frac{1}{Q_{xx}}$). The waiting times for each visit to a state are independent of each other and are also independent of all steps taken. The transition probabilities are given in terms of the infinitesimal generator by

$$P_x(X(t) = y) = E_x(\mathbb{1}_{X(t)=y}) = (e^{tQ})_{xy} \quad (t \geq 0). \quad (11.1.8)$$

Here P denotes the probability measure associated with X , and E is the corresponding expectation. The subscripts on P_x and E_x specify that the initial state of the random walk is $X(0) = x$.

Let $\beta = (\beta_{xy})$ be a $\Lambda \times \Lambda$ symmetric matrix with non-negative entries. As in (1.3.4), we define the Laplacian matrix Δ_β by

$$(\Delta_\beta f)_x = \sum_{y \in \Lambda} \beta_{xy} (f_y - f_x). \quad (11.1.9)$$

Equivalently,

$$\Delta_\beta : xy = \begin{cases} \beta_{xy} & (y \neq x) \\ -\sum_{z \in \Lambda : z \neq x} \beta_{xz} & (y = x). \end{cases} \quad (11.1.10)$$

Thus Δ_β is a Q -matrix. We fix β and consider the random walk X with generator Δ_β .

For example, if Δ is the nearest-neighbour Laplace operator on a finite discrete d -dimensional torus Λ approximating $\mathbb{Z}^d$, defined by $\beta_{xy} = \mathbb{1}_{x \sim y}$, then X is the continuous-time stochastic process X on Λ which takes steps uniformly to a nearest-neighbour of its current position, at the times of the events of a rate- $2d$ Poisson process. This follows from the fact that the events of a rate- λ Poisson process are separated by independent exponential random variables with mean $\frac{1}{\lambda}$. In fact, for this choice of β the above definition of the continuous time random walk applies directly also to the case where the state space of the walk is $\mathbb{Z}^d$ rather than a finite torus: at the times of a rate- $2d$ Poisson process the walk steps to a uniformly chosen one of the $2d$ neighbours. We will use this infinite-volume random walk in Section 11.1.3.

For the continuous-time weakly self-avoiding walk, we need two random variables. The first is the *local time of X at $u \in \Lambda$ up to time T* , defined by

$$L_{T,u} = \int_0^T \mathbb{1}_{X(s)=u} ds. \quad (11.1.11)$$

The second is the *self-intersection local time of X up to time T* , defined by

$$I(T) = \sum_{u \in \Lambda} L_{T,u}^2 = \int_0^T \int_0^T \mathbb{1}_{X(s)=X(t)} ds dt. \quad (11.1.12)$$

As its name suggests, $I(T)$ increases with the amount of time that the random walk path spends intersecting itself.

11.1.3 Continuous-time weakly self-avoiding walk

The continuous-time weakly self-avoiding walk is a modification of the self-avoiding walk of Section 11.1.1 in two respects. Firstly, an additional source of randomness is introduced by basing the model on the continuous-time simple random walk on $\mathbb{Z}^d$ whose infinitesimal generator is the standard Laplacian Δ on $\mathbb{Z}^d$, rather than on a discrete-time walk. Secondly, walks with self intersections are not eliminated, but instead receive lower probability. Thus, given $g > 0$ and $\nu \in \mathbb{R}$, we define the *two-point function*

$$G_{0x}(g, \nu) = \int_0^\infty E_0 \left(e^{-gI(T)} \mathbb{1}_{X(T)=x} \right) e^{-\nu T} dT. \quad (11.1.13)$$

In comparison with the two-point function $\sum_{n=0}^\infty c_n(x) z^n$ for the self-avoiding walk, now the integral over T plays the role of the sum over n , the variable z is replaced by $e^{-\nu}$, and $c_n(x)$ is replaced by $E_0(e^{-gI(T)} \mathbb{1}_{X(T)=x})$. This expectation gives positive weight to all walks X , but the factor $e^{-gI(T)}$ assigns reduced weight for self intersections.

The *susceptibility* is defined by

$$\chi(g, \nu) = \sum_{x \in \mathbb{Z}^d} G_{0x}(g, \nu). \quad (11.1.14)$$

A subadditivity argument [20, Lemma A.1] shows that there exists $\nu_c(g) \in (-\infty, 0]$, depending on d , such that

$$\chi(g, \nu) < \infty \quad \text{if and only if } \nu > \nu_c(g). \quad (11.1.15)$$

In particular, $\chi(g, \nu_c) = \infty$.

The continuous-time weakly self-avoiding walk is predicted to be in the same universality class as the strictly self-avoiding walk, for all $g > 0$. In particular, critical exponents and scaling limits are predicted to be the same for both models, including the powers of logarithmic corrections for $d = 4$. The following theorem is an example of this for small $g > 0$.

Theorem 11.1.5. [19, 20]. *Let $d = 4$, and consider the weakly self-avoiding walk on $\mathbb{Z}^4$ defined by the nearest-neighbour Laplacian. For small $g > 0$ and for $\nu = \nu_c + \varepsilon$, as $\varepsilon \downarrow 0$,*

$$\chi(g, \nu) \sim A_g \frac{1}{\varepsilon} (\log \varepsilon^{-1})^{1/4}. \quad (11.1.16)$$

As $|x| \rightarrow \infty$,

$$G_{0x}(g, \nu_c) = \frac{c_g}{|x|^2} \left(1 + O \left(\frac{1}{\log |x|} \right) \right). \quad (11.1.17)$$

As $g \downarrow 0$, the amplitude A_g and critical value obey $A_g \sim (g/2\pi^2)^{1/4}$ and $\nu_c(g) \sim -2N_4 g$ (with $N_4 = (-\Delta)_{00}^{-1}$).

The logarithmic factor for the susceptibility, and the absence of a logarithmic correction for the critical two-point function, are consistent with the predictions for self-avoiding walk in (11.1.5)–(11.1.6). Since the strictly self-avoiding walk corresponds to $g = \infty$ [44], Theorem 11.1.5 shows that the weakly self-avoiding walk demonstrates behaviour like the $g = \infty$ case, not the $g = 0$ case.

Theorem 11.1.5 is quantitatively similar to results for the 4-dimensional n -component $|\phi|^4$ model in Theorems 1.6.1 and 1.6.3. Indeed, (11.1.16) corresponds exactly to (1.6.12) with n replaced by $n = 0$, and the situation is similar for the asymptotic formulas for $A_{g,n}$ and $v_c(g,n)$ in Theorems 1.6.1: with $n = 0$ they give the corresponding results for the continuous-time weakly self-avoiding walk in Theorem 11.1.5. This is an instance of the observation of de Gennes [97] that spins with “ $n = 0$ ” components correspond to self-avoiding walk, which we discuss in more detail in Section 11.3. The “ $n = 0$ ” connection is an important element of the proof of Theorem 11.1.5.

Several extensions of Theorem 11.1.5 have been proved. These include the critical behaviour of the correlation length of order p in dimension 4 [26], the lack of effect of a small contact self-attraction in dimension 4 [27], the construction of the tricritical *theta point* for polymer collapse in dimension 3 [25], and the computation of non-Gaussian critical exponents for a long-range model below the upper critical dimension [118, 143]. In particular, versions of Theorems 1.6.1–1.6.4 have all been proved for the continuous-time weakly self-avoiding walk.

Related and stronger results have been proved for a 4-dimensional hierarchical version of the continuous-time weakly self-avoiding walk [40, 49, 50], including the predicted behaviour $T^{1/2}|\log T|^{1/8}$ for the mean end-to-end distance. This continuous-time weakly self-avoiding walk is defined in terms of the hierarchical random walk of Exercise 4.1.8 via a penalisation of self intersections using the self-interaction local time as in (11.1.13).

A model related to the 4-dimensional weakly self-avoiding walk is studied in [111] via a different renormalisation group approach.

11.2 Random walk representation of spin systems

Random walk representations of integrals arising in mathematical physics have been used for about half a century. Early references include the work of Symanzik [147] in quantum field theory and the work of Fisher on statistical mechanics [84]. Random walk representations have been used extensively in classical statistical mechanics, e.g., in [11, 46, 47, 74, 83]. In this section, we present an important example: the BFS–Dynkin isomorphism [47, 74]. The BFS–Dynkin isomorphism is the foundation upon which a supersymmetric version can be built. The supersymmetric version and its relation to the weakly self-avoiding walk are the topic of Section 11.3.

11.2.1 Continuous-time random walk and the Laplacian

This section is devoted to a special case of the BFS–Dynkin isomorphism, in Lemma 11.2.2. This special case is also a version of the *Feynman–Kac formula*.

Lemma 1.5.3 indicates that the Laplacian and simple random walk are closely related. The next exercise extends Lemma 1.5.3 to more general random walks on a finite set Λ .

Exercise 11.2.1. Let Λ be a finite set. Let $\beta = (\beta_{xy})$ be a symmetric $\Lambda \times \Lambda$ matrix with non-negative entries. Let V be a complex diagonal matrix V with $\operatorname{Re} v_x \geq c > 0$ for all $x \in \Lambda$. Let $\tilde{\beta}_x = \sum_{y \in \Lambda, y \neq x} \beta_{xy}$, and assume that $\tilde{\beta}_x > 0$ for all x . Then

$$(-\Delta_\beta + V)_{xy}^{-1} = \sum_{Y \in \mathcal{W}^*(x,y)} \prod_{i=1}^{|Y|} \beta_{Y_{i-1}Y_i} \prod_{j=0}^{|Y|} \frac{1}{\tilde{\beta}_{Y_j} + v_{Y_j}}, \quad (11.2.1)$$

where $\mathcal{W}^*(x,y)$ consists of the union, over non-negative integers n , of n -step walks $Y = (Y_0, Y_1, \dots, Y_n)$ with $Y_0 = x$, $Y_n = y$, and $Y_{j+1} \neq Y_j$ for each j . For the special case where Λ is a discrete d -dimensional torus and $\Delta_{\beta,xy} = \mathbb{1}_{x \sim y}$, the right-hand side of (11.2.1) gives a finite-volume version of (1.5.17). [Solution]

The following lemma provides a version of the relationship expressed by Exercise 11.2.1, but now in terms of the *continuous-time* random walk X with generator Δ_β . We denote expectation for X with $X(0) = x$ by E_x . Recall that the local time of X at $u \in \Lambda$ up to time $T \geq 0$ is the random variable $L_{T,u}$ given by (11.1.11). Since $\sum_{u \in \Lambda} L_{T,u} = T$, a special case of (11.2.3) is

$$(-\Delta_\beta + m^2)_{xy}^{-1} = \int_0^\infty E_x(\mathbb{1}_{X(T)=y}) e^{-m^2 T} dT. \quad (11.2.2)$$

Lemma 11.2.2. Let Λ be a finite set, and let V be a complex diagonal matrix with rows and columns indexed by Λ , whose elements obey $\operatorname{Re} v_x \geq c > 0$ for some positive c . Then

$$(-\Delta_\beta + V)_{xy}^{-1} = \int_0^\infty E_x(e^{-\sum_u v_u L_{T,u}} \mathbb{1}_{X(T)=y}) dT. \quad (11.2.3)$$

Proof. Let $\tilde{\beta}_x = \sum_{y \neq x} \beta_{xy}$. We can and do regard X as a discrete-time random walk Y whose steps have transition probabilities $p_{xy} = \beta_{xy}/\tilde{\beta}_x$ (for $x \neq y$), which are taken at rate $\tilde{\beta}_x$, as discussed in Section 11.1.2. Thus, at each visit to x , the time σ_x spent at x until the next step is an independent Exponential random variable with mean $1/\tilde{\beta}_x$. Given an n -step walk Y and $j \leq n$, we set $\gamma_j = \sum_{i=0}^j \sigma_{Y_i}$. We also write

$$p(Y) = \prod_{i=1}^n p_{Y_{i-1}Y_i}. \quad (11.2.4)$$

Then the right-hand side of (11.2.3) is equal to

$$\sum_{n=0}^{\infty} \sum_{Y \in \mathcal{W}_n(x,y)} p(Y) \int_0^{\infty} E_0 \left[e^{-\sum_{j=0}^{n-1} v_{Y_j} \sigma_{Y_j}} e^{-v_{Y_n}(t-\gamma_{n-1})} \mathbb{1}_{\gamma_{n-1} < t < \gamma_n} \mid Y \right] dt. \quad (11.2.5)$$

We use Fubini's theorem to interchange the expectation and integral. Since the holding times σ_x are independent of Y , given Y the integral in (11.2.5) is equal to

$$\begin{aligned} & E \left[e^{-\sum_{j=0}^{n-1} v_{Y_j} \sigma_{Y_j}} \int_{\gamma_{n-1}}^{\gamma_n} e^{-v_{Y_n}(t-\gamma_{n-1})} dt \right] \\ &= E \left[\left(e^{-\sum_{j=0}^{n-1} v_{Y_j} \sigma_{Y_j}} \right) \left(-\frac{1}{v_{Y_n}} \right) (e^{-v_{Y_n} \sigma_n} - 1) \right]. \end{aligned} \quad (11.2.6)$$

Since the random variables σ_x are independent, the expectation factors to become

$$\begin{aligned} & \left(\prod_{j=0}^{n-1} E \left[e^{-v_{Y_j} \sigma_{Y_j}} \right] \right) \left(\frac{1}{v_{Y_n}} \right) E \left[1 - e^{-v_{Y_n} \sigma_n} \right] \\ &= \left(\prod_{j=0}^{n-1} \frac{\bar{\beta}_{Y_j}}{\bar{\beta}_{Y_j} + v_{Y_j}} \right) \left(\frac{1}{v_{Y_n}} \right) \left(1 - \frac{\bar{\beta}_{Y_n}}{\bar{\beta}_{Y_n} + v_{Y_n}} \right) \\ &= \left(\prod_{j=0}^{n-1} \frac{\bar{\beta}_{Y_j}}{\bar{\beta}_{Y_j} + v_{Y_j}} \right) \left(\frac{1}{\bar{\beta}_{Y_n} + v_{Y_n}} \right). \end{aligned} \quad (11.2.7)$$

When we substitute this into (11.2.5), and use the definition of $p(Y)$, we find that

$$\int_0^{\infty} E_x \left(e^{-\sum_u v_u L_{T,u}} \mathbb{1}_{X(T)=y} \right) dT = \sum_{n=0}^{\infty} \sum_{Y \in \mathcal{W}_n(x,y)} \prod_{i=1}^{|\omega|} \beta_{\omega_{i-1} \omega_i} \prod_{j=0}^{|\omega|} \frac{1}{\bar{\beta}_{\omega_j} + v_{\omega_j}}. \quad (11.2.8)$$

The above right-hand side is $(-\Delta_{\beta} + V)_{xy}^{-1}$, by Exercise 11.2.1, and the proof is complete. ■

11.2.2 BFS–Dynkin isomorphism

We now prove the *BFS–Dynkin isomorphism* [47, 74], which relates the local time of the continuous-time random walk $X = (X(t))$ with generator Δ_{β} to the n -component GFF specified in terms of the same coupling constants β (see Section 1.5.1).

For an n -component field $(\varphi_x)_{x \in \Lambda}$, we define $(\tau_x)_{x \in \Lambda}$ by

$$\tau_x = \frac{1}{2} |\varphi_x|^2 = \frac{1}{2} (\varphi_x^1 \varphi_x^1 + \cdots + \varphi_x^n \varphi_x^n). \quad (11.2.9)$$

We again write E_x for the expectation when the initial condition is $X(0) = x$, and write $L_T = (L_{T,u})_{u \in \Lambda}$ for the local time field.

The term “isomorphism” is commonly used as an expression of (11.2.12) as the statement that

$$\tau \text{ under the signed measure } \varphi_x^1 \varphi_y^1 \mathbb{P}_{\text{GFF}} \quad (11.2.10)$$

and

$$L + \tau \text{ under the positive measure } P_{xy} \mathbb{P}_{\text{GFF}} \quad (11.2.11)$$

have the same distribution, where P_{xy} is the random walk measure (integrated over T) and $\mathbb{P}_{\text{GFF}}$ is the GFF measure. For a systematic development of the isomorphism theorem and its applications, see [148].

Theorem 11.2.3. *Let $n \geq 1$. Let $F : \mathbb{R}_+^\Lambda \rightarrow \mathbb{R}$ be such that there exists an $\varepsilon > 0$ such that $e^{\varepsilon \sum_{z \in \Lambda} t_z} F(t)$ is a bounded Borel function. Then*

$$\begin{aligned} \int_{\mathbb{R}^{n\Lambda}} e^{-\frac{1}{2}(\varphi, -\Delta_\beta \varphi)} F(\tau) \varphi_x^1 \varphi_y^1 d\varphi \\ = \int_{\mathbb{R}^{n\Lambda}} e^{-\frac{1}{2}(\varphi, -\Delta_\beta \varphi)} \int_0^\infty E_x(F(\tau + L_T) 1_{X(T)=y}) dT d\varphi. \end{aligned} \quad (11.2.12)$$

Proof. We first consider the special case $F(t) = e^{-(v,t)}$ with $\text{Re } v_z > 0$ for all $z \in \Lambda$. In this case, by Lemma 11.2.2, the right-hand side of (11.2.12) is

$$(-\Delta_\beta + V)_{xy}^{-1} \int_{\mathbb{R}^{n\Lambda}} e^{-\frac{1}{2}(\varphi, (-\Delta_\beta + V)\varphi)} d\varphi. \quad (11.2.13)$$

On the other hand, except for a missing normalisation, the left-hand side of (11.2.12) is a Gaussian correlation. By (2.1.12), it is also equal to (11.2.13). This proves (11.2.12) for the special case $F(t) = e^{-(v,t)}$. In the rest of the proof, we reduce the general case to this special case by writing F as a superposition of exponentials using the Fourier inversion theorem.

By hypothesis, $|F(t)| \leq C e^{-\varepsilon \sum_{z \in \Lambda} t_z}$ for some constant C . Therefore the integrands in the left- and right-hand sides of (11.2.12) are integrable by the previous paragraph. Integrability is all that the rest of the proof requires.

By considering the positive and negative parts of F it suffices to consider $F \geq 0$, and by replacing F by its product with a compactly supported characteristic function and using the monotone convergence theorem, we may assume that F has compact support in the quadrant $\mathbb{R}_+^\Lambda$. By extending F by zero outside the quadrant we regard it as a function of compact support in $\mathbb{R}^\Lambda$. By convolving F by a smooth approximate identity of compact support and using dominated convergence, we can further assume that F is smooth and compactly supported in $\mathbb{R}^\Lambda$. Finally, we define a smooth compactly supported function G by $G(t) = F(t) e^{\sum_{z \in \Lambda} t_z}$.

Since $G \in \mathcal{C}_0^\infty$, it is a Schwartz function. Let $\hat{G}(r)$ denote its Fourier transform. By applying the Fourier inversion theorem to G ,

$$F(t) = G(t) e^{-\sum_{z \in \Lambda} t_z} = (2\pi)^{-|\Lambda|} \int_{\mathbb{R}^\Lambda} e^{-\sum_{z \in \Lambda} (1 - ir_z) t_z} \hat{G}(r) dr, \quad (11.2.14)$$

and the integral converges absolutely because $\hat{G}$ is a Schwartz function. By inserting this formula into the left- and right-hand sides of (11.2.12) and bringing the integral over r outside all other integrals, (11.2.12) is reduced to the exponential case established in the first paragraph. The proof is complete. ■

Let $g > 0$ and $v \in \mathbb{R}$. With the choice $F(t) = e^{-\sum_{x \in A} (gt_x^2 + vt_x)}$, the left-hand side of (11.2.12) becomes the (unnormalised) two-point function of the $|\varphi|^4$ model. Thus the right-hand side provides a random walk representation for the $|\varphi|^4$ two-point function. This random walk representation can be a point of departure for the analysis of the $|\varphi|^4$ model and is used, e.g., in [41, 45, 89].

11.3 Supersymmetric representation

In this section, we derive a supersymmetric integral representation of the two-point function (11.1.13) for the continuous-time weakly self-avoiding walk in finite volume. The representation is given in (11.3.34). It involves the introduction of an anti-commuting fermion field, which we present as the differential of the boson field. We provide here a self-contained introduction to the fermion field.

11.3.1 The case $n = 0$

In 1972, de Gennes [97] argued that the self-avoiding walk corresponds to the case “ $n = 0$ ” of an n -component spin model. De Gennes’s observation has been very productive in physics, and leads to predictions for critical exponents for self-avoiding walk by setting $n = 0$ in the n -dependent formulas for the critical exponents of the n -component $|\varphi|^4$. However, it has been much less productive in mathematics, where the notion of a zero-component field raises obvious concerns, and a rigorous link between the critical behaviour of the self-avoiding walk and n -component spins has been elusive. An exception is Theorem 11.1.5 and its related results, where the $n = 0$ connection plays a central role.

As noted already above, the results of Theorem 11.1.5 agree with the result of setting $n = 0$ in Theorem 1.6.1, consistent with de Gennes’s prediction. In fact, the renormalisation group method used to prove Theorems 11.1.5 and 1.6.1 is mainly the same and the proofs are largely simultaneous. Here the correspondence between the self-avoiding walk and $n = 0$ arises from another mechanism. Roughly speaking, this mechanism is based on the observation that an n -component boson field contributes a factor n for every loop, but an n -component fermion (anti-commuting) field contributes $-n$. Combined, all loops cancel. This observation was first made in the physics literature [119, 123, 130], and mathematically rigorous versions are developed in [40, 51, 53, 116]. Applications in this spirit can be found in [18, 20, 126, 144].

Supersymmetric representations have had wider application than just to self-avoiding walks. Linearly reinforced walks are related to spin systems with hyperbolic symmetry. In particular, a relation between supersymmetric hyperbolic sigma models and reinforced walks were found in [136] and a hyperbolic analogue of the BFS–Dynkin isomorphism theorem in [24]. Supersymmetric hyperbolic sigma models have been studied in particular in [67–69]. For further references, see [24].

In the remainder of this chapter, we provide an introduction to supersymmetry and demonstrate the $n = 0$ correspondence, by obtaining a functional integral representation for the continuous-time weakly self-avoiding walk that is a supersymmetric version of the 2-component $|\varphi|^4$ model. The supersymmetric representation places the weakly self-avoiding walk within a similar framework as the $|\varphi|^4$ model, with the important new ingredient that a fermion (anti-commuting) field appears. It is via this framework that we are able to treat the self-avoiding walk as the $n = 0$ version of the $|\varphi|^4$ model.

11.3.2 Integration of differential forms

For our treatment of the fermion field, we require some minimal background on the integration of differential forms, which we discuss now. An elementary introduction to differential forms can be found in [134].

Let Λ be a finite set. For $x \in \Lambda$, let (u_x, v_x) be real coordinates. The 1-forms du_x, dv_x , for $x \in \Lambda$, generate the Grassmann algebra of differential forms on $\mathbb{R}^{2\Lambda}$, with multiplication given by the anti-commuting wedge product. In particular,

$$du_x \wedge du_y = -du_y \wedge du_x, \quad du_x \wedge dv_y = -dv_y \wedge du_x, \quad dv_x \wedge dv_y = -dv_y \wedge dv_x. \quad (11.3.1)$$

It follows that, e.g., $du_x \wedge du_x = 0$.

For $p \geq 0$, a p -form is a function of u, v times a product of p differentials, or any sum of such terms. A *form* K is a sum of p -forms with terms possibly having different values of p . The largest such p is called the *degree* of K , and the p -form contribution to K is called its *degree- p* part. A form which is a sum of p -forms for even p only is called *even*. The wedge product of any form with itself is zero, by anti-commutativity of the product. The *standard volume form* on $\mathbb{R}^{2\Lambda}$ is

$$du_1 \wedge dv_1 \wedge \cdots \wedge du_{|\Lambda|} \wedge dv_{|\Lambda|}, \quad (11.3.2)$$

where $1, \dots, |\Lambda|$ is any fixed enumeration of Λ . Any $2|\Lambda|$ -form K can be written as

$$K = f(u, v) du_1 \wedge dv_1 \wedge \cdots \wedge du_{|\Lambda|} \wedge dv_{|\Lambda|}. \quad (11.3.3)$$

There is no non-zero form of degree greater than $2|\Lambda|$, so degree $2|\Lambda|$ is naturally referred to as *top degree*.

We define the *integral* as a linear map from forms to $\mathbb{R}$, with

$$\int K = \begin{cases} 0 & (\deg K < 2|\Lambda|) \\ \int_{\mathbb{R}^{2\Lambda}} f(u, v) du_1 dv_1 \cdots du_{|\Lambda|} dv_{|\Lambda|} & (K \text{ is a } 2|\Lambda| \text{-form}), \end{cases} \quad (11.3.4)$$

where the integral on the right-hand side is the Lebesgue integral of f over $\mathbb{R}^{2\Lambda}$. It is natural to define the integral to be zero when $\deg K < 2|\Lambda|$, just as we do not give a significance to $\int_{\mathbb{R}^2} f(u_1, v_1) du_1$, but rather require $\int_{\mathbb{R}^2} f(u_1, v_1) du_1 dv_1$ instead. We define a form $f(u, v) du_{x_1} \wedge \cdots \wedge du_{x_k} \wedge dv_{y_1} \wedge \cdots \wedge dv_{y_l}$ to be *integrable* if f is Lebesgue integrable on $\mathbb{R}^{2\Lambda}$. Any form K is a sum of such forms and we define K to be integrable if all terms in this sum are integrable. In particular, the integral (11.3.4) exists.

The above formalism leads to attractive formulas when translated into complex variables. For this, we define

$$\phi_x = u_x + iv_x, \quad \bar{\phi}_x = u_x - iv_x, \quad d\phi_x = du_x + idv_x, \quad d\bar{\phi}_x = du_x - idv_x. \quad (11.3.5)$$

We call $(\phi, \bar{\phi})$ the complex *boson field*. By definition,

$$d\bar{\phi}_x \wedge d\phi_x = 2idu_x \wedge dv_x. \quad (11.3.6)$$

The product

$$\bigwedge_{x \in \Lambda} (d\bar{\phi}_x \wedge d\phi_x) = (2i)^{|\Lambda|} du_1 \wedge dv_1 \wedge \cdots \wedge du_{|\Lambda|} \wedge dv_{|\Lambda|} \quad (11.3.7)$$

defines a top degree form, which we abbreviate as $d\bar{\phi}d\phi$. Thus $d\bar{\phi}d\phi$ is $(2i)^{|\Lambda|}$ times the standard volume form, which becomes the Lebesgue measure under an integral over $\mathbb{R}^{2M}$. The order of the product on the left-hand side of (11.3.7) is unimportant, since each factor is an even form. However a change in the order of the product on the right-hand side may introduce a sign change.

We write (with any fixed choice of the square root)

$$\psi_x = \frac{1}{\sqrt{2\pi i}} d\phi_x, \quad \bar{\psi}_x = \frac{1}{\sqrt{2\pi i}} d\bar{\phi}_x, \quad (11.3.8)$$

and call $(\psi, \bar{\psi})$ the *fermion field*. Then

$$\bar{\psi}_x \wedge \psi_x = \frac{1}{2\pi i} d\bar{\phi}_x \wedge d\phi_x = \frac{1}{\pi} du_x \wedge dv_x. \quad (11.3.9)$$

Given a $\Lambda \times \Lambda$ complex matrix A , we define

$$S_A = \phi A \bar{\phi} + \psi A \bar{\psi} = \sum_{x, y \in \Lambda} (\phi_x A_{xy} \bar{\phi}_y + \psi_x \wedge A_{xy} \bar{\psi}_y). \quad (11.3.10)$$

For $J \in \mathbb{N}$, consider a C^∞ function $F : \mathbb{R}^J \rightarrow \mathbb{C}$. Let $K = (K_j)_{j \leq J}$ be a collection of even forms, and assume that the degree-zero part K_j^0 of each K_j is real. We define a form denoted $F(K)$ by Taylor series about the degree-zero part of K , i.e.,

$$F(K) = \sum_{\alpha} \frac{1}{\alpha!} F^{(\alpha)}(K^0)(K - K^0)^{\alpha}. \quad (11.3.11)$$

Here $\alpha = (\alpha_j)_{j \leq J}$ is a multi-index, with $\alpha! = \prod_{j=1}^J \alpha_j!$ and $(K - K^0)^{\alpha} = \bigwedge_{j=1}^J (K_j - K_j^0)^{\alpha_j}$. The order of the product does not matter since each $K_j - K_j^0$ is even by assumption. Also, the summation terminates as soon as $\sum_{j=1}^J \alpha_j = M$ since each non-zero $K_j - K_j^0$ has degree at least 2, so $(K - K^0)^{\alpha}$ is a sum of p -forms with $p > 2M$ when $\sum_{j=1}^J \alpha_j > M$, and forms beyond top degree vanish. Thus all forms $F(K)$ defined in this way are polynomials in the fermion field. For example,

$$\begin{aligned} e^{-(\phi_x \bar{\phi}_x + \psi_x \wedge \bar{\psi}_x)} &= e^{-\phi_x \bar{\phi}_x} e^{-\psi_x \wedge \bar{\psi}_x} \\ &= e^{-\phi_x \bar{\phi}_x} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} (\psi_x \wedge \bar{\psi}_x)^n = e^{-\phi_x \bar{\phi}_x} (1 - \psi_x \wedge \bar{\psi}_x), \end{aligned} \quad (11.3.12)$$

$$e^{-S_A} = e^{-\phi A \bar{\phi}} \sum_{n=0}^M \frac{(-1)^n}{n!} (\psi A \bar{\psi})^n. \quad (11.3.13)$$

Example 11.3.1. Let $|\Lambda| = 1$. By definition of the integral and (11.3.9),

$$\begin{aligned} \int e^{-a\phi\bar{\phi} - a\psi\wedge\bar{\psi}} &= \int e^{-a\phi\bar{\phi}} a(-\psi\wedge\bar{\psi}) \\ &= \int e^{-a(u^2+v^2)} \frac{a}{\pi} du dv = \left(\frac{1}{\sqrt{\pi}} \int e^{-t^2} dt \right)^2 = 1. \end{aligned} \quad (11.3.14)$$

The factors $\frac{1}{\sqrt{2\pi i}}$ are included in (11.3.8) precisely in order to normalise the above integral.

The scaling of the constant a in (11.3.14) could also have been done earlier, by scaling ϕ and ψ simultaneously:

$$\int e^{-a\phi\bar{\phi} - a\psi\wedge\bar{\psi}} = \int e^{-\phi\bar{\phi} - \psi\wedge\bar{\psi}}. \quad (11.3.15)$$

This properly accounts for the change of variables in the Lebesgue integral, since ψ is proportional to $d\phi$. This principle generalises to higher dimensional integrals and is used in the proof of the next lemma. Its hypothesis that A is an $M \times M$ matrix with positive definite Hermitian part means that $\frac{1}{2}(A + A^*)$ is a strictly positive definite matrix, or, more explicitly, that $\sum_{x,y \in \Lambda} \phi_x (A_{xy} + \bar{A}_{yx}) \bar{\phi}_y > 0$ for all nonzero ϕ .

Lemma 11.3.2. *Let ϕ have components ϕ_x for $x \in \Lambda$, and let A be a $\Lambda \times \Lambda$ matrix with positive definite Hermitian part. Then*

$$\int e^{-S_A} = 1. \quad (11.3.16)$$

Proof. Consider first the case where A is Hermitian, so there is a unitary matrix U and a diagonal matrix D such that $A = U^{-1}DU$, so $\phi A \bar{\phi} = w D \bar{w}$ with $w = \bar{U}\phi$. Then

the change of variables which replaces $\bar{U}\phi$ by ϕ and $\bar{U}\psi$ by ψ leads to

$$\int e^{-S_A} = \int e^{-\phi D\bar{\phi} - \psi D\bar{\psi}}. \quad (11.3.17)$$

The integral on the right-hand side factors into a product of $|\Lambda|$ 1-dimensional integrals which are all equal to 1 by Example 11.3.1. This proves the result in the Hermitian case.

For the general case, we write $A(z) = G + izH$ with $G = \frac{1}{2}(A + A^*)$, $H = \frac{1}{2i}(A - A^*)$ and $z = 1$. Since $\phi(iH)\bar{\phi}$ is imaginary, when G is positive definite the integral of e^{-S_A} converges and defines an analytic function of z in a neighborhood of the real axis. Furthermore, for z small and purely imaginary, $A(z)$ is Hermitian and positive definite, and hence (11.3.17) holds in this case. Therefore (11.3.17) holds for all real z (in particular for $z = 1$) by uniqueness of analytic extension. ■

The following exercise gives an instructive alternate proof of Lemma 11.3.2, via an argument involving cancellation of determinants.

Exercise 11.3.3. Let A be an $\Lambda \times \Lambda$ matrix with positive definite Hermitian part, and let ϕ be a complex field indexed by Λ .

(i) Show that

$$\int_{\mathbb{R}^{2\Lambda}} e^{-\phi A \bar{\phi}} d\bar{\phi} d\phi = \frac{(2\pi i)^{|\Lambda|}}{\det A}. \quad (11.3.18)$$

Since $d\bar{\phi} d\phi$ is a multiple of the standard volume form, the integral in (11.3.18) is the Lebesgue integral of a complex function.

(ii) Show that the degree- $2|\Lambda|$ part of $e^{-\psi A \bar{\psi}}$ is $(\det A) \bar{\psi}_1 \wedge \psi_1 \wedge \cdots \wedge \bar{\psi}_{|\Lambda|} \wedge \psi_{|\Lambda|}$, and use this with (11.3.13) to show that the degree- $2|\Lambda|$ part of e^{-S_A} is

$$(\det A) e^{-\phi A \bar{\phi}} \frac{d\bar{\phi} d\phi}{(2\pi i)^{|\Lambda|}}, \quad (11.3.19)$$

and hence that

$$\int e^{-S_A} = 1. \quad (11.3.20)$$

[Solution]

The following exercise makes a connection between integration of 0-forms and Gaussian integration as discussed in Chapter 2.

Exercise 11.3.4. Let C be a positive definite real symmetric $\Lambda \times \Lambda$ matrix, and let $A = C^{-1}$. Let $\phi = u + iv$ be a complex field indexed by Λ . We write $u_x = \frac{1}{\sqrt{2}}\phi_x^1$ and $v_x = \frac{1}{\sqrt{2}}\phi_x^2$, and regard $\phi = (\phi^1, \phi^2)$ as a 2-component real field. Let $\mathbb{E}_C$ be the Gaussian measure with respect to which ϕ is a 2-component Gaussian field with covariance C as in Example 2.1.4. Show that if f is a 0-form (function) then

$$\int e^{-S_A} f = \mathbb{E}_C f. \quad (11.3.21)$$

In particular,

$$\int e^{-S_A} \phi_x \bar{\phi}_y = C_{xy}. \quad (11.3.22)$$

[Solution]

Theorem 11.2.3 is a representation for the two-point function of an n -component boson field. The next proposition extends this representation to include the fermion field. Let

$$\tau_x = \phi_x \bar{\phi}_x + \psi_x \wedge \bar{\psi}_x. \quad (11.3.23)$$

Proposition 11.3.5. *Let $A = -\Delta_\beta$. Let $F : \mathbb{R}^\Lambda \rightarrow \mathbb{R}$ be such that $e^{\varepsilon \sum_{z \in \Lambda} t_z} F(t)$ is a Schwartz function for some $\varepsilon > 0$. Then*

$$\int e^{-S_A} F(\tau) \phi_x \bar{\phi}_y = \int e^{-S_A} \int_0^\infty E_x(F(\tau + L_T) 1_{X(T)=y}) dT. \quad (11.3.24)$$

Proof. We follow the same strategy as in the proof of Theorem 11.2.3, and first consider the special case $F(t) = e^{-(v,t)}$ with $\operatorname{Re} v_z > 0$ for all $z \in \Lambda$. In this case, the right-hand side of (11.3.24) is

$$\begin{aligned} \int e^{-S_A} \int_0^\infty E_x(F(\tau + L_T) 1_{X(T)=y}) dT &= \int e^{-S_{A+V}} E_x(e^{-\sum_u v_u L_{T,u}} 1_{X(T)=y}) dT \\ &= (A+V)_{xy}^{-1} \int e^{-S_{A+V}} \\ &= (A+V)_{xy}^{-1}, \end{aligned} \quad (11.3.25)$$

where we used Lemma 11.2.2 and then Lemma 11.3.2 for the last two equalities. On the other hand, by (11.3.22) the left-hand side of (11.3.24) is now

$$\int e^{-S_{A+V}} \phi_x \bar{\phi}_y = (A+V)_{xy}^{-1}. \quad (11.3.26)$$

This proves (11.3.24) for $F(t) = e^{-(v,t)}$.

For the general case, let G be the Schwartz function given by $F(t) = G(t) e^{-\varepsilon \sum_{z \in \Lambda} t_z}$. As in the proof of Theorem 11.2.3, we again write F in terms of the Fourier transform of G , as

$$F(t) = G(t) e^{-\sum_{z \in \Lambda} t_z} = (2\pi)^{-|\Lambda|} \int_{\mathbb{R}^\Lambda} e^{-\sum_{z \in \Lambda} (1 - ir_z) t_z} \hat{G}(r) dr. \quad (11.3.27)$$

This equation remains valid with τ in place of t because equality for all t implies both sides have the same Taylor expansions about t , and we can again interchange the order of integration to conclude the general case from the special case already verified. ■

Since the left-hand side of (11.3.24) depends only on the restriction of F to the quadrant $\mathbb{R}_+^\Lambda$, it is more natural to formulate Proposition 11.3.5 for a smooth function F defined on $\mathbb{R}_+^\Lambda$; such a formulation can be found in [51, Proposition 4.4].

Proposition 11.3.5 generalises to models with n -component boson fields and m -component fermion fields with m even, but we do not make this claim precise because we are in the special situation where the fermions have been identified with differential forms. The more general concept of Grassmann integration is needed for the case where $m \neq n$. In the present case Proposition 11.3.5 has a surprising simplification (see Corollary 11.3.7), via the *localisation theorem* discussed below.

11.3.3 Localisation theorem and weakly self-avoiding walk

The following theorem is the localisation theorem. A more general theorem is proved in Section 11.4, with a different and revealing proof.

Theorem 11.3.6. *For F as in Proposition 11.3.5 and any A with non-negative real part,*

$$\int e^{-S_A} F(\tau) = F(0). \quad (11.3.28)$$

Proof. The proof again follows by checking the case $F(t) = e^{-(v,t)}$, as in the proof of Proposition 11.3.5. Instead of Lemma 11.2.2, the special case is handled using (11.3.16). ■

Corollary 11.3.7. *With $A = -\Delta_\beta$ and F as in Proposition 11.3.5,*

$$\int e^{-S_A} F(\tau) \bar{\phi}_a \phi_b = \int_0^\infty E_x(F(L_T) 1_{X(T)=y}) dT. \quad (11.3.29)$$

Proof. The left-hand side of (11.3.29) is equal to the right-hand side of (11.3.24), and the latter is equal to $\int_0^\infty E_x(F(0 + L_T) 1_{X(T)=y}) dT$ by Theorem 11.3.6. ■

The right-hand side of (11.3.29) arises precisely because the outer integral on the right-hand side of (11.3.24) simply evaluates the inner integral at $\tau = 0$. This is a rigorous implementation of the idea that the case of “ $n = 0$ ” components corresponds to self-avoiding walk. As with (11.2.10)–(11.2.11), the identity (11.3.29) is commonly expressed as an “isomorphism,” in the sense that τ on the left-hand side can be loosely interpreted as having the same distribution under e^{-S_A} as does L_T under E_x on the right-hand side.

Corollary 11.3.7 provides the supersymmetric representation for the finite-volume weakly self-avoiding walk two-point function. The supersymmetric representation is actually a representation for a finite-volume version of the two-point function. The convergence of the finite-volume two-point function to its infinite-volume counterpart is not difficult, and can be found in [20]. Here we restrict attention to the finite-volume two-point function.

We fix N , let $\Lambda = \Lambda_N = \mathbb{Z}^d / L^N \mathbb{Z}^d$, and let E_0^N denote the expectation for the continuous-time simple random walk on the discrete torus Λ_N . For $g > 0$ and $\mathbf{v} \in \mathbb{R}$, as in (11.1.13), the *finite-volume* two-point function is defined by

$$G_{0x}^N(g, \mathbf{v}) = \int_0^\infty E_0^N \left(e^{-gI(T)} \mathbb{1}_{X(T)=x} \right) e^{-\mathbf{v}T} dT. \quad (11.3.30)$$

Exercise 11.3.8. Show that $G_{0x}^N(g, \mathbf{v})$ is finite for all $\mathbf{v} \in \mathbb{R}$, provided $g > 0$. This is clearly not the case when $g = 0$. Hint: use the Cauchy–Schwarz for $\sum_{u \in \Lambda} L_{T,u}$. [\[Solution\]](#)

Our analysis of the $|\phi|^4$ model relies on its formulation as a perturbation of a Gaussian free field, and on the convolution property of the corresponding Gaussian expectation. The extension to the supersymmetric setting relies on analogous properties of the Gaussian super-expectation.

Definition 11.3.9. Let C be a real symmetric positive definite $\Lambda \times \Lambda$ matrix. Let $A = C^{-1}$. The Gaussian *super-expectation* with covariance C , of a form K , is defined by

$$\mathbb{E}_C K = \int K e^{-S_A}, \quad (11.3.31)$$

where the integral on the right-hand side is defined by (11.3.4).

According to Exercise 11.3.4, the super-expectation of a 0-form f is equal to the usual Gaussian expectation, i.e.,

$$\mathbb{E}_C f = \mathbb{E}_C f. \quad (11.3.32)$$

However, the super-expectation can also be applied to an arbitrary differential form. Particular cases of (11.3.32), seen already in Lemma 11.3.2, are the self-normalising property $\mathbb{E}_C 1 = 1$ and the identity $\mathbb{E}_C \bar{\phi}_x \phi_y = C_{xy}$. In many ways, the properties of the Gaussian super-expectation parallel those of the ordinary Gaussian expectation. In particular, it satisfies a version of the convolution property. A systematic introduction is provided in [54].

Given any $m^2 > 0$, let

$$\mathbf{v}_0 = \mathbf{v} - m^2, \quad C = (-\Delta + m^2)^{-1}. \quad (11.3.33)$$

Then Corollary 11.3.7 can be restated in terms of the super-expectation as

$$G_{0x}^N(g, \mathbf{v}) = \mathbb{E}_C \left(e^{-\sum_{y \in \Lambda} (g\tau_y^2 + \mathbf{v}_0 \tau_y)} \bar{\phi}_0 \phi_x \right). \quad (11.3.34)$$

This gives a supersymmetric representation for $G_{g,\mathbf{v}}^N(x)$. Its origins include [116, 119, 123, 130] and, in the form presented here, [50, 51].

Note that there is no dependence on m^2 in (11.3.34), and its introduction is simply to regularise the Laplacian so that C is well-defined. The right-hand side of (11.3.34) is the two-point function of a supersymmetric field theory with boson field $(\phi, \bar{\phi})$

and fermion field $(\psi, \bar{\psi})$. The supersymmetric representation allows a unified treatment of both weakly self-avoiding walk and n -component $|\phi|^4$, with the former behaving as the $n = 0$ version of the latter. It bears a strong resemblance to the corresponding identity (4.3.3) for $|\phi|^4$, with the simplification that the denominator (partition function) in (4.3.3) is replaced here by 1.

11.3.4 Localisation theorem and strictly self-avoiding walk

Other models of self-avoiding walk also have integral representations. In this section, we use the localisation theorem to obtain integral representations for the strictly self-avoiding walk, from [51]. We also present a representation for the edge self-avoiding walks known as self-avoiding trails, from [131]. The proofs use the integration by parts formula given in the following exercise.

Exercise 11.3.10. Let Λ be a finite set. Extend Exercise 2.1.3 to the super-expectation defined in Definition 11.3.9, i.e., verify the Gaussian integration by parts formula

$$\mathbb{E}_C(\bar{\phi}_x K) = \sum_{y \in \Lambda} C_{xy} \mathbb{E}_C \left(\frac{\partial K}{\partial \phi_y} \right) \quad (11.3.35)$$

for any form K such that both sides converge absolutely. [Solution]

We define the two-point function of weighted strictly self-avoiding walk on an arbitrary finite set Λ , as follows. For $n \geq 1$, let $\mathcal{S}_n(x, y)$ denote the set of sequences $\omega = (\omega(0), \omega(1), \dots, \omega(n))$ with $\omega(i) \in \Lambda$ for all i , $\omega(0) = x$, $\omega(n) = y$, and with $\omega(i) \neq \omega(j)$ for all $i \neq j$. Let $\mathcal{S}_0(x, y)$ be empty if $x \neq y$, and let $\mathcal{S}_0(x, x)$ consist of the zero-step walk $\omega(0) = x$. Let $\mathcal{S}(x, y) = \bigcup_{n=0}^{\infty} \mathcal{S}_n(x, y)$. Given a symmetric $\Lambda \times \Lambda$ matrix W of edge weights, for $\omega \in \mathcal{S}_n(x, y)$ we set $W^\omega = \prod_{i=1}^n W_{\omega(i-1), \omega(i)}$. As usual, the empty product equals 1 when $n = 0$. We define the *weighted two-point function* to be

$$\sum_{\omega \in \mathcal{S}(x, y)} W^\omega. \quad (11.3.36)$$

The following proposition gives a representation for the two-point function (11.3.36) with weights given by *positive definite* matrix $W = C$.

Proposition 11.3.11. *Let A be positive definite and $C = A^{-1}$. Then for $x \neq y$*

$$\sum_{\omega \in \mathcal{S}(x, y)} C^\omega = \int \bar{\phi}_x \phi_y \prod_{z \in \Lambda \setminus \{x, y\}} (1 + \tau_z) e^{-S_A}. \quad (11.3.37)$$

Proof. The right-hand side of (11.3.37) is equal to $\mathbb{E}_C \bar{\phi}_x F$ with F given by $F = \phi_y \prod_{z \neq x, y} (1 + \tau_z)$. Computation of the derivative gives

$$\frac{\partial F}{\partial \phi_v} = \delta_{vy} \prod_{z \neq x, y} (1 + \tau_z) + \mathbb{1}_{v \neq x, y} \phi_y \bar{\phi}_v \prod_{z \neq x, y, v} (1 + \tau_z). \quad (11.3.38)$$

Substitution of (11.3.38) into the integration by parts formula (11.3.35), followed by application of the localisation theorem (11.3.28), gives

$$E_C \bar{\phi}_x F = C_{xy} + \sum_{v \neq x, y} C_{xv} E_C \bar{\phi}_v \phi_y \prod_{z \neq x, y, v} (1 + \tau_z). \quad (11.3.39)$$

After iteration, the right-hand side gives the left-hand side of (11.3.37). $\blacksquare$

The strictly self-avoiding walk requires that no vertex be visited more than once. An alternate model allows vertices to be revisited but prohibits edges from being visited more than once. These edge self-avoiding walks are commonly called *self-avoiding trails* [110]. A precise definition is as follows.

Let Λ be a finite set and let E denote the set of all unordered pairs of distinct points in Λ . Then (Λ, E) is the complete graph on $|\Lambda|$ vertices. For $n \geq 1$, let $\mathcal{T}_n(x, y)$ denote the set of sequences $\omega = (\omega(0), \omega(1), \dots, \omega(n))$ with $\omega(i) \in \Lambda$, $\omega(0) = x$, $\omega(n) = y$, and with the undirected edges $\{\omega(i), \omega(i+1)\}$ distinct for all $i < n$. Let $\mathcal{T}_0(x, y)$ be empty if $x \neq y$, and let $\mathcal{T}_0(x, x)$ consist of the zero-step walk. Let $\mathcal{T}(x, y) = \cup_{n=0}^{\infty} \mathcal{T}_n(x, y)$. Given a $\Lambda \times \Lambda$ matrix W , the *weighted two-point function* of self-avoiding trails is defined by

$$\sum_{\omega \in \mathcal{T}(x, y)} W^{\omega}. \quad (11.3.40)$$

The following exercise provides an integral representation for this weighted two-point function when the weights are given by a *symmetric* matrix $W = \beta$ (not necessarily positive definite). The representation is analogous to Proposition 11.3.11, and essentially appears in [131]. A similar formula holds for walks which do not revisit *directed* edges.

The representation is stated in terms of the form

$$\tau_{xy} = \frac{1}{2} (\phi_x \bar{\phi}_y + \psi_x \wedge \bar{\psi}_y + \phi_y \bar{\phi}_x + \psi_y \wedge \bar{\psi}_x). \quad (11.3.41)$$

The solution to the exercise uses the extension of the localisation theorem given in Theorem 11.4.5 below, which implies in particular that the identity (11.3.28), i.e., $\int e^{-S_A} F(\tau) = F(0)$, holds also when F is a function of (τ_{xy}) rather than just a function of (τ_x) as in Theorem 11.3.6.

Exercise 11.3.12. Let β be a symmetric matrix. Show that

$$\sum_{\omega \in \mathcal{T}(x, y)} \beta^{\omega} = \int \bar{\phi}_x \phi_y \prod_{\{u, v\} \in E} (1 + 2\beta_{uv} \tau_{uv}) \prod_{w \in \Lambda} e^{-\tau_w}, \quad (11.3.42)$$

where E is the set of edges in the complete graph (Λ, E) . [Solution]

11.4 Supersymmetry and the localisation theorem

Integrals such as $\int e^{-S_A} F(\tau)$ are unchanged if we formally interchange the pairs $\phi, \bar{\phi}$ and $\psi, \bar{\psi}$. By (11.3.28), it is also true that $\int e^{-S_A} F(\tau) \bar{\phi}_a \phi_b = \int e^{-S_A} F(\tau) \bar{\psi}_a \psi_b$ (the difference is $\int e^{-S_A} \tau F(\tau) = 0$). This is a manifestation of a symmetry between bosons and fermions, called *supersymmetry*. In this section, we use methods of supersymmetry to provide an alternate proof of the localisation theorem, Theorem 11.3.6. Ideas of this nature are discussed in much more generality in [157, Section 2]. The localisation theorem is related to the Duistermaat–Heckman formula and equivariant cohomology; see, e.g. [12, 70, 140, 157].

11.4.1 The localisation theorem

We start with some definitions. An *anti-derivation* Γ is a linear map from the space of forms to itself which obeys

$$\Gamma(K_1 \wedge K_2) = (\Gamma K_1) \wedge K_2 + (-1)^{p_1} K_1 \wedge (\Gamma K_2) \quad (11.4.1)$$

when K_1 is a p_1 -form.

For $x \in \Lambda$, we define

$$\frac{\partial}{\partial \phi_x} = \frac{1}{2} \left(\frac{\partial}{\partial u_x} - i \frac{\partial}{\partial v_x} \right), \quad \frac{\partial}{\partial \bar{\phi}_x} = \frac{1}{2} \left(\frac{\partial}{\partial u_x} + i \frac{\partial}{\partial v_x} \right). \quad (11.4.2)$$

The following definition provides a notion of differentiation of a form with respect to the fermion field. This is a standard notion in Grassmann calculus (see, e.g., [33, 81, 139]). For $x \in \Lambda$, the derivatives $\frac{\partial}{\partial \psi_x}$ and $\frac{\partial}{\partial \bar{\psi}_x}$ are defined as the anti-derivations which obey the conditions:

$$\frac{\partial \psi_y}{\partial \psi_x} = \frac{\partial \bar{\psi}_y}{\partial \bar{\psi}_x} = \delta_{xy}, \quad \frac{\partial \bar{\psi}_y}{\partial \psi_x} = \frac{\partial \psi_y}{\partial \bar{\psi}_x} = 0, \quad \frac{\partial f}{\partial \psi_x} = \frac{\partial f}{\partial \bar{\psi}_x} = 0, \quad (11.4.3)$$

for any 0-form f . It follows from the definition that the derivatives anti-commute, i.e.,

$$\frac{\partial}{\partial \psi_x} \frac{\partial}{\partial \psi_y} = - \frac{\partial}{\partial \psi_y} \frac{\partial}{\partial \psi_x}, \quad \frac{\partial}{\partial \bar{\psi}_x} \frac{\partial}{\partial \bar{\psi}_y} = - \frac{\partial}{\partial \bar{\psi}_y} \frac{\partial}{\partial \bar{\psi}_x}, \quad \frac{\partial}{\partial \psi_x} \frac{\partial}{\partial \bar{\psi}_y} = - \frac{\partial}{\partial \bar{\psi}_y} \frac{\partial}{\partial \psi_x}. \quad (11.4.4)$$

Example 11.4.1. For notational simplicity, let $\partial_{\psi_x} = \frac{\partial}{\partial \psi_x}$ and $\partial_{\bar{\psi}_x} = \frac{\partial}{\partial \bar{\psi}_x}$, and let $\partial_{\psi} \partial_{\bar{\psi}}$ denote the product $\prod_{x \in \Lambda} \partial_{\psi_x} \partial_{\bar{\psi}_x}$. For any $x \in \Lambda$,

$$K = \psi_x \partial_{\psi_x} K = \bar{\psi}_x \partial_{\bar{\psi}_x} K. \quad (11.4.5)$$

In particular,

$$\int K = \frac{1}{\pi^{|\Lambda|}} \int_{\mathbb{R}^{2\Lambda}} \partial_\psi \partial_{\bar{\psi}} K \frac{d\bar{\phi} d\phi}{(2i)^{|\Lambda|}}, \quad (11.4.6)$$

so $\int K$ is the Lebesgue integral over $\mathbb{R}^{2\Lambda}$ of the function (0-form) $\pi^{-|\Lambda|} \partial_\psi \partial_{\bar{\psi}} K$.

The *supersymmetry generator* Q is the anti-derivation defined by

$$Q = \sum_{x \in \Lambda} \left(\psi_x \frac{\partial}{\partial \phi_x} + \bar{\psi}_x \frac{\partial}{\partial \bar{\phi}_x} - \phi_x \frac{\partial}{\partial \psi_x} + \bar{\phi}_x \frac{\partial}{\partial \bar{\psi}_x} \right). \quad (11.4.7)$$

In particular,

$$Q\phi_x = \psi_x, \quad Q\bar{\phi}_x = \bar{\psi}_x, \quad Q\psi_x = -\phi_x, \quad Q\bar{\psi}_x = -\bar{\phi}_x. \quad (11.4.8)$$

An form K is said to be *supersymmetric* or *Q -closed* if $QK = 0$. A form K that is in the image of Q is called *Q -exact*. Note that the integral of any *Q -exact* form is zero (assuming that the form decays appropriately at infinity), since integration acts only on forms of top degree $2N$ and the ψ -derivatives in (11.4.7) reduce the degree to at most $2N - 1$, while the integral of the ϕ -derivatives is zero by the Fundamental Theorem of Calculus (recall (11.4.2) and the fact that integrals ultimately are evaluated as Lebesgue integrals).

Example 11.4.2. The form

$$\tau_{xy} = \frac{1}{2} (\phi_x \bar{\phi}_y + \psi_x \wedge \bar{\psi}_y + \phi_y \bar{\phi}_x + \psi_y \wedge \bar{\psi}_x) \quad (11.4.9)$$

is both *Q -exact* and *Q -closed*. (Note that the degree-zero part of τ_{xy} is real.) In fact, it follows from (11.4.8) that $Q(\phi_x \bar{\psi}_y) = \phi_x \bar{\phi}_y + \psi_x \wedge \bar{\psi}_y$ and hence

$$\tau_{xy} = Q\lambda_{xy}, \quad \text{where} \quad \lambda_{xy} = \frac{1}{2} (\phi_x \bar{\psi}_y + \phi_y \bar{\psi}_x). \quad (11.4.10)$$

Similarly, it is easily verified that $Q\tau_{xy} = 0$.

Exercise 11.4.3. Prove that Q obeys the chain rule for even forms, in the sense that if $K = (K_j)_{j \leq J}$ is a finite collection of even forms, and if $F : \mathbb{R}^J \rightarrow \mathbb{C}$ is C^∞ , then

$$Q(F(K)) = \sum_{i=1}^J F_i(K) QK_i, \quad (11.4.11)$$

where F_i denotes the partial derivative. [Solution]

Example 11.4.4. Let $F : \mathbb{R}^{\Lambda \times \Lambda} \rightarrow \mathbb{C}$ be a smooth function, and let $\tau = (\tau_{xy})$. Then $Q(F(\tau)) = 0$. In particular, $Qe^{-S_A} = 0$ for any symmetric $\Lambda \times \Lambda$ matrix A .

Proof. This follows from Exercise 11.4.3 and $Q\tau_{xy} = 0$. For e^{-S_A} , it can alternatively be seen by expanding e^{-S_A} and applying Q term by term using $Q\tau_{xy} = 0$. ■

The following version of the localisation theorem generalises Theorem 11.3.6 and provides an alternative proof.

Theorem 11.4.5. *Let K be a smooth integrable Q -closed form, so $QK = 0$. Then*

$$\int K = K^0(0), \quad (11.4.12)$$

where $K^0(0)$ is the evaluation of the degree-zero part K^0 of K at $\varphi = 0$.

Proof. Any integrable form K can be written as $K = \sum_{\alpha} K^{\alpha} \psi^{\alpha}$, where ψ^{α} is a monomial in $\psi_x, \bar{\psi}_x, x \in \Lambda$, and K^{α} is an integrable function of $\phi, \bar{\phi}$. To emphasise this, we write $K = K(\phi, \bar{\phi}, \psi, \bar{\psi})$. Let $S = \sum_{x \in \Lambda} (\phi_x \bar{\phi}_x + \psi_x \wedge \bar{\psi}_x)$. Thus $S = S_A$ with $A = \text{Id}$.

Step 1. We prove the following version of Laplace's Principle:

$$\lim_{t \rightarrow \infty} \int e^{-tS} K = K^0(0). \quad (11.4.13)$$

Let $t > 0$. We make the change of variables $\phi_x = \frac{1}{\sqrt{t}} \phi'_x$ and $\psi_x = \frac{1}{\sqrt{t}} \psi'_x$; since ψ_x is proportional to $d\phi_x$ this correctly implements the change of variables. Let $\omega = -\sum_{x \in \Lambda} \psi_x \wedge \bar{\psi}_x$. After dropping the primes, we obtain

$$\int e^{-tS} K = \int e^{-\sum_x \phi_x \bar{\phi}_x + \omega} K\left(\frac{1}{\sqrt{t}} \phi, \frac{1}{\sqrt{t}} \bar{\phi}, \frac{1}{\sqrt{t}} \psi, \frac{1}{\sqrt{t}} \bar{\psi}\right). \quad (11.4.14)$$

To evaluate the right-hand side, we expand e^{ω} and obtain

$$\int e^{-tS} K = \sum_{n=0}^{|\Lambda|} \int e^{-\sum_x \phi_x \bar{\phi}_x} \frac{1}{n!} \omega^n K\left(\frac{1}{\sqrt{t}} \phi, \frac{1}{\sqrt{t}} \bar{\phi}, \frac{1}{\sqrt{t}} \psi, \frac{1}{\sqrt{t}} \bar{\psi}\right). \quad (11.4.15)$$

We write $K = K^0 + G$, where $G = K - K^0$ contains no degree-zero part. The contribution of K^0 to (11.4.15) involves only the $n = |\Lambda|$ term and equals

$$\int e^{-tS} K^0 = \int e^{-\sum_x \phi_x \bar{\phi}_x} \frac{1}{|\Lambda|!} \omega^{|\Lambda|} K^0\left(\frac{1}{\sqrt{t}} \phi, \frac{1}{\sqrt{t}} \bar{\phi}\right), \quad (11.4.16)$$

so by the continuity of K^0 ,

$$\lim_{t \rightarrow \infty} \int e^{-tS} K^0 = K^0(0) \int e^{-\sum_x \phi_x \bar{\phi}_x} \frac{1}{|\Lambda|!} \omega^{|\Lambda|} = K^0(0) \int e^{-S}. \quad (11.4.17)$$

By Lemma 11.3.2 (with $A = \text{Id}$), this proves that

$$\lim_{t \rightarrow \infty} \int e^{-tS} K^0 = K^0(0). \quad (11.4.18)$$

To complete the proof of (11.4.13), it remains to show that $\lim_{t \rightarrow \infty} \int e^{-tS} G = 0$. As above,

$$\int e^{-tS} G = \sum_{n=0}^{|\Lambda|} \int e^{-\sum_x \phi_x \bar{\phi}_x} \frac{1}{n!} \omega^n G \left(\frac{1}{\sqrt{t}} \phi, \frac{1}{\sqrt{t}} \bar{\phi}, \frac{1}{\sqrt{t}} \psi, \frac{1}{\sqrt{t}} \bar{\psi} \right). \quad (11.4.19)$$

Since G has no degree-zero part, the term with $n = |\Lambda|$ is zero. Terms with smaller n require factors $\psi \bar{\psi}$ from G , which carry inverse powers of t . They therefore vanish in the limit, and the proof of (11.4.13) is complete.

Step 2. The Laplace approximation is exact:

$$\int e^{-tS} K \text{ is independent of } t \geq 0. \quad (11.4.20)$$

To prove this, recall from Example 11.4.2 that $\tau_x = Q\lambda_x$ where $\lambda_x = \lambda_{xx}$. Let $\lambda = \sum_{x \in \Lambda} \lambda_x$. Then

$$S = \sum_{x \in \Lambda} \tau_x = \sum_{x \in \Lambda} Q\lambda_x = Q\lambda. \quad (11.4.21)$$

Also, $Qe^{-S} = 0$ by Example 11.4.4, and $QK = 0$ by assumption. Therefore,

$$\frac{d}{dt} \int e^{-tS} K = - \int e^{-tS} SK = - \int e^{-tS} (Q\lambda) K = - \int Q(e^{-tS} \lambda K) = 0, \quad (11.4.22)$$

since the integral of any Q -exact form is zero.

Step 3. Finally, we combine Laplace's Principle (11.4.13) and the exactness of the Laplace approximation (11.4.20), to obtain the desired result

$$\int K = \lim_{t \rightarrow \infty} \int e^{-tS} K = K^0(0). \quad (11.4.23)$$

This completes the proof. ■

Alternate proof of Theorem 11.3.6. By Example 11.4.4, $Q(F(\tau)) = 0$ and $Qe^{-S_A} = 0$. Since Q is an anti-derivation, this gives $Q(e^{-S_A} F) = (Qe^{-S_A})F + e^{-S_A} QF = 0$. Also, $e^{-S_A} F$ is integrable by the decay assumption in Theorem 11.3.6. The claim therefore follows from Theorem 11.4.5. ■

11.4.2 Supersymmetry and exterior calculus

As a final observation, we indicate how the supersymmetry generator Q can be expressed in terms of standard operations in differential geometry, namely the exterior derivative, the interior product, and the Lie derivative.

The *exterior derivative* d is the anti-derivation that maps a form of degree p to a form of degree $p+1$, defined by $d^2 = 0$ and, for a zero form f ,

$$df = \sum_{x \in \Lambda} \left(\frac{\partial f}{\partial \phi_x} d\phi_x + \frac{\partial f}{\partial \bar{\phi}_x} d\bar{\phi}_x \right). \quad (11.4.24)$$

Consider the flow acting on $\mathbb{C}^A$ defined by $\phi_x \mapsto e^{-2\pi i \theta} \phi_x$. This flow is generated by the vector field X defined by $X(\phi_x) = -2\pi i \phi_x$, and $X(\bar{\phi}_x) = 2\pi i \bar{\phi}_x$. The action by pullback of the flow on forms is

$$d\phi_x \mapsto d(e^{-2\pi i \theta} \phi_x) = e^{-2\pi i \theta} d\phi_x, \quad d\bar{\phi}_x \mapsto e^{2\pi i \theta} d\bar{\phi}_x. \quad (11.4.25)$$

The *interior product* $\underline{i} = \underline{i}_X$ with the vector field X is the anti-derivation that maps forms of degree p to forms of degree $p - 1$ (and maps forms of degree zero to zero), given by

$$\underline{i}d\phi_x = -2\pi i \phi_x, \quad \underline{i}d\bar{\phi}_x = 2\pi i \bar{\phi}_x. \quad (11.4.26)$$

The interior product obeys $\underline{i}^2 = 0$.

The *Lie derivative* $\mathcal{L} = \mathcal{L}_X$ is the infinitesimal flow obtained by differentiating with respect to the flow at $\theta = 0$. Thus, for example,

$$\mathcal{L}d\phi_x = \frac{d}{d\theta} e^{-2\pi i \theta} d\phi_x \Big|_{\theta=0} = -2\pi i d\phi_x. \quad (11.4.27)$$

A form K is defined to be *invariant under the flow of X* if $\mathcal{L}K = 0$. For example, the form

$$u_{xy} = \phi_x d\bar{\phi}_y \quad (11.4.28)$$

is invariant since it is constant under the flow of X .

Proposition 11.4.6. *The supersymmetry generator is given by $Q = \frac{1}{\sqrt{2\pi i}}(d + \underline{i})$, and $Q^2 = \frac{1}{2\pi i}\mathcal{L}$. In particular, a supersymmetric form is invariant under the flow of X .*

Proof. By the definitions of d and $\underline{i}$, and of $\psi, \bar{\psi}$,

$$d = \sqrt{2\pi i} \sum_{x=1}^M \left(\psi_x \frac{\partial}{\partial \phi_x} + \bar{\psi}_x \frac{\partial}{\partial \bar{\phi}_x} \right), \quad (11.4.29)$$

$$\underline{i} = \sqrt{2\pi i} \sum_{x=1}^M \left(-\phi_x \frac{\partial}{\partial \psi_x} + \bar{\phi}_x \frac{\partial}{\partial \bar{\psi}_x} \right). \quad (11.4.30)$$

The identity $Q = \frac{1}{\sqrt{2\pi i}}(d + \underline{i})$ then follows immediately from the definition of Q in (11.4.7).

Cartan's formula asserts that $\mathcal{L} = d\underline{i} + \underline{i}d$ (see, e.g., [150, Prop. 2.25] or [103, p. 146]). Since $d^2 = 0$ and $\underline{i}^2 = 0$, it follows that $\mathcal{L} = 2\pi i Q^2$. ■

Part V

Appendices

Appendix A

Extension to Euclidean models

In this book, we have applied the renormalisation group method to analyse the 4-dimensional hierarchical model. We now briefly describe some of the modifications needed to extend the method from the hierarchical to the Euclidean setting, and also point out where in the literature these extensions are carried out in detail.

The Euclidean setting refers to models defined on $\mathbb{Z}^d$. As usual, we work first with finite volume, followed by an infinite volume limit. (The renormalisation group map can however be defined directly in infinite volume, as explained in [57, Section 1.8.3].) To preserve translation invariance in finite volume, we use the d -dimensional discrete torus Λ of period L^N . The decomposition (4.1.15) of the covariance in the hierarchical setting is supplied directly by the definition of the hierarchical Laplacian. In the Euclidean setting, we instead use the finite-range decomposition of the Laplacian on the Euclidean torus (3.4.4). As discussed in Section 4.1.1, the finite-range decomposition bears similarities to the decomposition of the hierarchical Laplacian, but it is less simple and details differ.

The main issues discussed below are:

- Unlike the hierarchical expectation at a given scale, the finite-range expectation does not factorise over blocks. A more general version of local coordinates is needed.
- Unlike the hierarchical fields at a given scale, the finite-range fields are not constant within blocks, but only approximately so. This requires careful control not only of large fields but also of large gradients.
- Unlike the hierarchical fields at a given scale, the covariances do not have the zero-sum property (4.1.13). This necessitates an analogue of the term W as in (5.2.22) not only in the last step, but throughout all renormalisation group steps.

In particular, the above points require the following generalisations:

- The generalisations of the T_ϕ -seminorms must account for the fact that fields are only approximately constant.
- The generalisation of the $\mathcal{W}$ -norm must account for the fact that the analogue of K does not factorise over blocks.

- In addition to the coupling constants (g, v) , the marginal monomial $\varphi(\Delta\varphi)$ with coupling constant denoted by z must be tracked carefully. This coupling constant z is associated to the field strength or stiffness of the field.

A.1 Perturbative renormalisation group coordinate

For the hierarchical model, the interaction $I : \mathcal{B} \rightarrow \mathcal{N}$ is defined in Section 5.2.1 as

$$I(b) = e^{-V(b)}, \quad V(b) = \sum_{x \in b} \left(\frac{1}{4} g |\varphi|^4 + \frac{1}{2} v |\varphi|^2 \right). \quad (\text{A.1.1})$$

This requires two generalisations: field-strength renormalisation and the second-order irrelevant contribution to the interaction.

Field-strength renormalisation

For the Euclidean model, the n -component field $\varphi = (\varphi_x)_{x \in \Lambda}$ is an element of $(\mathbb{R}^n)^\Lambda$. Since fields are not constant on blocks, the polynomial V acquires a marginal monomial $\varphi \cdot (-\Delta\varphi)$, and a corresponding coupling constant z called the *field-strength renormalisation*. There are three coupling constants (g, v, z) , and V has the form

$$V(b) = \sum_{x \in b} \left(\frac{1}{4} g |\varphi_x|^4 + \frac{1}{2} v |\varphi_x|^2 + \frac{1}{2} z \varphi_x \cdot (-\Delta\varphi_x) \right). \quad (\text{A.1.2})$$

Now there are two marginal monomials: $|\varphi|^4$ and $\varphi \cdot (-\Delta\varphi)$. As a consequence, the clever Bleher–Sinai argument applied in Section 8.3 to construct the critical point is replaced by a more robust dynamical systems argument [22].

Second-order irrelevant contribution

For the hierarchical model, the second-order irrelevant term W arises from the expectation in (5.2.22), but it vanishes at all scales except the last scale due to the zero-sum condition (4.1.13) on the covariance. For the finite-range decomposition, the zero-sum condition does not hold, and the term W occurs at all nonzero scales and must be incorporated into the interaction I . Thus we set

$$I_j(V, b) = e^{-V_j(b)} (1 + W_j(V, b)). \quad (\text{A.1.3})$$

The term W is an explicit quadratic polynomial in V and is defined by

$$W_j(V, b) = \frac{1}{2} (1 - \text{Loc}_b) F_{w_j}(V(b), V(\Lambda)) \quad (\text{A.1.4})$$

where F_C is given by (2.2.9) and

$$w_j = C_1 + \cdots + C_j \quad (\text{A.1.5})$$

is the total covariance that has been integrated so far. The above formula for W involves the Euclidean localisation operator Loc_X , which is defined for arbitrary subsets $X \subset \Lambda$ on smooth functions of the field ϕ . Its output is a local polynomial summed over X , of the form $\sum_{x \in X} V(\phi_x)$, with V in a class of local polynomials which includes the $\phi \cdot (-\Delta \phi)$ term. See [18, Section 2.4] for more details, and [55] for the general theory.

The definition (5.2.17) of the renormalised polynomial V_{pt} must be generalised. For the Euclidean model, it is given by

$$V_{\text{pt}} = \mathbb{E}_+ \theta V - P, \quad (\text{A.1.6})$$

where P is the local polynomial defined for $B \in \mathcal{B}_+$ by

$$P(B) = \text{Loc}_B \left(\mathbb{E}_+ \theta W(V, B) + \frac{1}{2} \mathbb{E}_+ (\theta V(B); \theta V(\Lambda)) \right). \quad (\text{A.1.7})$$

The formulas (A.1.6)–(A.1.7) reduce to (5.2.17) when the field is hierarchical because there $W = 0$ except for the last step and also $V(\Lambda \setminus B)$ is independent of $V(B)$ and hence drops out of the covariance term. The definition (A.1.6) of V_{pt} is discussed at length in [21] (see also [18, Section 3.2]). In particular, the equation for P_x in [21, (2.12)] is equivalent to (A.1.7) because of the relation between F_{C_+} and $\mathbb{E}_+$ given in Exercise 2.2.4, and because of the fact that $\sum_{x \in B} \text{Loc}_x$ can be replaced here by Loc_B due to [55, Proposition 1.8].

The main achievement of (A.1.6) and (A.1.7) is the following lemma, which is an extension of Lemma 5.2.6. It differs from (5.2.22) in the sense that the left-hand side of (5.2.22) is of the form $e^{-U(B)}$ rather than $e^{-U(B)}(1 + W(B))$. In the statement of the lemma, $O(U^p)$ denotes an error of p^{th} order in the coupling constants U , which need not be uniform in the field ϕ or the volume Λ . A proof of the lemma (in the supersymmetric case) is given in [21, Proposition 2.1].

Lemma A.1.1. *For any polynomial V as in (A.1.2) such that the expectation exists, and for $B \in \mathcal{B}_+$,*

$$\mathbb{E}_+ \theta I(\Lambda) = I_+(V_{\text{pt}}, \Lambda) + O(V^3). \quad (\text{A.1.8})$$

A.2 Approximate factorisation

The conceptually most significant generalisation that is required is that of only approximate factorisation.

A.2.1 Factorisation of expectation

From Definition 4.1.1, recall that $\mathcal{B}$ denotes the set of *blocks* at scale j .

Definition A.2.1. A *polymer* is a union of blocks from $\mathcal{B}$. We define $\mathcal{P}$ to be the set of polymers. We write $\mathcal{B}(X)$ and $\mathcal{P}(X)$ for the sets of blocks and polymers contained in the polymer X . We say that X and Y are *disjoint* if $X \cap Y = \emptyset$ and that X and Y are *disconnected* if there is no pair of blocks $B \in X$ and $B' \in Y$ that touch (as in Definition 4.1.1). A polymer is *connected* if it is not the union of two disconnected polymers; this gives a partition $\text{Comp}(X)$ of a polymer X into its connected components.

Let $\mathcal{N}$ denote the algebra of sufficiently smooth functions of the field $\phi = (\phi_x)_{x \in \Lambda}$, i.e., maps from $(\mathbb{R}^n)^\Lambda$ to $\mathbb{R}$. We say that $F : \mathcal{P} \rightarrow \mathcal{N}$ is *strictly local* if $F(X)$ depends only on $\phi|_X$. Let F, G be strictly local. The hierarchical expectation has the factorisation property:

$$\mathbb{E}_+(F(X)G(Y)) = \mathbb{E}_+(F(X))\mathbb{E}_+(G(Y)) \quad \text{if } X, Y \in \mathcal{P}_+ \text{ are disjoint.} \quad (\text{A.2.1})$$

The finite-range expectation has the weaker factorisation property:

$$\mathbb{E}_+(F(X)G(Y)) = \mathbb{E}_+(F(X))\mathbb{E}_+(G(Y)) \quad \text{if } X, Y \in \mathcal{P}_+ \text{ are disconnected.} \quad (\text{A.2.2})$$

In fact, for (A.2.2), F and G do not need to be strictly local: the finite-range property (3.3.6) ensures that the expectation factorises as long as X and Y are subsets of $\mathbb{Z}^d$ that are separated by a distance at least $\frac{1}{2}L^j$.

A.2.2 Circle product

The hierarchical model is written in the *factorised* form (see (5.1.14), (5.2.2))

$$\prod_{b \in \mathcal{B}} e^{-u|b|} (I(b) + K(b)) = e^{-u|\Lambda|} \prod_{b \in \mathcal{B}} (I(b) + K(b)), \quad (\text{A.2.3})$$

and this form is preserved by the hierarchical expectation due to (A.2.1) (see (5.2.6)). The expectation with *finite-range* covariance does not preserve this strong factorisation and a generalisation is required.

Definition A.2.2. For $F, G : \mathcal{P} \rightarrow \mathcal{N}$ we define the *circle product* $F \circ G : \mathcal{P} \rightarrow \mathcal{N}$ by

$$(F \circ G)(X) = \sum_{Y \in \mathcal{P}(X)} F(Y)G(X \setminus Y) \quad (X \in \mathcal{P}). \quad (\text{A.2.4})$$

Let $F : \mathcal{P} \rightarrow \mathcal{N}$. We say that:

- F *factorises over blocks* if $F(X) = \prod_{b \in \mathcal{B}(X)} F(b)$ holds for any $X \in \mathcal{P}$;
- F *factorises over connected components* if $F(X) = \prod_{Y \in \text{Comp}(X)} F(Y)$.

When F factorises over blocks we write $F^X = F(X)$. The circle product has the following properties which we use below (see [57]), namely:

- Commutativity: $F \circ G = G \circ F$.
- Associativity: $((F \circ G) \circ H) = F \circ (G \circ H)$.
- Suppose that F and G factorise over blocks. Then

$$(F \circ G)(X) = \sum_{Y \in \mathcal{P}(X)} F^Y G^{X \setminus Y} = \prod_{b \in \mathcal{B}(X)} (F(b) + G(b)) = (F + G)^X. \quad (\text{A.2.5})$$

By (A.2.5), the representation (A.2.3) equals $e^{-u|\Lambda|}(I \circ K)(\Lambda)$ when I and K both factorise over blocks, as they do in the hierarchical setting. In the Euclidean setting, I does factorise over blocks, but K only factorises over connected components. Hence we work with $(I \circ K)(\Lambda)$, and we must maintain this form after taking the expectation. More precisely, in the Euclidean setting the hierarchical formula (5.2.6) becomes

$$Z_+ = e^{-u_+|\Lambda|}(I_+ \circ K_+)(\Lambda) = e^{-u|\Lambda|}\mathbb{E}_+ \theta(I \circ K)(\Lambda) = \mathbb{E}_+ \theta Z. \quad (\text{A.2.6})$$

The circle product is scale dependent: in (A.2.6) $I \circ K$ is a scale- j product whereas $I_+ \circ K_+$ is at scale- $(j+1)$.

A.3 Change of coordinates

The hierarchical representation $Z = \prod_b (I(b) + K(b))$ is not unique because it permits division of $I(b) + K(b)$ into terms I and K in different ways. The circle product representation $Z = (I \circ K)(\Lambda)$ of a given $Z \in \mathcal{N}$ in terms of the coordinates I and K is further from being unique under the constraints that I factors over blocks and K over connected components, as it allows parts of $K(X)$ to be redistributed over different polymers.

The essential difficulty is to obtain a representation $Z = (I \circ K)(\Lambda)$ with the property that K does not grow with the scale. This requires the transfer of dangerous parts of K into I , via exploitation of the nonuniqueness of the circle product representation. This is done via two mechanisms of change of coordinates, which we now demonstrate.

A.3.1 Block cancellation

A change of coordinates for the hierarchical model is performed in (10.1.4), where, given $V, K, \hat{V}$, we find $\hat{K}$ such that

$$e^{-V(B)} + K(B) = e^{-\hat{V}(B)} + \hat{K}(B). \quad (\text{A.3.1})$$

Indeed $\hat{K}(B)$ is simply given in (10.1.5) as the solution to this equation. For the Euclidean model, the corresponding step would be easy to perform if we only wished to alter K on blocks and not on larger polymers. Indeed, by the associative property of the circle product and the identity (A.2.5), given any $I, \tilde{I}, K$, we can set $\delta I = \tilde{I} - I$ and $\tilde{K} = \delta I \circ K$ and obtain

$$I \circ K = (\tilde{I} + \delta I) \circ K = (\tilde{I} \circ \delta I) \circ K = \tilde{I} \circ \tilde{K}. \quad (\text{A.3.2})$$

In particular,

$$\tilde{K}(B) = \delta I(B) + K(B). \quad (\text{A.3.3})$$

By choosing δI appropriately, we can cancel the relevant and marginal parts from $K(B)$ by transferring them into $\tilde{I}$. This is what we did in the hierarchical setting in (10.1.4), and it was sufficient.

A.3.2 Small set cancellation

The procedure used in (A.3.2) does not cancel the relevant parts from $K(X)$ when X is not a single block. It turns out to be necessary to cancel the relevant parts from $K(X)$ only for the restricted class of *small sets* $X \in \mathcal{S}$, where $\mathcal{S}$ is the set of connected polymers which consist of at most 2^d blocks. Indeed, $K(X)$ contracts for geometric reasons when the polymer X is not a small set, making it unnecessary to extract relevant parts (see Lemma A.4.3 below).

The small set cancellation lies at the heart of the non-hierarchical problem, and is achieved by a different mechanism than (A.3.2). Instead, given any I, K , we produce K' so that

$$(I \circ K)(\Lambda) = (I \circ K')(\Lambda). \quad (\text{A.3.4})$$

Note that the *same* I appears on both sides of (A.3.4). (Unlike (A.3.2), we do *not* have equality of $(I \circ K)(X)$ and $(I \circ K')(X)$ for every polymer X .) The new coordinate K' will effectively move the unwanted part of $K(X)$ when X is a small set that is not a block into $K(B)$; in particular $K(B)$ will not undergo a cancellation. However, we can subsequently apply a version of (A.3.2) to deal with $K(B)$. A precursor of (A.3.4) appears in [58, Theorem A].

For simplicity, we illustrate (A.3.4) for the case $I = 1$. For a small set X that is not a block, let $\tilde{J}(X)$ be the portion of $K(X)$ that we wish to cancel. A key example is to have $\tilde{J}(X)$ equal to $\text{Loc}_X K(X)$. This is a local polynomial in the field, summed over the polymer X . It can therefore be written as $\tilde{J}(X) = \sum_{B \in \mathcal{B}(X)} J(X, B)$ where $J(X, B)$ is the restriction of $\tilde{J}(X)$ to summation over the block B . Now we *define* $J(B, B)$ by

$$J(B, B) = - \sum_{X \supset B: X \neq B} J(X, B). \quad (\text{A.3.5})$$

Thus we assume that we are given $\tilde{J}(X) = \sum_{B \in \mathcal{B}(X)} J(X, B)$ with

$$J(X, B) = 0 \quad \text{if } X \notin \mathcal{S} \text{ or } B \not\subset X, \quad \sum_{X \supset B} J(X, B) = 0. \quad (\text{A.3.6})$$

A change of coordinates in this situation is given by [57, Proposition D.1], whose conclusion is that, given (A.3.6), the identity (A.3.4) holds with K' obeying component factorisation, good estimates, and the desirable property

$$K'(X) = K(X) - \bar{J}(X) + \text{remainder} \quad (X \in \mathcal{S}). \quad (\text{A.3.7})$$

Thus, for $X \in \mathcal{S} \setminus \mathcal{B}$, $K'(X)$ is approximately equal to $K(X)$ with its relevant and marginal parts subtracted. The price to be paid for this is that

$$K'(B) = K(B) - \bar{J}(B) + \text{remainder} = K(B) + \sum_{X \supset B: X \neq B} J(X, B) + \text{remainder}. \quad (\text{A.3.8})$$

Thus $K'(B)$ not only fails to make a cancellation in $K(B)$, but it also receives the dangerous parts of $K(X)$ from small sets X that contain but do not equal B .

To fix this defect in $K'(B)$, as mentioned already above, we can use (A.3.2). Moreover, that repair does not do harm to $K'(X)$ for polymers that are not a single block.

A.3.3 Application

In Sections A.4–A.5, we will apply each of the changes of coordinates (A.3.2)–(A.3.4) twice, as follows.

Perturbation theory

In Section A.4, we choose $\tilde{I}$ suggested by perturbation theory, apply (A.3.2) and take the expectation to obtain (A.2.6) in the form $\mathbb{E}_+ \theta(I \circ K)(\Lambda) = (\tilde{I} \circ \tilde{K})(\Lambda)$. The resulting $\tilde{K}$ is unsatisfactory, as it contains second-order contributions. When $I = 1$ these bad contributions are called $h(U)$. Because of our choice of V_{pt} , to second order we can find $h(U, B)$ such that $h(U)$ equals $\bar{h}(U) = \sum_{B \in \mathcal{B}(U)} h(U, B)$ with $\sum_{U \supset B} h(U, B) = 0$. From (A.3.7) with $\bar{J} = \bar{h}$, we obtain now

$$h'(X) = h(X) - \bar{h}(X) + \text{remainder} = \text{remainder} \quad (X \in \mathcal{S}). \quad (\text{A.3.9})$$

(This holds also for $X = B$.) Now h' is third order. This is carried out in Section A.5.1. There we indicate why (A.3.4) holds with (A.3.9) for this easier special case which has $h \approx \bar{h}$.

Relevant and marginal parts

After the above has been carried out, we have a third-order K , but it contains relevant and marginal parts which would grow uncontrollably as the scale advances. For X a small set that is not a block, we transfer these parts from $K(X)$ to I using the mechanism described in Section A.3.2 with $\tilde{J} = \text{Loc } K$. The details are given in Section A.5.2, where we indicate why (A.3.4) holds with (A.3.7) in the general case. Finally, the unwanted parts of $K(B)$ for blocks B are removed by an application of (A.3.2).

A.4 Expectation, change of scale, and reblocking

For the hierarchical model (see (5.2.4)), we showed that for any choice of U_+ (in fact any choice of I_+) we could choose K_+ as in (5.2.5) to obtain the representation

$$e^{-u|\Lambda|} \mathbb{E}_+ \prod_{b \in \mathcal{B}} (I(b) + K(b)) = e^{-u_+|\Lambda|} \prod_{B \in \mathcal{B}_+} (I_+(B) + K_+(B)) \quad (\text{A.4.1})$$

where $\mathbb{E}_+$ is the hierarchical expectation. A Euclidean version of this is given in the following proposition. The proposition shows that given any choice of $\tilde{I}_+$ we can find an appropriate $\tilde{K}_+$, in the more general setting of the circle product. The scale of the circle product becomes increased in this operation, and this requires a reblocking step. Proposition A.4.2 provides the defining element of *Map 3* in [57, Section 5.1]. For its statement and proof, we need the following definition.

Definition A.4.1. The *closure* of a polymer $X \in \mathcal{P}$ is the smallest polymer $\bar{X} \in \mathcal{P}_+$ such that $X \subset \bar{X}$.

Proposition A.4.2. Let $I, \tilde{I}_+$ factorise over blocks $b \in \mathcal{B}_j$ and let $\delta I(b) = \theta I(b) - \tilde{I}_+(b)$. Let K factorise over connected components at scale j . Then

$$\mathbb{E}_+ \theta(I \circ K)(\Lambda) = (\tilde{I}_+ \circ \tilde{K}_+)(\Lambda) \quad (\text{A.4.2})$$

with

$$\tilde{K}_+(U) = \sum_{X \in \mathcal{P}(U)} \tilde{I}_+^{U \setminus X} \mathbb{E}_+ (\delta I \circ \theta K)(X) \mathbb{1}_{\bar{X}=U} \quad (U \in \mathcal{P}_+), \quad (\text{A.4.3})$$

and $\tilde{K}_+$ factorises over connected components at scale $j+1$.

Proof. Let $P = \delta I \circ \theta K$. By (A.3.2) at scale j ,

$$(\theta I) \circ (\theta K) = \tilde{I} \circ P. \quad (\text{A.4.4})$$

Since $\tilde{I}_+$ does not depend on the fluctuation field,

$$\begin{aligned}
\mathbb{E}_+ \theta(I \circ K)(\Lambda) &= (\tilde{I}_+ \circ \mathbb{E}_+ P)(\Lambda) = \sum_{X \in \mathcal{P}} \tilde{I}_+^{\Lambda \setminus X} \mathbb{E}_+(P(X)) \\
&= \sum_{U \in \mathcal{P}_+} \tilde{I}_+^{\Lambda \setminus U} \sum_{X \in \mathcal{P}} \tilde{I}_+^{U \setminus X} \mathbb{E}_+(P(X)) \mathbb{1}_{\bar{X}=U}.
\end{aligned} \tag{A.4.5}$$

The right-hand side is (A.4.2) with $\tilde{K}_+$ given by (A.4.3), and the proof of the identity is complete.

It is not difficult to verify that $\tilde{K}_+$ factorises over connected components. The geometry of the identity (A.4.3) defining $\tilde{K}_+(U)$ is illustrated in Figure A.1, which is helpful for the verification of factorisation. ■

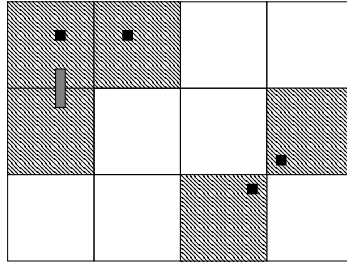


Fig. A.1 The five large shaded blocks represent U , which is the closure of the union of the four small dark blocks (the support of δI) and the small shaded polymer (the support of K).

For $X \in \mathcal{P}$, let $|\mathcal{B}(X)|$ denote the number of scale- j blocks in X . Similarly, we write $|\mathcal{B}_+(U)|$ for the number of scale- $(j+1)$ blocks in $U \in \mathcal{P}_+$. In the formula (A.4.3) for $\tilde{K}_+(U)$, it is helpful if $|\mathcal{B}(X)|$ is large, as this brings small factors from $(\delta I \circ K)(X)$. The following lemma shows that for large connected sets $X \in \mathcal{C} \setminus \mathcal{S}$, the constraint $\bar{X} = U$ in (A.4.3) forces $|\mathcal{B}(X)|$ to be strictly larger than $|\mathcal{B}_+(U)|$; for small sets $|\mathcal{B}(X)| = |\mathcal{B}_+(U)|$ is possible and the choice of 2^d in the definition of $\mathcal{S}$ is precisely due to this possibility. It is this geometric fact—the excess of $|\mathcal{B}(X)|$ over $|\mathcal{B}_+(U)|$ for large connected sets—that allows the main focus to be placed on the control of small sets. Large sets are *irrelevant*. A proof of Lemma A.4.3 is given in [57, Lemma C.3], and an earlier statement is [66, Lemma 2]. Its application in the Euclidean setting occurs in [57, Lemma 5.6].

Lemma A.4.3. *Let $d \geq 1$. There is an $\eta = \eta(d) > 1$ such that for all $L \geq 2^d + 1$ and for all $X \in \mathcal{C} \setminus \mathcal{S}$,*

$$|\mathcal{B}(X)| \geq \eta |\mathcal{B}_+(\bar{X})|. \tag{A.4.6}$$

The following example indicates a mechanism in which Lemma A.4.3 is applied. It illustrates why the focus can be restricted to small sets.

Example A.4.4. Let $A > 1$ and define a norm on $F : \mathcal{C} \rightarrow \mathcal{N}$ by

$$\|F\| = \sup_{X \in \mathcal{C}} A^{|\mathcal{B}(X)|} |F(X)|. \quad (\text{A.4.7})$$

We extend $F : \mathcal{C} \rightarrow \mathcal{N}$ to $F : \mathcal{P} \rightarrow \mathcal{N}$ by component factorisation, and define $\bar{F} : \mathcal{C}_+ \rightarrow \mathcal{N}$ by

$$F(U) = \sum_{\bar{X}=U} F(X). \quad (\text{A.4.8})$$

The map $F \mapsto \bar{F}$ is a prototype for the map $K \mapsto K_+$ that captures the reblocking aspect. Suppose that $F(X) = 0$ if $X \in \mathcal{S}$. We claim that

$$A^{|\mathcal{B}_+(U)|} |F(U)| \leq \left(A^{|\mathcal{B}_+(U)|} \sum_{\bar{X}=U} A^{-|\mathcal{B}(X)|} \mathbb{1}_{X \notin \mathcal{S}} \right) \|F\| \leq o(1) \|F\|, \quad (\text{A.4.9})$$

with the second inequality valid as $A \rightarrow \infty$. Therefore, with A sufficiently large, there exists $\kappa < 1$ such that

$$\|\bar{F}\|_+ \leq \kappa \|F\| \quad (\text{A.4.10})$$

for all F with $F(X) = 0$ for $X \in \mathcal{S}$. The inequality (A.4.10) shows that large sets are not important for the simple prototype $F \mapsto \bar{F}$ for the map $K \mapsto K_+$.

It remains to prove (A.4.9). The first inequality holds by definition of the norm. For the second, we bound the number of terms X in the sum by $2^{|\mathcal{B}(U)|} = 2^{L^d |\mathcal{B}_+(U)|}$, and apply Lemma A.4.3 to obtain $A^{-|\mathcal{B}(X)|} \mathbb{1}_{X \notin \mathcal{S}} \leq A^{-\eta |\mathcal{B}_+(U)|}$. This gives

$$A^{|\mathcal{B}_+(U)|} \sum_{\bar{X}=U} A^{-|\mathcal{B}(X)|} \mathbb{1}_{X \notin \mathcal{S}} \leq (A 2^{L^d} A^{-\eta})^{|\mathcal{B}_+(U)|} \leq 2^{L^d} A^{1-\eta}, \quad (\text{A.4.11})$$

with the last inequality valid assuming $A^{\eta-1} \geq 2^{L^d}$ (which does hold for large A since $\eta > 1$). The right-hand side becomes arbitrarily small for A sufficiently large.

A.5 Cancellation via change of coordinates

A.5.1 Local cancellation: perturbative

The formula for $\tilde{K}_+$ in (A.4.3) is not adequate even when $\tilde{I}_+$ is well chosen as $\tilde{I}_+(V_{\text{pt}})$, due to the presence of perturbative contributions to $\tilde{K}_+$ that are manifestly second order in V . In this section, we sketch an argument to explain how the change of coordinates (A.3.4) can be used to correct this problem. We also sketch a proof of (A.3.4) in this special case. This discussion reveals what lies at the heart of Map 4 in [57, Section 5.3].

Second-order contribution to $\tilde{K}_+$

Let U be a connected polymer in $\mathcal{C}_+$. There is a contribution to the right-hand side of (A.4.3) of the form

$$H(U) = \tilde{I}_+^U h(U), \quad h(U) = \sum_{X \in \mathcal{P}(U): |X|=1,2} \tilde{I}_+^{-X} \mathbb{E}_+ \delta \tilde{I}^X \mathbb{1}_{\bar{X}=U}, \quad (\text{A.5.1})$$

with the closure $\bar{X}$ of X defined in Definition A.4.1. For the terms in (A.5.1) where X consists of a single block, $h(U) = 0$ unless U is a single block, and when X consists of two blocks then $h(U) = 0$ unless U consists of one or two blocks. We extend the definition of H and h to disconnected polymers by imposing component factorisation. The contribution of H to $(\tilde{I}_+ \circ \tilde{K}_+)(\Lambda)$ is

$$(\tilde{I}_+ \circ H)(\Lambda) = \tilde{I}_+^\Lambda (1 \circ h)(\Lambda). \quad (\text{A.5.2})$$

The terms in the formula for $\tilde{K}_+$ that are first or second order in δI are isolated in h . Naively, we expect each factor of $\delta I(b)$ to provide a factor $O(V)$, so that three or more factors of $\delta I(b)$ will ensure an estimate $O(V^3)$. In $h(U)$ there are only one or two such factors when $U \in \mathcal{C}_+$. The apparently first-order terms with $|X| = 1$ are in fact second order in V , because in $\mathbb{E}_+ \delta I(b) = \mathbb{E}_+ \theta I - I_+$ there is cancellation of the first-order term in (A.1.6) due to our use of V_{pt} to define $\tilde{I}_+$. Thus h is $O(V^2)$.

As we will argue at the end of Section A.5.1, the second-order part of $h(U)$ has the form

$$h(U) \equiv \sum_{B \in \mathcal{B}_+(U)} h(U, B), \quad (\text{A.5.3})$$

with $h(U, B)$ second order and obeying the local cancellation

$$\sum_{U \in \mathcal{C}_+: U \supset B} h(U, B) \equiv 0, \quad (\text{A.5.4})$$

where $F \equiv G$ denotes that $F = G + O(V^3)$.

Local cancellation in $\tilde{K}_+$

We now apply (A.5.3)–(A.5.4) and exploit the non-uniqueness of the circle product representation, to show that it is possible to reappportion the second-order contributions to h in such a way that there is a third-order h_+ such that

$$(1 \circ h)(\Lambda) = (1 \circ h_+)(\Lambda), \quad (\text{A.5.5})$$

with $h_+(U) \equiv 0$ when $U \in \mathcal{C}_+$. This gives a version of (A.3.4) at scale $j+1$, with J and K both given by h on small sets and with $I = 1$.

We use the component factorisation property of h and (A.5.3) to obtain

$$\begin{aligned}
(1 \circ h)(\Lambda) &= \sum_{Y \in \mathcal{P}_+} h(Y) = \sum_{Y \in \mathcal{P}_+} \prod_{Y_i \in \text{Comp}(Y)} h(Y_i) \\
&= \sum_{Y \in \mathcal{P}_+} \prod_{Y_i \in \text{Comp}(Y)} \sum_{B_i \in \mathcal{B}_+(Y_i)} h(Y_i, B_i). \tag{A.5.6}
\end{aligned}$$

Given a block B , let $B^{(2)}$ denote the polymer which is the union of B and all two-block connected polymers that contain B . For example, when the dimension is $d = 2$ then $B^{(2)}$ is the union of B with the eight blocks that touch B . We partition the summation on the right-hand side according to the polymer $\cup_i B_i^{(2)}$, to obtain

$$\begin{aligned}
(1 \circ h)(\Lambda) &= \sum_{U \in \mathcal{P}_+} \sum_{Y \in \mathcal{P}_+} \prod_{Y_i \in \text{Comp}(Y)} \sum_{B_i \in \mathcal{B}_+(Y_i)} h(Y_i, B_i) \mathbb{1}_{\cup_i B_i^{(2)} = U} \\
&= \sum_{U \in \mathcal{P}_+} h_+(U) \\
&= (1 \circ h_+)(\Lambda), \tag{A.5.7}
\end{aligned}$$

where the second equality defines the terms $h_+(U)$. It can be checked that h_+ has the component factorisation property.

A second-order contribution to $h_+(U)$ can occur only when U is connected (otherwise $h_+(U)$ factors into contributions from each connected component of U , each of which is second order). Therefore the only possible second-order contribution to $h_+(U)$ is

$$\sum_{Y \in \mathcal{P}_+} \sum_{B \in \mathcal{B}_+(Y)} h(Y, B) \mathbb{1}_{B^{(2)} = U}. \tag{A.5.8}$$

Given U , the condition $B^{(2)} = U$ uniquely determines B (or there is no such B). With that particular $B = B(U)$, the above is equal to

$$\sum_{Y \in \mathcal{P}} h(Y, B), \tag{A.5.9}$$

which vanishes by (A.5.4). Thus we have achieved the goal (A.5.5) with third-order h_+ . The calculations here illustrate part of what occurs in the proof of [57, Proposition D.1], in a simplified setting.

Verification of (A.5.3)–(A.5.4)

We now verify (A.5.3)–(A.5.4). That is, we will identify second-order quantities $h(U, B)$, for U a two-block polymer containing B , with the properties that

$$h(U) \equiv \sum_{B \in \mathcal{B}_+(U)} h(U, B), \tag{A.5.10}$$

and that, due to our choice of V_{pt} , there is the *local cancellation*

$$\sum_{U \in \mathcal{C}_+ : U \supset B} h(U, B) \equiv 0 \quad (\text{A.5.11})$$

(a version of (A.5.11) with *equality* appears in [56, (2.22)]).

To keep the focus on the main ideas, let us simplify the problem and assume that $I_+(U) = e^{-V_{\text{pt}}(U)}$. With $\delta V = \theta V - V_{\text{pt}}$, we can then rewrite h as

$$h(U) = \sum_{X \in \mathcal{P}(U) : |X|=1,2} \left(\prod_{b \in \mathcal{B}(X)} e^{-\delta V(b)} - 1 \right) \mathbb{1}_{\bar{X}=U}. \quad (\text{A.5.12})$$

To uncover the lower-order terms in $h(U)$, we expand the exponential in a Taylor series and obtain (A.5.10) with

$$h(B, B) = \sum_{b \in \mathcal{B}(B)} \mathbb{E}_+ \left(\delta V(b) + \frac{1}{2} (\delta V(b))^2 \right) + \frac{1}{2} \sum_{b, b' \in \mathcal{B}(B) : b \neq b'} \mathbb{E}_+ \left(\delta V(b) \delta V(b') \right), \quad (\text{A.5.13})$$

$$h(U, B) = \frac{1}{2} \sum_{b \in \mathcal{B}(B)} \sum_{b' \in \mathcal{B}(B')} \mathbb{E}_+ \left(\delta V(b) \delta V(b') \right). \quad (\text{A.5.14})$$

The $\delta V(b)$ term in $h(B, B)$ is actually second order, not first order, because V_{pt} is equal to $\mathbb{E}_+ \theta V$ minus quadratic terms in V , and the linear term in V therefore cancels in $\mathbb{E}_+ \delta V(b)$. Thus all terms in $h(B, B)$ and $h(U, B)$ are second order.

To derive (A.5.11), we continue to neglect W , and recast Lemma A.1.1 as $\mathbb{E}_+ \theta e^{-V(\Lambda)} \equiv e^{-V_{\text{pt}}(\Lambda)}$, i.e.,

$$e^{-V_{\text{pt}}(\Lambda)} \mathbb{E}_+ \left(\prod_{b \in \mathcal{B}(\Lambda)} e^{-\delta V(b)} - 1 \right) \equiv 0. \quad (\text{A.5.15})$$

Again only the case where the product is over one or two small blocks can lead to a second-order contribution, and these small blocks must either lie in the same large block or in adjacent large blocks, because otherwise the finite-range property of the expectation produces a product of two second-order factors and hence is fourth order. The same Taylor expansion used above then leads to

$$\begin{aligned} \mathbb{E}_+ \left(\prod_{b \in \mathcal{B}(\Lambda)} e^{-\delta V(b)} - 1 \right) &\equiv \sum_{U \in \mathcal{C}_+(\Lambda)} h(U) \\ &= \sum_{U \in \mathcal{C}_+(\Lambda)} \sum_{B \in \mathcal{B}_+(U)} h(U, B) \\ &= \sum_{B \in \mathcal{B}_+(\Lambda)} \sum_{U \in \mathcal{C}_+ : U \supset B} h(U, B). \end{aligned} \quad (\text{A.5.16})$$

It is natural that the right-hand side would be third order because each term in the sum over B is, and this indeed turns out to be the case and gives (A.5.11).

A.5.2 Local cancellation: nonperturbative

For the hierarchical model, the marginal and relevant directions in K are absorbed into U_+ via the term $\text{Loc}(e^V K)$ in (5.2.31). In the Euclidean setting, the analogous manoeuvre is more delicate because I has only one degree of freedom for each block B (it factorises over blocks), while now K is a function of arbitrary polymers X . Two steps are used: (i) we apply the change of coordinates (A.3.4) to move the contributions from small sets into blocks, and (ii) we use the simpler change of coordinates (A.3.2) for single blocks, as done in (10.1.4) in the hierarchical setting.

To explain how the cancellation on small sets is arranged, we first write $\text{Loc}_U F$ as $\text{Loc}_U F = \sum_{x \in U} P_x$ and use P to define $\text{Loc}_{U,B} F = \sum_{x \in B} P_x$ for $U \supset B$. In particular, for $U \in \mathcal{P}$ and $B \in \mathcal{B}$,

$$\sum_{B \in \mathcal{B}(U)} \text{Loc}_{U,B} F = \text{Loc}_U F. \quad (\text{A.5.17})$$

Then we define $J(U, B) = 0$ if U is not a small set containing B , and otherwise

$$J(U, B) = \text{Loc}_{U,B} I^{-U} K(U) \quad \text{for } U \in \mathcal{S} \text{ with } U \supsetneq B, \quad (\text{A.5.18})$$

$$J(B, B) = - \sum_{U \in \mathcal{S}: U \supsetneq B} J(U, B). \quad (\text{A.5.19})$$

By construction,

$$\sum_{U: U \supset B} J(U, B) = 0. \quad (\text{A.5.20})$$

The local cancellation in (A.5.20), which holds by definition of J , is as in (A.3.6).

For $U \in \mathcal{S}$, let

$$\bar{J}(U) = \sum_{B \in \mathcal{B}_+(U)} I^U J(U, B), \quad M(U) = K(U) - \bar{J}(U). \quad (\text{A.5.21})$$

The new feature compared to our analysis of h in Section A.5.1 is that here for small sets U the role of $h(U)$ is played by $I^{-U} K(U) = I^{-U} M(U) + I^{-U} \bar{J}(U)$; the analysis for h corresponds to $M = 0$ which we no longer have. This requires more sophisticated combinatorics.

Cancellation on small sets other than blocks

To illustrate the main idea we make the following simplifications:

- We assume that $I = 1$.
- We assume that among connected polymers K is supported on small sets only.

Then, with V the union of the components of $\hat{V}$,

$$\begin{aligned}
(1 \circ K)(\Lambda) &= \sum_{U \in \mathcal{P}} \prod_{U_i \in \text{Comp}(U)} (M(U_i) + \bar{J}(U_i)) \\
&= \sum_{U \in \mathcal{P}} \sum_{\hat{V} \subset \text{Comp}(U)} M^{U \setminus \hat{V}} \prod_{U \in \hat{V}} \sum_{B \in \mathcal{B}(U)} J(U, B). \tag{A.5.22}
\end{aligned}$$

Given $X \in \mathcal{P}$, let $B_1, \dots, B_n$ be a list of the blocks in $\mathcal{B}(X)$, and let

$$\begin{aligned}
\mathcal{U}(X) &= \{ \{ (U_{B_1}, B_1), \dots, (U_{B_n}, B_n) \} : \\
&\quad U_{B_i} \in \mathcal{S}, U_{B_i} \supset B_i, U_{B_i} \text{ does not touch } U_{B_j} \text{ for } i \neq j \}. \tag{A.5.23}
\end{aligned}$$

Given an element of $\mathcal{U}(X)$, we write $Y_J = \cup_{B \in \mathcal{B}(X)} U_B$, and write $\mathcal{P}^-(Y_J)$ for the set of polymers that do not touch Y_J . The *small-set neighbourhood* $X^\square$ of a polymer X is the union of all small sets that contain a block in X . By interchanging the sums over blocks B and polymers U_B , we obtain

$$\begin{aligned}
(1 \circ K)(\Lambda) &= \sum_{X \in \mathcal{P}} \sum_{\{(U_B, B)\} \in \mathcal{U}(X)} \left(\prod_{B \in \mathcal{B}(X)} J(U_B, B) \right) \sum_{Y \in \mathcal{P}^-(Y_J)} M^Y \\
&= \sum_{W \in \mathcal{P}} \sum_{X \in \mathcal{P}} \sum_{\{(U_B, B)\} \in \mathcal{U}(X)} \left(\prod_{B \in \mathcal{B}(X)} J(U_B, B) \right) \sum_{Y \in \mathcal{P}^-(Y_J)} M^Y \mathbb{1}_{X^\square \cup Y = W}, \tag{A.5.24}
\end{aligned}$$

where the last equality is just a conditioning of the sums over X and Y according to the constraint $X^\square \cup Y = W$. Then we define $K'(W)$ to be the summand in the sum over W . It can be verified that K' has the component factorisation property, and it is proved in [57, Proposition D.1] that K' obeys good estimates.

We examine two special cases:

- If W is a small set S then we must have $X = \emptyset$ (because otherwise $X^\square$ cannot be contained in S , as $X^\square$ is not a small set even if X is a single block) and also $Y = S$, so

$$K'(S) = M(S). \tag{A.5.25}$$

Therefore, for $S = U \notin \mathcal{B}_+$ or $S = B \in \mathcal{B}_+$,

$$K'(U) = K(U) - \sum_{B \in \mathcal{B}_+(U)} J(U, B) = K(U) - \text{Loc}_U K(U) \quad (U \notin \mathcal{B}_+), \tag{A.5.26}$$

$$K'(B) = K(B) + \sum_{U \not\supseteq B} J(U, B) = K(B) + \sum_{U \not\supseteq B} \text{Loc}_{U, B} K(U) \quad (B \in \mathcal{B}_+). \tag{A.5.27}$$

In (A.5.26), the subtracted term is simply $\text{Loc}_U K(U)$ by (A.5.17). Thus the relevant and marginal parts of $K(U)$ are subtracted on small sets that are not a single block. The price to pay is that those subtractions have been transferred into $K'(B)$, which additionally fails to have the relevant and marginal parts of $K(B)$ subtracted.

- If $X = B$ and $Y = \emptyset$ then we must have $W = X^\square$ and X is uniquely determined by W , and the contribution from this case to $K(W) = K(B^\square)$ is

$$\sum_{U \supset B} J(U, B) = 0. \quad (\text{A.5.28})$$

This cancellation has the good consequence that there is no contribution to $K'(W)$, for any polymer W that is not a single block, that consists solely of J terms. The net effect of this is that there is no connected polymer W such that $K'(W)$ is a linear function of J without any compensating K factors.

The details of the above analysis can be found in the proof of [56, Proposition D.1]. It leads to a representation

$$(I \circ K)(\Lambda) = (I \circ K')(\Lambda) \quad (\text{A.5.29})$$

where in $K'(X)$ the relevant and marginal parts of $K(X)$ have been removed from all small sets X except single blocks. This is carried out in detail in *Map 1* of [57, Section 4.2].

Cancellation on blocks

It remains to remove the relevant and marginal parts of $K'(B)$ (which incorporate the relevant and marginal parts of $K(X)$ for all small sets X), and to transfer them into V_+ . This is achieved by replacing $V_{\text{pt}}(V)$ by $V_{\text{pt}}(\hat{V})$, where

$$\hat{V} = V - \sum_{U \in \mathcal{S}: U \supset B} \text{Loc}_{U,B} I^{-U} K(U). \quad (\text{A.5.30})$$

Let $\hat{I} = I(\hat{V})$, $\delta \hat{I} = I - \hat{I}$, $\hat{K} = \delta \hat{I} \circ K'$. By (A.3.2),

$$(I \circ K')(\Lambda) = (\hat{I} \circ \hat{K})(\Lambda). \quad (\text{A.5.31})$$

The relevant and marginal parts of K' are thereby transferred to $\hat{V}$ and removed from $\hat{K}$. The details of this operation are outlined in *Map 2* of [57, Section 4.3]. The corresponding step for the hierarchical model is performed at (10.1.3).

A.6 Norms

The norms applied in this book for the hierarchical model require modification and extension in the Euclidean setting. We discuss some aspects of this here. Full details can be found in [57], and a general development of properties of the norms is presented in [54].

A.6.1 T_φ -seminorms

For the Euclidean model, a counterpart of the hierarchical T_ε -seminorm of (7.1.9) is defined in [54]. For simplicity, we consider the 1-component Euclidean φ^4 model, and do not include an auxiliary space $\mathcal{Y}$. The field $\varphi = (\varphi_x)_{x \in \Lambda}$ is a point in $\mathbb{R}^\Lambda$, and we will define the Euclidean T_φ -seminorm on the space of functions $F : \mathbb{R}^\Lambda \rightarrow \mathbb{R}$. An example of such an F is the nonperturbative coordinate $K(X)$ evaluated on a polymer X ; in this case the dependence is only on φ_x for x in or near X .

Given a function $F : \mathbb{R}^\Lambda \rightarrow \mathbb{R}$, the derivative $F^{(p)}(\varphi)$ is a p -linear function on the space $\mathbb{R}^\Lambda$ of directions. Let Φ be a normed vector subspace of $\mathbb{R}^\Lambda$. We denote a direction in Φ by $\dot{\varphi}$ and a p -tuple of directions by $\dot{\varphi}^p$. Let $\Phi(1)$ be the unit ball in Φ . Then $\|F^{(p)}(\varphi)\| = \sup_{\dot{\varphi}^p \in \Phi(1)^p} |F^{(p)}(\varphi; \dot{\varphi}^p)|$. The Euclidean T_φ -seminorm is defined by

$$\|F\|_{T_\varphi} = \sum_{p=0}^{p_N} \frac{1}{p!} \|F^{(p)}(\varphi)\|, \quad (\text{A.6.1})$$

where $p_N \in [0, \infty]$ is a parameter at our disposal. The example $\Phi = \mathbb{R}_h^\Lambda$ with norm $\|\dot{\varphi}\| = \frac{1}{h} \max\{|\dot{\varphi}_x| : x \in \Lambda\}$ gives a T_φ -seminorm with the product property $\|FG\|_{T_\varphi} \leq \|F\|_{T_\varphi} \|G\|_{T_\varphi}$ of (7.1.10). Restrictions on the spaces of directions that are consistent with the product property are discussed in [54].

The freedom to choose allows us to take into account the properties that are imposed on typical fields by their probability distribution. For example, hierarchical fields are constant on blocks. Suppose that $F(\varphi)$ depends only on fields in a block B . Let Φ be the subspace of directions in $\mathbb{R}_h^\Lambda$ such that $\dot{\varphi}$ is constant on B , i.e., $\dot{\varphi}_x = \dot{\varphi}_y$ for all x, y in B . For φ constant on B the Euclidean T_φ -seminorm with this choice of Φ equals the T_φ -seminorm of Definition 7.1.2. This is true by virtue of chain rule formulas like

$$\sum_{x \in \Lambda} \frac{\partial}{\partial \varphi_x} F(\varphi) = \sum_{x \in B} \frac{\partial}{\partial \varphi_x} F(\varphi) = \frac{\partial}{\partial u} f(u) \quad (\text{A.6.2})$$

which is valid when the left-hand side is evaluated at φ such that $\varphi_x = u$ for all $x \in B$ and, by definition, $f(u) = F(\varphi)$.

In the Euclidean setting, we use Φ which takes into account the spatial variation of fields. After j renormalisation group steps, the remaining field to be integrated is $\varphi = \zeta_{j+1} + \dots + \zeta_N$, with increments as in Corollary 3.4.1. The scaling estimates (3.3.7) indicate that the variance of $\nabla^\alpha \varphi$ typically scales down with j like $L^{-j(d-2)} L^{-2j|\alpha|_1}$. Fix a positive integer p_Φ . Let the norm on Φ be the lattice $\mathcal{C}^{p_\Phi}$ -norm

$$\|\dot{\varphi}\|_{\Phi(h)} = \max\{h_j^{-1} L^{j|\alpha|_1} |\nabla^\alpha \dot{\varphi}_x| : x \in \Lambda, |\alpha|_1 \leq p_\Phi\}. \quad (\text{A.6.3})$$

With the choice $h = \ell_j = \ell_0 L^{-j(d-2)/2}$, as in (8.1.4), the T_φ -seminorm of F tests the response of F to typical fluctuations of the field, in particular fluctuations around being constant on blocks. The choice $h = h_j = k_0 L^{-jd/4} \tilde{g}_j^{-1/4}$, as in (8.1.7), is used to test the response of F to typical large fields. This is all as it is for the hierarchi-

cal model, apart from the fact that now spatial gradients of the field are taken into account.

Both parameters ℓ and h are combined in the hierarchical $\mathcal{W}$ -norm defined in (8.2.9). A Euclidean counterpart of the $\mathcal{W}$ -norm is defined in [57, (1.45)]. The latter also involves *regulators*, which we discuss next.

A.6.2 Regulators

For the hierarchical model, the crucial Proposition 10.5.1 asserts that the renormalisation group map is contractive in T_∞ -norm. The proof uses the following fact: if the $+$ scale field φ is large in a block b , then it is large on $B \setminus b$ where B is the $+$ scale block that contains b , because φ is constant on B . This is used in (10.5.20) where the factor $P_{h_+}^6(\varphi)$ arises from the growth of $K(b) - \text{Loc}K(b)$ as φ becomes large in b . If φ is large in b then the exponential $\exp[-V(B \setminus b)]$ is small and more than compensates for the growth of $P_{h_+}^6(\varphi)$.

For the Euclidean model the $+$ field can be large in b without being large in $B \setminus b$. We have to prove that typical fields do not do this. Let $\varphi_B = |B|^{-1} \sum_{x \in B} \varphi_x$ be the average of φ over the block B and let $\delta_B = \delta_B(\varphi)$ be the supremum over $x, y \in B$ of $|\varphi_x - \varphi_y|/h_+$. Then $|\varphi_x - \varphi_B|/h_+ \leq \delta_B$. In other words φ_x/h_+ is constant to within δ_B . We will show that for typical φ , $\delta_B = O_L(g^{1/4})$. Thus typical fields are very close to constants in this sense and it should be plausible that the hierarchical bound (10.5.20) continues to hold for fields with $\delta_B = O_L(\tilde{g}^{1/4})$.

What does it mean for a field to be typical? For intuition, recall from (3.3.7) that the standard deviation of $\nabla \varphi_x$ is $O_L(L^{-j} \ell_+)$, where ℓ_+ is defined in (8.1.4). We say a field φ is *typical* if the maximum over B of $|\nabla \varphi|$ is $O_L(L^{-j} \ell_+)$. Since $|\varphi_x - \varphi_y|$ is bounded by the length L^{j+1} of a path joining x to y times the maximum gradient, we find, using (8.1.7), that δ_B is bounded by $O_L(L \ell_+/h_+) = O_L(\tilde{g}^{1/4})$ as claimed above.

Although (10.5.20) does not hold for all Euclidean φ , the inequality obtained by including an extra factor $\exp[-\delta_B]$ might hold for all φ because it holds for typical φ by the arguments above and the decay of $\exp[-\delta_B]$ might compensate for atypical fields with large δ_B . This example leads to the idea that the T_∞ -norm for the hierarchical model should be replaced by a weighted T_∞ -norm where the weight will allow the hierarchical proofs that work for $\delta = 0$ to extend to the Euclidean model. The T_∞ -norm of a function $F(\varphi)$ tests $F(\varphi)$ on all possible φ but a weighted T_∞ -norm focuses on the fields that $F(\varphi)$ actually encounters when taking its expectation.

Given $w(X, \varphi) > 0$, a general weighted T_∞ -norm is defined by

$$\|K(X)\|_w = \sup_{\varphi \in \mathbb{R}^A} \frac{\|K(X)\|_{T_\varphi}}{w(X, \varphi)}. \quad (\text{A.6.4})$$

For the Euclidean $|\varphi|^4$ model, we use two choices of weight function, or *regulators*, corresponding to the two choices $\mathfrak{h} = \ell$ and $\mathfrak{h} = h$ of the parameter $\mathfrak{h}$ in the defini-

tion of the T_φ -seminorm. The specific choices are discussed in [56, Section 1.1.6]. The systematic use of such regulators originated in [58]; in [93, p.216, (1)] the breakdown of estimates for fields with large gradients is instead put into inductive assumptions.

We discuss here the most important case: the *large-field regulator* $w = \tilde{G}$. Desirable properties of the regulator are:

- (i) $\tilde{G}(X \cup Y, \varphi) = \tilde{G}(X, \varphi) \tilde{G}(Y, \varphi)$ if X, Y are disjoint.
- (ii) $\tilde{G}(X, \varphi + \varphi') \leq \tilde{G}^2(X, \varphi) \tilde{G}^2(X, \varphi')$.
- (iii) $\mathbb{E}_+ \tilde{G}^t(X) \leq 2^{|\mathcal{B}(X)|}$ for bounded powers t .
- (iv) $\tilde{G}' \leq \tilde{G}_+$ for bounded powers t .

Property (i) extends the product property of the T_φ -seminorm to the norm (A.6.4) when X, Y are disjoint. Properties (ii)-(iv) allow estimates to be advanced from one scale to the next, as in the following lemma. The proof of the lemma uses the general inequality

$$\|\mathbb{E}_+ \theta F\|_{T_\varphi(h_+)} \leq \mathbb{E}_+ \|F\|_{T_{\varphi+\zeta}(h)}, \quad (\text{A.6.5})$$

which follows from [54, Proposition 3.19] (see also [56, (7.2)–(7.3)]). The inequality (A.6.5) is reminiscent of Proposition 7.3.1 for the hierarchial case.

Lemma A.6.1. *Suppose that $\tilde{G}$ obeys (ii), (iii) and (iv) above. Then*

$$\|\mathbb{E}_+ \theta K(X)\|_{\tilde{G}_+} \leq 2^{|\mathcal{B}(X)|} \|K(X)\|_{\tilde{G}}. \quad (\text{A.6.6})$$

Proof. We first apply (A.6.5), and then the definition of the weighted norm, to obtain

$$\begin{aligned} \|\mathbb{E}_+ \theta K(X)\|_{T_\varphi(h_+)} &\leq \mathbb{E}_+ \|K(X)\|_{T_{\varphi+\zeta}(h)} \\ &\leq \|K(X)\|_{\tilde{G}} \mathbb{E}_+ \tilde{G}(X, \varphi + \zeta). \end{aligned} \quad (\text{A.6.7})$$

Using properties (ii), (iii) and (iv), we find that

$$\begin{aligned} \|\mathbb{E}_+ \theta K(X)\|_{T_\varphi(h_+)} &\leq \|K(X)\|_{\tilde{G}} \tilde{G}^2(X, \varphi) \mathbb{E}_+ \tilde{G}^2(X, \zeta) \\ &\leq \|K(X)\|_{\tilde{G}} \tilde{G}^2(X, \varphi) 2^{|\mathcal{B}(X)|} \\ &\leq \|K(X)\|_{\tilde{G}} \tilde{G}_+(X, \varphi) 2^{|\mathcal{B}(X)|}. \end{aligned} \quad (\text{A.6.8})$$

This proves (A.6.6). ■

To implement the above, we require a regulator $\tilde{G}$ which obeys properties (i)-(iv). A trivial choice is of course given by $\tilde{G} = 1$. However, for the weight to be helpful, $\tilde{G}$ should be as large as possible. Other authors have used regulators based on lattice Sobolev norms, e.g., [66, (47)]. Our choice is the regulator $\tilde{G}$ given in [56, (1.41)]. We conclude by presenting its definition. Further details, including a discussion of the *fluctuation-field regulator*, can be found in [56, Section 1.1.6].

First, for $X \subset \Lambda$ with diameter less than the period of the torus, we define

$$\|\varphi\|_{\Phi(X)} = \inf\{\|\varphi - f\|_{\Phi} : f \text{ restricted to } X \text{ is a linear polynomial}\}. \quad (\text{A.6.9})$$

The restriction on the diameter of X is present so that it makes sense to consider f as a linear polynomial in (A.6.9). The *large-field regulator* is then given by

$$\tilde{G}(X, \varphi) = \prod_{x \in X} \exp\left(L^{-dj} \|\varphi\|_{\Phi(b_x^{\square})}^2\right), \quad (\text{A.6.10})$$

where $b_x \in \mathcal{B}$ is the unique block which contains the point x .

The above construction of $\tilde{G}$ factors out linear polynomials. This is a way to examine the size of $|\nabla^2 \varphi|$, and in that sense is related to a Sobolev norm. Thus the regulator can bound $\nabla^2 \varphi$, but not φ . In our motivation of the weighted T_{∞} -norm, we estimated how close φ is to being constant in a block B . However, the regulator (A.6.10) only enables us to estimate how close φ is to being a linear function. Of course linear functions include constants and in fact we expect that fields are close to being constants, but it is easier to prove the weaker statement that they are close to linear. Also it is sufficient: if φ is linear on B and it is large on b then it is large on roughly half of B so the factor $\|e^{-V(B \setminus b)}\|_{T_{\varphi}}$ in (10.5.20) is still exponentially small and bounds the polynomial that depends on the field in b .

An advantage of the regulator (A.6.10) is that its weighted norm leads to a complete Banach space after an additional weighted supremum over polymers X is taken in (A.6.4). This is discussed in detail in [57, Appendix A]. The Sobolev regulator was erroneously claimed to produce a complete space, e.g., in [52]; this error was pointed out and corrected in [1] in a manner than maintained the Sobolev regulator.

Finally, we note that properties (i)-(iv) hold for (A.6.10). Property (i) holds by definition, and property (ii) is a consequence of the elementary inequality $(a+b)^2 \leq 2a^2 + 2b^2$. Property (iii) is a consequence of [54, Proposition 3.20] together with the fact that the large-field regulator is less than or equal to the fluctuation-field regulator. According to [56, Lemma 1.2], property (iv) holds if L is large enough.

Appendix B

Solutions to exercises

B.1 Chapter 1 exercises

Solution to Exercise 1.4.5. By replacing $V(\varphi)$ by $V(\varphi) - V(\varphi_0)$, $g(\varphi) - g(\varphi_0)$ by $g(\varphi)$, and φ by $\varphi + \varphi_0$, we can assume that $\varphi_0 = 0$, $V(\varphi_0) = 0$ and $g(\varphi_0) = 0$. For $E \subset \mathbb{R}^n$, let

$$I_N(E, f) = \int_E f(\varphi) e^{-NV(\varphi)} d\varphi. \quad (\text{B.1.1})$$

With the above assumptions, we must prove that

$$\lim_{N \rightarrow \infty} \frac{I_N(\mathbb{R}^n, g)}{I_N(\mathbb{R}^n, 1)} = 0. \quad (\text{B.1.2})$$

Given $t \in (0, 1)$, let

$$M_t = \{\varphi : V(\varphi) \leq t\}. \quad (\text{B.1.3})$$

By assumption, M_t is compact, and it clearly contains the set $\{\varphi : V(\varphi) < t\}$ which is open by continuity of V . This set is not empty because it contains 0. Therefore the integral $I_1(M_t, 1) = \int_{M_t} e^{-V} d\varphi$ is nonzero. On M_t^c , we have $e^{-NV} = e^{-(N-1)V} e^{-V} \leq e^{-(N-1)t} e^{-V}$. With this and a similar but reversed inequality on $M_{t/2}$, we obtain

$$|I_N(M_t^c, g)| \leq \|g\|_\infty e^{-(N-1)t} I_1(\mathbb{R}^n, 1), \quad (\text{B.1.4})$$

$$I_N(\mathbb{R}^n, 1) \geq I_N(M_{t/2}, 1) \geq e^{-(N-1)t/2} I_1(M_{t/2}, 1). \quad (\text{B.1.5})$$

Thus, for a t -dependent constant c_t ,

$$\frac{|I_N(M_t^c, g)|}{I_N(\mathbb{R}^n, 1)} \leq c_t e^{-Nt/2}. \quad (\text{B.1.6})$$

Given $\varepsilon > 0$, choose $\delta > 0$ small enough that $|g(\varphi)| < \varepsilon$ if $|\varphi| < \delta$. Since $\cap_{t>0} M_t = \{0\}$ we have $\cap_{t>0} M_t \cap \{|\varphi| \geq \delta\} = \emptyset$. By $M_t \subset M_{t'}$ for $t > t'$ and the finite

intersection property for compact sets, there exists t_δ such that $M_{t_\delta} \cap \{|\varphi| \geq \delta\} = \emptyset$. Therefore $M_{t_\delta} \subset \{|\varphi| < \delta\}$. Then, with $t = t_\delta$,

$$|I_N(M_t, g)| \leq \varepsilon I_N(M_t, 1) \leq \varepsilon I_N(\mathbb{R}^n, 1), \quad (\text{B.1.7})$$

and hence

$$\frac{|I_N(\mathbb{R}^n, g)|}{I_N(\mathbb{R}^n, 1)} \leq \frac{|I_N(M_t, g)|}{I_N(\mathbb{R}^n, 1)} + c_t e^{-Nt/2} \leq \varepsilon + c_t e^{-Nt/2}. \quad (\text{B.1.8})$$

Consequently the $\limsup_{N \rightarrow \infty}$ of the left-hand side is at most ε . Since ε is arbitrary, the limit must exist and equal zero.

Solution to Exercise 1.4.7. By definition and since $|\sigma| = 1$,

$$\begin{aligned} V(\varphi) &= -\log \int_{S^2} e^{-\frac{\beta}{2}|\varphi - \sigma|^2 + h \cdot \sigma} \mu(d\sigma) \\ &= \frac{\beta}{2}|\varphi|^2 + \frac{\beta}{2} - \log \int_{S^2} e^{(\beta\varphi + h) \cdot \sigma} \mu(d\sigma). \end{aligned} \quad (\text{B.1.9})$$

In spherical coordinates,

$$\begin{aligned} \int_{S^2} e^{(\beta\varphi + h) \cdot \sigma} \mu(d\sigma) &= \frac{1}{2} \int_0^\pi e^{|\beta\varphi + h| \cos \theta} \sin \theta d\theta \\ &= \frac{1}{2} \int_{-1}^1 e^{|\beta\varphi + h| u} du = \frac{\sinh(|\beta\varphi + h|)}{|\beta\varphi + h|}. \end{aligned} \quad (\text{B.1.10})$$

Solution to Exercise 1.4.8. Let $\varphi \in \mathbb{R}^n$, $h \in \mathbb{R}^n$, $e \in S^{n-1}$. Denote by μ_φ^h the measure from (1.4.15) with external field $h \in \mathbb{R}^n$. From (1.4.13) it follows that

$$e \cdot \text{Hess}V(\varphi)e = \beta - \beta^2 \text{Var}_{\mu_\varphi^h}(e \cdot \sigma). \quad (\text{B.1.11})$$

(i) By [76, Theorem D.2],

$$\text{Var}_{\mu_\varphi^h}(e \cdot \sigma) \leq \text{Var}_{\mu_0^0}(e \cdot \sigma) = \frac{1}{n}. \quad (\text{B.1.12})$$

Therefore, $e \cdot \text{Hess}V(\varphi)e \geq \beta - \beta^2/n$. When $\beta \leq n$, the right-hand side is indeed non-negative.

(ii) Note that

$$\frac{\beta}{2}(\varphi - \sigma, \varphi - \sigma) - (h, \sigma) = \frac{\beta}{2}(\varphi, \varphi) - (\beta\varphi + h, \sigma) + \text{constant}. \quad (\text{B.1.13})$$

Hence, for $\beta\varphi + h = 0$, the measure μ_φ^h on S^{n-1} is uniform and thus

$$\text{Var}_{\mu_\phi^h}(e \cdot \sigma) = \frac{1}{n}. \quad (\text{B.1.14})$$

For $\phi = -h/\beta$ and any $e \in S^{n-1}$, this implies that $e \cdot \text{Hess}V(\phi)e = \beta - \beta^2/n$. The right-hand side is negative if $\beta > n$, so V is non-convex.

Solution to Exercise 1.5.2. For $f \in \mathbb{R}^\Lambda$ define $(f, g) = \sum_{x \in \Lambda} f_x g_x$ and $(\nabla f)_{xy} = f_x - f_y$. By (1.3.4)

$$\begin{aligned} (-\Delta_\beta f, f) &= - \sum_{x \in \Lambda} \left(\sum_{y \in \Lambda} \beta_{xy} (\nabla f)_{yx} \right) f_x = -\frac{1}{2} \sum_{x, y \in \Lambda} \beta_{xy} (\nabla f)_{yx} (f_x - f_y) \\ &= \frac{1}{2} \sum_{x, y \in \Lambda} \beta_{xy} (\nabla f)_{yx}^2 \geq 0. \end{aligned} \quad (\text{B.1.15})$$

The second equality is obtained by interchanging x, y , $(\nabla f)_{yx} = -(\nabla f)_{xy}$ and recalling that $\beta_{xy} = \beta_{yx}$. Also, $(-\Delta_\beta \mathbb{1})_x = -\sum_{y \in \Lambda} \beta_{xy} (\nabla \mathbb{1})_{yx} = 0$, and (1.5.3) follows by applying the inverse operator to $\mathbb{1} = m^{-2}(-\Delta_\beta + m^2)\mathbb{1}$.

Solution to Exercise 1.5.4. Since $\lambda(k) \sim |k|^2$ as $k \rightarrow 0$, we see from (1.5.29) that B_0 is finite if and only if $d > 4$. So it remains to prove that

$$B_{m^2} \sim b_d \times \begin{cases} m^{-(4-d)} & (d < 4) \\ \log m^{-2} & (d = 4), \end{cases} \quad (\text{B.1.16})$$

with $b_1 = \frac{1}{8}$, $b_2 = \frac{1}{4\pi}$, $b_3 = \frac{1}{8\pi}$, $b_4 = \frac{1}{16\pi^2}$.

Let $d \leq 4$. By (1.5.29),

$$B_{m^2} = \int_{[-\pi, \pi]^d} \left| \frac{1}{4 \sum_{j=1}^d \sin^2(\frac{k_j}{2}) + m^2} \right|^2 \frac{dk}{(2\pi)^d}. \quad (\text{B.1.17})$$

Let U_1 be the ball of radius 1 in $\mathbb{R}^d$. Then, uniformly as $m^2 \downarrow 0$,

$$\int_{[-\pi, \pi]^d \setminus U_1} \left| \frac{1}{4 \sum_{j=1}^d \sin^2(\frac{k_j}{2}) + m^2} \right|^2 \frac{dk}{(2\pi)^d} = O(1), \quad (\text{B.1.18})$$

and, uniformly in $k \in U_1$,

$$\frac{1}{4 \sum_{j=1}^d \sin^2(\frac{k_j}{2}) + m^2} = \frac{1}{|k|^2 + m^2} + O(1). \quad (\text{B.1.19})$$

Therefore,

$$B_{m^2} \sim \int_{U_1} \left(\frac{1}{|k|^2 + m^2} \right)^2 \frac{dk}{(2\pi)^d}. \quad (\text{B.1.20})$$

We use polar coordinates to obtain

$$\int_{U_1} \left(\frac{1}{|k|^2 + m^2} \right)^2 \frac{dk}{(2\pi)^d} = \frac{\omega_{d-1}}{(2\pi)^d} \int_0^1 \left(\frac{1}{r^2 + m^2} \right)^2 r^{d-1} dr \quad (\text{B.1.21})$$

where $\omega_0 = 1$, $\omega_1 = 2\pi$, $\omega_2 = 4\pi$, $\omega_3 = 2\pi^2$ arise from the area of the unit $(d-1)$ -sphere. With the change of variables $r = sm$, this gives

$$B_{m^2} \sim \frac{\omega_{d-1}}{(2\pi)^d} m^{d-4} \int_0^{m^{-1}} \left(\frac{1}{s^2 + 1} \right)^2 s^{d-1} ds. \quad (\text{B.1.22})$$

If $d < 4$ then the integral converges to a finite limit as $m \downarrow 0$, and if $d = 4$ then it is asymptotic to $\log m^{-1}$. For $d < 4$, the value of the integral is given by [102, 3.241] as

$$\int_0^\infty \left(\frac{1}{s^2 + 1} \right)^2 s^{d-1} ds = \frac{1}{2} \frac{\Gamma(d/2)\Gamma(2-d/2)}{\Gamma(2)}. \quad (\text{B.1.23})$$

This leads to

$$\int_0^\infty \left(\frac{1}{s^2 + 1} \right)^2 s^{d-1} ds = \begin{cases} \frac{\pi}{4} & (d=1) \\ \frac{1}{2} & (d=2) \\ \frac{\pi}{4} & (d=3). \end{cases} \quad (\text{B.1.24})$$

From this we obtain the constants b_d reported below (B.1.16).

Solution to Exercise 1.5.5. (i) Let $T_0 = 0$ and $T_k = \inf\{n > T_{k-1} : S_n = 0\}$. Then $u = P(T_1 < \infty)$, and by induction and the strong Markov property, $P(T_k < \infty) = u^k$. Therefore,

$$m = EN = \sum_{k \geq 0} P(T_k < \infty) = (1 - u)^{-1}. \quad (\text{B.1.25})$$

(ii) Let S_n denote simple random walk. The equality $EN = \sum_{n \geq 0} p_n(0)$ follows from the identity $N = \sum_{n=0}^\infty \mathbb{1}_{S_n=0}$. Let $D(x) = \frac{1}{2d} \mathbb{1}_{|x|=1}$. Then $\hat{D}(k) = \frac{1}{d} \sum_{j=1}^d \cos k_j$ and $1 - \hat{D}(k) = \frac{1}{2d} \lambda(k)$. Also,

$$P(S_n = 0) = \int_{[-\pi, \pi]^d} \hat{D}(k)^n \frac{d^d k}{(2\pi)^d}. \quad (\text{B.1.26})$$

Some care is required to perform the sum over n since the best uniform bound on $\hat{D}^n$ is 1 which is not summable. By monotone convergence, and then by the dominated convergence theorem,

$$m = \lim_{t \nearrow 1} \sum_{n \geq 0} P(S_n = 0) t^n = \lim_{t \nearrow 1} \int_{[-\pi, \pi]^d} \frac{1}{1 - t \hat{D}(k)} \frac{d^d k}{(2\pi)^d}. \quad (\text{B.1.27})$$

The function $\hat{D}$ is real valued, and

$$\frac{1}{1-t\hat{D}(k)} \leq \frac{2}{1-\hat{D}(k)} \quad \text{for } t \in [1/2, 1]. \quad (\text{B.1.28})$$

If $(1-\hat{D})^{-1} \in L^1$, the claim then follows by dominated convergence. If $(1-\hat{D})^{-1} \notin L^1$, the claim follows from Fatou's lemma.

(iii) This follows from the fact that $\lambda(k) \asymp |k|^2$ as $k \rightarrow 0$, and thus $1/\lambda(k)$ is integrable if and only if $d > 2$.

Solution to Exercise 1.5.6. By definition,

$$I = \sum_{x \in \mathbb{Z}^d} \left(\sum_{m=0}^{\infty} \mathbb{1}_{S_m^1=x} \right) \left(\sum_{n=0}^{\infty} \mathbb{1}_{S_n^2=x} \right). \quad (\text{B.1.29})$$

By Lemma 1.5.3 (with monotone convergence to take the limit $m^2 \downarrow 0$),

$$E \sum_{i \geq 0} \mathbb{1}_{X_i^1=x} = 2dC_{0x}(0). \quad (\text{B.1.30})$$

Therefore, by using the independence, we have

$$EI = (2d)^2 \sum_{x \in \mathbb{Z}^d} (C_{0x}(0))^2 = (2d)^2 B_0. \quad (\text{B.1.31})$$

The sum is infinite if and only if both sides are infinite. The latter happens if and only if $d \leq 4$, by Exercise 1.5.4.

B.2 Chapter 2 exercises

Solution to Exercise 2.1.3. For notational convenience, we consider the case where C is strictly positive definite. The semi-definite case can be handled by replacing C by C' as in (2.1.3). Let $A = C^{-1}$, so that $P_C(d\varphi)$ is proportional to $e^{-\frac{1}{2}(\varphi, A\varphi)} d\varphi$. Then standard integration by parts and the symmetry of the matrix C give

$$\int \frac{\partial F}{\partial \varphi_y} e^{-\frac{1}{2}(\varphi, A\varphi)} d\varphi = \int (A\varphi)_y F e^{-\frac{1}{2}(\varphi, A\varphi)} d\varphi. \quad (\text{B.2.1})$$

Now we multiply by C_{xy} , sum over y , and use $CA = I$. This gives

$$\sum_y C_{xy} \int \frac{\partial F}{\partial \varphi_y} e^{-\frac{1}{2}(\varphi, A\varphi)} d\varphi = \int F \varphi_x e^{-\frac{1}{2}(\varphi, A\varphi)} d\varphi, \quad (\text{B.2.2})$$

as required.

Solution to Exercise 2.1.7. By definition,

$$\text{Cov}_C(\varphi_x^p, \varphi_{x'}^{p'}) = \mathbb{E}_C(\varphi_x^p \varphi_{x'}^{p'}) - (\mathbb{E}_C \varphi_x^p)(\mathbb{E}_C \varphi_{x'}^{p'}). \quad (\text{B.2.3})$$

The estimate is obtained by bounding each expectation on the right-hand side using (2.1.11), without any attention to cancellation between the two terms. The operator $e^{\frac{1}{2}\Delta_C}$ is defined by power series expansion, and in using (2.1.11) nonzero contributions can arise only when all fields are differentiated. For the term $\mathbb{E}_C(\varphi_x^p \varphi_{x'}^{p'})$, this differentiation leads to $(p + p')/2$ factors of the covariance, which can be factors C_{xx} , $C_{xx'}$, or $C_{x'x'}$. The covariance is maximal on the diagonal since it is positive definite, so these factors are all bounded by $\|C\|$ and hence the term $\mathbb{E}_C(\varphi_x^p \varphi_{x'}^{p'})$ obeys the desired estimate. The subtracted term $(\mathbb{E}_C \varphi_x^p)(\mathbb{E}_C \varphi_{x'}^{p'})$ is similar.

Solution to Exercise 2.1.10. This follows from (2.1.17), using (with $A = C^{-1}$)

$$\begin{aligned} \int e^{(f, \varphi)} Z_0(\varphi) e^{-\frac{1}{2}(\varphi, A \varphi)} d\varphi &= e^{\frac{1}{2}(f, C f)} \int Z_0(\varphi) e^{-\frac{1}{2}(\varphi - C f, A(\varphi - C f))} d\varphi \\ &= e^{\frac{1}{2}(f, C f)} \int Z_0(\varphi + C f) e^{-\frac{1}{2}(\varphi, A \varphi)} d\varphi. \end{aligned} \quad (\text{B.2.4})$$

Solution to Exercise 2.1.13. (i) Let $X = \{(x, i) : x \in \Lambda, i = 1, \dots, n\}$ and $\hat{C}_{(x, i), (y, j)} = \delta_{ij} C_{xy}$. According to Example 2.1.4 and Proposition 2.1.9, the n -component Gaussian field $\varphi = (\varphi_x^i)_{x \in \Lambda, i=1, \dots, n}$ with covariance C is characterised by

$$\mathbb{E}_C(e^{\langle \hat{f}, \varphi \rangle}) = e^{\frac{1}{2}(\hat{f}, \hat{C} \hat{f})} \quad \text{for } \hat{f} = (f_x^i) \in \mathbb{R}^X. \quad (\text{B.2.5})$$

The form of $\hat{C}$ implies this is the same as

$$\mathbb{E}_C(e^{\langle \hat{f}, \varphi \rangle}) = \prod_i e^{\frac{1}{2}(f^i, C f^i)} \quad \text{for } f^i = (f_x^i)_{x \in \Lambda} \in \mathbb{R}^\Lambda \text{ and } i = 1, \dots, n. \quad (\text{B.2.6})$$

The factorisation on the right-hand side implies that the components φ^i are independent and by Proposition 2.1.9 applied to each component the components are identically distributed Gaussian fields on Λ with covariance C , as desired.

(ii) The set of functions F for which

$$\mathbb{E}_C(F(\varphi)) = \mathbb{E}_C(F(T\varphi)) \quad (\text{B.2.7})$$

holds is a vector space closed under bounded convergence and under monotone convergence. Exponential functions generate the Borel σ -algebra in $\mathbb{R}^X$ and form a class closed under multiplication. Hence if (B.2.7) holds for exponential functions then it holds for all bounded Borel functions F . For exponential functions we evaluate and compare both sides of (B.2.7) using Proposition 2.1.9.

$$\mathbb{E}_C(e^{(f,T\varphi)}) = \mathbb{E}_C(e^{(T^t f, \varphi)}) = e^{\frac{1}{2}(T^t \hat{f}, \hat{C} T^t \hat{f})} = e^{\frac{1}{2}(\hat{f}, \hat{C} \hat{f})} = \mathbb{E}_C(e^{(f, \varphi)}). \quad (\text{B.2.8})$$

The formula $\mathbb{E}_C \theta \circ T = T \circ \mathbb{E}_C \theta$ is obtained from (B.2.7) by renaming the random variable φ to ζ followed by replacing $F(\zeta)$ by $F(\varphi + \zeta)$ where φ is a fixed element of $\mathbb{R}^X$.

Solution to Exercise 2.1.15. We apply (e.g.) [87, Theorem 10.18] with $A = \mathbb{Z}^d$. By Corollary 2.1.14, the finite-dimensional distributions are consistent, and the permutation hypothesis of [87, Theorem 10.18] follows from the definition of the Gaussian measure with covariance $C_{X \times X}$ for finite X .

Solution to Exercise 2.2.2. Let I be a finite nonempty subset of natural numbers. A partition π of I is a collection of disjoint nonempty subsets of I whose union is I . In particular, $\{I\}$ is a partition of I . Let $\Pi(I)$ be the set of all partitions of I . Given a natural number n and coefficients μ_I for all I of cardinality $|I| \leq n$, define coefficients κ_J for all finite subsets J with $|J| \leq n$ to be the unique solution of the system of equations

$$\mu_I = \sum_{\pi \in \Pi(I)} \prod_{J \in \pi} \kappa_J, \quad (\text{B.2.9})$$

where there is one equation for each I with $|I| \leq n$. To show that this system has a unique solution, rewrite it as

$$\kappa_I = \mu_I - \sum_{\pi \in \Pi(I) \setminus \{I\}} \prod_{J \in \pi} \kappa_J. \quad (\text{B.2.10})$$

For any finite I , this defines κ_I in terms of μ_I and recursively in terms of κ_J , where J runs over proper subsets of I . Thus we obtain a formula for κ_I in terms of μ_J by inserting the recursion into itself. Since J is a proper subset of I , the recursion terminates after a finite number of steps determined by the cardinality $|I|$. For $I = \{i\}$ the recursion reduces to $\kappa_{\{i\}} = \mu_{\{i\}}$ because $\{I\}$ has no proper subsets and empty sums are by definition zero. By induction on $|I|$, the coefficient κ_I is a finite sum of finite products of μ_J with $|J| \leq |I|$. Conversely, given κ_J for $|J| \leq n$ the formula (B.2.9) constructs μ_I for $|I| \leq n$.

We assume the existence of exponential moments as required by the definition of cumulants in (2.2.1), and set $\mu_I = \mathbb{E}(A_{i_1} \cdots A_{i_n})$ for all $I = \{i_1, \dots, i_n\}$. We claim that $\kappa_{i_1, \dots, i_n} = \mathbb{E}(A_{i_1}; \dots; A_{i_n})$. This claim proves the desired result. In particular, the cumulant of order n exists precisely when expectations up to order n exist.

To prove the claim, for arbitrary $I = \{i_1, \dots, i_n\}$ let

$$\partial_I = \frac{\partial^n}{\partial t_{i_1} \cdots \partial t_{i_n}},$$

and define f_I by $f_I(t_{j_1}, \dots, t_{j_n}) = \log \mathbb{E}(e^{t_{i_1} A_{i_1} + \cdots + t_{i_n} A_{i_n}})$. By the chain rule and induction on $|I|$,

$$\partial_I e^{f_I} = \left(\sum_{\pi \in \Pi(I)} \prod_{J \in \pi} \partial_J f_J \right) e^{f_I}.$$

Set $t_{i_1}, \dots, t_{i_n} = 0$. By the definition of f_I , the left-hand side is $\mu_I = \mathbb{E}(A_{i_1} \cdots A_{i_n})$. By comparing the above equation with (B.2.9), and noting that $e^{f_I} = 1$ at $t_{i_1}, \dots, t_{i_n} = 0$, we have $\kappa_I = \partial_I f_I$ for all I . By the definition of the truncated expectation, $\partial_I f_I = \mathbb{E}(A_{i_1}; \dots; A_{i_n})$, so we have proved the claim that $\mathbb{E}(A_{i_1}; \dots; A_{i_n}) = \kappa_I$, as desired.

Solution to Exercise 2.2.3. Suppose first that φ is Gaussian with covariance C . By Proposition 2.1.9,

$$\mathbb{E}_C(e^{\sum_{i=1}^p t_i \varphi_{x_i}}) = e^{\frac{1}{2} \sum_{i,j=1}^p t_i t_j C_{x_i x_j}}. \quad (\text{B.2.11})$$

By (2.2.1), the cumulants are derivatives of the right-hand side, and therefore (2.2.4) holds, as desired.

Suppose next that for all $p \in \mathbb{N}$ and $x_1, \dots, x_p \in X$,

$$\mathbb{E}(\varphi_{x_1}; \dots; \varphi_{x_p}) = \begin{cases} C_{x_1 x_2} & (p = 2) \\ 0 & (p \neq 2). \end{cases} \quad (\text{B.2.12})$$

By Exercise 2.2.2, the truncated expectations up to order n determine the expectations up to order n . Therefore all moments are the same as those of a Gaussian with covariance C . This implies that

$$\sum_{n=0}^{\infty} \frac{1}{n!} \mathbb{E}((f, \varphi)^n) \quad (\text{B.2.13})$$

is equal to the sum over even n (because odd Gaussian moments are zero) and therefore converges by monotone convergence to $\mathbb{E}(e^{(f, \varphi)})$. However, since the moments are Gaussian, the above sum is equal to $e^{\frac{1}{2}(f, Cf)}$. It follows that $\mathbb{E}(e^{(f, \varphi)}) = e^{\frac{1}{2}(f, Cf)}$. By Proposition 2.1.9, this proves that the field is Gaussian with covariance C , and the proof is complete.

Solution to Exercise 2.2.4. Let A, B be polynomials in φ degree at most p . By definition,

$$F_C(A, B) = e^{\frac{1}{2} \Delta_C} ((e^{-\frac{1}{2} \Delta_C} A) (e^{-\frac{1}{2} \Delta_C} B)) - AB. \quad (\text{B.2.14})$$

We must show that

$$F_C(A, B) = \sum_{n=1}^p \frac{1}{n!} \sum_{x_1, y_1} \cdots \sum_{x_n, y_n} C_{x_1, y_1} \cdots C_{x_n, y_n} \frac{\partial^n A}{\partial \varphi_{x_1} \cdots \partial \varphi_{x_n}} \frac{\partial^n B}{\partial \varphi_{y_1} \cdots \partial \varphi_{y_n}}. \quad (\text{B.2.15})$$

Define

$$\mathcal{L}_C = \frac{1}{2} \Delta_C = \frac{1}{2} \sum_{u,v \in \Lambda} C_{u,v} \partial_{\varphi_u} \partial_{\varphi_v} \quad \overset{\leftrightarrow}{\mathcal{L}}_C = \sum_{u,v \in \Lambda} C_{u,v} \partial_{\varphi'_u} \partial_{\varphi''_v} \quad (\text{B.2.16})$$

$$\mathcal{L}'_C = \frac{1}{2} \sum_{u,v \in \Lambda} C_{u,v} \partial_{\varphi'_u} \partial_{\varphi'_v} \quad \mathcal{L}''_C = \frac{1}{2} \sum_{u,v \in \Lambda} C_{u,v} \partial_{\varphi''_u} \partial_{\varphi''_v}. \quad (\text{B.2.17})$$

Then (B.2.14) becomes

$$\begin{aligned} F_C(A, B) &= e^{\mathcal{L}'_C + \mathcal{L}''_C + \overset{\leftrightarrow}{\mathcal{L}}_C} ((e^{-\mathcal{L}'_C} A(\varphi')) (e^{-\mathcal{L}''_C} B(\varphi''))) \Big|_{\varphi' = \varphi'' = \varphi} - AB \\ &= e^{\overset{\leftrightarrow}{\mathcal{L}}_C} (A(\varphi') B(\varphi'')) \Big|_{\varphi' = \varphi'' = \varphi} - AB, \end{aligned} \quad (\text{B.2.18})$$

and (B.2.15) follows by expanding the exponential.

B.3 Chapter 3 exercises

Solution to Exercise 3.1.2. Two random variables X and Y are independent if their distribution is a product measure. Provided that both random variables have exponential moments, this is equivalent to the factorisation of the Laplace transform:

$$\mathbb{E}(e^{tX+sY}) = \mathbb{E}(e^{tX}) \mathbb{E}(e^{sY}), \quad (\text{B.3.1})$$

since the distribution of (X, Y) is characterised by the Laplace transform and the Laplace transform of independent random variables factorises.

Consider now the special case $X = \varphi_x$ and $Y = \varphi_y$, and let $C_{xy} = \mathbb{E}(\varphi_x \varphi_y)$. By assumption, $C_{xy} = 0$ for $x \neq y$. The above factorisation now follows from (2.1.16), which implies that

$$\mathbb{E}(e^{t\varphi_x + s\varphi_y}) = e^{\frac{1}{2}(t^2 C_{xx} + s^2 C_{yy} + 2st C_{xy})} = e^{\frac{1}{2}(t^2 C_{xx} + s^2 C_{yy})} = \mathbb{E}(e^{t\varphi_x}) \mathbb{E}(e^{s\varphi_y}). \quad (\text{B.3.2})$$

This completes the proof.

Solution to Exercise 3.2.2. If h is even then $h * h$ is even since

$$\begin{aligned} h * h(x) &= \int_{\mathbb{R}^d} h(x-y) h(y) dy = \int_{\mathbb{R}^d} h(-x+y) h(y) dy \\ &= \int_{\mathbb{R}^d} h(-x-y) h(-y) dy = \int_{\mathbb{R}^d} h(-x-y) h(y) dy = h * h(-x). \end{aligned} \quad (\text{B.3.3})$$

Since $\widehat{h * h} = \hat{h}^2$, and since $\hat{h}$ is real because h is even, we see that $\widehat{h * h} \geq 0$. Thus the positive definiteness of $h * h$ follows from the more general statement about f .

To prove the more general statement, suppose f has non-negative Fourier transform. Then for $v \in \mathbb{R}^n$ we have

$$\begin{aligned} \sum_{l,m} v_l f(x_l - x_m) v_m &= \sum_{l,m} v_l v_m \int_{\mathbb{R}^d} \hat{f}(k) e^{ik \cdot (x_l - x_m)} \frac{dk}{(2\pi)^d} \\ &= \int_{\mathbb{R}^d} \hat{f}(k) \sum_l |v_l e^{ik \cdot x_l}|^2 \frac{dk}{(2\pi)^d} \geq 0. \end{aligned} \quad (\text{B.3.4})$$

Solution to Exercise 3.2.4. The Schwartz–Paley–Wiener Theorem states that a Schwartz distribution g on $\mathbb{R}^d$ has support in a ball of radius R if its Fourier transform $\hat{g}$ is entire on $\mathbb{C}^d$ and satisfies the growth estimate

$$|\hat{g}(k)| \leq C(1 + |k|)^N e^{R|\operatorname{Im}(k)|} \quad (k \in \mathbb{C}^d) \quad (\text{B.3.5})$$

for some constants C and N . Thus it suffices to prove that the function $f(|k|) = \frac{1}{2\pi} \int_{-1}^1 \hat{f}(s) \cos(|k|s) ds$ is entire in k and obeys an estimate of the form (B.3.5). Let $\hat{g}(k) = f(|k|)$. The function $\cos(|k|s) = \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m)!} (s^2 k^2)^m$ is a convergent series in powers of the components $k = (k_1, \dots, k_d)$ and therefore is entire in k . By Morera's theorem, with interchange of integrals over k and s , $\hat{g}$ is indeed entire. We will prove below that

$$|\cos(|k|)| \leq e^{|\operatorname{Im}(k)|}. \quad (\text{B.3.6})$$

Given this, it follows, as desired, that

$$\begin{aligned} |\hat{g}(k)| &= \frac{1}{2\pi} \left| \int_{-1}^1 \hat{f}(s) \cos(|k|s) ds \right| \\ &\leq \frac{1}{2\pi} \int_{-1}^1 |\hat{f}(s)| e^{|\operatorname{Im}(k)|s} ds \leq C e^{|\operatorname{Im}(k)|}. \end{aligned} \quad (\text{B.3.7})$$

It remains only to prove (B.3.6). We use the branch of the square root with branch cut $(-\infty, 0)$ and with positive real numbers having positive square root. This branch of the square root is analytic on the cut plane $\mathbb{C} \setminus (-\infty, 0)$. It suffices to prove that for k in the cut plane,

$$|\cos(\sqrt{k_1^2 + \dots + k_d^2})| \leq e^{|\operatorname{Im}(k)|}. \quad (\text{B.3.8})$$

Let $k_1^2 + \dots + k_d^2 = A + Bi$ with $A, B \in \mathbb{R}$. For $j = 1, \dots, d$, let $k_j = u_j + iv_j$ with $u_j, v_j \in \mathbb{R}$, and let $u = (u_1, \dots, u_d)$, $v = (v_1, \dots, v_d)$. Since $\cos \sqrt{A + iB} = \frac{1}{2}(e^{i\sqrt{A+iB}} + e^{-i\sqrt{A+iB}})$, it suffices to prove that

$$|\operatorname{Im} \sqrt{A + iB}| \leq |v|. \quad (\text{B.3.9})$$

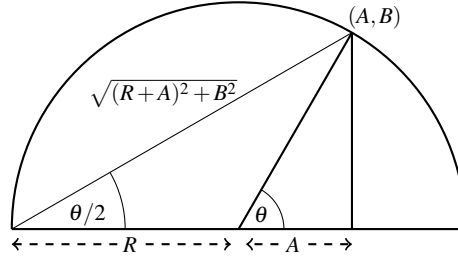


Fig. B.1 Illustration of (B.3.10).

We use polar coordinates to write $A + iB = Re^{i\theta}$ with $\theta \in (-\pi, \pi)$ and $R = \sqrt{A^2 + B^2}$. From Figure B.1, we see that

$$|\operatorname{Im} \sqrt{A + iB}| = \sqrt{R} |\sin(\theta/2)| = \frac{\sqrt{R}|B|}{\sqrt{(R+A)^2 + B^2}} = \frac{1}{\sqrt{2}} \frac{|B|}{\sqrt{R+A}}. \quad (\text{B.3.10})$$

It therefore suffices to prove that

$$\frac{B^2}{R+A} \leq 2v \cdot v. \quad (\text{B.3.11})$$

By construction,

$$A = u \cdot u - v \cdot v, \quad B = 2(u \cdot v), \quad R^2 = A^2 + B^2. \quad (\text{B.3.12})$$

Thus (B.3.11) is equivalent to $B^2 - 2v^2A \leq 2v^2R$, which is implied by $(B^2 - 2v^2A)^2 \leq 4(v^2)^2R^2$. The latter is equivalent to $B^2 - 4v^2A \leq 4(v^2)^2$, which in turn is equivalent to $4(u \cdot v)^2 - 4v^2(u^2 - v^2) \leq 4(v^2)^2$. This last inequality follows from the Cauchy–Schwarz inequality $|u \cdot v| \leq |u||v|$. This proves (B.3.8) and completes the proof.

Solution to Exercise 3.3.4. For $p \in \mathbb{Z}$,

$$\begin{aligned} \int_0^{2\pi} f_t^*(x) \cos(px) dx &= \sum_{n \in \mathbb{Z}} \int_0^{2\pi} f(xt - 2\pi nt) e^{ipx} dx \\ &= \int_0^\infty f(xt) e^{ipx} dx = \frac{1}{t} \hat{f}(p/t), \end{aligned}$$

and then (3.3.25) follows by Fourier series inversion for the even periodic function f_t^* .

B.4 Chapter 4 exercises

Solution to Exercise 4.1.3. We assign a *generation* to each vertex in the tree as follows: a vertex at distance k from the root has generation $N - k$. Thus each leaf is at generation 0, a vertex adjacent to a leaf has generation 1, and the root has generation N . Any edge in the tree joins two vertices at subsequent generations $j - 1$ and j (say), and we say this edge has generation j . Given $j \in \{1, \dots, N\}$, we assign to each edge at generation j an independent Gaussian random variable ζ_j with a covariance C_j . Random variables from different generations are independent. A leaf corresponds to a point $x \in \Lambda_N$. Then we set $\varphi_x = \zeta_1 + \dots + \zeta_N$.

Solution to Exercise 4.1.6. We first show that the range of Q_j is X_j . Let $\varphi \in \ell^2$ and let $x \in B$ for some j -block B . Then

$$(Q_j \varphi)_x = \sum_{y \in A} Q_{j,xy} \varphi_y = \sum_{y \in B} L^{-dj} \varphi_y, \quad (\text{B.4.1})$$

and since the right-hand side is the same for every $x \in B$, we see that $Q_j \varphi \in X_j$. Also, if $\varphi \in X_j$, so that for all $x \in B$ we have $\varphi_x = c_B$ for some constant c_B , the above calculation gives $(Q_j \varphi)_x = c_B$, so $Q_j \varphi = \varphi$. This proves that the range of Q_j is X_j .

Next we show that the range of P_j is orthogonal to X_j . Let $\psi \in X_j$, so there are constants c_B such that $\psi_x = c_{B_x}$. Then

$$\begin{aligned} (\psi, P_j \varphi) &= (\psi, Q_{j-1} \varphi) - (\psi, Q_j \varphi) \\ &= \sum_{B \in \mathcal{B}_j} \sum_{b \in \mathcal{B}_{j-1}(B)} \sum_{x \in b} c_B \sum_{y \in b} L^{-d(j-1)} \varphi_y - \sum_{B \in \mathcal{B}_j} \sum_{x \in B} c_B \sum_{y \in B} L^{-dj} \varphi_y \\ &= \sum_{B \in \mathcal{B}_j} c_B \sum_{y \in B} \varphi_y - \sum_{B \in \mathcal{B}_j} c_B \sum_{y \in B} \varphi_y = 0. \end{aligned} \quad (\text{B.4.2})$$

Finally, we prove that the range of P_j is $X_{j-1} \cap X_j^\perp$. Clearly the range of P_j is contained in X_{j-1} , since X_{j-1} is the range of Q_{j-1} and the range of Q_j is $X_j \subset X_{j-1}$. We have the direct sum decomposition

$$X_{j-1} = (X_{j-1} \cap X_j) \oplus (X_{j-1} \cap X_j^\perp) = X_j \oplus (X_{j-1} \cap X_j^\perp), \quad (\text{B.4.3})$$

so $\varphi \in X_{j-1}$ can be written uniquely as $\varphi = \psi + \eta$ with $\psi \in X_j$ and $\eta \in X_{j-1} \cap X_j^\perp$. Then $Q_{j-1} \psi = \psi$ since $\psi \in X_j \subset X_{j-1}$, $Q_j \psi = \psi$ since $\psi \in X_j$, $Q_{j-1} \eta = \eta$ since $\eta \in X_{j-1}$, and $Q_j \eta = 0$ since $\eta \in X_j^\perp$. Therefore,

$$P_j \varphi = Q_{j-1} \psi - Q_j \psi + Q_{j-1} \eta - Q_j \eta = \psi - \psi + \eta - 0 = \eta. \quad (\text{B.4.4})$$

This completes the proof.

Solution to Exercise 4.1.8. By (4.1.7), (4.1.3), (4.1.2),

$$\begin{aligned}\Delta_{H;0,0} &= - \sum_{j=1}^N L^{-2(j-1)} (L^{-dj} - L^{-d(j-1)}) \\ &= -(1 - L^{-d}) \sum_{j=1}^N L^{-(d+2)(j-1)} = -\frac{1 - L^{-d}}{1 - L^{-(d+2)}} (1 - L^{-(d+2)N}).\end{aligned}\quad (\text{B.4.5})$$

Also, for $x \neq 0$,

$$\begin{aligned}\Delta_{H;0,x} &= - \sum_{j=1}^N L^{-2(j-1)} (L^{-d(j-1)} \mathbb{1}_{j_x \leq j-1} - L^{-dj} \mathbb{1}_{j_x \leq j}) \\ &= - \sum_{j=j_x+1}^N L^{-2(j-1)} L^{-d(j-1)} + \sum_{j=j_x}^N L^{-2(j-1)} L^{-dj} \\ &= -(1 - L^{-d}) \sum_{j=j_x+1}^N L^{-(d+2)(j-1)} + L^2 L^{-(d+2)j_x} \\ &= -\frac{1 - L^{-d}}{1 - L^{-(d+2)}} (L^{-(d+2)j_x} - L^{-(d+2)N}) + L^2 L^{-(d+2)j_x} \\ &= \frac{L^2 - 1}{1 - L^{-(d+2)}} L^{-(d+2)j_x} + \frac{1 - L^{-d}}{1 - L^{-(d+2)}} L^{-(d+2)N}.\end{aligned}\quad (\text{B.4.6})$$

For $x \neq 0$, let n_k be the cardinality of $\{x : j_x = k\}$, namely $n_k = L^{dk} - L^{d(k-1)} = L^{dk}(1 - L^{-d})$. Then

$$\begin{aligned}\sum_{x \neq 0} \Delta_{H;0,x} &= \frac{L^2 - 1}{1 - L^{-(d+2)}} \sum_{k=1}^N L^{dk} (1 - L^{-d}) L^{-(d+2)k} \\ &\quad + (L^{dN} - 1) \frac{1 - L^{-d}}{1 - L^{-(d+2)}} L^{-(d+2)N} \\ &= \frac{1 - L^{-d}}{1 - L^{-(d+2)}} (1 - L^{-2N}) + (L^{dN} - 1) \frac{1 - L^{-d}}{1 - L^{-(d+2)}} L^{-(d+2)N} \\ &= \frac{1 - L^{-d}}{1 - L^{-(d+2)}} - \frac{1 - L^{-d}}{1 - L^{-(d+2)}} L^{-(d+2)N} \\ &= -\Delta_{H;0,0}.\end{aligned}\quad (\text{B.4.7})$$

A random walk with infinitesimal generator Q takes steps from a site x at rate $-Q_{x,x}$, and when the step is taken it is a step to y with probability $-Q_{x,y}/Q_{x,x}$. Here $Q = \Delta_{H,N}$ as in (4.1.8). The random walk can make a step to any site, and the probability to step from x to y decays with a factor $L^{-(d+2)j_{x-y}}$ where j_{x-y} is the smallest scale such that x and y are in the same block at that scale.

Solution to Exercise 4.1.10. By (4.1.15), $C = \sum_{j=1}^N C_j + C_{\hat{N}}$. By (4.1.10) and (4.1.16) we have $\sum_x C_{j;0x} = 0$, and by (4.1.10) and (4.1.2) we have $\sum_x C_{\hat{N};0x} = m^{-2}$.

Solution to Exercise 4.1.11. Let $M_j = (1 + m^2 L^{2j})^{-1}$. Then

$$\begin{aligned} c_j^{(2)} &= (L^{-(d-2)j} M_j)^2 \left(L^{dj} (1 - L^{-d})^2 + (L^{d(j+1)} - L^{dj}) (-L^{-d})^2 \right) \\ &= L^{-(d-4)j} M_j^2 (1 - L^{-d}). \end{aligned} \quad (\text{B.4.8})$$

$$\begin{aligned} c_j^{(3)} &= (L^{-(d-2)j} M_j)^3 \left(L^{dj} (1 - L^{-d})^3 + (L^{d(j+1)} - L^{dj}) (-L^{-d})^3 \right) \\ &= L^{-(2d-6)j} M_j^3 (1 - 3L^{-d} + 2L^{-2d}). \end{aligned} \quad (\text{B.4.9})$$

$$\begin{aligned} c_j^{(4)} &= (L^{-(d-2)j} M_j)^4 \left(L^{dj} (1 - L^{-d})^4 + (L^{d(j+1)} - L^{dj}) (-L^{-d})^4 \right) \\ &= L^{-(3d-8)j} M_j^4 (1 - 4L^{-d} + 6L^{-2d} - 4L^{-3d} + L^{-4d} + (L^d - 1)L^{-4d}) \\ &= L^{-(3d-8)j} M_j^4 (1 - 4L^{-d} + 6L^{-2d} - 3L^{-3d}). \end{aligned} \quad (\text{B.4.10})$$

Solution to Exercise 4.1.12. The infinite-volume hierarchical bubble diagram is given by

$$\begin{aligned} B_{m^2}^H &= \sum_x \left(\sum_{j=0}^{\infty} C_{j+1;0x}(m^2) \right)^2 \\ &= 2 \sum_{0 \leq j < k} \sum_x C_{j+1;0x} C_{k+1;0x} + \sum_{j=0}^{\infty} \sum_x C_{j+1;0x}^2, \end{aligned} \quad (\text{B.4.11})$$

with the sum over all $x \in \mathbb{Z}^d$. The sum over x in the first sum on the right-hand side is zero by (4.1.11). The second sum is $\sum_{j=0}^{\infty} c_j^{(2)}$, as required.

By Exercise 4.1.11,

$$\sum_{j=0}^{\infty} c_j^{(2)} = (1 - L^{-d}) \sum_{j=0}^{\infty} \frac{L^{(4-d)j}}{(1 + L^{2j} m^2)^2}, \quad (\text{B.4.12})$$

which converges for $d > 4$. For $d \leq 4$, the above is asymptotically $(1 - L^{-d})$ times (use change of variables $y = L^{2x}$ followed by $z = ym^2$)

$$\begin{aligned}
\int_0^\infty \frac{L^{(4-d)x}}{(1+L^{2x}m^2)^2} dx &= \frac{1}{\log L} \int_1^\infty \frac{y^{(4-d)/2}}{(1+ym^2)^2} \frac{dy}{y} \\
&= m^{d-4} \frac{1}{\log L} \int_{m^2}^\infty \frac{z^{(4-d)/2}}{(1+z)^2} \frac{dz}{z}.
\end{aligned} \tag{B.4.13}$$

The desired asymptotic behaviour then follows from the fact that the integral converges with lower limit zero if $d < 4$, whereas it diverges logarithmically if $d = 4$.

Solution to Exercise 4.1.13. (i) By definition,

$$\begin{aligned}
C(m^2) &= \sum_{j=1}^N \gamma_j P_j + m^{-2} Q_N = \sum_{j=1}^N \gamma_j (Q_{j-1} - Q_j) + m^{-2} Q_N \\
&= \gamma_1 Q_0 + \sum_{j=1}^{N-1} (\gamma_{j+1} - \gamma_j) Q_j + (m^{-2} - \gamma_N) Q_N.
\end{aligned} \tag{B.4.14}$$

(ii) The coalescence scale j_x is the smallest j such that $B_x = B_0$. Since 0 is at the corner of B_0 by Definition 4.1.1, $L^{j_x-1} < |x|_\infty \leq L^{j_x}$. In particular, $L^{j_x} \asymp |x|$, and also $\log_L |x|_\infty \leq j_x \leq \log_L |x|_\infty + 1$ so $j_x = \log_L |x| + O(1)$. We use the fact that if $j \geq j_x$, then $Q_{j;0x} = L^{-dj}$ and otherwise is it zero.

For $d > 2$, we are interested in the limit as $N \rightarrow \infty$ and then $m^2 \downarrow 0$ of (B.4.14). We take the limit of the right-hand side and obtain

$$\begin{aligned}
\lim_{m^2 \downarrow 0} \lim_{N \rightarrow \infty} C_{0x}(m^2) &= \lim_{m^2 \downarrow 0} \sum_{j=j_x}^\infty (\gamma_{j+1} - \gamma_j) L^{-dj} \\
&= \sum_{j=j_x}^\infty (L^{2j} - L^{2(j-1)}) L^{-dj} = (1 - L^{-2}) \sum_{j=j_x}^\infty L^{-(d-2)j} \\
&\asymp L^{-(d-2)j_x} \asymp |x|^{-(d-2)}.
\end{aligned} \tag{B.4.15}$$

For $d \leq 2$, we have instead

$$\begin{aligned}
\lim_{m^2 \downarrow 0} \lim_{N \rightarrow \infty} (C_{0x}(m^2) - C_{00}(m^2)) &= - \lim_{m^2 \downarrow 0} \sum_{j=1}^{j_x-1} (\gamma_{j+1} - \gamma_j) L^{-dj} \\
&= - \sum_{j=j_x-1}^\infty (L^{2j} - L^{2(j-1)}) L^{-dj} \\
&= -(1 - L^{-2}) \sum_{j=1}^{j_x-1} L^{(2-d)j}.
\end{aligned} \tag{B.4.16}$$

For $d = 1$, the right-hand side is $\asymp -L^{j_x} \asymp -|x|$, whereas for $d = 2$ it is equal to $-(1 - L^{-2})(j_x - 1) = -(1 - L^{-2}) \log_L |x| + O(1)$.

Solution to Exercise 4.3.1. By evaluation of the derivative, we see that

$$D^2 \Sigma_N(0; \mathbb{1}, \mathbb{1}) = \sum_{x, y \in \Lambda} \mathbb{E}_C(\varphi_x^1 \varphi_y^1 Z_0(\varphi)) = |\Lambda| \sum_{x \in \Lambda} \mathbb{E}_C(\varphi_0^1 \varphi_x^1 Z_0(\varphi)), \quad (\text{B.4.17})$$

which proves the first equality of (4.3.8). We may also compute the above derivative using the identity (4.3.6), which states that

$$\Sigma_N(f) = e^{\frac{1}{2}(f, Cf)} (\mathbb{E}_C \theta Z_0)(Cf). \quad (\text{B.4.18})$$

In this way, since $C\mathbb{1} = (m^{-2}, 0, \dots, 0)\mathbb{1}$ by Exercise 4.1.10, we obtain

$$D^2 \Sigma_N(0; \mathbb{1}, \mathbb{1}) = \frac{1}{m^2} |\Lambda| Z_N(0) + \frac{1}{m^4} D^2 Z_N(0; \mathbb{1}, \mathbb{1}). \quad (\text{B.4.19})$$

For example, the factor $m^{-2}|\Lambda|$ in the first term on the right-hand side arises from

$$\left. \frac{d^2}{ds dt} \right|_{s=t=0} e^{\frac{1}{2}((s+t)\mathbb{1}, C(s+t)\mathbb{1})} = \left. \frac{d^2}{ds dt} \right|_{s=t=0} e^{\frac{1}{2}(s+t)^2 m^{-2} |\Lambda|}. \quad (\text{B.4.20})$$

This proves the second equality of (4.3.8).

Solution to Exercise 4.3.2. (i) By definition,

$$\sum_{x, y, z \in \Lambda} \langle \varphi_0 \varphi_x \varphi_y \varphi_z \rangle_N = \frac{1}{|\Lambda|} \frac{D^4 \Sigma_N(0; \mathbb{1}, \mathbb{1}, \mathbb{1}, \mathbb{1})}{\Sigma_N(0)}. \quad (\text{B.4.21})$$

We write $\lambda = |\Lambda| m^{-2}$, $t_j = s_j + \dots s_4$, and $u_j = m^{-2} t_j$ (for $j = 1, 2, 3, 4$). We compute the numerator on the right-hand side using (4.3.6) with $f = t_1 \mathbb{1}$ and $Cf = u_1 \mathbb{1}$. This gives (with derivatives D^j having all j directions equal to $\mathbb{1}$)

$$\begin{aligned} & D^4 \Sigma_N(0; \mathbb{1}, \mathbb{1}, \mathbb{1}, \mathbb{1}) \\ &= \left. \frac{d^4}{ds_1 ds_2 ds_3 ds_4} \right|_0 e^{\frac{1}{2} t_1^2 \lambda} Z_{\hat{N}}(u_1 \mathbb{1}) \\ &= \left. \frac{d^3}{ds_2 ds_3 ds_4} \right|_0 e^{\frac{1}{2} t_2^2 \lambda} (\lambda t_2 Z_{\hat{N}}(u_2 \mathbb{1}) + m^{-2} D^1 Z_{\hat{N}}(u_2 \mathbb{1})) \\ &= \left. \frac{d^2}{ds_3 ds_4} \right|_0 e^{\frac{1}{2} t_3^2 \lambda} ((\lambda^2 t_3^2 + \lambda) Z_{\hat{N}}(u_3 \mathbb{1}) + 2\lambda t_3 m^{-2} D^1 Z_{\hat{N}}(u_3 \mathbb{1}) + m^{-4} D^2 Z_{\hat{N}}(u_3 \mathbb{1})) \\ &= \left. \frac{d}{ds_4} \right|_0 e^{\frac{1}{2} t_4^2 \lambda} \left((\lambda^3 t_4^3 + 3\lambda^2 t_4) Z_{\hat{N}}(u_4 \mathbb{1}) + 3(\lambda^2 t_4^2 + \lambda) m^{-2} D^1 Z_{\hat{N}}(u_4 \mathbb{1}) \right. \\ &\quad \left. + 3\lambda t_4 m^{-4} D^2 Z_{\hat{N}}(u_4 \mathbb{1}) + m^{-6} D^3 Z_{\hat{N}}(u_4 \mathbb{1}) \right) \\ &= 3\lambda^2 Z_{\hat{N}}(0) + 6\lambda m^{-4} D^2 Z_{\hat{N}}(0; \mathbb{1}, \mathbb{1}) + m^{-8} D^4 Z_{\hat{N}}(0; \mathbb{1}, \mathbb{1}, \mathbb{1}, \mathbb{1}). \end{aligned} \quad (\text{B.4.22})$$

Therefore,

$$\sum_{x,y,z \in \Lambda} \langle \varphi_0 \varphi_x \varphi_y \varphi_z \rangle_N = \frac{3|\Lambda|}{m^4} + \frac{6}{m^6} \frac{D^2 Z_{\hat{N}}(0; \mathbb{1}, \mathbb{1})}{Z_{\hat{N}}(0)} + \frac{1}{m^8 |\Lambda|} \frac{D^4 Z_{\hat{N}}(0; \mathbb{1}, \mathbb{1}, \mathbb{1}, \mathbb{1})}{Z_{\hat{N}}(0)}. \quad (\text{B.4.23})$$

Using Exercise 4.3.1, we subtract from this the quantity

$$3|\Lambda| \chi_N^2 = 3|\Lambda| \left(\frac{1}{m^4} + \frac{2}{m^6 |\Lambda|} \frac{D^2 Z_{\hat{N}}(0; \mathbb{1}, \mathbb{1})}{Z_{\hat{N}}(0)} + \frac{1}{m^8 |\Lambda|^2} \left(\frac{D^2 Z_{\hat{N}}(0; \mathbb{1}, \mathbb{1})}{Z_{\hat{N}}(0)} \right)^2 \right). \quad (\text{B.4.24})$$

This gives the desired formula for $\bar{u}_4$.

(ii) Direct calculation gives

$$\begin{aligned} \frac{D^4 e^{-V_N}(0; \mathbb{1}, \mathbb{1}, \mathbb{1}, \mathbb{1})}{e^{-V_N}(0)} - 3 \left(\frac{D^2 e^{-V_N}(0; \mathbb{1}, \mathbb{1})}{e^{-V_N}(0)} \right)^2 \\ = [-6g_N |\Lambda| + 3(v_N |\Lambda|)^2] - 3(-v_N |\Lambda|)^2 = -6g_N |\Lambda|. \end{aligned} \quad (\text{B.4.25})$$

Therefore,

$$\tilde{g}_{\text{ren},N} = g_N, \quad (\text{B.4.26})$$

as desired. The factor $\frac{1}{6}$ accounts for the fact that the natural prefactor of φ^4 in this context is $\frac{1}{4!}g$ rather than our convention $\frac{1}{4}g$.

B.5 Chapter 5 exercises

Solution to Exercise 5.1.2. By Proposition 2.1.6,

$$\mathbb{E}_C \theta U = U + \frac{1}{2} \Delta_C \left(\frac{1}{4} g |\varphi|^4 + \frac{1}{2} v |\varphi|^2 \right) + \frac{1}{8} \Delta_C^2 \frac{1}{4} g |\varphi|^4. \quad (\text{B.5.1})$$

The terms involving $\Delta_C |\varphi|^2$ and $\Delta_C^2 |\varphi|^4$ produce constants. The remaining term involves

$$\Delta_C |\varphi|^4 = C_{xx} \sum_{i=1}^n \frac{\partial^2}{\partial (\varphi^i)^2} (|\varphi|^2)^2 = C_{xx} \sum_{i=1}^n (8(\varphi^i)^2 + 4|\varphi|^2), \quad (\text{B.5.2})$$

which produces a $|\varphi|^2$ term. A complete calculation of $\mathbb{E}_C \theta U$ is given in the proof of Lemma 5.3.6.

Solution to Exercise 5.2.7. As in (2.1.2), we have

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2} dx = 1. \quad (\text{B.5.3})$$

Since $e^{-gx^4} \leq 1$ for $g \geq 0$ the first bound follows. On the other hand,

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{+gx^4} e^{-\frac{1}{2}x^2} dx = \infty. \quad (\text{B.5.4})$$

If the series (5.2.29) were to converge absolutely for some $g \neq 0$, then by dominated convergence this also would imply the convergence of (B.5.4). Since (B.5.4) does not converge, we conclude that neither does (5.2.29).

Solution to Exercise 5.3.7. Since the field is constant, we drop subscripts x, y , and for notational convenience use subscripts (rather than superscripts) for component indices. To begin, we observe that

$$\frac{\partial}{\partial \varphi_i} |\varphi|^2 = 2\varphi_i, \quad (\text{B.5.5})$$

$$\frac{\partial^2 |\varphi|^2}{\partial \varphi_i \partial \varphi_j} = 2\delta_{ij}, \quad (\text{B.5.6})$$

$$\frac{\partial}{\partial \varphi_i} |\varphi|^4 = 4|\varphi|^2 \varphi_i, \quad (\text{B.5.7})$$

$$\frac{\partial^2 |\varphi|^4}{\partial \varphi_i \partial \varphi_j} = 8\varphi_i \varphi_j + 4|\varphi|^2 \delta_{ij}, \quad (\text{B.5.8})$$

$$\frac{\partial^3 |\varphi|^4}{\partial \varphi_i \partial \varphi_j \partial \varphi_k} = 8(\varphi_i \delta_{jk} + \varphi_j \delta_{ik} + \varphi_k \delta_{ij}), \quad (\text{B.5.9})$$

$$\frac{\partial^4 |\varphi|^4}{\partial \varphi_i \partial \varphi_j \partial \varphi_k \partial \varphi_l} = 8(\delta_{il} \delta_{jk} + \delta_{jl} \delta_{ik} + \delta_{kl} \delta_{ij}). \quad (\text{B.5.10})$$

The two terms that were not computed in the proof of Lemma 5.3.6 are $\kappa'_v = \frac{1}{4} \Delta_C |\varphi|^2$ and $\kappa'_g = \frac{1}{32} \Delta_C^2 |\varphi|^4$. From (B.5.6), we have

$$\kappa'_v = \frac{1}{4} c \sum_i \frac{\partial^2 |\varphi|^2}{\partial \varphi_i^2} = \frac{1}{4} c 2n = \frac{1}{2} nc. \quad (\text{B.5.11})$$

Similarly,

$$\frac{\partial^4 |\varphi|^4}{\partial \varphi_i^2 \partial \varphi_j^2} = 16\delta_{ij} + 8, \quad (\text{B.5.12})$$

and hence

$$\begin{aligned} \kappa'_g &= \frac{1}{32} \Delta_C^2 |\varphi|^4 = \frac{1}{32} c^2 \sum_{i,j} (16\delta_{ij} + 8) \\ &= \frac{1}{32} c^2 (16n + 8n^2) = \frac{1}{4} n(n+2)c^2. \end{aligned} \quad (\text{B.5.13})$$

Now we turn to the more difficult quadratic term. We first compute the sum over $y \in B$ of (5.3.31), which is

$$\begin{aligned} & \frac{1}{16} g^2 F_C(|\varphi|^4; |\varphi|^4) \\ & + \frac{1}{4} \left(g^2 (\eta')^2 + 2g\nu\eta' + \nu^2 \right) F_C(|\varphi|^2; |\varphi|^2) \\ & + \frac{1}{4} \left(g^2 \eta' + g\nu \right) F_C(|\varphi|^2; |\varphi|^4); \end{aligned} \quad (\text{B.5.14})$$

here subscripts x, y and $\sum_{y \in B}$ are implicit in the notation. From (5.3.32) we obtain

$$\begin{aligned} F_C(|\varphi|^2; |\varphi|^2) &= \frac{1}{1!} c^{(1)} \sum_i 2\varphi_i 2\varphi_i + \frac{1}{2!} c^{(2)} \sum_{i,j} 2\delta_{ij} 2\delta_{ij} \\ &= 4c^{(1)} |\varphi|^2 + 2c^{(2)} n, \end{aligned} \quad (\text{B.5.15})$$

$$\begin{aligned} F_C(|\varphi|^2; |\varphi|^4) &= \frac{1}{1!} c^{(1)} \sum_i 2\varphi_i 4|\varphi|^2 \varphi_i + \frac{1}{2!} c^{(2)} \sum_{i,j} 2\delta_{ij} (8\varphi_i \varphi_j + 4|\varphi|^2 \delta_{ij}) \\ &= 8c^{(1)} |\varphi|^4 + c^{(2)} (4n + 8) |\varphi|^2, \end{aligned} \quad (\text{B.5.16})$$

$$\begin{aligned} F_C(|\varphi|^4; |\varphi|^4) &= \frac{1}{1!} c^{(1)} \sum_i 4|\varphi|^2 \varphi_i 4|\varphi|^2 \varphi_i + \frac{1}{2!} c^{(2)} \sum_{i,j} (8\varphi_i \varphi_j + 4|\varphi|^2 \delta_{ij})^2 \\ &+ \frac{1}{3!} c^{(3)} \sum_{i,j,k} 64(\varphi_i \delta_{jk} + \varphi_j \delta_{ik} + \varphi_k \delta_{ij})^2 \\ &+ \frac{1}{4!} c^{(4)} \sum_{i,j,k,l} 64(\delta_{il} \delta_{jk} + \delta_{jl} \delta_{ik} + \delta_{kl} \delta_{ij})^2 \\ &= 16c^{(1)} |\varphi|^6 + 8c^{(2)} (n + 8) |\varphi|^4 + 32c^{(3)} (n + 2) |\varphi|^2 + 8c^{(4)} n(n + 2). \end{aligned} \quad (\text{B.5.17})$$

Substitution into (B.5.14) gives

$$\begin{aligned}
& \frac{1}{16}g^2 \left(16c^{(1)}|\varphi|^6 + 8c^{(2)}(n+8)|\varphi|^4 + 32c^{(3)}(n+2)|\varphi|^2 + 8c^{(4)}n(n+2) \right) \\
& + \frac{1}{4} \left(g^2(\eta')^2 + 2g\nu\eta' + \nu^2 \right) \left(4c^{(1)}|\varphi|^2 + 2c^{(2)}n \right) \\
& + \frac{1}{4} \left(g^2\eta' + g\nu \right) \left(8c^{(1)}|\varphi|^4 + c^{(2)}(4n+8)|\varphi|^2 \right) \\
& = g^2c^{(1)}|\varphi|^6 \\
& + \left(\frac{1}{2}g^2c^{(2)}(n+8) + 2 \left(g^2\eta' + g\nu \right) c^{(1)} \right) |\varphi|^4 \\
& + \left(2g^2c^{(3)}(n+2) + \frac{1}{4} \left(g^2\eta' + g\nu \right) c^{(2)}(4n+8) + \left(g^2(\eta')^2 + 2g\nu\eta' + \nu^2 \right) c^{(1)} \right) |\varphi|^2 \\
& + \left(\frac{1}{2}g^2c^{(4)}n(n+2) + \frac{1}{2} \left(g^2(\eta')^2 + 2g\nu\eta' + \nu^2 \right) c^{(2)}n \right) \\
& = 8g^2c^{(1)}\tau^3 \\
& + 4 \left(\frac{1}{2}g^2c^{(2)}(n+8) + 2 \left(g^2\eta' + g\nu \right) c^{(1)} \right) \tau^2 \\
& + 2 \left(2g^2c^{(3)}(n+2) + \left(g^2\eta' + g\nu \right) c^{(2)}(n+2) + \left(g^2(\eta')^2 + 2g\nu\eta' + \nu^2 \right) c^{(1)} \right) \tau \\
& + \left(\frac{1}{2}g^2c^{(4)}n(n+2) + \frac{1}{2} \left(g^2(\eta')^2 + 2g\nu\eta' + \nu^2 \right) c^{(2)}n \right). \tag{B.5.18}
\end{aligned}$$

The variance term enters U_{pt} with factor $-\frac{1}{2}$, and with this factor the above becomes

$$\begin{aligned}
& -4g^2c^{(1)}\tau^3 \\
& - \left(g^2c^{(2)}(n+8) + 4 \left(g^2\eta' + g\nu \right) c^{(1)} \right) \tau^2 \\
& - \left(2g^2c^{(3)}(n+2) + \left(g^2\eta' + g\nu \right) c^{(2)}(n+2) + \left(g^2(\eta')^2 + 2g\nu\eta' + \nu^2 \right) c^{(1)} \right) \tau \\
& - \frac{1}{4} \left(g^2c^{(4)}n(n+2) + \left(g^2(\eta')^2 + 2g\nu\eta' + \nu^2 \right) c^{(2)}n \right). \tag{B.5.19}
\end{aligned}$$

According to our definitions (5.3.19)–(5.3.20) and (5.3.7)–(5.3.9) of the coefficients, and by the identity $c^{(2)}(n+2) = \gamma\beta'$, the above is equal to

$$\begin{aligned}
& -4g^2c^{(1)}\tau^3 \\
& - (\beta'g^2 + s'_{\tau^2})\tau^2 \\
& - (\xi'g^2 + \gamma\beta'g\nu + s'_\tau)\tau \\
& - \kappa'_{gg}g^2 - \kappa'_{g\nu}g\nu - \kappa'_{\nu\nu}\nu^2. \tag{B.5.20}
\end{aligned}$$

The τ^3 term is the one term that is not in the range of Loc, and hence it is equal to W_+ .

B.6 Chapter 6 exercises

Solution to Exercise 6.1.1. Let $\varepsilon = \min\{1, a/(2M), 1/(2A + 2M)\}$. Let $0 < g_0 < \varepsilon$. We assume by induction that $0 < g_j < \varepsilon$. Then

$$g_j - g_{j+1} = a_j g_j^2 - e_j \geq a g_j^2 (1 - \frac{M}{a} g_j) > \frac{1}{2} a g_j^2, \quad (\text{B.6.1})$$

so $g_{j+1} < g_j$. Also, $g_{j+1} \geq g_j(1 - A g_j - M g_j^2)$ and the second factor on the right-hand side is greater than $\frac{1}{2}$, so $g_{j+1} > \frac{1}{2} g_j$. The induction is complete, the strict monotonicity follows, as does the inequality $0 < \frac{1}{2} g_j < g_{j+1} < g_j$.

Let $g_\infty = \lim_{n \rightarrow \infty} g_j$, which is nonnegative. We take the limit $j \rightarrow \infty$ in the inequality

$$a g_j^2 \leq a_j g_j^2 = g_j - g_{j+1} + e_j \leq g_j - g_{j+1} + M g_j^3 \quad (\text{B.6.2})$$

to obtain $a g_\infty^2 \leq M g_\infty^3$. One solution is $g_\infty = 0$. A positive solution requires $g_\infty \geq a/M$ which is not possible because it exceeds g_0 .

Solution to Exercise 6.1.2. Let $m^2 > 0$. For $j \leq j_m$,

$$\begin{aligned} \sum_{i=0}^{j-1} \frac{1}{(1 + L^{2i} m^2)^2} &= \sum_{i=0}^{j-1} 1 + \sum_{i=0}^{j-1} \left(\frac{1}{(1 + L^{2i} m^2)^2} - 1 \right) \\ &= j - \sum_{i=0}^{j-1} \frac{2m^2 L^{2i} + m^4 L^{4i}}{(1 + L^{2i} m^2)^2} \\ &= j + O(L^{-2(j_m - j)_+}). \end{aligned} \quad (\text{B.6.3})$$

Therefore, for $j > j_m$,

$$\begin{aligned} \sum_{i=0}^{j-1} \frac{1}{(1 + L^{2i} m^2)^2} &= \sum_{i=0}^{j_m} \frac{1}{(1 + L^{2i} m^2)^2} + \sum_{i=j_m}^{j-1} \frac{1}{(1 + L^{2i} m^2)^2} \\ &= (j_m + O(1)) + O(1). \end{aligned} \quad (\text{B.6.4})$$

This proves the result for A_j .

Secondly, for large j ,

$$\begin{aligned} t_j &= \frac{g_0}{1 + g_0 \beta_0^0(j \wedge j_m) + O(1)} \\ &= \frac{g_0}{1 + g_0 \beta_0^0(j \wedge j_m)} \left(1 + \frac{O(1)}{1 + g_0 \beta_0^0(j \wedge j_m)} \right) \\ &\asymp \frac{g_0}{1 + g_0 \beta_0^0(j \wedge j_m)}. \end{aligned} \quad (\text{B.6.5})$$

Finally, for the last inequality it suffices (by the previous result) to prove it for $s_i = t_{j \wedge j_m}(0) = \frac{g_0}{1+g_0\beta_0^0(j \wedge j_m)}$. If $j < j_m$ then

$$\sum_{i=0}^j \vartheta_i s_i = \sum_{i=0}^j s_i = O(|\log s_j|), \quad (\text{B.6.6})$$

while if $j \geq j_m$ then

$$\sum_{i=0}^j \vartheta_i s_i = \sum_{i=0}^{j_m} s_i + \sum_{i=j_m+1}^j \vartheta_i s_i = O(|\log s_{j_m}|) + O(s_{j_m}) = O(|\log s_j|). \quad (\text{B.6.7})$$

Solution to Exercise 6.1.4. By Proposition 6.1.3 it suffices to verify the claims for the sequence t_j . By definition, $t_j - t_{j+1} = t_j t_{j+1} \beta_{j+1}$, so

$$1 - \frac{t_{j+1}}{t_j} = t_{j+1} \beta_{j+1} = O(g_0), \quad (\text{B.6.8})$$

which proves that $t_{j+1} = t_j(1 + O(g_0))$.

Since $A_j(m^2)$ decreases as m^2 increases, $t_j(m^2) \leq t_j(0)$. This proves the first inequality when $j \leq j_m$. For $j \geq j_m$, we note instead that

$$\vartheta_j(m^2) t_j(m^2) \leq 2^{-(j-j_m)} t_{j_m}(m^2) \leq g_0 O((1 + g_0 \beta_0^0 j)^{-1}). \quad (\text{B.6.9})$$

For the remaining inequality, by (6.1.9) it suffices (as in the solution to Exercise 6.1.2) to verify the inequality with $t_j(m^2)$ replaced by $t_{j \wedge j_m}(0)$. Let $p > 1$. If $j < j_m$ then by comparison of the sum with an integral,

$$\sum_{i=j}^{\infty} \vartheta_i s_i^p = \sum_{i=j}^{j_m} s_i^p + \sum_{i=j_m+1}^{\infty} \vartheta_i s_i^p \leq O(s_j^{p-1} + s_{j_m}^p) \leq O(s_j^{p-1}) = O(\vartheta_j s_j^{p-1}), \quad (\text{B.6.10})$$

while if $j \geq j_m$ then

$$\sum_{i=j}^{\infty} \vartheta_i s_i^p \leq O(\vartheta_j s_j^p). \quad (\text{B.6.11})$$

Alternate solution to Exercise 6.1.4. The following alternative solution is adapted from [22, Lemma 2.1]. The identity (B.6.12) follows directly from the recursion (6.1.3). The desired bounds, including the logarithmic bound for $p = 1$ are corollaries. The useful identity (B.6.12) gives an alternative way to analyse the recursion (6.1.3).

We first show that if $\psi : \mathbb{R}_+ \rightarrow \mathbb{R}$ is absolutely continuous and the coefficients a_l in (6.1.3) are uniformly bounded, $|a_l| \leq A$, then

$$\sum_{l=j}^k (a_l g_l^2 - e_l) \psi(g_l) = \int_{g_{k+1}}^{g_j} \psi(t) dt + O\left(\int_{g_{k+1}}^{g_j} t^2 |\psi'(t)| dt\right). \quad (\text{B.6.12})$$

To prove (B.6.12), we apply (6.1.3) to obtain

$$\sum_{l=j}^k (a_l g_l^2 - e_l) \psi(g_l) = \sum_{l=j}^k \psi(g_l) (g_l - g_{l+1}). \quad (\text{B.6.13})$$

We wish to replace the Riemann sum on the right-hand side by the corresponding integral. For this we use

$$\psi(g_l) (g_l - g_{l+1}) = \int_{g_{l+1}}^{g_l} \psi(t) dt + \int_{g_{l+1}}^{g_l} \int_t^{g_l} \psi'(s) ds dt, \quad (\text{B.6.14})$$

which follows by applying the fundamental theorem of calculus to the last term. After inserting this into (B.6.13) we have

$$\sum_{l=j}^k (a_l g_l^2 - e_l) \psi(g_l) = \int_{g_{k+1}}^{g_j} \psi(t) dt + \sum_{l=j}^k \int_{g_{l+1}}^{g_l} \int_{g_{l+1}}^s \psi'(s) dt ds, \quad (\text{B.6.15})$$

where we have inverted the order of integration. Upon evaluating the t integral we obtain a factor $s - g_{l+1}$ so (B.6.12) holds if $|s - g_{l+1}| = O(s^2)$. This is proved as follows: by (6.1.1) and $g_l \leq g_0$, for s in the domain of integration we have

$$|s - g_{l+1}| \leq |g_l - g_{l+1}| = |a_l| \bar{g}_l^2 \leq (1 + O(\bar{g}_0)) |a_l| \bar{g}_{l+1}^2 \leq O(s^2), \quad (\text{B.6.16})$$

where we used the hypothesis $|a_l| \leq A$. This concludes the proof of (B.6.12).

Direct evaluation of the integrals in (B.6.12) with $\psi(t) = t^{p-2}$ and $a_l = \beta_l$ gives

$$\sum_{l=j}^k (\beta_l g_l^p - e_l g_l^{p-2}) \beta_l g_l^p \leq C_p \begin{cases} |\log g_k| & p = 1 \\ g_j^{p-1} & p > 1. \end{cases} \quad (\text{B.6.17})$$

We only deduce (6.1.17), as the proof of (6.1.18) is similar. Suppose first that $j \leq j_m$. Then $1 = O(\beta_j)$ and $|e_l| \leq M_l g_l^3$, therefore

$$\sum_{l=j}^k \vartheta_l g_l^p \leq \sum_{l=j}^{j_m} O(\beta_l g_l^p - e_l g_l^{p-2}) + \sum_{l=j_m+1}^k 2^{-(l-j_m)+} g_l^p. \quad (\text{B.6.18})$$

By (B.6.17), the first term is bounded by $O(g_j^{p-1})$. The second term (which is only present when $j_m < \infty$) obeys the same bound using monotonicity of $\bar{g}_j$ in j . This proves (6.1.17) for the case $j \leq j_m$. On the other hand, if $j > j_m$, then again using the exponential decay of ϑ_l and $\bar{g}_{l+1} \leq \bar{g}_l$, we obtain

$$\sum_{l=j}^k \vartheta_l g_l^p \leq C \vartheta_j g_j^p \leq C \bar{g}_0 \vartheta_j g_j^{p-1}. \quad (\text{B.6.19})$$

This completes the proof of (6.1.17).

Remark B.6.1. By choosing $\psi(t) = t^{-2}$, $j = 0$ and replacing k by $k - 1$ in (B.6.12) we obtain

$$\sum_{l=0}^{k-1} (a_l - e_l g_l^{-2}) = \frac{1}{g_k} - \frac{1}{g_0} + O(|\log g_k|). \quad (\text{B.6.20})$$

Let $a_l = \beta_l$. Recall from (6.1.6) that $A_k = \sum_{l=0}^{k-1} \beta_l$ and insert $|e_l| \leq M_l g_l^3$ to obtain

$$A_k - O\left(\sum_{l=0}^{k-1} g_l\right) = \frac{1}{g_k} - \frac{1}{g_0} + O(|\log g_k|). \quad (\text{B.6.21})$$

Solving for g_k leads to (6.1.11).

Solution to Exercise 6.1.5. By Proposition 6.1.3 it suffices to verify the claim for the sequence t_j . Let $m^2 > 0$. Let $t_j = t_j(m^2)$ and $\tilde{t}_j = t_j(\tilde{m}^2)$, and similarly for $A_j, \tilde{A}_j$. By the definition (6.1.6),

$$t_j - \tilde{t}_j = t_j \tilde{t}_j (\tilde{A}_j - A_j) = t_j \tilde{t}_j \beta_0^j \sum_{i=0}^{j-1} \frac{2(m^2 - \tilde{m}^2)L^{2i} + (m^4 - \tilde{m}^4)L^{4i}}{(1 + \tilde{m}^2 L^{2i})^2 (1 + m^2 L^{2i})^2}. \quad (\text{B.6.22})$$

For case $\tilde{m}^2 = 0$, the condition $m^2 \in \mathbb{I}_j(0)$ implies that $m^2 L^{2j} \leq 1$ so the sum is roughly geometric, dominated by its largest term, and therefore of order one. Similarly, for case $\tilde{m}^2 > 0$ the condition $m^2 \in \mathbb{I}_j(\tilde{m}^2)$ implies that $m^2 = O(\tilde{m}^2)$ and now the terms in the sum such that $L^{2i} \tilde{m}^2 > 1$ are negligible and again the sum is of order one. Therefore in both cases $t_j - \tilde{t}_j = O(t_j \tilde{t}_j) = O(\tilde{t}_j^2)$ as desired.

Solution to Exercise 6.2.6. The desired conclusion is obtained by applying the following lemma, which is [20, Lemma 4.3], to $u(t) = (B\chi(v_c + t))^{-1}$. In fact, (B.6.23) is the hypothesis (6.2.33), and (B.6.24) is the conclusion (6.2.34).

Lemma. Let $\gamma \in \mathbb{R}$ and $\delta > 0$. Suppose that $u : [0, \delta) \rightarrow [0, \infty)$ is continuous, differentiable on $(0, \delta)$, that $u(0) = 0$ and $u(t) > 0$ for $t > 0$, and that

$$u'(t) = (-\log u(t))^{-\gamma} (1 + o(1)) \quad (\text{as } t \downarrow 0). \quad (\text{B.6.23})$$

Then

$$u(t) = t(-\log t)^{-\gamma} (1 + o(1)) \quad (\text{as } t \downarrow 0). \quad (\text{B.6.24})$$

Proof. By hypothesis,

$$\int_0^t u'(t) (-\log u(t))^\gamma dt = \int_0^t (1 + o(1)) dt = t(1 + o(1)). \quad (\text{B.6.25})$$

Since $u(t) > 0$ implies that $u'(t) > 0$ for small t , we see that u is monotone. By a change of variables, followed by integration by parts,

$$\begin{aligned} \int_0^t u'(t)(-\log u(t))^\gamma dt &= \int_0^{u(t)} (-\log v)^\gamma dv \\ &= u(t)(-\log u(t))^\gamma (1 + O((-\log u(t))^{-1})). \end{aligned} \quad (\text{B.6.26})$$

Since the above two right-hand sides are equal,

$$u(t)(-\log u(t))^\gamma = t(1 + o(1)). \quad (\text{B.6.27})$$

Let $f(x) = x(-\log x)^\gamma$ and $g(y) = y(-\log y)^{-\gamma}$. Then f and g are approximate inverses in the sense that $f(g(y)) = y(1 + o(1))$. Thus $u(t) = t(-\log t)^{-\gamma}(1 + o(1))$. This completes the proof. ■

B.7 Chapter 7 exercises

Solution to Exercise 7.1.4. There is no dependence on $\mathcal{Y}$ so we work with the T_ϕ -seminorm. By Example 7.1.1, for $p \leq k$,

$$\|F^{(p)}(\phi)\| \leq \frac{k!}{(k-p)!} \|M\|_{\mathcal{Z}_1} \left(\frac{|\phi|}{\mathfrak{h}} \right)^{k-p}, \quad (\text{B.7.1})$$

and $F^{(p)}$ is zero if $p > k$. We insert this bound into Definition 7.1.2 and obtain

$$\|F\|_{T_\phi(\mathfrak{h})} \leq \|M\|_{\mathcal{Z}_1} \sum_{p \leq k} \binom{k}{p} \left(\frac{|\phi|}{\mathfrak{h}} \right)^{k-p} = \|M\|_{\mathcal{Z}_1} \left(1 + \frac{|\phi|}{\mathfrak{h}} \right)^k. \quad (\text{B.7.2})$$

This proves (7.1.15).

To prove $\|(\phi \cdot \phi)^p\|_{T_\phi} \leq (|\phi| + \mathfrak{h})^{2p}$, by the product property Lemma 7.1.3 it suffices to consider the case $p = 1$. For this we apply (B.7.2) with $F = \phi \cdot \phi$ and $M(\phi, \psi) = \phi \cdot \psi$. By the definition (7.1.1) and the Cauchy-Schwarz inequality $|\phi \cdot \psi| \leq |\phi| |\psi|$, we have $\|M\|_{\mathcal{Z}_1} \leq \mathfrak{h}^2$. Therefore, by (B.7.2), $\|\phi \cdot \phi\|_{T_\phi} \leq (|\phi| + \mathfrak{h})^2$ as desired.

For the last part, which is $\|(\zeta \cdot \phi)(\phi \cdot \phi)^p\|_{T_\phi} \leq |\zeta|(|\phi| + \mathfrak{h})^{2p+1}$, the product property and the previous estimate reduce the desired bound to $\|(\zeta \cdot \phi)\|_{T_\phi} \leq |\zeta|(|\phi| + \mathfrak{h})$. This follows easily from (B.7.2) and the Cauchy-Schwarz inequality.

Solution to Exercise 7.5.2. This is an immediate corollary of Lemma 7.5.1, since $F = \text{Tay}_k F$ and hence

$$\|F\|_{T_{\phi,y}} = \|\text{Tay}_k F\|_{T_{\phi,y}} \leq \|F\|_{T_{0,y}} P_{\mathfrak{h}}^k(\phi). \quad (\text{B.7.3})$$

Solution to Exercise 7.6.2. By Taylor's Theorem, $U(\varphi + \zeta_x) = \sum_{|\alpha| \leq 4} \frac{1}{\alpha!} U^{(\alpha)}(\varphi) \zeta_x^\alpha$ and hence, since $U^{(\alpha)} = V^{(\alpha)}$ for $|\alpha| \geq 1$,

$$\mathbb{E}_{C_+}(\theta U(B) - U(B)) = \sum_{x \in B} \sum_{1 \leq |\alpha| \leq 4} \frac{1}{\alpha!} V^{(\alpha)}(\varphi) \mathbb{E}_{C_+} \zeta_x^\alpha. \quad (\text{B.7.4})$$

Therefore, by (7.6.5) and Exercise 2.1.7, and since $\mathfrak{h} \geq \mathfrak{c}_+$,

$$\begin{aligned} \|\mathbb{E}_{C_+}(\theta U(B) - U(B))\|_{T_{\varphi,y}(\mathfrak{h}, \lambda)} &\leq O(1) \sum_{|\alpha| \in \{2,4\}} \mathfrak{h}^{-|\alpha|} \|V(B)\|_{T_{0,y}} P_{\mathfrak{h}}^{4-|\alpha|}(\varphi) \mathfrak{c}_+^{|\alpha|} \\ &\leq O(1) \left(\frac{\mathfrak{c}_+}{\mathfrak{h}} \right)^2 \|V(B)\|_{T_{0,y}} P_{\mathfrak{h}}^2(\varphi), \end{aligned} \quad (\text{B.7.5})$$

as required.

B.8 Chapter 8 exercises

Solution to Exercise 8.1.1. By the definition of U and completing the square,

$$\begin{aligned} U(\varphi) &= \frac{1}{8}g|\varphi|^4 + \frac{1}{8}g|\varphi|^4 + \nu \frac{1}{2}|\varphi|^2 + u \\ &= \frac{1}{8}g|\varphi|^4 + \frac{1}{8}g \left(|\varphi|^2 + \frac{2\nu}{g} \right)^2 - \frac{1}{8}g \left(\frac{2\nu}{g} \right)^2 + u \\ &\geq \frac{1}{8}g|\varphi|^4 - \frac{1}{8}g \left(\frac{2\nu}{g} \right)^2 + u = \frac{1}{8}g|\varphi|^4 - \frac{1}{2} \frac{\nu^2}{g} + u. \end{aligned} \quad (\text{B.8.1})$$

The bounds on coupling constants due to $U \in \mathcal{D}^{\text{st}}$ (defined in (8.1.8)) imply that

$$-\frac{1}{2} \frac{\nu^2}{g} + u \geq -\frac{1}{2} k_0^3 L^{-dj} - k_0^4 L^{-dj} \geq -\frac{3}{2} k_0^3 L^{-dj}, \quad (\text{B.8.2})$$

since $k_0 \leq 1$. Combining this with the previous bound and using $g \geq k_0 \tilde{g}$ gives the desired bound,

$$U(\varphi) \geq \frac{1}{8} k_0 \tilde{g} |\varphi|^4 - \frac{3}{2} k_0^3 L^{-dj}, \quad (\text{B.8.3})$$

which, by the definition (8.1.7) of h is equivalent to the first inequality in (8.1.10).

Solution to Exercise 8.2.2. Let

$$U(\varphi_x) = \frac{1}{4}g(\varphi_x \cdot \varphi_x)^2 + \frac{1}{2}\nu \varphi_x \cdot \varphi_x + u. \quad (\text{B.8.4})$$

We write $\varphi = \varphi_x$, because φ_x is constant in the block that contains x . By Definition 7.1.2, the T_0 -seminorm equals the sum of the seminorms of the monomials in F , because they have different degrees. Applying the definition of the norm to the monomial $\frac{1}{4}g(\varphi \cdot \varphi)^2$, we have

$$\|\frac{1}{4}g(\varphi \cdot \varphi)^2\|_{T_0(\mathfrak{h})} = \sup \frac{1}{4}g\frac{1}{4!} \sum_{i_1, \dots, i_4} |\dot{\varphi}_{i_1} \cdot \dot{\varphi}_{i_2}| |\dot{\varphi}_{i_3} \cdot \dot{\varphi}_{i_4}|, \quad (\text{B.8.5})$$

where the supremum is over unit norm directions $\dot{\varphi}_1, \dot{\varphi}_2, \dot{\varphi}_3, \dot{\varphi}_4$, and $i_1, \dots, i_4$ is summed over permutations of 1, 2, 3, 4. Note that the sum is normalised by $\frac{1}{4!}$. By (7.1.7), unit norm in the space X means that the Euclidean norm is $|\dot{\varphi}_i| = \mathfrak{h}$. Therefore

$$\|\frac{1}{4}g(\varphi \cdot \varphi)^2\|_{T_0(\mathfrak{h})} \leq \frac{1}{4}g\mathfrak{h}^4. \quad (\text{B.8.6})$$

This upper bound is actually equality because the right-hand side is also a lower bound on the supremum by testing the case where $\dot{\varphi}_1, \dot{\varphi}_2, \dot{\varphi}_3, \dot{\varphi}_4$ all equal. By a similar easier argument we find that the norm of $\frac{1}{2}\mathbf{v}\varphi \cdot \varphi$ is $\frac{1}{2}|\mathbf{v}|\mathfrak{h}^2$ and obtain

$$\|U_x\|_{T_0(\mathfrak{h})} = \frac{1}{4}|g|\mathfrak{h}^4 + \frac{1}{2}|\mathbf{v}|\mathfrak{h}^2 + |u|, \quad (\text{B.8.7})$$

as desired. Since derivatives are taken in directions $\dot{\varphi}$ that are constant on the block b the norm of $U(b)$ is $|b| = L^{d_j}$ times as large and (8.2.5) immediately follows from (8.1.4).

B.9 Chapter 9 exercises

Solution to Exercise 9.4.2. We consider some examples; higher-order and mixed derivatives can be handled similarly.

Let $\dot{Q}(b) = \text{Loc}(e^{V(b)}\dot{K})$, and note that $D_K Q(K; \dot{K}) = \dot{Q}$ since Loc is linear. Recall from (9.1.5) that

$$R_+^U(B) = -\mathbb{E}_{C_+} \theta Q(B) + \text{Cov}_+(\theta(V(B) - \frac{1}{2}Q(B)), \theta Q(B)). \quad (\text{B.9.1})$$

Since Cov_+ is bilinear, differentiation gives

$$D_K R_+^U(B; \dot{K}) = -\mathbb{E}_+ \theta \dot{Q} + \text{Cov}_+(\theta(V - Q), \theta \dot{Q}(B)), \quad (\text{B.9.2})$$

$$D_K^2 R_+^U(B; \dot{K}, \ddot{K}) = -\text{Cov}_+(\theta \dot{Q}, \theta \ddot{Q}(B)), \quad (\text{B.9.3})$$

and higher-order K -derivatives are zero. Similarly,

$$\begin{aligned} D_V R_+^U(B; \dot{V}) &= -\mathbb{E}_+ \theta \dot{V} Q(B) + \text{Cov}_+(\theta \dot{V}(1 - \frac{1}{2}Q(B)), \theta Q(B)) \\ &\quad + \text{Cov}_+(\theta(V(B) - \frac{1}{2}Q(B)), \theta \dot{V}(B)Q(B)). \end{aligned} \quad (\text{B.9.4})$$

It is now straightforward to estimate the derivatives. For example,

$$\begin{aligned}\|D_K R_+^U(B; \dot{K})\|_+ &\leq \|\dot{Q}(B)\|_+ + O(1)\|V(B) - Q(B)\|_+ \|\dot{Q}(B)\|_+ \\ &\leq O(\|\dot{K}\|),\end{aligned}\tag{B.9.5}$$

which gives an $O(1)$ bound for $D_K R_+^U$. Similarly, we obtain an $O(\|K\|)$ bound for $D_V R_+^U$ from the fact that each term on the right-hand side of (B.9.4) contains a factor Q .

B.10 Chapter 11 exercises

Solution to Exercise 11.1.2. There are d^n n -step walks that take steps only in positive coordinate directions, and such walks are self-avoiding. Also, every walk that avoids reversing its previous step is self-avoiding, and there are $(2d)(2d-1)^{n-1}$ such n -step walks. Therefore $d^n \leq c_n \leq (2d)(2d-1)^{n-1}$, and the result follows.

Solution to Exercise 11.2.1. We denote the right-hand side of (11.2.1) by W_{xy} , and write $r_x = \frac{1}{\bar{\beta}_x + v_x}$. The condition on V guarantees that W_{xy} converges, since

$$|W_{xy}| = \left| \sum_{\omega \in \mathcal{W}^*(x,y)} r_{Y_{|\omega|}} \prod_{i=0}^{|\omega|-1} r_{Y_i} \beta_{Y_i Y_{i+1}} \right| \leq \sum_{n=0}^{\infty} \frac{1}{c} (\max_x r_x \bar{\beta}_x)^n, \tag{B.10.1}$$

and the sum on the right-hand side converges because $|r_x \bar{\beta}_x| \leq \bar{\beta}_x / (\bar{\beta}_x + c) < 1$, and hence the maximum is strictly less than 1 since there are finitely many points x in Λ .

We extract the term with $|\omega| = 0$, and condition on the first step for the remaining terms, to get

$$W_{xy} = r_x \delta_{xy} + \sum_{u \neq x} r_x \beta_{xu} W_{uy}. \tag{B.10.2}$$

This can be rearranged to give

$$(-\Delta_\beta W)_{xy} + \left(\frac{1}{r_x} - \sum_{u \neq x} \beta_{xu} \right) W_{xy} = \delta_{xy}, \tag{B.10.3}$$

which is the same as

$$((-\Delta_\beta + V)W)_{xy} = \delta_{xy}. \tag{B.10.4}$$

Therefore $W = (-\Delta_\beta + V)^{-1}$. The special case follows from the fact that the first product in (11.2.1) then selects the nearest-neighbour walks, and $\bar{\beta}_x = 2d$ for all x .

Solution to Exercise 11.3.3. (i) This is taken from [51, Lemma 2.1], and involves steps used in the proof of Lemma 11.3.2. Consider first the case where A is Hermitian. Then there is a unitary matrix U and a diagonal matrix D such that $A = U^{-1}DU$, so $\phi A \bar{\phi} = w D \bar{w}$ with $w = \bar{U} \phi$, and

$$\frac{1}{(2\pi i)^M} Z_C = \prod_{x=1}^M \left(\frac{1}{\pi} \int_{-\infty}^{\infty} e^{-d_x(u_x^2 + v_x^2)} du_x dv_x \right) = \prod_{x=1}^M \frac{1}{d_x} = \frac{1}{\det A}. \quad (\text{B.10.5})$$

For the general case, we write $A(z) = G + izH$ with $G = \frac{1}{2}(A + A^*)$, $H = \frac{1}{2i}(A - A^*)$ and $z = 1$. Since $\phi(iH)\bar{\phi}$ is imaginary, when G is positive definite the integral in (11.3.18) converges and defines an analytic function of z in a neighborhood of the real axis. For z small and purely imaginary, $A(z)$ is Hermitian and positive definite, and hence (11.3.18) holds in this case. Since $(\det A(z))^{-1}$ is a meromorphic function of z , (11.3.18) follows from the uniqueness of analytic extension.

(ii) We expand the exponential and obtain

$$e^{-\psi A \bar{\psi}} = \sum_{n=0}^M \frac{(-1)^n}{n!} (\psi A \bar{\psi})^n = \frac{(-1)^M}{M!} (\psi A \bar{\psi})^M + (\text{forms of degree} < 2M). \quad (\text{B.10.6})$$

Only the forms of top degree $(2M)$ contribute to the integral. In the following, for simplicity we drop the symbol $\wedge$ for the wedge product. By definition, $\psi A \bar{\psi} = \sum_x \psi_x (A \bar{\psi})_x = \sum_{x,y} A_{xy} \psi_x \bar{\psi}_y$, and hence

$$\begin{aligned} (\psi A \bar{\psi})^M &= \sum_{x_1, y_1} \cdots \sum_{x_M, y_M} A_{x_1 y_1} \cdots A_{x_M y_M} \psi_{x_1} \bar{\psi}_{y_1} \cdots \psi_{x_M} \bar{\psi}_{y_M} \\ &= \sum_{\eta \in S_M} \sum_{\sigma \in S_M} A_{\eta(1)\sigma(1)} \cdots A_{\eta(M)\sigma(M)} \psi_{\eta(1)} \bar{\psi}_{\sigma(1)} \cdots \psi_{\eta(M)} \bar{\psi}_{\sigma(M)} \\ &= M! \sum_{\sigma \in S_M} A_{1\sigma(1)} \cdots A_{M\sigma(M)} \psi_1 \bar{\psi}_{\sigma(1)} \cdots \psi_M \bar{\psi}_{\sigma(M)} \\ &= M! \sum_{\sigma \in S_M} \text{sgn}(\sigma) A_{1\sigma(1)} \cdots A_{M\sigma(M)} \psi_1 \bar{\psi}_1 \cdots \psi_M \bar{\psi}_M \end{aligned} \quad (\text{B.10.7})$$

In the above the second equality follows from the fact that any product with two identical ψ factors vanishes, the third follows by rearranging $\psi \bar{\psi}$ pairs (which does not introduce signs), and the fourth follows by reordering the ψ factors. This proves that $(\psi A \bar{\psi})^M = (-1)^M M! (\det A) \bar{\psi}_1 \psi_1 \cdots \bar{\psi}_M \psi_M$ and the top degree part of $e^{-\psi A \bar{\psi}}$ is equal to $(\det A) \bar{\psi}_1 \psi_1 \cdots \bar{\psi}_M \psi_M$. Finally,

$$\int e^{-S_A} = (\det A) \int e^{-\phi A \bar{\phi}} \bar{\psi}_1 \psi_1 \cdots \bar{\psi}_M \psi_M, \quad (\text{B.10.8})$$

and the right-hand side is 1 by (11.3.18).

Solution to Exercise 11.3.4. By Exercise 11.3.3,

$$\int e^{-S_A} f = \frac{\int_{\mathbb{R}^{2M}} f e^{-\phi A \bar{\phi}} d\bar{\phi} d\phi}{\int_{\mathbb{R}^{2M}} e^{-\phi A \bar{\phi}} d\bar{\phi} d\phi}. \quad (\text{B.10.9})$$

By the symmetry of A ,

$$\phi A \bar{\phi} = \frac{1}{2}(u, Au) + \frac{1}{2}(v, Av), \quad (\text{B.10.10})$$

so

$$\int e^{-S_A} f = \frac{\int_{\mathbb{R}^{2M}} f e^{-\frac{1}{2}(u, Au) + \frac{1}{2}(v, Av)} du dv}{\int_{\mathbb{R}^{2M}} e^{-\frac{1}{2}(u, Au) + \frac{1}{2}(v, Av)} du dv} = \mathbb{E}_C f. \quad (\text{B.10.11})$$

This proves (11.3.21), and (11.3.22) then follows from

$$\begin{aligned} \int e^{-S_A} \phi_x \bar{\phi}_y &= \mathbb{E}_C \frac{1}{2} (u_x v_y + v_x v_y - i u_x v_y + i v_x u_y) \\ &= \frac{1}{2} (C_{xy} + C_{xy} - 0 + 0) = C_{xy}. \end{aligned} \quad (\text{B.10.12})$$

Solution to Exercise 11.3.8. By the Cauchy–Schwarz inequality, $T = \sum_{x \in \Lambda} L_{T,x} \leq (|\Lambda| I(T))^{1/2}$, and hence

$$G_{0x}^N(g, \nu) \leq \int_0^\infty e^{-gT^2/|\Lambda_N|} e^{-\nu T} dT < \infty \quad \text{for all } \nu \in \mathbb{R}. \quad (\text{B.10.13})$$

Solution to Exercise 11.3.10. By linearity of both sides, we may assume that K is a p -form. It follows from the definition of the super-expectation that both sides vanish unless K contains the same number of factors of ψ and $\bar{\psi}$. We can therefore assume that $K = f(\phi, \bar{\phi}) \bar{\psi}_{x_1} \psi_{y_1} \cdots \bar{\psi}_{x_p} \psi_{y_p}$. Then

$$\bar{\phi}_x K e^{-S_A} = e^{-\phi A \bar{\phi}} \bar{\phi}_x f \sum_{N=0}^{|\Lambda|} \frac{(-1)^N}{N!} (\bar{\psi} A \psi)^N \bar{\psi}_{x_1} \psi_{y_1} \cdots \bar{\psi}_{x_p} \psi_{y_p}. \quad (\text{B.10.14})$$

Since only the top-degree part of this form contributes to its integral,

$$\int \bar{\phi}_x K e^{-S_A} = T_{A, x_1, y_1, \dots, x_p, y_p} \mathbb{E}_C \bar{\phi}_x f \quad (\text{B.10.15})$$

for some constants $T_{A, x_1, y_1, \dots, x_p, y_p}$ not depending on the function f . Therefore, by standard Gaussian integration by parts (Exercise 2.1.3),

$$\int \bar{\phi}_x K e^{-S_A} = T_{A, x_1, y_1, \dots, x_p, y_p} \sum_{y \in \Lambda} C_{xy} \mathbb{E}_C \left(\frac{\partial f}{\partial \phi_y} \right). \quad (\text{B.10.16})$$

Since the constants $T_{A,x_1,y_1,\dots,x_p,y_p}$ do not depend on f , it is also the case that

$$\int \frac{\partial K}{\partial \phi_y} e^{-S_A} = T_{A,x_1,y_1,\dots,x_p,y_p} \mathbb{E}_C \left(\frac{\partial f}{\partial \phi_y} \right). \quad (\text{B.10.17})$$

Insertion of (B.10.17) into (B.10.16) gives

$$\int \bar{\phi}_x K e^{-S_A} = \sum_{y \in \Lambda} C_{xy} \int \frac{\partial K}{\partial \phi_y} e^{-S_A}, \quad (\text{B.10.18})$$

as claimed.

Solution to Exercise 11.3.12. Let C be the identity matrix and set

$$G_{xy}(\beta) = \mathbb{E}_C \left(\bar{\phi}_x \phi_y \prod_{\{u,v\} \in E} (1 + 2\beta_{uv} \tau_{uv}) \right). \quad (\text{B.10.19})$$

Then $G_{xy}(\beta)$ is the right-hand side of (11.3.42). By Gaussian integration by parts (11.3.35),

$$G_{xy}(\beta) = \delta_{xy} \mathbb{E}_C \left(\prod_{\{u,v\}} (1 + 2\beta_{uv} \tau_{uv}) \right) + \mathbb{E}_C \left(\phi_y \frac{\partial}{\partial \phi_x} \prod_{\{u,v\}} (1 + 2\beta_{uv} \tau_{uv}) \right). \quad (\text{B.10.20})$$

The first term involves the expectation of a function of (τ_{uv}) which evaluates to 1 by the localisation theorem (11.4.12). Given $x, w \in \Lambda$, let $\beta_{uv}^{(xw)} = \beta_{uv}$ for $\{u, v\} \neq \{x, w\}$ and $\beta_{xw}^{(xw)} = \beta_{wx}^{(xw)} = 0$. The second expectation is

$$\begin{aligned} & \mathbb{E}_C \left(\phi_y \frac{\partial}{\partial \phi_x} \prod_{\{u,v\}} (1 + 2\beta_{uv} \tau_{uv}) \right) \\ &= \sum_{w \in \Lambda} \beta_{xw} \mathbb{E}_C \left(\bar{\phi}_w \phi_y \prod_{\{u,v\}} (1 + 2\beta_{uv}^{(xw)} \tau_{uv}) \right) = \sum_{w \in \Lambda} \beta_{xw} G_{wy}(\beta^{(xw)}). \end{aligned} \quad (\text{B.10.21})$$

Thus we have shown that

$$G_{xy}(\beta) = \delta_{xy} + \sum_{w \in \Lambda} \beta_{xw} G_{wy}(\beta^{(xw)}). \quad (\text{B.10.22})$$

This recursion characterises the weighted two-point function $\sum_{\omega \in \mathcal{T}(x,y)} \beta^\omega$ appearing on the left-hand side of (11.3.42).

Solution to Exercise 11.4.3. Suppose first that K is a collection of zero forms. Then

$$QF(K) = \sum_{x \in \Lambda} \left[\psi_x \frac{\partial F(K)}{\partial \phi_x} + \bar{\psi}_x \frac{\partial F(K)}{\partial \bar{\phi}_x} \right] = \sum_{j=1}^J F_j(K) \sum_{x \in \Lambda} \left[\psi_x \frac{\partial K_j}{\partial \phi_x} + \bar{\psi}_x \frac{\partial K_j}{\partial \bar{\phi}_x} \right], \quad (\text{B.10.23})$$

where the second equality follows from the chain rule for zero-forms. The right-hand side is $\sum_j F_j(K) QK_j$, so this proves (11.4.11) for zero-forms and we may assume now that K is higher degree.

Let ε_j be the multi-index that has j^{th} component 1 and all other components 0. Let $K^0 = (K_j^0)_{j \in J}$ denote the zero-degree part of K . By (11.3.11), the fact that Q is an anti-derivation, and the chain rule applied to zero-forms,

$$\begin{aligned} QF(K) &= \sum_{\alpha} \frac{1}{\alpha!} [QF^{(\alpha)}(K^0)] (K - K^0)^{\alpha} + \sum_{\alpha} \frac{1}{\alpha!} F^{(\alpha)}(K^0) Q[(K - K^0)^{\alpha}] \\ &= \sum_{\alpha} \frac{1}{\alpha!} \sum_{j=1}^J F^{(\alpha + \varepsilon_j)}(K^0) [QK_j^0] (K - K^0)^{\alpha} \\ &\quad + \sum_{\alpha} \frac{1}{\alpha!} F^{(\alpha)}(K^0) Q[(K - K^0)^{\alpha}]. \end{aligned} \quad (\text{B.10.24})$$

Since Q is an anti-derivation,

$$Q(K - K^0)^{\alpha} = \sum_{j=1}^J \alpha_j (K - K^0)^{\alpha - \varepsilon_j} [QK_j - QK_j^0]. \quad (\text{B.10.25})$$

The first term on the right-hand side of (B.10.24) is cancelled by the contribution to the second term of (B.10.24) due to the second term of (B.10.25). The remaining contribution to the second term of (B.10.24) due to the first term of (B.10.25) then gives

$$QF(K) = \sum_{j=1}^J \left(\sum_{\alpha} \frac{1}{\alpha!} F^{(\alpha)}(K^0) \alpha_j (K - K^0)^{\alpha - \varepsilon_j} \right) QK_j = \sum_{j=1}^J F_j(K) QK_j \quad (\text{B.10.26})$$

as required.

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