Extension of the generalised inductive approach to the lace expansion: Full proof

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Abstract

This paper extends the result of [2] in order to use the inductive approach to prove Gaussian asymptotic behaviour for models with critical dimension other than 4. The results are applied in [3] to study sufficiently spread-out lattice trees in dimensions $d > 8$ and may also be applicable to percolation in dimensions $d > 6$.

1 Introduction

This paper consists of large parts of the material in [2], reproduced verbatim, but with the introduction of parameters $\theta(d) > 2$ and $p^* \geq 1$ as described in [1]. The case $\theta = \frac{d}{2}$, and $p^* = 1$ is that dealt with in [2]. The main conclusion is one of Gaussian asymptotic behaviour for models with critical dimension other than 4, satisfying certain properties. We do not include the proof of the local central limit theorem [2, Theorem 1.3], which does require $\theta = \frac{d}{2}$. The result of this paper is applied in [3] to lattice trees with $d > 8$, $\theta = \frac{d-4}{2}$ and $p^* = 2$. We also expect the result to be applicable to other models where the analysis uses the lace expansion above a critical dimension $d_c \geq 4$. In such cases the lace expansion for $d > d_c$ suggests setting $\theta = \frac{d-(d_c-4)}{2}$. In particular the above statement for percolation in dimensions $d > d_c = 6$ would give $\theta = \frac{d-2}{2}$.

This paper simply provides the details of the proof described in [1], and we refer the reader to [1] and [2] for a more thorough introduction to the inductive approach to the lace expansion. In Section 2 we state the form of the recursion relation, and the assumptions S, D, E$_\theta$, and G$_\theta$ on the quantities appearing in the recursion equation. We also state the "θ-theorem" to be proved. In Section 3, we introduce the induction hypotheses on f_n that will be used to prove the θ-theorem, and derive some consequences of the induction hypotheses. The induction is advanced in Section 4. In Section 5, the θ-theorem stated in Section 2 are proved.

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2 Assumptions on the Recursion Relation

When applied to self-avoiding walks, oriented percolation and lattice trees, the lace expansion gives rise to a convolution recursion relation of the form

$$f_{n+1}(k;z) = \sum_{m=1}^{n+1} g_m(k;z) f_{n+1-m}(k;z) + e_{n+1}(k;z) \qquad (n \geq 0), \tag{2.1}$$

with $f_0(k;z) = 1$. Here, $k \in [-\pi,\pi]^d$ is a parameter dual to a spatial lattice variable $x \in \mathbb{Z}^d$, and z is a positive parameter. The functions g_m and e_m are to be regarded as given, and the goal is to understand the behaviour of the solution $f_n(k;z)$ of (2.1).

2.1 Assumptions S,D,E$_\theta$,G$_\theta$

The first assumption, Assumption S, requires that the functions appearing in the recursion equation (2.1) respect the lattice symmetries of reflection and rotation, and that f_n remains bounded in a weak sense. We have strengthened this assumption from that appearing in [2], as one requires smoothness of f_n and g_n which holds in all of the applications.

Assumption S. For every $n \in \mathbb{N}$ and $z > 0$, the mapping $k \mapsto f_n(k;z)$ is symmetric under replacement of any component k_i of k by $-k_i$, and under permutations of the components of k. The same holds for $e_n(\cdot;z)$ and $g_n(\cdot;z)$. In addition, for each n, $|f_n(k;z)|$ is bounded uniformly in $k \in [-\pi,\pi]^d$ and z in a neighbourhood of 1 (which may depend on n). We also assume that f_n and g_n have continuous second derivatives in a neighbourhood of 0 for every n. It is an immediate consequence of Assumption S that the mixed partials of f_n and g_n at $k = 0$ are equal to zero.

The next assumption, Assumption D, incorporates a "spread-out" aspect to the recursion equation. It introduces a function D which defines the underlying random walk model, about which Equation (2.1) is a perturbation. The assumption involves a non-negative parameter L, which will be taken to be large, and which serves to spread out the steps of the random walk over a large set. We write $D = D_L$ in the statement of Assumption D to emphasise this dependence, but the subscript will not be retained elsewhere. An example of a family of D's obeying the assumption is taking $D(\cdot)$ uniform on a box side length $2L$, centred at the origin. In particular Assumption D implies that D has a finite second moment and we define

$$\sigma^2 \equiv -\nabla^2 \hat{D}(0) = -\left[\sum_j \frac{\partial^2}{\partial k_j^2} \sum_x e^{ik\cdot x} D(x)\right]_{k=0} = -\left[\sum_j \sum_x (ix_j)^2 e^{ik\cdot x} D(x)\right]_{k=0} = \sum_x |x|^2 D(x). \tag{2.2}$$

The assumptions involve a parameter d, which corresponds to the spatial dimension in our applications, and a parameter $\theta > 2$ which will be model dependent.

Let

$$a(k) = 1 - \hat{D}(k). \tag{2.3}$$

Assumption D. We assume that $D(x) \geq 0$ and

$$f_1(k;z) = z\hat{D}_L(k), \quad e_1(k;z) = 0. \tag{2.4}$$

In particular, this implies that $g_1(k;z) = z\hat{D}_L(k)$. As part of Assumption D, we also assume:
(i) D_L is normalised so that $\hat{D}_L(0) = 1$, and has $2+2\epsilon$ moments for some $\epsilon \in (0, \theta - 2)$, i.e.,

$$\sum_{x \in \mathbb{Z}^d} |x|^{2+2\epsilon} D_L(x) < \infty. \tag{2.5}$$

(ii) There is a constant C such that, for all $L \geq 1$,

$$\|D_L\|_\infty \leq CL^{-d}, \qquad \sigma^2 = \sigma_L^2 \leq CL^2, \tag{2.6}$$

(iii) There exist constants $\eta, c_1, c_2 > 0$ such that

$$c_1 L^2 k^2 \leq a_L(k) \leq c_2 L^2 k^2 \quad (\|k\|_\infty \leq L^{-1}), \tag{2.7}$$

$$a_L(k) > \eta \quad (\|k\|_\infty \geq L^{-1}), \tag{2.8}$$

$$a_L(k) < 2 - \eta \quad (k \in [-\pi, \pi]^d). \tag{2.9}$$

Assumptions E and G of [2] are now adapted to general $\theta > 2$ as follows. The relevant bounds on f_m, which *a priori* may or may not be satisfied, are that for some $p^* \geq 1$, some nonempty $B \subset [1, p^*]$ and

$$\beta = \beta(p^*) = L^{-\frac{d}{p^*}} \tag{2.10}$$

we have for every $p \in B$,

$$\|\hat{D}^2 f_m(\cdot; z)\|_p \leq \frac{K}{L^{\frac{d}{p}} m^{\frac{d}{2p} \wedge \theta}}, \quad |f_m(0;z)| \leq K, \quad |\nabla^2 f_m(0;z)| \leq K\sigma^2 m, \tag{2.11}$$

for some positive constant K. The full generality in which this has been presented is not required for our application to lattice trees where we have $p^* = 2$ and $B = \{2\}$. This is because we require only the $p = 2$ case in (2.11) to estimate the diagrams arising from the lace expansion for lattice trees and verify the assumptions E_θ, G_θ which follow. In other applications it may be that a larger collection of $\|\cdot\|_p$ norms are required to verify the assumptions and the set B is allowing for this possibility. The parameter p^* serves to make this set bounded so that $\beta(p^*)$ is small for large L.

The bounds in (2.11) are identical to the ones in [2], except for the first bound, which only appears for $p = 1$ and $\theta = \frac{d}{2}$.

Assumption $\mathbf{E}_\theta$. There is an L_0, an interval $I \subset [1-\alpha, 1+\alpha]$ with $\alpha \in (0,1)$, and a function $K \mapsto C_e(K)$, such that if (2.11) holds for some $K > 1$, $L \geq L_0$, $z \in I$ and for all $1 \leq m \leq n$, then for that L and z, and for all $k \in [-\pi, \pi]^d$ and $2 \leq m \leq n+1$, the following bounds hold:

$$|e_m(k;z)| \leq C_e(K)\beta m^{-\theta}, \quad |e_m(k;z) - e_m(0;z)| \leq C_e(K) a(k) \beta m^{-\theta+1}. \tag{2.12}$$

Assumption G$_\theta$. There is an L_0, an interval $I \subset [1-\alpha, 1+\alpha]$ with $\alpha \in (0,1)$, and a function $K \mapsto C_g(K)$, such that if (2.11) holds for some $K > 1$, $L \geq L_0$, $z \in I$ and for all $1 \leq m \leq n$, then for that L and z, and for all $k \in [-\pi, \pi]^d$ and $2 \leq m \leq n+1$, the following bounds hold:

$$|g_m(k;z)| \leq C_g(K)\beta m^{-\theta}, \quad |\nabla^2 g_m(0;z)| \leq C_g(K)\sigma^2\beta m^{-\theta+1}, \tag{2.13}$$

$$|\partial_z g_m(0;z)| \leq C_g(K)\beta m^{-\theta+1}, \tag{2.14}$$

$$|g_m(k;z) - g_m(0;z) - a(k)\sigma^{-2}\nabla^2 g_m(0;z)| \leq C_g(K)\beta a(k)^{1+\epsilon'} m^{-\theta+(1+\epsilon')}, \tag{2.15}$$

with the last bound valid for any $\epsilon' \in [0, \epsilon]$.

Theorem 2.1. *Let $d > d_c$ and $\theta(d) > 2$, and assume that Assumptions S, D, E$_\theta$ and G$_\theta$ all hold. There exist positive $L_0 = L_0(d, \epsilon)$, $z_c = z_c(d, L)$, $A = A(d, L)$, and $v = v(d, L)$, such that for $L \geq L_0$, the following statements hold.*
(a) Fix $\gamma \in (0, 1 \wedge \epsilon)$ and $\delta \in (0, (1 \wedge \epsilon) - \gamma)$. Then

$$f_n\Big(\frac{k}{\sqrt{v\sigma^2 n}}; z_c\Big) = Ae^{-\frac{k^2}{2d}}[1 + \mathcal{O}(k^2 n^{-\delta}) + \mathcal{O}(n^{-\theta+2})], \tag{2.16}$$

with the error estimate uniform in $\{k \in \mathbb{R}^d : a(k/\sqrt{v\sigma^2 n}) \leq \gamma n^{-1}\log n\}$.
(b)

$$-\frac{\nabla^2 f_n(0;z_c)}{f_n(0;z_c)} = v\sigma^2 n[1 + \mathcal{O}(\beta n^{-\delta})]. \tag{2.17}$$

(c) For all $p \geq 1$,

$$\|\hat{D}^2 f_n(\cdot; z_c)\|_p \leq \frac{C}{L^{\frac{d}{p}} n^{\frac{d}{2p}\wedge\theta}}. \tag{2.18}$$

(d) The constants z_c, A and v obey

$$\begin{aligned} 1 &= \sum_{m=1}^{\infty} g_m(0;z_c), \\ A &= \frac{1 + \sum_{m=1}^{\infty} e_m(0;z_c)}{\sum_{m=1}^{\infty} m g_m(0;z_c)}, \\ v &= -\frac{\sum_{m=1}^{\infty} \nabla^2 g_m(0;z_c)}{\sigma^2 \sum_{m=1}^{\infty} m g_m(0;z_c)}. \end{aligned} \tag{2.19}$$

It follows immediately from Theorem 2.1(d) and the bounds of Assumptions E$_\theta$ and G$_\theta$ that

$$z_c = 1 + \mathcal{O}(\beta), \quad A = 1 + \mathcal{O}(\beta), \quad v = 1 + \mathcal{O}(\beta). \tag{2.20}$$

With modest additional assumptions, the critical point z_c can be characterised in terms of the *susceptibility*

$$\chi(z) = \sum_{n=0}^{\infty} f_n(0;z). \tag{2.21}$$

Theorem 2.2. *Let $d > d_c$, $\theta(d) > 2$, $p^* \geq 1$ and assume that Assumptions S, D, E_θ and G_θ all hold. Let L be sufficiently large. Suppose there is a $z'_c > 0$ such that the susceptibility (2.21) is absolutely convergent for $z \in (0, z'_c)$, with $\lim_{z \uparrow z'_c} \chi(z) = \infty$ (if $\chi(z)$ is a power series in z then z'_c is the radius of convergence of $\chi(z)$). Suppose also that the bounds of (2.11) for $z = z_c$ and all $m \geq 1$ imply the bounds of Assumptions E_θ and G_θ for all $m \geq 2$, uniformly in $z \in [0, z_c]$. Then $z_c = z'_c$.*

3 Induction hypotheses

We will analyse the recursion relation (2.1) using induction on n, as done in [2]. In this section, we introduce the induction hypotheses, verify that they hold for $n = 1$, discuss their motivation, and derive some of their consequences.

3.1 Statement of induction hypotheses (H1–H4)

The induction hypotheses involve a sequence v_n, which is defined as follows. We set $v_0 = b_0 = 1$, and for $n \geq 1$ we define

$$b_n = -\frac{1}{\sigma^2} \sum_{m=1}^{n} \nabla^2 g_m(0; z), \quad c_n = \sum_{m=1}^{n} (m-1) g_m(0; z), \quad v_n = \frac{b_n}{1 + c_n}. \tag{3.1}$$

The z–dependence of b_n, c_n, v_n will usually be left implicit in the notation. We will often simplify the notation by dropping z also from e_n, f_n and g_n, and write, e.g., $f_n(k) = f_n(k; z)$.

Remark 3.1. *Note that the above definition and assumption D gives*

$$b_1 = -\frac{1}{\sigma^2} \nabla^2 g_1(0; z) = -\frac{1}{\sigma^2} \nabla^2 z \widehat{D}(0) = -\frac{z}{\sigma^2} . \left(-\sigma^2\right) = z. \tag{3.2}$$

Obviously we also have $c_1 = 0$ so that $v_1 = z$.

The induction hypotheses also involve several constants. Let $d > d_c$, $\theta > 2$, and recall that ϵ was specified in (2.5). We fix $\gamma, \delta > 0$ and $\lambda > 2$ according to

$$\begin{aligned} 0 < \gamma &< 1 \wedge \epsilon \\ 0 < \delta &< (1 \wedge \epsilon) - \gamma \\ \theta - \gamma < \lambda &< \theta. \end{aligned} \tag{3.3}$$

We also introduce constants $K_1, \ldots, K_5$, which are independent of β. We define

$$K'_4 = \max\{C_e(cK_4), C_g(cK_4), K_4\}, \tag{3.4}$$

where c is a constant determined in Lemma 3.6 below. To advance the induction, we will need to assume that

$$K_3 \gg K_1 > K'_4 \geq K_4 \gg 1, \quad K_2 \geq K_1, 3K'_4, \quad K_5 \gg K_4. \tag{3.5}$$

Here $a \gg b$ denotes the statement that a/b is sufficiently large. The amount by which, for instance, K_3 must exceed K_1 is independent of β (but may depend on p^*) and will be determined during the course of the advancement of the induction in Section 4.

Let $z_0 = z_1 = 1$, and define z_n recursively by

$$z_{n+1} = 1 - \sum_{m=2}^{n+1} g_m(0; z_n), \qquad n \geq 1. \tag{3.6}$$

For $n \geq 1$, we define intervals

$$I_n = [z_n - K_1 \beta n^{-\theta+1}, z_n + K_1 \beta n^{-\theta+1}]. \tag{3.7}$$

In particular this gives $I_1 = [1 - K_1\beta, 1 + K_1\beta]$.

Recall the definition $a(k) = 1 - \hat{D}(k)$ from (2.3). Our induction hypotheses are that the following four statements hold for all $z \in I_n$ and all $1 \leq j \leq n$.

(H1) $|z_j - z_{j-1}| \leq K_1 \beta j^{-\theta}$.

(H2) $|v_j - v_{j-1}| \leq K_2 \beta j^{-\theta+1}$.

(H3) For k such that $a(k) \leq \gamma j^{-1} \log j$, $f_j(k; z)$ can be written in the form

$$f_j(k; z) = \prod_{i=1}^{j} \left[1 - v_i a(k) + r_i(k)\right],$$

with $r_i(k) = r_i(k; z)$ obeying

$$|r_i(0)| \leq K_3 \beta i^{-\theta+1}, \quad |r_i(k) - r_i(0)| \leq K_3 \beta a(k) i^{-\delta}.$$

(H4) For k such that $a(k) > \gamma j^{-1} \log j$, $f_j(k; z)$ obeys the bounds

$$|f_j(k; z)| \leq K_4 a(k)^{-\lambda} j^{-\theta}, \quad |f_j(k; z) - f_{j-1}(k; z)| \leq K_5 a(k)^{-\lambda+1} j^{-\theta}.$$

Note that, for $k = 0$, (H3) reduces to $f_j(0) = \prod_{i=1}^{j} [1 + r_i(0)]$.

3.2 Initialisation of the induction

We now verify that the induction hypotheses hold when $n = 1$. This remains unchanged from the $p = 1$ case. Fix $z \in I_1$.

(H1) We simply have $z_1 - z_0 = 1 - 1 = 0$.

(H2) From Remark 3.1 we simply have $|v_1 - v_0| = |z - 1|$, so that (H2) is satisfied provided $K_2 \geq K_1$.

(H3) We are restricted to $a(k) = 0$. By (2.7), this means $k = 0$. By Assumption D, $f_1(0; z) = z$, so that $r_1(0) = z - 1 = z - z_1$. Thus (H3) holds provided we take $K_3 \geq K_1$.

(H4) We note that $|f_1(k; z)| \leq z \leq 2$ for β sufficiently small (i.e. so that $\beta K_1 \leq 1$), $|f_1(k; z) - f_0(k; z)| \leq 3$, and $a(k) \leq 2$. The bounds of (H4) therefore hold provided we take $K_4 \geq 2^{\lambda+1}$ and $K_5 \geq 3 \cdot 2^{\lambda-1}$.

3.3 Discussion of induction hypotheses

(H1) and the critical point. The critical point can be formally identified as follows. We set $k = 0$ in (2.1), then sum over n, and solve for the susceptibility

$$\chi(z) = \sum_{n=0}^{\infty} f_n(0; z). \tag{3.8}$$

The result is

$$\chi(z) = \frac{1 + \sum_{m=2}^{\infty} e_m(0; z)}{1 - \sum_{m=1}^{\infty} g_m(0; z)}. \tag{3.9}$$

The critical point should correspond to the smallest zero of the denominator and hence should obey the equation

$$1 - \sum_{m=1}^{\infty} g_m(0; z_c) = 1 - z_c - \sum_{m=2}^{\infty} g_m(0; z_c) = 0. \tag{3.10}$$

However, we do not know *a priori* that the series in (3.9) or (3.10) converge. We therefore approximate (3.10) with the recursion (3.6), which bypasses the convergence issue by discarding the $g_m(0)$ for $m > n + 1$ that cannot be handled at the n^{th} stage of the induction argument. The sequence z_n will ultimately converge to z_c.

In dealing with the sequence z_n, it is convenient to formulate the induction hypotheses for a small interval I_n approximating z_c. As we will see in Section 3.4, (H1) guarantees that the intervals I_j are decreasing: $I_1 \supset I_2 \supset \cdots \supset I_n$. Because the length of these intervals is shrinking to zero, their intersection $\cap_{j=1}^{\infty} I_j$ is a single point, namely z_c. Hypothesis (H1) drives the convergence of z_n to z_c and gives some control on the rate. The rate is determined from (3.6) and the ansatz that the difference $z_j - z_{j-1}$ is approximately $-g_{j+1}(0, z_c)$, with $|g_j(k; z_c)| = \mathcal{O}(\beta j^{-\theta})$ as in Assumption G.

3.4 Consequences of induction hypotheses

In this section we derive important consequences of the induction hypotheses. The key result is that the induction hypotheses imply (2.11) for all $1 \leq m \leq n$, from which the bounds of Assumptions E_θ and G_θ then follow, for $2 \leq m \leq n + 1$.

Here, and throughout the rest of this paper:

- C denotes a strictly positive constant that may depend on $d, \gamma, \delta, \lambda$, but *not* on the K_i, *not* on k, *not* on n, and *not* on β (provided β is sufficiently small, possibly depending on the K_i). The value of C may change from line to line.

- We frequently assume $\beta \ll 1$ without explicit comment.

The first lemma shows that the intervals I_j are nested, assuming (H1).

Lemma 3.2. *Assume (H1) for $1 \le j \le n$. Then $I_1 \supset I_2 \supset \cdots \supset I_n$.*

Proof. Suppose $z \in I_j$, with $2 \le j \le n$. Then by (H1) and (3.7),

$$|z - z_{j-1}| \le |z - z_j| + |z_j - z_{j-1}| \le \frac{K_1\beta}{j^{\theta-1}} + \frac{K_1\beta}{j^{\theta}} \le \frac{K_1\beta}{(j-1)^{\theta-1}}, \tag{3.11}$$

and hence $z \in I_{j-1}$. Note that here we have used the fact that

$$\frac{1}{j^a} + \frac{1}{j^b} \le \frac{1}{(j-1)^a} \iff 1 + \frac{1}{j^{b-a}} \le \left(\frac{j}{j-1}\right)^a \tag{3.12}$$

which holds if $a \ge 1$ and $b - a \ge 1$ since then

$$1 + \frac{1}{j^{b-a}} \le 1 + \frac{1}{j} \le 1 + \frac{1}{j-1} \le \left(1 + \frac{1}{j-1}\right)^a. \tag{3.13}$$

□

By Lemma 3.2, if $z \in I_j$ for $1 \le j \le n$, then $z \in I_1$ and hence, by (3.7),

$$|z - 1| \le K_1\beta. \tag{3.14}$$

It also follows from (H2) that, for $z \in I_n$ and $1 \le j \le n$,

$$|v_j - 1| \le CK_2\beta. \tag{3.15}$$

Define

$$s_i(k) = [1 + r_i(0)]^{-1}[v_i a(k) r_i(0) + (r_i(k) - r_i(0))]. \tag{3.16}$$

We claim that the induction hypothesis (H3) has the useful alternate form

$$f_j(k) = f_j(0) \prod_{i=1}^{j} [1 - v_i a(k) + s_i(k)]. \tag{3.17}$$

Firstly $f_j(0) = \prod_{i=1}^{j}[1 + r_i(0)]$. Therefore the RHS of (3.17) is

$$\prod_{i=1}^{j} (1 - v_i a(k)) [1 + r_i(0)] + v_i a(k) r_i(0) + (r_i(k) - r_i(0)) \tag{3.18}$$

which after cancelling terms gives the result. Note that (3.17) shows that the $s_i(k)$ are symmetric with continuous second derivative in a neighbourhood of 0 (since each $f_i(k)$ and $a(k)$ have these properties). To see this note that $f_1(k)$ and $a(k)$ symmetric implies that $s_1(k)$ is symmetric. Next, $f_2(k), a(k)$, and $s_1(k)$ symmetric implies that $s_2(k)$ symmetric etc.

We further claim that

$$|s_i(k)| \leq K_3(2 + C(K_2 + K_3)\beta)\beta a(k) i^{-\delta}. \tag{3.19}$$

This is different to that appearing in [2, (2.19)] in that the constant is now 2 rather than 1. This is a correction to [2, (2.19)] but it does not affect the analysis. To verify (3.19) we use the fact that $\frac{1}{1-x} \leq 1 + 2x$ for $x \leq \frac{1}{2}$ to write for small enough β,

$$\begin{aligned} |s_i(k)| &\leq [1 + 2K_3\beta]\,[(1 + |v_i - 1|)a(k)r_i(0) + |r_i(k) - r_i(0)|] \\ &\leq [1 + 2K_3\beta]\left[(1 + CK_2\beta)a(k)\frac{K_3\beta}{i^{\theta-1}} + \frac{K_3\beta a(k)}{i^{\delta}}\right] \\ &\leq \frac{K_3\beta a(k)}{i^{\delta}}[1 + 2K_3\beta][2 + CK_2\beta] \leq \frac{K_3\beta a(k)}{i^{\delta}}[2 + C(K_2 + K_3)\beta]. \end{aligned} \tag{3.20}$$

Where we have used the bounds of (H3) as well as the fact that $\theta - 1 > \delta$. The next lemma provides an important upper bound on $f_j(k; z)$, for k small depending on j, as in (H3).

Lemma 3.3. *Let $z \in I_n$ and assume (H2–H3) for $1 \leq j \leq n$. Then for k with $a(k) \leq \gamma j^{-1} \log j$,*

$$|f_j(k; z)| \leq e^{CK_3\beta} e^{-(1 - C(K_2 + K_3)\beta) j a(k)}. \tag{3.21}$$

Proof. We use H3, and conclude from the bound on $r_i(0)$ of (H3) that

$$|f_j(0)| = \prod_{i=1}^{j} |1 + r_i(0)| \leq \prod_{i=1}^{j} \left|1 + \frac{K_3\beta}{i^{\theta-1}}\right| \leq e^{CK_3\beta},$$

using $1 + x \leq e^x$ for each factor. Then we use (3.15), (3.17) and (3.19) to obtain

$$\prod_{i=1}^{j} |1 - v_i a(k) + s_i(k)| \leq \prod_{i=1}^{j} \left|1 - (1 - CK_2\beta)a(k) + CK_3\beta a(k) i^{-\delta}\right|. \tag{3.22}$$

The desired bound then follows, again using $1 + x \leq e^x$ for each factor on the right side, and by (3.17). □

The middle bound of (2.11) follows, for $1 \leq m \leq n$ and $z \in I_m$, directly from Lemma 3.3. We next prove two lemmas which provide the other two bounds of (2.11). This will supply the hypothesis (2.11) for Assumptions E_θ and G_θ, and therefore plays a crucial role in advancing the induction.

Lemma 3.4. *Let $z \in I_n$ and assume (H2), (H3) and (H4). Then for all $1 \leq j \leq n$, and $p \geq 1$,*

$$\|\hat{D}^2 f_j(\cdot; z)\|_p \leq \frac{C(1 + K_4)}{L^{\frac{d}{p}} j^{\frac{d}{2p} \wedge \theta}}, \tag{3.23}$$

where the constant C may depend on p, d.

Proof. We show that

$$\|\hat{D}^2 f_j(\cdot; z)\|_p^p \leq \frac{C(1+K_4)^p}{L^d j^{\frac{d}{2}\wedge\theta p}}. \tag{3.24}$$

For $j = 1$ the result holds since $|f_1(k)| = |z\widehat{D}(k)| \leq z \leq 2$ and by using (2.6) and the fact that $p \geq 1$. We may therefore assume that $j \geq 2$ where needed in what follows, so that in particular $\log j \geq \log 2$.

Fix $z \in I_n$ and $1 \leq j \leq n$, and define

$$\begin{aligned}
R_1 &= \{k \in [-\pi,\pi]^d : a(k) \leq \gamma j^{-1}\log j,\ \|k\|_\infty \leq L^{-1}\},\\
R_2 &= \{k \in [-\pi,\pi]^d : a(k) \leq \gamma j^{-1}\log j,\ \|k\|_\infty > L^{-1}\},\\
R_3 &= \{k \in [-\pi,\pi]^d : a(k) > \gamma j^{-1}\log j,\ \|k\|_\infty \leq L^{-1}\},\\
R_4 &= \{k \in [-\pi,\pi]^d : a(k) > \gamma j^{-1}\log j,\ \|k\|_\infty > L^{-1}\}.
\end{aligned}$$

The set R_2 is empty if j is sufficiently large. Then

$$\|\hat{D}^2 f_j\|_p^p = \sum_{i=1}^4 \int_{R_i} \left(\hat{D}(k)^2 |f_j(k)|\right)^p \frac{d^d k}{(2\pi)^d}. \tag{3.25}$$

We will treat each of the four terms on the right side separately.

On R_1, we use (2.7) in conjunction with Lemma 3.3 and the fact that $\hat{D}(k)^2 \leq 1$, to obtain for all $p > 0$,

$$\int_{R_1} \left(\hat{D}(k)^2\right)^p |f_j(k)|^p \frac{d^d k}{(2\pi)^d} \leq \int_{R_1} Ce^{-cpj(Lk)^2} \frac{d^d k}{(2\pi)^d} \leq \prod_{i=1}^d \int_{-\frac{1}{L}}^{\frac{1}{L}} Ce^{-cpj(Lk_i)^2} dk_i \leq \frac{C}{L^d (pj)^{d/2}} \leq \frac{C}{L^d j^{d/2}}. \tag{3.26}$$

Here we have used the substitution $k_i' = Lk_i\sqrt{pj}$. On R_2, we use Lemma 3.3 and (2.8) to conclude that for all $p > 0$, there is an $\alpha(p) > 1$ such that

$$\int_{R_2} \left(\hat{D}(k)^2 |f_j(k)|\right)^p \frac{d^d k}{(2\pi)^d} \leq C \int_{R_2} \alpha^{-j} \frac{d^d k}{(2\pi)^d} = C\alpha^{-j}|R_2|, \tag{3.27}$$

where $|R_2|$ denotes the volume of R_2. This volume is maximal when $j = 3$, so that

$$|R_2| \leq |\{k : a(k) \leq \tfrac{\gamma \log 3}{3}\}| \leq |\{k : \hat{D}(k) \geq 1 - \tfrac{\gamma \log 3}{3}\}| \leq (\tfrac{1}{1-\frac{\gamma\log 3}{3}})^2 \|\hat{D}^2\|_1 \leq (\tfrac{1}{1-\frac{\gamma\log 3}{3}})^2 CL^{-d}, \tag{3.28}$$

using (2.6) in the last step. Therefore $\alpha^{-j}|R_2| \leq CL^{-d}j^{-d/2}$ since $\alpha^{-j}j^{\frac{d}{2}} \leq C(\alpha, d)$ for every j (using L'Hôpital's rule for example with $\alpha^j = e^{j\log\alpha}$), and

$$\int_{R_2} \left(\hat{D}(k)^2 |f_j(k)|\right)^p \frac{d^d k}{(2\pi)^d} \leq CL^{-d}j^{-d/2}. \tag{3.29}$$

On R_3 and R_4, we use (H4). As a result, the contribution from these two regions is bounded above by

$$\left(\frac{K_4}{j^\theta}\right)^p \sum_{i=3}^{4} \int_{R_i} \frac{\hat{D}(k)^{2p}}{a(k)^{\lambda p}} \frac{d^dk}{(2\pi)^d}. \tag{3.30}$$

On R_3, we use $\hat{D}(k)^2 \leq 1$ and (2.7). Define $R_3^C = \{k : \|k\|_\infty < L^{-1},\ |k|^2 > Cj^{-1}\log j\}$ to obtain the upper bound

$$\begin{aligned} \frac{CK_4^p}{j^{\theta p}L^{2\lambda p}} \int_{R_3} \frac{1}{|k|^{2\lambda p}} d^dk &\leq \frac{CK_4^p}{j^{\theta p}L^{2\lambda p}} \int_{R_3^C} \frac{1}{|k|^{2\lambda p}} d^dk \\ &= \frac{CK_4^p}{j^{\theta p}L^{2\lambda p}} \int_{\sqrt{\frac{C\log j}{L^2 j}}}^{\frac{d}{L}} r^{d-1-2\lambda p} dr. \end{aligned} \tag{3.31}$$

Since $\log 1 = 0$, this integral will not be finite if both $j = 1$ and $p \geq \frac{d}{2\lambda}$, but recall that we can restrict our attention to $j \geq 2$. Thus we have an upper bound of

$$\frac{CK_4^p}{j^{\theta p}L^{2\lambda p}} \cdot \begin{cases} \int_0^{\frac{d}{L}} r^{d-1-2\lambda p} dr, & d > 2\lambda p \\ \int_{\sqrt{\frac{C\log j}{L^2 j}}}^{\frac{d}{L}} \frac{1}{r} dr, & d = 2\lambda p \\ \int_{\sqrt{\frac{C\log j}{L^2 j}}}^{\infty} r^{d-1-2\lambda p} dr, & d < 2\lambda p \end{cases} \leq \frac{CK_4^p}{j^{\theta p}L^{2\lambda p}} \cdot \begin{cases} \left(\frac{d}{L}\right)^{d-2\lambda p} & , d > 2\lambda p \\ \log\left(\frac{d\sqrt{L^2 j}}{CL\sqrt{\log j}}\right) = \frac{1}{2}\log\left(\frac{C' j}{\log j}\right) & , d = 2\lambda p \\ \left(\frac{C' L^2 j}{\log j}\right)^{\frac{2\lambda p - d}{2}} & , d < 2\lambda p. \end{cases} \tag{3.32}$$

Now use the fact that $\lambda < \theta$ to see that each term on the right is bounded by $\frac{CK_4^p}{j^{\frac{d}{2}}L^d}$.

On R_4, we use (2.6) and (2.8) to obtain the bound

$$\frac{CK_4^p}{j^{\theta p}} \int_{[-\pi,\pi]^d} \hat{D}(k)^{2p} \frac{d^dk}{(2\pi)^d} \leq \frac{CK_4^p}{j^{\theta p}} \int_{[-\pi,\pi]^d} \hat{D}(k)^2 \frac{d^dk}{(2\pi)^d} \leq \frac{CK_4}{j^{\theta p}L^d}, \tag{3.33}$$

where we have used the fact that $p \geq 1$ and $\hat{D}(k)^2 \leq 1$. Since $K_4^p \leq (1+K_4)^p$, this completes the proof. □

Lemma 3.5. *Let $z \in I_n$ and assume (H2) and (H3). Then, for $1 \leq j \leq n$,*

$$|\nabla^2 f_j(0; z)| \leq (1 + C(K_2 + K_3)\beta)\sigma^2 j. \tag{3.34}$$

Proof. Fix $z \in I_n$ and j with $1 \leq j \leq n$. Using the product rule multiple times and the symmetry of all of the quantities in (3.17) to get cross terms equal to 0,

$$\nabla^2 f_j(0) = f_j(0) \sum_{i=1}^{j} \left[-\sigma^2 v_i + \nabla^2 s_i(0)\right]. \tag{3.35}$$

By (3.15), $|v_i - 1| \le CK_2\beta$. For the second term on the right side, we let $e_1, \ldots, e_d$ denote the standard basis vectors in $\mathbb{R}^d$. Since $s_i(k)$ has continuous second derivative in a neighbourhood of 0, we use the extended mean value theorem $s(t) = s(0) + ts'(0) + \frac{1}{2}t^2 s''(t^*)$ for some $t^* \in (0, t)$, together with (3.19) to see that for all $i \le n$ we have

$$|\nabla^2 s_i(0)| = 2\Big|\sum_{l=1}^{d} \lim_{t\to 0} \frac{s_i(te_l)}{t^2}\Big| \le CK_3\beta i^{-\delta} \sum_{l=1}^{d} \lim_{t\to 0} \frac{a(te_l)}{t^2} = CK_3\sigma^2\beta i^{-\delta}. \tag{3.36}$$

Note the constant 2 here that is a correction to [2].

Thus, by (3.35) and Lemma 3.3

$$|\nabla^2 f_j(0)| \le f_j(0) \sum_{i=1}^{j} \left[\sigma^2 (1 + CK_2\beta) + \frac{CK_3\sigma^2\beta}{i^\delta}\right] \le e^{CK_3\beta}\sigma^2 j\Big(1 + C(K_2 + K_3)\beta\Big). \tag{3.37}$$

This completes the proof. ☐

The next lemma is the key to advancing the induction, as it provides bounds for e_{n+1} and g_{n+1}.

Lemma 3.6. *Let $z \in I_n$, and assume (H2), (H3) and (H4). For $k \in [-\pi, \pi]^d$, $2 \le j \le n+1$, and $\epsilon' \in [0, \epsilon]$, the following hold:*
(i) $|g_j(k; z)| \le K_4'\beta j^{-\theta}$,
(ii) $|\nabla^2 g_j(0; z)| \le K_4'\sigma^2\beta j^{-\theta+1}$,
(iii) $|\partial_z g_j(0; z)| \le K_4'\beta j^{-\theta+1}$,
(iv) $|g_j(k; z) - g_j(0; z) - a(k)\sigma^{-2}\nabla^2 g_j(0; z)| \le K_4'\beta a(k)^{1+\epsilon'} j^{-\theta+1+\epsilon'}$,
(v) $|e_j(k; z)| \le K_4'\beta j^{-\theta}$,
(vi) $|e_j(k; z) - e_j(0; z)| \le K_4' a(k)\beta j^{-\theta+1}$.

Proof. The bounds (2.11) for $1 \le m \le n$ follow from Lemmas 3.3–3.5, with $K = cK_4$ (this defines c), assuming that β is sufficiently small. The bounds of the lemma then follow immediately from Assumptions E_θ and G_θ, with K_4' given in (3.4). ☐

4 The induction advanced

In this section we advance the induction hypotheses (H1–H4) from n to $n+1$. Throughout this section, in accordance with the uniformity condition on (H2–H4), we fix $z \in I_{n+1}$. We frequently assume $\beta \ll 1$ without explicit comment.

4.1 Advancement of (H1)

By (3.6) and the mean-value theorem,

$$z_{n+1} - z_n = -\sum_{m=2}^{n} [g_m(0; z_n) - g_m(0; z_{n-1})] - g_{n+1}(0; z_n)$$
$$= -(z_n - z_{n-1}) \sum_{m=2}^{n} \partial_z g_m(0; y_n) - g_{n+1}(0; z_n),$$

for some y_n between z_n and z_{n-1}. By (H1) and (3.7), $y_n \in I_n$. Using Lemma 3.6 and (H1), it then follows that

$$|z_{n+1} - z_n| \leq K_1 \beta n^{-\theta} \sum_{m=2}^{n} K_4' \beta m^{-\theta+1} + K_4' \beta (n+1)^{-\theta}$$

$$\leq K_4' \beta (1 + C K_1 \beta)(n+1)^{-\theta}.$$

Thus (H1) holds for $n+1$, for β small and $K_1 > K_4'$.

Having advanced (H1) to $n+1$, it then follows from Lemma 3.2 that $I_1 \supset I_2 \supset \cdots \supset I_{n+1}$.

For $n \geq 0$, define

$$\zeta_{n+1} = \zeta_{n+1}(z) = \sum_{m=1}^{n+1} g_m(0; z) - 1 = \sum_{m=2}^{n+1} g_m(0; z) + z - 1. \tag{4.1}$$

The following lemma, whose proof makes use of (H1) for $n+1$, will be needed in what follows.

Lemma 4.1. *For all $z \in I_{n+1}$,*

$$|\zeta_{n+1}| \leq C K_1 \beta (n+1)^{-\theta+1}. \tag{4.2}$$

Proof. By (3.6) and the mean-value theorem,

$$|\zeta_{n+1}| = \Big| (z - z_{n+1}) + \sum_{m=2}^{n+1} [g_m(0; z) - g_m(0; z_n)] \Big|$$
$$= \Big| (z - z_{n+1}) + (z - z_n) \sum_{m=2}^{n+1} \partial_z g_m(0; y_n) \Big|,$$

for some y_n between z and z_n. Since $z \in I_{n+1} \subset I_n$ and $z_n \in I_n$, we have $y_n \in I_n$. Therefore, by Lemma 3.6,

$$|\zeta_{n+1}| \leq K_1 \beta (n+1)^{-\theta+1} + K_1 \beta n^{-\theta+1} \sum_{m=2}^{n+1} K_4' \beta m^{-\theta+1} \leq K_1 \beta (1 + C K_4' \beta)(n+1)^{-\theta+1}. \tag{4.3}$$

The lemma then follows, for β sufficiently small. $\square$

4.2 Advancement of (H2)

Let $z \in I_{n+1}$. As observed in Section 4.1, this implies that $z \in I_j$ for all $j \leq n+1$. The definitions in (3.1) imply that

$$v_{n+1} - v_n = \frac{1}{1+c_{n+1}}(b_{n+1} - b_n) - \frac{b_n}{(1+c_n)(1+c_{n+1})}(c_{n+1} - c_n), \tag{4.4}$$

with

$$b_{n+1} - b_n = -\frac{1}{\sigma^2}\nabla^2 g_{n+1}(0), \quad c_{n+1} - c_n = n g_{n+1}(0). \tag{4.5}$$

By Lemma 3.6, both differences in (4.5) are bounded by $K_4'\beta(n+1)^{-\theta+1}$, and, in addition,

$$|b_j - 1| \leq CK_4'\beta, \quad |c_j| \leq CK_4'\beta \tag{4.6}$$

for $1 \leq j \leq n+1$. Therefore

$$|v_{n+1} - v_n| \leq K_2\beta(n+1)^{-\theta+1}, \tag{4.7}$$

provided we assume $K_2 \geq 3K_4'$. This advances (H2).

4.3 Advancement of (H3)

4.3.1 The decomposition

The advancement of the induction hypotheses (H3–H4) is the most technical part of the proof. For (H3), we fix k with $a(k) \leq \gamma(n+1)^{-1}\log(n+1)$, and $z \in I_{n+1}$. The induction step will be achieved as soon as we are able to write the ratio $f_{n+1}(k)/f_n(k)$ as

$$\frac{f_{n+1}(k)}{f_n(k)} = 1 - v_{n+1}a(k) + r_{n+1}(k), \tag{4.8}$$

with $r_{n+1}(0)$ and $r_{n+1}(k) - r_{n+1}(0)$ satisfying the bounds required by (H3).

To begin, we divide the recursion relation (2.1) by $f_n(k)$, and use (4.1), to obtain

$$\frac{f_{n+1}(k)}{f_n(k)} = 1 + \sum_{m=1}^{n+1}\Big[g_m(k)\frac{f_{n+1-m}(k)}{f_n(k)} - g_m(0)\Big] + \zeta_{n+1} + \frac{e_{n+1}(k)}{f_n(k)}. \tag{4.9}$$

By (3.1),

$$v_{n+1} = b_{n+1} - v_{n+1}c_{n+1} = -\sigma^{-2}\sum_{m=1}^{n+1}\nabla^2 g_m(0) - v_{n+1}\sum_{m=1}^{n+1}(m-1)g_m(0). \tag{4.10}$$

Thus we can rewrite (4.9) as

$$\frac{f_{n+1}(k)}{f_n(k)} = 1 - v_{n+1}a(k) + r_{n+1}(k), \tag{4.11}$$

where

$$r_{n+1}(k) = X(k) + Y(k) + Z(k) + \zeta_{n+1} \tag{4.12}$$

with

$$\begin{aligned} X(k) &= \sum_{m=2}^{n+1} \Big[\big(g_m(k) - g_m(0)\big) \frac{f_{n+1-m}(k)}{f_n(k)} - a(k)\sigma^{-2}\nabla^2 g_m(0) \Big], \\ Y(k) &= \sum_{m=2}^{n+1} g_m(0) \left[\frac{f_{n+1-m}(k)}{f_n(k)} - 1 - (m-1)v_{n+1}a(k) \right], \\ Z(k) &= \frac{e_{n+1}(k)}{f_n(k)}. \end{aligned} \tag{4.13}$$

The $m = 1$ terms in X and Y vanish and have not been included.

We will prove that

$$|r_{n+1}(0)| \le \frac{C(K_1 + K_4')\beta}{(n+1)^{\theta-1}}, \qquad |r_{n+1}(k) - r_{n+1}(0)| \le \frac{CK_4'\beta a(k)}{(n+1)^{\delta}}. \tag{4.14}$$

This gives (H3) for $n+1$, provided we assume that $K_3 \gg K_1$ and $K_3 \gg K_4'$. To prove the bounds on r_{n+1} of (4.14), it will be convenient to make use of some elementary convolution bounds, as well as some bounds on ratios involving f_j. These preliminary bounds are given in Section 4.3.2, before we present the proof of (4.14) in Section 4.3.3.

4.3.2 Convolution and ratio bounds

The proof of (4.14) will make use of the following elementary convolution bounds. To keep the discussion simple, we do not obtain optimal bounds.

Lemma 4.2. *For $n \ge 2$,*

$$\sum_{m=2}^{n} \frac{1}{m^a} \sum_{j=n-m+1}^{n} \frac{1}{j^b} \le \begin{cases} Cn^{-(a\wedge b)+1} & \text{for } a,b>1 \\ Cn^{-(a-2)\wedge b} & \text{for } a>2, b>0 \\ Cn^{-(a-1)\wedge b} & \text{for } a>2, b>1 \\ Cn^{-a\wedge b} & \text{for } a,b>2. \end{cases} \tag{4.15}$$

Proof. Since $m + j \ge n$, either m or j is at least $\frac{n}{2}$. Therefore

$$\sum_{m=2}^{n} \frac{1}{m^a} \sum_{j=n-m+1}^{n} \frac{1}{j^b} \le \left(\frac{2}{n}\right)^a \sum_{m=2}^{n} \sum_{j=n-m+1}^{n} \frac{1}{j^b} + \left(\frac{2}{n}\right)^b \sum_{m=2}^{n} \sum_{j=n-m+1}^{n} \frac{1}{m^a}. \tag{4.16}$$

If $a, b > 1$, then the first term is bounded by Cn^{1-a} and the second by Cn^{1-b}.
If $a > 2, b > 0$, then the first term is bounded by Cn^{2-a} and the second by Cn^{-b}.

If $a > 2, b > 1$, then the first term is bounded by Cn^{1-a} and the second by Cn^{-b}.
If $a, b > 2$, then the first term is bounded by Cn^{-a} and the second by Cn^{-b}. □

We also will make use of several estimates involving ratios. We begin with some preparation. Given a vector $x = (x_l)$ with $\sup_l |x_l| < 1$, define $\chi(x) = \sum_l \frac{|x_l|}{1-|x_l|}$. The bound $(1-t)^{-1} \leq \exp[t(1-t)^{-1}]$, together with Taylor's Theorem applied to $f(t) = \prod_l \frac{1}{1-tx_l}$, gives

$$\left|\prod_l \frac{1}{1-x_l} - 1\right| \leq \chi(x)e^{\chi(x)}, \quad \left|\prod_l \frac{1}{1-x_l} - 1 - \sum_l x_l\right| \leq \chi(x)^2 e^{\chi(x)} \tag{4.17}$$

as follows. Firstly,

$$\frac{df}{dt} = f(t)\sum_{j=1}^d \frac{x_j}{1-tx_j} = \left[\prod_{l=1}^d \frac{1}{1-tx_l}\right]\sum_{j=1}^d \frac{x_j}{1-tx_j} \leq \left[\prod_{l=1}^d e^{\frac{|tx_j|}{1-|tx_j|}}\right]\sum_{j=1}^d \frac{|x_j|}{1-|tx_j|}, \tag{4.18}$$

which gives $f'(0) = \sum_{j=1}^d x_j$, and for $|t| \leq 1$, $|f'(t)| \leq \chi(x)e^{\chi(x)}$. This gives the first bound by Taylor's Theorem. The second bound can be obtained in the same way using the fact that

$$\frac{d^2 f}{dt^2} = f(t)\left[\sum_{j=1}^d \frac{x_j^2}{(1-tx_j)^2} + \left(\sum_{j=1}^d \frac{x_j}{1-tx_j}\right)^2\right]. \tag{4.19}$$

We assume throughout the rest of this section that $a(k) \leq \gamma(n+1)^{-1}\log(n+1)$ and $2 \leq m \leq n+1$, and define

$$\psi_{m,n} = \sum_{j=n+2-m}^{n} \frac{|r_j(0)|}{1-|r_j(0)|}, \quad \chi_{m,n}(k) = \sum_{j=n+2-m}^{n} \frac{v_j a(k) + |s_j(k)|}{1 - v_j a(k) - |s_j(k)|}. \tag{4.20}$$

By (3.15) and (3.19),

$$\chi_{m,n}(k) \leq (m-1)a(k)Q(k) \quad \text{with} \quad Q(k) = [1 + C(K_2+K_3)\beta][1 + Ca(k)], \tag{4.21}$$

where we have used the fact that for $|x| \leq \frac{1}{2}$, $\frac{1}{1-x} \leq 1 + 2|x|$. In our case $x = v_j a(k) + |s_j(k)|$ satisfies $|x| \leq (1+CK_2\beta)a(k) + CK_3\beta a(k)$. Since $a(k) \leq \gamma(n+1)^{-1}\log(n+1)$, we have $Q(k) \leq [1+C(K_2+K_3)\beta][1+C\gamma(n+1)^{-1}\log(n+1)]$. Therefore

$$\begin{aligned} e^{\chi_{m,n}(k)} &\leq e^{\gamma\log(n+1)Q(k)} \leq e^{\gamma\log(n+1)[1+C(K_2+K_3)\beta]} e^{\frac{C\gamma^2(\log(n+1))^2}{n+1}} \\ &\leq e^{\gamma\log(n+1)[1+C(K_2+K_3)\beta]} e^{4C\gamma^2} \leq C(n+1)^{\gamma q}, \end{aligned} \tag{4.22}$$

where we have used the fact that $\log x \leq 2\sqrt{x}$, and where $q = 1 + C(K_2+K_3)\beta$ may be taken to be as close to 1 as desired, by taking β to be small.

We now turn to the ratio bounds. It follows from (H3) and the first inequality of (4.17) that

$$\begin{aligned}\left|\frac{f_{n+1-m}(0)}{f_n(0)}-1\right| &= \left|\prod_{i=n+2-m}^{n}\frac{1}{1-(-r_i(0))}-1\right| \\ &\le \psi_{m,n}e^{\psi_{m,n}} \le \sum_{j=n+2-m}^{n}\frac{CK_3\beta}{j^{\theta-1}} \le \frac{CK_3\beta}{(n+2-m)^{\theta-2}}\end{aligned} \tag{4.23}$$

Therefore

$$\left|\frac{f_{n+1-m}(0)}{f_n(0)}\right| \le 1+CK_3\beta. \tag{4.24}$$

By (3.17),

$$\begin{aligned}\left|\frac{f_{n+1-m}(k)}{f_n(k)}-1\right| &= \left|\frac{f_{n+1-m}(0)}{f_n(0)}\prod_{j=n+2-m}^{n}\frac{1}{[1-v_ja(k)+s_j(k)]}-\frac{f_{n+1-m}(0)}{f_n(0)}+\frac{f_{n+1-m}(0)}{f_n(0)}-1\right| \\ &\le \left|\frac{f_{n+1-m}(0)}{f_n(0)}\right|\left|\prod_{j=n+2-m}^{n}\frac{1}{[1-v_ja(k)+s_j(k)]}-1\right|+\left|\frac{f_{n+1-m}(0)}{f_n(0)}-1\right|.\end{aligned} \tag{4.25}$$

The first inequality of (4.17), together with (4.21–4.24), then gives

$$\left|\frac{f_{n+1-m}(k)}{f_n(k)}-1\right| \le C(m-1)a(k)(n+1)^{\gamma q}+\frac{CK_3\beta}{(n+2-m)^{\theta-2}}. \tag{4.26}$$

Similarly,

$$\left|\frac{f_n(0)}{f_n(k)}-1\right| = \left|\prod_{i=1}^{n}\frac{1}{1-v_ja(k)+s_j(k)}-1\right| \le \chi_{n+1,n}(k)e^{\chi_{n+1,n}(k)} \le Ca(k)(n+1)^{1+\gamma q}. \tag{4.27}$$

Next, we estimate the quantity $R_{m,n}(k)$, which is defined by

$$R_{m,n}(k) = \prod_{j=n+2-m}^{n}[1-v_ja(k)+s_j(k)]^{-1}-1-\sum_{j=n+2-m}^{n}[v_ja(k)-s_j(k)]. \tag{4.28}$$

By the second inequality of (4.17), together with (4.21) and (4.22), this obeys

$$|R_{m,n}(k)| \le \chi_{m,n}(k)^2e^{\chi_{m,n}(k)} \le Cm^2a(k)^2(n+1)^{\gamma q}. \tag{4.29}$$

Finally, we apply (H3) with $\frac{1}{1-x}-1=\frac{x}{1-x}\le\frac{|x|}{1-|x|}$ to obtain for $m\le n$,

$$\left|\frac{f_{m-1}(k)}{f_m(k)}-1\right| = \left|[1-v_ma(k)+(r_m(k)-r_m(0))+r_m(0)]^{-1}-1\right| \le Ca(k)+\frac{CK_3\beta}{m^{\theta-1}}. \tag{4.30}$$

Note that for example, $1-(|v_ma(k)|+|r_m(k)-r_m(0)|+|r_m(0)|)>c$ for small enough β (depending on γ, among other things).

4.3.3 The induction step

By definition,

$$r_{n+1}(0) = Y(0) + Z(0) + \zeta_{n+1} \tag{4.31}$$

and

$$r_{n+1}(k) - r_{n+1}(0) = X(k) + \Big(Y(k) - Y(0)\Big) + \Big(Z(k) - Z(0)\Big). \tag{4.32}$$

Since $|\zeta_{n+1}| \leq CK_1\beta(n+1)^{-\theta+1}$ by Lemma 4.1, to prove (4.14) it suffices to show that

$$|Y(0)| \leq CK_4'\beta(n+1)^{-\theta+1}, \quad |Z(0)| \leq CK_4'\beta(n+1)^{-\theta+1} \tag{4.33}$$

and

$$|X(k)| \leq CK_4'\beta a(k)(n+1)^{-\delta}, \quad |Y(k) - Y(0)| \leq CK_4'\beta a(k)(n+1)^{-\delta},$$
$$|Z(k) - Z(0)| \leq CK_4'\beta a(k)(n+1)^{-\delta}.$$

The remainder of the proof is devoted to establishing (4.33) and (4.34).

Bound on X. We write X as $X = X_1 + X_2$, with

$$\begin{aligned} X_1 &= \sum_{m=2}^{n+1} \Big[g_m(k) - g_m(0) - a(k)\sigma^{-2}\nabla^2 g_m(0)\Big], \\ X_2 &= \sum_{m=2}^{n+1} \Big[g_m(k) - g_m(0)\Big]\Big[\frac{f_{n+1-m}(k)}{f_n(k)} - 1\Big]. \end{aligned} \tag{4.34}$$

The term X_1 is bounded using Lemma 3.6(iv) with $\epsilon' \in (\delta, \epsilon)$, and using the fact that $a(k) \leq \gamma(n+1)^{-1}\log(n+1)$, so that $a(k)^{\epsilon'} \leq \left(\frac{\gamma\log(n+1)}{n+1}\right)^{\epsilon'} \leq \frac{C}{(n+1)^\delta}$ by

$$|X_1| \leq K_4'\beta a(k)^{1+\epsilon'} \sum_{m=2}^{n+1} \frac{1}{m^{\theta-1-\epsilon'}} \leq CK_4'\beta a(k)^{1+\epsilon'} \leq \frac{CK_4'\beta a(k)}{(n+1)^\delta}. \tag{4.35}$$

For X_2, we first apply Lemma 3.6(ii,iv), with $\epsilon' = 0$, to obtain

$$|g_m(k) - g_m(0)| \leq 2K_4'\beta a(k) m^{-\theta+1}. \tag{4.36}$$

Applying (4.26) then gives

$$|X_2| \leq CK_4'\beta a(k) \sum_{m=2}^{n+1} \frac{1}{m^{\theta-1}} \left((m-1)a(k)(n+1)^{\gamma q} + \frac{K_3\beta}{(n+2-m)^{\theta-2}}\right). \tag{4.37}$$

By the elementary estimate

$$\sum_{m=2}^{n+1} \frac{1}{m^{\theta-1}} \frac{1}{(n+2-m)^{\theta-2}} \leq \frac{C}{(n+1)^{\theta-2}}, \tag{4.38}$$

which is proved easily by breaking the sum up according to $m \leq \lfloor \frac{n+1}{2} \rfloor$, the contribution from the second term on the right side is bounded above by $CK_3K_4'\beta^2 a(k)(n+1)^{-\theta+2}$. The first term is bounded above by

$$CK_4'\beta a(k)(n+1)^{\gamma q-1} \log(n+1) \times \begin{cases} (n+1)^{0\vee(3-\theta)} & (\theta \neq 3) \\ \log(n+1) & (\theta = 3). \end{cases} \tag{4.39}$$

Since we may choose q to be as close to 1 as desired, and since $\delta + \gamma < 1 \wedge (\theta - 2)$ by (3.3), this is bounded above by $CK_4'\beta a(k)(n+1)^{-\delta}$. With (4.35), this proves the bound on X in (4.34).

Bound on Y. By (3.17),

$$\frac{f_{n+1-m}(k)}{f_n(k)} = \frac{f_{n+1-m}(0)}{f_n(0)} \prod_{j=n+2-m}^{n} [1 - v_j a(k) + s_j(k)]^{-1}. \tag{4.40}$$

Recalling the definition of $R_{m,n}(k)$ in (4.28), we can therefore decompose Y as $Y = Y_1 + Y_2 + Y_3 + Y_4$ with

$$\begin{aligned} Y_1 &= \sum_{m=2}^{n+1} g_m(0) \frac{f_{n+1-m}(0)}{f_n(0)} R_{m,n}(k), \\ Y_2 &= \sum_{m=2}^{n+1} g_m(0) \frac{f_{n+1-m}(0)}{f_n(0)} \sum_{j=n+2-m}^{n} \left[(v_j - v_{n+1})a(k) - s_j(k)\right], \\ Y_3 &= \sum_{m=2}^{n+1} g_m(0) \left[\frac{f_{n+1-m}(0)}{f_n(0)} - 1\right] (m-1) v_{n+1} a(k), \\ Y_4 &= \sum_{m=2}^{n+1} g_m(0) \left[\frac{f_{n+1-m}(0)}{f_n(0)} - 1\right]. \end{aligned} \tag{4.41}$$

Then

$$Y(0) = Y_4 \quad \text{and} \quad Y(k) - Y(0) = Y_1 + Y_2 + Y_3. \tag{4.42}$$

For Y_1, we use Lemma 3.6, (4.24) and (4.29) to obtain

$$|Y_1| \leq CK_4'\beta a(k)^2 (n+1)^{\gamma q} \sum_{m=2}^{n+1} \frac{1}{m^{\theta-2}}. \tag{4.43}$$

As in the analysis of the first term of (4.37), we therefore have

$$|Y_1| \leq \frac{CK_4'\beta a(k)}{(n+1)^{\delta}}. \tag{4.44}$$

For Y_2, we use $\theta-2>\delta>0$ with Lemma 3.6, (4.24), (H2) (now established up to $n+1$), (3.19) and Lemma 4.2 to obtain

$$|Y_2| \le \sum_{m=2}^{n+1} \frac{K_4'\beta}{m^\theta} C \sum_{j=n+2-m}^{n} \left[\frac{K_2\beta a(k)}{j^{\theta-2}} + \frac{K_3\beta a(k)}{j^\delta}\right] \le \frac{CK_4'(K_2+K_3)\beta^2 a(k)}{(n+1)^\delta}. \tag{4.45}$$

The term Y_3 obeys

$$|Y_3| \le \sum_{m=2}^{n+1} \frac{K_4'\beta}{m^{\theta-1}} \frac{CK_3\beta}{(n+2-m)^{\theta-2}} a(k) \le \frac{CK_4'K_3\beta^2 a(k)}{(n+1)^{\theta-2}}, \tag{4.46}$$

where we used Lemma 3.6, (4.23), (3.15), and an elementary convolution bound. This proves the bound on $|Y(k)-Y(0)|$ of (4.34), if β is sufficiently small.

We bound Y_4 in a similar fashion, using Lemma 4.2 and the intermediate bound of (4.23) to obtain

$$|Y_4| \le \sum_{m=2}^{n+1} \frac{K_4'\beta}{m^\theta} \sum_{j=n+2-m}^{n} \frac{CK_3\beta}{j^{\theta-1}} \le \frac{CK_4'K_3\beta^2}{(n+1)^{\theta-1}}. \tag{4.47}$$

Taking β small then gives the bound on $Y(0)$ of (4.33).

Bound on Z. We decompose Z as

$$Z = \frac{e_{n+1}(0)}{f_n(0)} + \frac{1}{f_n(0)}\left[e_{n+1}(k) - e_{n+1}(0)\right] + \frac{e_{n+1}(k)}{f_n(0)}\left[\frac{f_n(0)}{f_n(k)} - 1\right] = Z_1 + Z_2 + Z_3. \tag{4.48}$$

Then

$$Z(0) = Z_1 \quad \text{and} \quad Z(k) - Z(0) = Z_2 + Z_3. \tag{4.49}$$

Using Lemma 3.6(v,vi), and (4.24) with $m=n+1$, we obtain

$$|Z_1| \le CK_4'\beta(n+1)^{-\theta} \quad \text{and} \quad |Z_2| \le CK_4'\beta a(k)(n+1)^{-\theta+1}. \tag{4.50}$$

Also, by Lemma 3.6, (4.24) and (4.27), we have

$$|Z_3| \le CK_4'\beta(n+1)^{-\theta} a(k)(n+1)^{1+\gamma q} \le CK_4'\beta a(k)(n+1)^{-(1+\delta)}, \tag{4.51}$$

for small enough q, where we again use $\gamma+\delta<\theta-2$.

This completes the proof of (4.14), and hence completes the advancement of (H3) to $n+1$.

4.4 Advancement of (H4)

In this section, we fix $a(k) > \gamma(n+1)^{-1}\log(n+1)$. To advance (H4) to $j=n+1$, we first recall the definitions of b_{n+1}, ζ_{n+1} and X_1 from (3.1), (4.1) and (4.34). After some algebra, (2.1) can be rewritten as

$$f_{n+1}(k) = f_n(k)\Big(1 - a(k)b_{n+1} + X_1 + \zeta_{n+1}\Big) + W + e_{n+1}(k), \tag{4.52}$$

with

$$W=\sum_{m=2}^{n+1} g_m(k)\left[f_{n+1-m}(k)-f_n(k)\right]. \tag{4.53}$$

We already have estimates for most of the relevant terms. By Lemma 4.1, we have $|\zeta_{n+1}| \leq CK_1\beta(n+1)^{-\theta+1}$. By (4.35), $|X_1| \leq CK_4'\beta a(k)^{1+\epsilon'}$, for any $\epsilon' \in (\delta, \epsilon)$. By Lemma 3.6(v), $|e_{n+1}(k)| \leq K_4'\beta(n+1)^{-\theta}$. It remains to estimate W. We will show below that W obeys the bound

$$|W| \leq \frac{CK_4'\beta}{a(k)^{a-1}(n+1)^{\theta}}(1+K_3\beta+K_5). \tag{4.54}$$

Before proving (4.54), we will first show that it is sufficient for the advancement of (H4).

In preparation for this, we first note that it suffices to consider only large n. In fact, since $|f_n(k;z)|$ is bounded uniformly in k and in z in a compact set by Assumption S, and since $a(k) \leq 2$, it is clear that both inequalities of (H4) hold for all $n \leq N$, if we choose K_4 and K_5 large enough (depending on N). We therefore assume in the following that $n \geq N$ with N large.

Also, care is required to invoke (H3) or (H4), as applicable, in estimating the factor $f_n(k)$ of (4.52). Given k, (H3) should be used for the value n for which $\gamma(n+1)^{-1}\log(n+1) < a(k) \leq \gamma n^{-1}\log n$ ((H4) should be used for larger n). We will now show that the bound of (H3) actually implies the first bound of (H4) in this case. To see this, we use Lemma 3.3 to see that there are q, q' arbitrarily close to 1 such that

$$|f_n(k)| \leq Ce^{-qa(k)n} \leq \frac{C}{(n+1)^{q\gamma n/(n+1)}} \leq \frac{C}{n^{q'\gamma}} \leq \frac{C}{n^{\theta}}\frac{n^{\lambda}}{n^{q'\gamma+\lambda-\theta}} \leq \frac{C}{n^{\frac{d}{2p}}a(k)^{\lambda}}, \tag{4.55}$$

where we used the fact that $\gamma+\lambda-\theta > 0$ by (3.3). Thus, taking $K_4 \gg 1$, we may use the first bound of (H4) also for the value of n to which (H3) nominally applies. We will do so in what follows, without further comment.

Advancement of the second bound of (H4) assuming (4.54). To advance the second estimate in (H4), we use (4.52), (H4), and the bounds found above, to obtain

$$\begin{aligned}\Big|f_{n+1}(k)-f_n(k)\Big| &\leq \big|f_n(k)\big|\,\big|-a(k)b_{n+1}+X_1+\zeta_{n+1}\big|+|W|+|e_{n+1}(k)| \\ &\leq \frac{K_4}{n^{\theta}a(k)^{\lambda}}\left(a(k)b_{n+1}+CK_4'\beta a(k)^{1+\epsilon'}+\frac{CK_1\beta}{(n+1)^{\theta-1}}\right) \\ &\quad+\frac{CK_4'\beta(1+K_3\beta+K_5)}{(n+1)^{\theta}a(k)^{\lambda-1}}+\frac{K_4'\beta}{(n+1)^{\theta}}.\end{aligned}$$

Since $b_{n+1}=1+\mathcal{O}(\beta)$ by (4.6), and since $(n+1)^{-\theta+1} < [a(k)/\gamma\log(n+1)]^{\theta-1} \leq Ca(k)$, the second estimate in (H4) follows for $n+1$ provided $K_5 \gg K_4$ and β is sufficiently small.

Advancement of the first bound of (H4) assuming (4.54). To advance the first estimate of (H4), we argue as in (4.56) to obtain

$$\begin{aligned}\left|f_{n+1}(k)\right| &\leq \left|f_n(k)\right| \left|1 - a(k)b_{n+1} + X_1 + \zeta_{n+1}\right| + |W| + |e_{n+1}(k)| \\ &\leq \frac{K_4}{n^\theta a(k)^\lambda}\left(|1 - a(k)b_{n+1}| + CK_4'\beta a(k)^{1+\epsilon'} + \frac{CK_1\beta}{(n+1)^{\theta-1}}\right) \\ &\quad + \frac{CK_4'\beta(1+K_3\beta+K_5)}{(n+1)^\theta a(k)^{\lambda-1}} + \frac{K_4'\beta}{(n+1)^\theta}.\end{aligned}$$

We need to argue that the right-hand side is no larger than $K_4(n+1)^{-\theta}a(k)^{-\lambda}$. To achieve this, we will use separate arguments for $a(k) \leq \frac{1}{2}$ and $a(k) > \frac{1}{2}$. These arguments will be valid only when n is large enough.

Suppose that $a(k) \leq \frac{1}{2}$. Since $b_{n+1} = 1 + \mathcal{O}(\beta)$ by (4.6), for β sufficiently small we have

$$1 - b_{n+1}a(k) \geq 0. \tag{4.56}$$

Hence, the absolute value signs on the right side of (4.56) may be removed. Therefore, to obtain the first estimate of (H4) for $n+1$, it now suffices to show that

$$1 - ca(k) + \frac{CK_1\beta}{(n+1)^{\theta-1}} \leq \frac{n^\theta}{(n+1)^\theta}, \tag{4.57}$$

for c within order β of 1. The term $ca(k)$ has been introduced to absorb $b_{n+1}a(k)$, the order β term in (4.56) involving $a(k)^{1+\epsilon'}$, and the last two terms of (4.56). However, $a(k) > \gamma(n+1)^{-1}\log(n+1)$. From this, it can be seen that (4.57) holds for n sufficiently large and β sufficiently small.

Suppose, on the other hand, that $a(k) > \frac{1}{2}$. By (2.9), there is a positive η, which we may assume lies in $(0, \frac{1}{2})$, such that $-1+\eta < 1 - a(k) < \frac{1}{2}$. Therefore $|1-a(k)| \leq 1-\eta$ and

$$|1 - b_{n+1}a(k)| \leq |1-a(k)| + |b_{n+1}-1|\,|a(k)| \leq 1 - \eta + 2|b_{n+1}-1|. \tag{4.58}$$

Hence

$$|1-a(k)b_{n+1}| + CK_4'\beta a(k)^{1+\epsilon'} + \frac{CK_1\beta}{(n+1)^{\theta-1}} \leq 1-\eta+C(K_1+K_4')\beta, \tag{4.59}$$

and the right side of (4.56) is at most

$$\begin{aligned}&\frac{K_4}{n^\theta a(k)^\lambda}\left[1-\eta+C(K_1+K_4')\beta\right] + \frac{CK_4'(1+K_3\beta+K_5)\beta}{(n+1)^\theta a(k)^\lambda} \\ &\quad \leq \frac{K_4}{n^\theta a(k)^\lambda}\left[1-\eta+C(K_5K_4'+K_1)\beta\right].\end{aligned}$$

This is less than $K_4(n+1)^{-\theta}a(k)^{-\lambda}$ if n is large and β is sufficiently small.
This advances the first bound in (H4), assuming (4.54).

Bound on W. We now obtain the bound (4.54) on W. As a first step, we rewrite W as

$$W = \sum_{j=0}^{n-1} g_{n+1-j}(k) \sum_{l=j+1}^{n} [f_{l-1}(k) - f_l(k)]. \tag{4.60}$$

Let

$$m(k) = \begin{cases} 1 & (a(k) > \gamma 3^{-1} \log 3) \\ \max\{l \in \{3, \ldots, n\} : a(k) \leq \gamma l^{-1} \log l\} & (a(k) \leq \gamma 3^{-1} \log 3). \end{cases} \tag{4.61}$$

For $l \leq m(k)$, f_l is in the domain of (H3), while for $l > m(k)$, f_l is in the domain of (H4). By hypothesis, $a(k) > \gamma(n+1)^{-1} \log(n+1)$. We divide the sum over l into two parts, corresponding respectively to $l \leq m(k)$ and $l > m(k)$, yielding $W = W_1 + W_2$. By Lemma 3.6(i),

$$\begin{aligned} |W_1| &\leq \sum_{j=0}^{m(k)} \frac{K_4' \beta}{(n+1-j)^\theta} \sum_{l=j+1}^{m(k)} |f_{l-1}(k) - f_l(k)| \\ |W_2| &\leq \sum_{j=0}^{n-1} \frac{K_4' \beta}{(n+1-j)^\theta} \sum_{l=(m(k)\vee j)+1}^{n} |f_{l-1}(k) - f_l(k)|. \end{aligned} \tag{4.62}$$

The term W_2 is easy, since by (H4) and Lemma 4.2 we have

$$|W_2| \leq \sum_{j=0}^{n-1} \frac{K_4' \beta}{(n+1-j)^\theta} \sum_{l=j+1}^{n} \frac{K_5}{a(k)^{\lambda-1}\, l^\theta} \leq \frac{C K_5 K_4' \beta}{a(k)^{\lambda-1} (n+1)^\theta}. \tag{4.63}$$

For W_1, we have the estimate

$$|W_1| \leq \sum_{j=0}^{m(k)} \frac{K_4' \beta}{(n+1-j)^\theta} \sum_{l=j+1}^{m(k)} |f_{l-1}(k) - f_l(k)|. \tag{4.64}$$

For $1 \leq l \leq m(k)$, it follows from Lemma 3.3 and (4.30) that

$$|f_{l-1}(k) - f_l(k)| \leq C e^{-qa(k)l} \left(a(k) + \frac{K_3 \beta}{l^{\theta-1}} \right), \tag{4.65}$$

with $q = 1 - \mathcal{O}(\beta)$. We fix a small $r > 0$, and bound the summation over j in (4.64) by summing separately over j in the ranges $0 \leq j \leq (1-r)n$ and $(1-r)n \leq j \leq m(k)$ (the latter range may be empty). We denote the contributions from these two sums by $W_{1,1}$ and $W_{1,2}$ respectively.

To estimate $W_{1,1}$, we will make use of the bound

$$\sum_{l=j+1}^{\infty} e^{-qa(k)l} l^{-b} \leq C e^{-qa(k)j} \qquad (b > 1). \tag{4.66}$$

With (4.64) and (4.65), this gives

$$
\begin{aligned}
|W_{1,1}| &\le \frac{CK_4'\beta}{(n+1)^\theta} \sum_{j=0}^{(1-r)n} e^{-qa(k)j}\,(1+K_3\beta) \\
&\le \frac{CK_4'\beta}{(n+1)^\theta}\frac{1+K_3\beta}{a(k)} \le \frac{CK_4'\beta}{(n+1)^\theta}\frac{1+K_3\beta}{a(k)^{\lambda-1}}.
\end{aligned}
$$

For $W_{1,2}$, we have

$$
|W_{1,2}| \le \sum_{j=(1-r)n}^{m(k)} \frac{CK_4'\beta}{(n+1-j)^\theta} \sum_{l=j+1}^{m(k)} e^{-qa(k)l}\left(a(k)+\frac{K_3\beta}{l^{\theta-1}}\right). \tag{4.67}
$$

Since l and $m(k)$ are comparable ($(1-r)(n+1) < (1-r)n+1 \le l \le m(k) < n+1$) and large, it follows as in (4.55) that

$$
e^{-qa(k)l}\left(a(k)+\frac{K_3\beta}{l^{\theta-1}}\right) \le \frac{C}{a(k)^\lambda l^\theta}\left(a(k)+\frac{K_3\beta}{l^{\theta-1}}\right) \le \frac{C(1+K_3\beta)}{a(k)^{\lambda-1} l^\theta}, \tag{4.68}
$$

where we have used the definition of $m(k)$ in the form $\frac{\gamma\log(m(k)+1)}{m(k)+1} < a(k) \le \frac{\gamma\log(m(k))}{m(k)}$ as well as the facts that $\lambda > \theta-\gamma$ and that $q(1-r)$ can be chosen as close to 1 as we like to obtain the intermediate inequality, and the same bound on $a(k)$ together with the fact that $\theta > 2$ to obtain the last inequality. Hence, by Lemma 4.2,

$$
|W_{1,2}| \le \frac{C(1+K_3\beta)K_4'\beta}{a(k)^{\lambda-1}} \sum_{j=(1-r)n}^{m(k)} \frac{1}{(n+1-j)^\theta} \sum_{l=j+1}^{m(k)} \frac{1}{l^\theta} \le \frac{C(1+K_3\beta)K_4'\beta}{a(k)^{\lambda-1}(n+1)^\theta}. \tag{4.69}
$$

Summarising, by (4.67), (4.69), and (4.63), we have

$$
|W| \le |W_{1,1}|+|W_{1,2}|+|W_2| \le \frac{CK_4'\beta}{a(k)^{\lambda-1}(n+1)^\theta}(1+K_3\beta+K_5), \tag{4.70}
$$

which proves (4.54).

5 Proof of the main results

As a consequence of the completed induction, it follows from Lemma 3.2 that $I_1 \supset I_2 \supset I_3 \supset \cdots$, so $\cap_{n=1}^\infty I_n$ consists of a single point $z = z_c$. Since $z_0 = 1$, it follows from (H1) that $z_c = 1+\mathcal{O}(\beta)$. We fix $z = z_c$ throughout this section. The constant A is defined by $A = \prod_{i=1}^\infty [1+r_i(0)] = 1+\mathcal{O}(\beta)$. By (H2), the sequence $v_n(z_c)$ is a Cauchy sequence. The constant v is defined to be the limit of this Cauchy sequence. By (H2), $v = 1+\mathcal{O}(\beta)$ and

$$
|v_n(z_c) - v| \le \mathcal{O}(\beta n^{-\theta+2}). \tag{5.1}
$$

5.1 Proof of Theorem 2.1

Proof of Theorem 2.1(a). By (H3),

$$|f_n(0;z_c) - A| = \prod_{i=1}^{n}[1+r_i(0)]\big|1 - \prod_{i=n+1}^{\infty}[1+r_i(0)]\big| \leq \mathcal{O}(\beta n^{-\theta+2}). \tag{5.2}$$

Suppose k is such that $a(k/\sqrt{\sigma^2 v n}) \leq \gamma n^{-1}\log n$, so that (H3) applies. Here, we use the γ of (3.3). By (2.5), $a(k) = \sigma^2 k^2/2d + \mathcal{O}(k^{2+2\epsilon})$ with $\epsilon > \delta$, where we now allow constants in error terms to depend on L. Using this, together with (3.17–3.19), 5.1, and $\delta < 1 \wedge (\theta - 2) \wedge \epsilon$, we obtain

$$\begin{aligned}\frac{f_n(k/\sqrt{v\sigma^2 n};z_c)}{f_n(0;z_c)} &= \prod_{i=1}^{n}\left[1 - v_i a\big(\frac{k}{\sqrt{v\sigma^2 n}}\big) + \mathcal{O}(\beta a\big(\frac{k}{\sqrt{v\sigma^2 n}}\big) i^{-\delta})\right] \\ &= e^{-k^2/2d}[1 + \mathcal{O}(k^{2+2\epsilon}n^{-\epsilon}) + \mathcal{O}(k^2 n^{-\delta})].\end{aligned} \tag{5.3}$$

With (5.2), this gives the desired result.

Proof of Theorem 2.1(b). Since $\delta < 1 \wedge (\theta - 2)$, it follows from (3.35–3.36) and (5.1–5.2) that

$$\frac{\nabla^2 f_n(0;z_c)}{f_n(0;z_c)} = -v\sigma^2 n[1 + \mathcal{O}(\beta n^{-\delta})]. \tag{5.4}$$

Proof of Theorem 2.1(c). The claim is immediate from Lemma 3.4, which is now known to hold for all n.

Proof of Theorem 2.1(d). Throughout this proof, we fix $z = z_c$ and drop z_c from the notation. The first identity of (2.19) follows after we let $n \to \infty$ in (4.1), using Lemma 4.1.

To determine A, we use a summation argument. Let $\chi_n = \sum_{k=0}^{n} f_k(0)$. By (2.1),

$$\begin{aligned}\chi_n &= 1 + \sum_{j=1}^{n} f_j(0) = 1 + \sum_{j=1}^{n}\sum_{m=1}^{j} g_m(0) f_{j-m}(0) + \sum_{j=1}^{n} e_j(0) \\ &= 1 + z\chi_{n-1} + \sum_{m=2}^{n} g_m(0)\chi_{n-m} + \sum_{m=1}^{n} e_m(0).\end{aligned}$$

Using (4.1) to rewrite z, this gives

$$f_n(0) = \chi_n - \chi_{n-1} = 1 + \zeta_n \chi_{n-1} - \sum_{m=2}^{n} g_m(0)(\chi_{n-1} - \chi_{n-m}) + \sum_{m=1}^{n} e_m(0). \tag{5.5}$$

By Theorem 2.1(a), $\chi_n \sim nA$ as $n \to \infty$. Therefore, using Lemma 4.1 to bound the ζ_n term, taking the limit $n \to \infty$ in the above equation gives

$$A = 1 - A\sum_{m=2}^{\infty}(m-1)g_m(0) + \sum_{m=1}^{\infty} e_m(0). \tag{5.6}$$

With the first identity of (2.19), this gives the second.

Finally, we use (5.1), (3.1) and Lemma 3.6 to obtain

$$v = \lim_{n\to\infty} v_n = \frac{-\sigma^{-2}\sum_{m=2}^{\infty}\nabla^2 g_m(0)}{1+\sum_{m=2}^{\infty}(m-1)g_m(0)}. \tag{5.7}$$

The result then follows, once we rewrite the denominator using the first identity of (2.19).

5.2 Proof of Theorem 2.2

By Theorem 2.1(a), $\chi(z_c) = \infty$. Therefore $z_c \geq z_c'$. We need to rule out the possibility that $z_c > z_c'$. Theorem 2.1 also gives (2.11) at $z = z_c$. By assumption, the series

$$G(z) = \sum_{m=2}^{\infty} g_m(0;z), \quad E(z) = \sum_{m=2}^{\infty} e_m(0;z) \tag{5.8}$$

therefore both converge absolutely and are $\mathcal{O}(\beta)$ uniformly in $z \leq z_c$. For $z < z_c'$, since the series defining $\chi(z)$ converges absolutely, the basic recursion relation (2.1) gives

$$\chi(z) = 1 + z\chi(z) + G(z)\chi(z) + E(z), \tag{5.9}$$

and hence

$$\chi(z) = \frac{1+E(z)}{1-z-G(z)}, \quad (z < z_c'). \tag{5.10}$$

It is implicit in the bound on $\partial_z g_m(k;z)$ of Assumption G that $g_m(k;\cdot)$ is continuous on $[0, z_c]$. By dominated convergence, G is also continuous on $[0, z_c]$. Since $E(z) = \mathcal{O}(\beta)$ and $\lim_{z\uparrow z_c'} \chi(z) = \infty$, it then follows from (5.10) that

$$1 - z_c' - G(z_c') = 0. \tag{5.11}$$

By the first identity of (2.19), (5.11) holds also when z_c' is replaced by z_c. If $z_c' \neq z_c$, then it follows from the mean-value theorem that

$$z_c - z_c' = G(z_c') - G(z_c) = -(z_c - z_c')\sum_{m=2}^{\infty} \partial_z g_m(0;t) \tag{5.12}$$

for some $t \in (z_c', z_c)$. However, by a bound of Assumption G, the sum on the right side is $\mathcal{O}(\beta)$ uniformly in $t \leq z_c$. This is a contradiction, so we conclude that $z_c = z_c'$. $\square$

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References

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