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Renormalisation group analysis of weakly self-avoiding walk in dimensions four and higher

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Abstract. We outline a proof, by a rigorous renormalisation group method, that the critical two-point function for continuous-time weakly self-avoiding walk on $\mathbb{Z}^d$ decays as $|x|^{-(d-2)}$ in the critical dimension $d = 4$, and also for all $d > 4$.

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1. Introduction

We prove $|x|^{-(d-2)}$ decay for the critical two-point function of the continuous-time weakly self-avoiding walk in dimensions $d \geq 4$. This is a summary of the ideas and the steps in the proof. The details are provided in [12]. The proof is based on a rigorous renormalisation group argument. For the case $d > 4$, this provides an approach completely different from the lace expansion methods of [18, 19]. But our main contribution is that our method applies also in the case of the *critical* dimension $d = 4$, where lace expansion methods do not apply.

Renormalisation group methods have been applied previously to study weakly self-avoiding walk on a 4-dimensional *hierarchical* lattice. The continuous-time version of the model has been studied in the series of papers [4, 16, 8, 9]; see [5] for a review. More recently, a completely different renormalisation group approach to the discrete-time weakly self-avoiding walk on a 4-dimensional hierarchical lattice has been developed in [20].

The $|x|^{-(d-2)}$ decay for the two-point function for a continuum 4-dimensional Edwards model, with a smoothed delta function, has been proved in [24]; unlike our model, this is not a model of walks taking nearest neighbour steps in the lattice, but it is expected to be in the same universality class as our model. The relation between our model and the Edwards model is discussed in [26]. A big step towards an understanding of the behaviour in dimension $d = 4 - \epsilon$ is taken in [27] (their

model is formulated on a lattice in dimension 3 but it mimics the behaviour of the nearest-neighbour model in dimension $4-\epsilon$).

Our renormalisation group method is a greatly extended and generalised form of work in [4, 8, 9] for the hierarchical lattice and [13, 14, 3, 11] for continuum quantum field theory. Details will appear in [12]. Our method is based on an exact functional integral representation of the two-point function of the continuous-time self-avoiding walk as the two-point function of a quantum field theory containing both bosonic and fermionic fields. Such representations have been recently summarised in [10].

1.1. Background. A self-avoiding walk on the simple cubic lattice $\mathbb{Z}^d$ is an *injective* map

$$\omega : \{0, 1, \ldots, n\} \to \mathbb{Z}^d \tag{1}$$

such that for all i, $\omega(i)$ and $\omega(i+1)$ are nearest neighbours in $\mathbb{Z}^d$. We call n the number of steps. The main result of this article will actually be a statement about about random maps $X : [0, T] \to \mathbb{Z}^d$, but to explain the background we start with self-avoiding walk.

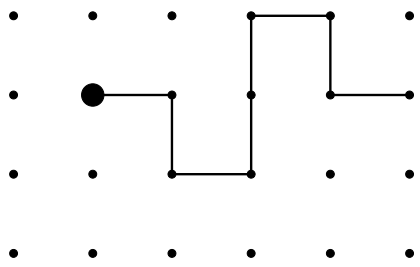

Figure 1. An 8 step self-avoiding walk on $\mathbb{Z}^d$, $d = 2$.

Let $\mathcal{S}_n$ be the set of all self-avoiding walks with n steps and with $\omega(0) = 0$. Let c_n be the number of elements in $\mathcal{S}_n$. By declaring that all ω in $\mathcal{S}_n$ have equal probability $1/c_n$ we make $\mathcal{S}_n$ into a probability space with expectation $\mathbb{E}_n$. The subscript n reminds us that the probability space depends on n. In the sequel "model" means a choice of probability space and law.

This model arose in statistical mechanics. It is, for example, a natural model when one is interested in the conformation of linear polymer molecules. There is another natural model called the *true* or *myopic* self-avoiding walk. Unlike our model, true self-avoiding walk is a stochastic process which at each step looks at its neighbours and chooses uniformly from those visited least often in the past. Recent progress on this model is reported in [23].

The key problem is to determine the growth in n of the mean-square displacement,

$$\mathbb{E}_n |\omega(n)|^2 = c_n^{-1} \sum_{\omega \in \mathcal{S}_n} |\omega(n)|^2, \tag{2}$$

where $|\omega(n)|$ is the Euclidean norm of $\omega(n)$ as an element of $\mathbb{Z}^d$. More precisely, we want to prove the existence of ν such that

$$\lim_{n\to\infty} n^{-2\nu}\mathbb{E}_n|\omega(n)|^2 \in (0,\infty), \tag{3}$$

and we want to calculate ν. We will call this the ν problem.

As explained in [26, page 16], there is an easier version of this problem that we will call the *Abelian* ν problem, because proving the existence of ν after solving the Abelian problem is a Tauberian problem. Let $\mathcal{S} = \bigcup_n \mathcal{S}_n$ and let $n(\omega) = n$ for $\omega \in \mathcal{S}_n$. For $z > 0$ we define the *two-point function*

$$G_z(x) = \sum_{\omega\in\mathcal{S}} z^{n(\omega)} \mathbb{1}_{\omega(n(\omega))=x}. \tag{4}$$

Let

$$\chi^{(p)} = \sum_{\omega\in\mathcal{S}} z^{n(\omega)}|\omega(n(\omega))|^p = \sum_{x\in\mathbb{Z}^d} G_z(x)|x|^p. \tag{5}$$

The Abelian version of the ν problem is to determine the growth of $\sqrt{\chi^{(2)}}/\chi^{(0)}$ as $z \uparrow z_c$, where z_c is the common radius of convergence of the power series in this ratio. If ν exists then it equals the Abelian ν. In dimensions $d \geq 5$, according to the following theorem, $\nu = 1/2$.

Theorem 1.1. *[21, 22] For $d \geq 5$, there are positive constants A, D, c, ϵ such that*

$$c_n = A\mu^n[1 + O(n^{-\epsilon})], \tag{6}$$

$$\mathbb{E}_n|\omega(n)|^2 = Dn[1 + O(n^{-\epsilon})], \tag{7}$$

and the rescaled self-avoiding walk converges weakly to Brownian motion:

$$\frac{\omega(\lfloor nt\rfloor)}{\sqrt{Dn}} \Rightarrow B_t. \tag{8}$$

Also [18], as $|x| \to \infty$,

$$G_{z_c}(x) = c|x|^{-(d-2)}[1 + O(|x|^{-\epsilon})]. \tag{9}$$

The limit in (8) is called a scaling limit. The identification of scaling limits for dimensions $d = 2, 3, 4$ is the grand goal, but the ν problem is a key intermediate objective because $n^{-\nu}\omega(\lfloor nt\rfloor)$ is the candidate sequence for the scaling limit.

If we set up the probability space without imposing the injective condition in the definition of ω, then the mean-square displacement is exactly n, because then the law for ω is that of simple random walk. According to Donsker's Theorem, the scaling limit of simple random walk, with $D = 1$, is also Brownian motion. Thus, in dimensions $d \geq 5$ self-avoiding walk and simple random walk have the same scaling limit. When different models have the same scaling limit, we say the models are in the same *universality class*. One of the goals of mathematical statistical mechanics is to classify universality classes.

Theorem 1.1 will not hold with $\nu = 1/2$ for dimensions four and less. There is a conjecture going back to [2] that, for $d = 4$,

$$c_n \sim A\mu^n (\log n)^{1/4}, \quad \mathbb{E}_n|\omega(n)|^2 \sim Dn(\log n)^{1/4}. \tag{10}$$

This and the next paragraph motivates our interest in four dimensions.

In dimension $d = 3$, nothing is known rigorously about the ν problem. The existence of ν is not proved. It is not known that self-avoiding walk moves away from the origin faster than simple random walk, $\mathbb{E}_n|\omega(n)|^2 \geq n$, nor is it known that self-avoiding walk is slower than ballistic, $\mathbb{E}_n|\omega(n)/n|^2 \to 0$. In dimension $d = 2$, there is the same basic lack of control as in $d = 3$, but the good news is that there is a candidate for the scaling limit, which tells us that if ν exists it should be equal to $3/4$. In [25], the process known as $\mathrm{SLE}_{8/3}$ is identified as the scaling limit of self-avoiding walk subject to the unproven hypothesis that the scaling limit exists and is conformally invariant.

SLE is a breakthrough discovery because it is provides a comprehensive list of possible scaling limits in $d = 2$. It has separated off the issues of existence of limits and universality and made it possible to study candidate limits without first having proved they are limits. On the other hand, theoretical physicists have a profound calculus called the *Renormalisation Group* (RG) that naturally explains when different models are in the same universality class and that can also prove the existence of limits. We will follow this path. RG, in the form that we will develop, was largely invented by Ken Wilson [28, 30, 29]. RG as a rigorous tool originated with [1, 15]. Later developments are reviewed in [6]. The hierarchical lattices mentioned earlier have special properties that greatly simplify RG. The $n(\log n)^{1/4}$ growth of (10) has been shown to hold for continuous-time weakly self-avoiding walk on a four dimensional hierarchical lattice in [4, 8, 9]. Very recently, the corresponding Abelian ν problem has been solved in [20] for a *discrete-time* model on the hierarchical lattice.

1.2. Continuous-time weakly self-avoiding walk and the main result. We now describe a probability law on a space of maps $X : [0, T] \to \mathbb{Z}^d$. We use the word "time" for the parameter $t \in [0, T]$, but as for the discrete-time case there is a different space and law for each T. It is not a stochastic process which reveals more about itself as "time" advances, so it is better to think of the interval $[0, T]$ as a continuous parametrisation of a path in $\mathbb{Z}^d$.

Fix a dimension $d \geq 4$. Let X be the continuous-time simple random walk on $\mathbb{Z}^d$ with Exp(1) holding times and right-continuous sample paths. In other words, the walk takes its nearest neighbour steps at the events of a rate-1 Poisson process. Let P_a and E_a be the probability law and the expectation for this process started in $X(0) = a$. The local time at x up to time T is given by

$$L_{x,T} = \int_0^T \mathbb{1}_{X(s)=x}\, ds, \tag{11}$$

and we can measure the amount of self-intersection experienced by X up to time

T by

$$I(0,T) = \int_0^T ds_1 \int_0^T ds_2 \mathbb{1}_{X(s_1)=X(s_2)} = \int_0^T ds_1 \int_0^T ds_2 \sum_{x\in\mathbb{Z}^d} \mathbb{1}_{X(s_1)=x}\mathbb{1}_{X(s_2)=x} = \sum_{x\in\mathbb{Z}^d} L_{x,T}^2. \tag{12}$$

Then, for $g > 0$, $e^{-gI(0,T)}$ is our substitute for the indicator function supported on self-avoiding X. For $g > 0$, we define a new probability law

$$P_{g,a}(A) = E_a(e^{-gI(0,T)}\mathbb{1}_A)/E_a(e^{-gI(0,T)}) \tag{13}$$

on measurable subsets A of the set of all maps $X : [0,T] \to \mathbb{Z}^d$ with $X(0) = a$. For this model there is a ν problem[1], but only the Abelian ν problem for $\mathbb{Z}^d$ is currently within the reach of the methods of this paper.

The continuous-time weakly self-avoiding walk two-point function is defined by

$$G_{g,\nu}(a,b) = \int_0^\infty E_a(e^{-gI(0,T)}\mathbb{1}_{X(T)=b})e^{-\nu T}dT, \tag{14}$$

where ν is a parameter (possibly negative) which is chosen in such a way that the integral converges. For $p \geq 0$ define

$$\chi_g^{(p)}(\nu) = \sum_{b\in\mathbb{Z}^d} G_{g,\nu}(a,b)|b-a|^p. \tag{15}$$

By subadditivity, cf.[26], there exists $\nu_c = \nu_c(g)$ such that $\chi_g^{(0)}(\nu) < \infty$ if and only if $\nu > \nu_c$. We call this ν_c the *critical value of* ν. Our main result is the following theorem.

Theorem 1.2. *Let $d \geq 4$. There exists $g_{\max} > 0$ such that for each $g \in [0, g_{\max}]$ there exists $c_g > 0$ such that as $|a-b| \to \infty$,*

$$G_{g,\nu_c(g)}(a,b) = \frac{c_g}{|a-b|^{d-2}}\left(1 + o(1)\right). \tag{16}$$

This is the analogue of (9) in Theorem 1.1, but now including dimension $d = 4$. There are no log corrections. Log corrections are only expected in the singular behaviour of $\chi_g^{(p)}(\nu)$ as $\nu \downarrow \nu_c$ for $p \geq 0$. The case $g = 0$ is a standard fact about simple random walk; our proof is given for case $g > 0$.

2. Finite volume approximation

In this section we describe the first step in our proof, which is to approximate the *infinite volume* $\mathbb{Z}^d$ by *finite volume*, namely a discrete torus.

[1] solved on the hierarchical lattice for g small in [4, 8, 9]

We do not make explicit the dependence on g, which is fixed and positive. Let $R \geq 3$ be an integer, and let $\Lambda = \mathbb{Z}^d/R\mathbb{Z}^d$ denote the discrete torus of side R. For $a, b \in \Lambda$, let

$$G_{\Lambda,\nu}(a,b) = \int_0^\infty E_{a,\Lambda}\left(e^{-gI(0,T)}\mathbb{1}_{X(T)=b}\right)e^{-\nu T}dT, \tag{17}$$

where $E_{a,\Lambda}$ denotes the continuous-time simple random walk on Λ, started from a. The following theorem shows that it is possible to study the critical two-point function in the double limit in which first $\Lambda \uparrow \mathbb{Z}^d$ and then $\nu \downarrow \nu_c$. We will follow this route, focusing our analysis on the subcritical finite volume model with sufficient uniformity to take the limits.

Theorem 2.1. *Let $d \geq 1$ and $\nu \geq \nu_c$. Then*

$$G_\nu(a,b) = \lim_{\nu' \downarrow \nu} \lim_{\Lambda \uparrow \mathbb{Z}^d} G_{\Lambda,\nu'}(a,b). \tag{18}$$

3. Integral representation

The next step in the proof is to represent the two-point function in finite volume by an integral that we will approximate by a Gaussian integral.

Recall that Λ denotes a discrete torus in $\mathbb{Z}^d$. Given $\varphi \in \mathbb{C}^\Lambda$ and writing $\varphi = (\varphi_x)$, $x \in \Lambda$, we write $d\varphi_x$ and $d\bar{\varphi}_x$ for the differentials, we fix a choice of the square root $\sqrt{2\pi i}$, and we set

$$\psi_x = \frac{1}{\sqrt{2\pi i}}d\varphi_x, \quad \bar{\psi}_x = \frac{1}{\sqrt{2\pi i}}d\bar{\varphi}_x. \tag{19}$$

Define the differential forms

$$\tau_x = \varphi_x\bar{\varphi}_x + \psi_x \wedge \bar{\psi}_x \qquad (x \in \Lambda), \tag{20}$$

and

$$\tau_{\Delta,x} = \frac{1}{2}\Big(\varphi_x(-\Delta\bar{\varphi})_x + (-\Delta\varphi)_x\bar{\varphi}_x + \psi_x \wedge (-\Delta\bar{\psi})_x + (-\Delta\psi)_x \wedge \bar{\psi}_x\Big), \tag{21}$$

where Δ is the lattice Laplacian on Λ defined by $\Delta\varphi_x = \sum_{y:|y-x|=1}(\varphi_y - \varphi_x)$, and $\wedge$ is the standard wedge product. From now on, for differential forms u, v, we will abbreviate by writing $uv = u \wedge v$. In particular $\psi_x\psi_y = -\psi_y\psi_x$ and likewise $\bar{\psi}_x$ anticommutes with $\bar{\psi}_y$ and with ψ_y. The proof of the following proposition is given in [4, 9]; see also [10] for a self-contained proof.

Proposition 3.1. *Given $g > 0$, let ν be such that $G_{\Lambda,\nu}(a,b)$ is finite. Then*

$$G_{\Lambda,\nu}(a,b) = \int_{\mathbb{C}^\Lambda} e^{-\sum_{x\in\Lambda}(\tau_{\Delta,x} + g\tau_x^2 + \nu\tau_x)}\bar{\varphi}_a\varphi_b. \tag{22}$$

The definition of an integral such as the right-hand side of (22) is as follows:

1. Expand the entire integrand in a power series about its degree-zero part (this is a *finite* sum due to the anti-commutativity of the wedge product, and the order of factors in the resulting products is immaterial due to the even degree), e.g.,

$$e^{-\nu\tau_x} = e^{-\nu\varphi_x\bar{\varphi}_x - \frac{\nu}{2\pi i}d\varphi_x d\bar{\varphi}_x} = e^{-\nu\varphi_x\bar{\varphi}_x}\left(1 - \frac{\nu}{2\pi i}d\varphi_x d\bar{\varphi}_x\right). \tag{23}$$

   In general, any function of the differentials is defined by its formal power series about its degree-zero part.

2. Keep only terms with one factor $d\varphi_x$ and one $d\bar{\varphi}_x$ for each $x \in \Lambda$, write $\varphi_x = u_x + iv_x$, $\bar{\varphi}_x = u_x - iv_x$ and similarly for the differentials.

3. Rearrange the differentials to $\prod_{x\in\Lambda} du_x dv_x$, using the anti-commutativity of the wedge product.

4. Finally, perform the Lebesgue integral over $\mathbb{R}^{2|\Lambda|}$.

This is explained in more detail in [10]. These integrals have the remarkable self-normalisation property that

$$\int e^{-\sum_{x\in\Lambda}(a_x\tau_{\Delta,x} + b_x\tau_x^2 + c_x\tau_x)} = 1, \qquad a_x \geq 0,\, b_x > 0,\, c_x \in \mathbb{R},\, x \in \Lambda. \tag{24}$$

Self-contained proofs of this, and of generalisations, can be found in [10]. The variables φ_x and the forms ψ_x are called *fields*.

4. Quadratic or Gaussian approximation

The integral representation of Proposition 3.1 opens a natural route for approximation by non-interacting walk with different parameters. To do this we split the exponent $\tau_{\Delta,x} + g\tau_x^2 + \nu\tau_x$ in (22) into a part which is quadratic in the variables φ and a remainder. When the remainder is ignored the rest of the integral becomes Gaussian and the Gaussian integral represents a non-interacting walk. It is important not to assume that the best approximation is the quadratic terms $\tau_{\Delta,x} + \nu\tau_x$. We even want to allow τ_Δ to be divided up. To see what a different coefficient in front of τ_Δ means we make the change of variable $\varphi_x \mapsto \sqrt{1+z_0}\varphi_x$, with $z_0 > -1$. This gives

$$G_{\Lambda,\nu}(a,b) = (1+z_0)\int_{\mathbb{C}^\Lambda} e^{-\sum_{x\in\Lambda}\left((1+z_0)\tau_{\Delta,x} + g(1+z_0)^2\tau_x^2 + \nu(1+z_0)\tau_x\right)}\bar{\varphi}_a\varphi_b, \tag{25}$$

where the Jacobian is contained in the transformation of $\psi, \bar{\psi}$. Then, for any $m^2 \geq 0$, simple algebra allows us to rewrite this as

$$G_{\Lambda,\nu}(a,b) = (1+z_0)\int e^{-S(\Lambda) - \tilde{V}_0(\Lambda)}\bar{\varphi}_a\varphi_b, \tag{26}$$

where

$$S(\Lambda) = \sum_{x\in\Lambda} \big(\tau_{\Delta,x} + m^2\tau_x\big), \tag{27}$$

$$\tilde{V}_0(\Lambda) = \sum_{x\in\Lambda} \big(g_0\tau_x^2 + \nu_0\tau_x + z_0\tau_{\Delta,x}\big), \tag{28}$$

$$g_0 = (1+z_0)^2 g, \qquad \nu_0 = (1+z_0)\nu_c, \qquad m^2 = (1+z_0)(\nu-\nu_c), \tag{29}$$

and ν_c was defined below (15). The two-point function $G_{\Lambda,\nu}(a,b)$ in (26) does not depend on (z_0, m^2) so, in the next theorem, these are free parameters that do not get fixed until Section 12. In view of Theorem 2.1 and Proposition 3.1, to prove Theorem 1.2 it suffices to prove the following theorem.

Theorem 4.1. *Let $d \geq 4$. There exists $g_{\max} > 0$ such that for each $g \in [0, g_{\max}]$ there exist $c(g) > 0$ such that as $|a-b| \to \infty$,*

$$\lim_{\nu\downarrow\nu_c} \lim_{\Lambda\uparrow\mathbb{Z}^d} (1+z_0) \int_{\mathbb{C}^\Lambda} e^{-S(\Lambda)-\tilde{V}_0(\Lambda)} \bar{\varphi}_a\varphi_b = \frac{c(g)}{|a-b|^{d-2}} \left(1+o(1)\right). \tag{30}$$

To prove Theorem 4.1, we study the integral on the left-hand side via a renormalisation group analysis, without making further direct reference to its connection with self-avoiding walks. In order to calculate this integral we define, for $\sigma \in \mathbb{C}$,

$$V_0(\Lambda) = \tilde{V}_0(\Lambda) + \sigma\bar{\varphi}_a + \bar{\sigma}\varphi_b \tag{31}$$

and use

$$\int_{\mathbb{C}^\Lambda} e^{-S(\Lambda)-\tilde{V}_0(\Lambda)} \bar{\varphi}_a\varphi_b = -\left.\frac{\partial}{\partial\sigma}\frac{\partial}{\partial\bar{\sigma}}\right|_0 \int_{\mathbb{C}^\Lambda} e^{-S(\Lambda)-V_0(\Lambda)}. \tag{32}$$

We will call σ an *external field*.

5. Forms and test functions

In this section we introduce notation for handling the differential forms that appear in Theorem 4.1. We will write *form* in place of "differential forms" from now on. We focus on dimension $d = 4$, but leave d in various formulas since 4 can also appear for other reasons.

5.1. The space $\mathcal{N}$. A form is a polynomial in $\psi, \bar{\psi}$ with coefficients that are functions of $(\varphi, \sigma) \in \mathbb{C}^\Lambda \times \mathbb{C}$.

Given $\sigma \in \mathbb{C}$ we define $\sigma_1 = \sigma$ and $\sigma_2 = \bar{\sigma}$ so that σ can be identified with a function $\sigma : \{1,2\} \to \mathbb{C}$. Similarly, let $\Lambda_2 = \Lambda \times \{1,2\}$ so that given $\varphi \in \mathbb{C}^\Lambda$ we have the function on $x = (s,i) \in \Lambda_2$ defined by

$$\phi_x = \begin{cases} \varphi_s & i = 1, \\ \bar{\varphi}_s & i = 2. \end{cases} \tag{33}$$

Since ϕ and φ are in one to one correspondence and since we are only interested in functions on Λ_2 that arise from some φ we write $\phi \in \mathbb{C}^\Lambda$.

Forms are elements of the algebra $\mathcal{N}$ whose generators are the degree one forms $(\psi_x, \bar{\psi}_x,\, x \in \Lambda)$ subject to the relations that all generators mutually anticommute. For $x = (s, i) \in \Lambda_2$, we write

$$\psi_x = \begin{cases} \psi_s & i = 1, \\ \bar{\psi}_s & i = 2. \end{cases} \tag{34}$$

Then we introduce the space $\Lambda^* = \cup_{q=0}^\infty \Lambda_2^q$ of all sequences in Λ_2 with finitely many terms so that every monomial in ψ can be written in the form, for some $y \in \Lambda^*$,

$$\psi^y = \begin{cases} 1 & \text{if } q = 0 \\ \psi_{y_1} \cdots \psi_{y_q} & \text{if } q \geq 1. \end{cases} \tag{35}$$

The $q = 0$ term in Λ^* is a set consisting of a single element called the "empty sequence", which by definition has length zero. Given a sequence $y \in \Lambda^*$, $q = q(y)$ is the length of the sequence and $y! = q(y)!$. Every element of $\mathcal{N}$ has the form

$$F = F(\phi, \sigma) = \sum_{y \in \Lambda^*} \frac{1}{y!} F_y(\phi, \sigma) \psi^y. \tag{36}$$

Given $x = (x_1, \ldots, x_p) \in \Lambda_2^p$ and $z = (z_1, \ldots, z_r) \in \{1, 2\}^r$, we write

$$F_{x,y,z}(\phi, \sigma) = \frac{\partial^p}{\partial \phi_{x_p} \cdots \partial \phi_{x_1}} \frac{\partial^r}{\partial \sigma_{z_r} \cdots \partial \sigma_{z_1}} F_y(\phi, \sigma). \tag{37}$$

For $X \subset \Lambda$, we define $\mathcal{N}(X)$, which is a subspace of $\mathcal{N}$, by

$$\mathcal{N}(X) = \{F \in \mathcal{N} : F_{x,y} = 0 \text{ if any component of } x, y \text{ is not in } X\}. \tag{38}$$

For example $\tau_x \in \mathcal{N}(\{x\})$ and $\tau_{\Delta,x} \in \mathcal{N}(X)$ where $X = \{y : |y - x| \leq 1\}$.

By introducing

$$\phi^y = \begin{cases} 1 & \text{if } q = 0 \\ \phi_{y_1} \cdots \phi_{y_q} & \text{if } q \geq 1, \end{cases} \tag{39}$$

we write the formal Taylor expansion of $F(\phi + \xi)$ in powers of ξ and σ as

$$\sum_{x,y \in \Lambda^*, z \in \{1,2\}^*} \frac{1}{x!y!z!} F_{x,y,z}(\phi, 0) \xi^x \psi^y \sigma^z. \tag{40}$$

Functions $f : \Lambda^* \times \Lambda^* \times \{1, 2\}^* \to \mathbb{C}$ are called *test functions*. We define a pairing between elements of $\mathcal{N}$ and the set of test functions as follows: for a test function f, for $\phi \in \mathbb{C}^\Lambda$, let

$$\langle F, f \rangle_\phi = \sum_{x,y \in \Lambda^*, z \in \{1,2\}^*} \frac{1}{x!y!z!} F_{x,y,z}(\phi, 0) f_{x,y,z}. \tag{41}$$

For example, let $F = \varphi_k$ and $F' = \varphi_0 + (k \cdot \nabla\varphi)_0$. Then

$$\langle F, f\rangle_0 = f_k, \qquad \langle F', f\rangle_0 = f_0 + (k \cdot \nabla f)_0, \tag{42}$$

and more generally when $\phi = 0$ the effect of the pairing is to replace fields by the test function.

5.2. Local polynomials and localisation. For a function $f : \Lambda \to \mathbb{C}$ and e a unit vector in $\mathbb{Z}^d$ we define the *finite difference derivative* $(\nabla_e f)_x = f(x+e) - f(x)$. Repeated differences such as $(\nabla_e \nabla_{e'} f)_x$ are called *derivatives*.

A *local monomial* is a product of finitely many fields and derivatives of fields such as $M = \psi\bar{\psi}\nabla_e\bar{\varphi}$. Using this example to introduce a general notation, given $x \in \Lambda$ let $M_x = \psi_x\bar{\psi}_x(\nabla_e\bar{\varphi})_x$, and given $X \subset \Lambda$ let $M(X) = \sum_{x\in X} M_x$. *Local polynomials* are finite sums of local monomials with constant coefficients.

An important example of a local polynomial is

$$V = g\tau^2 + \nu\tau + z\tau_{\Delta,x} + \lambda\mathbb{1}_a\bar{\sigma}\varphi + \lambda\mathbb{1}_b\sigma\bar{\varphi} + (q/2)(\mathbb{1}_a + \mathbb{1}_b)\bar{\sigma}\sigma, \tag{43}$$

which extends the local polynomial of (31) by the addition of the $\bar{\sigma}\sigma$ term. The indicator function $\mathbb{1}_a : \Lambda \to \{0, 1\}$ equals 1 when evaluated on a and is zero otherwise. The parameters (g, ν, z, λ, q) are called *coupling constants*.

Euclidean symmetry: The lattice $\mathbb{Z}^d$ has automorphisms $E : \mathbb{Z}^d \to \mathbb{Z}^d$. An example for $d = 1$ is $Ex = 1 - x$. By letting an automorphism E act on the spatial labels on fields, $\varphi_x \mapsto \varphi_{Ex}$, E induces an action, $E : \mathcal{N} \to \mathcal{N}$. A local polynomial P is *Euclidean invariant* if automorphisms of $\mathbb{Z}^d$ that fix x also fix P_x. For example, $\psi\bar{\psi}\nabla_e\bar{\varphi}$ is not Euclidean invariant because there is a reflection that changes φ_{x+e} into φ_{x-e} so that $(\nabla_e\bar{\varphi})_x \mapsto (\nabla_{-e}\bar{\varphi})_x$. On the other hand, the term τ_Δ in (43) is a Euclidean invariant local monomial.

Gauge invariance: A local polynomial is gauge invariant if it is invariant under the *gauge flow*: $(\sigma, \varphi) \to (e^{i\theta}\sigma, e^{i\theta}\varphi)$. Thus V of (43) is gauge invariant.

Supersymmetry: There is an antiderivation $\hat{Q} : \mathcal{N} \to \mathcal{N}$ characterised by

$$\hat{Q}\varphi_x = \psi_x, \qquad \hat{Q}\psi_x = -\varphi_x, \qquad \hat{Q}\bar{\varphi}_x = \bar{\psi}_x, \qquad \hat{Q}\bar{\psi}_x = \bar{\varphi}_x. \tag{44}$$

An element of $F \in \mathcal{N}$ is said to be *supersymmetric* if $\hat{Q}F = 0$. The terms $\tau, \tau_\Delta, \tau^2$ in V are supersymmetric local monomials. The forms $\bar{\sigma}\varphi, \sigma\bar{\varphi}, \bar{\sigma}\sigma$ are gauge invariant, but not supersymmetric. It is straightforward to check that $\hat{Q}^2$ generates the gauge flow. Therefore supersymmetry implies gauge invariance. Further details can be found in [10].

The pairing (41) defines $F \in \mathcal{N}$ as a linear function, $f \mapsto \langle F, f\rangle_0$, on test functions. The subscript means that we set $\phi = 0$. Let Π be a set of test functions. Two elements F_1 and F_2 of $\mathcal{N}$ are equivalent when they define the same linear function on Π. We say they are *separated* if they are not equivalent.

Example 1. Let Π be the set of test functions that are linear in their Λ arguments. Fix a point $k \in \mathbb{Z}^d$. Let $F = \varphi_k$, and let $F' = \varphi_0 + (k \cdot \nabla\varphi)_0$. Then F and F' are

equivalent because a linear test function $f(x) = a + b \cdot x$ cannot separate them, since by (42),

$$\langle F, f\rangle = a + b \cdot k = \langle F', f\rangle. \tag{45}$$

To avoid confusion let us emphasise that two different contexts for "polynomial" are in use: a test functions can be a polynomial in $x \in \Lambda$, while local polynomials are polynomial in fields.

The choice for Π in this example is not the one we want. The details in the definition given below are less important than the objective of the definition, which is that Π should be a minimal space of test functions that separates the terms in (43).

We define Π to be the set of test functions $f(x, y, z)$ that are polynomial in the Λ arguments of $(x, y) \in \Lambda^* \times \Lambda^*$ with restrictions on degree listed below. For $f \in \Pi$, as a polynomial in the x, y components in Λ:

1. The restriction of f to (x, y, z) with $r(z) = 0$ has total degree at most $d - p(x)[\phi] - q(y)[\phi]$; $f(x, y, z) = 0$ when $d - p(x)[\phi] - q(y)[\phi] < 0$. Here

$$[\phi] = (d - 2)/2. \tag{46}$$

   For dimension $d = 4$, $[\phi] = 1$.

2. The restriction of f to (x, y, z) with $r(z) = r \in \{1, 2\}$ has total degree at most $r - p(x) - q(y)$; $f(x, y, z) = 0$ if $r - p(x) - q(y) < 0$ or $r > 2$.

Let $\mathcal{V}$ be the vector space of gauge invariant local polynomials that are separated by Π and, for $X \subset \Lambda$, let $\mathcal{V}(X) = \{P(X) : P \in \mathcal{V}\}$. The following proposition associates to any form $F \in \mathcal{N}$ an equivalent local polynomial in $\mathcal{V}(X)$ [12].

Proposition 5.1. *For nonempty $X \subset \mathbb{Z}^d$ there exists a linear map $\overline{\mathrm{Loc}}_X : \mathcal{N} \to \mathcal{V}(X)$ such that*

$$(a) \quad \langle \overline{\mathrm{Loc}}_X F, f\rangle_0 = \langle F, f\rangle_0 \quad \textit{for } f \in \Pi,\ F \in \mathcal{N}, \tag{47}$$

$$(b) \quad E\big(\overline{\mathrm{Loc}}_X F\big) = \overline{\mathrm{Loc}}_{EX}(EF) \quad \textit{for automorphisms } E : \mathbb{Z}^d \to \mathbb{Z}^d,\ F \in \mathcal{N}, \tag{48}$$

$$(c) \quad \overline{\mathrm{Loc}}_{X'} \circ \overline{\mathrm{Loc}}_X = \overline{\mathrm{Loc}}_{X'} \quad \textit{for } X, X' \subset \Lambda. \tag{49}$$

Let $\mathcal{V}_H \subset \mathcal{V}$ be the subspace generated by monomials that are not divisible by σ or $\bar{\sigma}$, and let $\mathcal{V}_O \subset \mathcal{V}$ be the subspace generated by monomials that are divisible by σ or $\bar{\sigma}$. Then $\mathcal{V} = \mathcal{V}_H \oplus \mathcal{V}_O$, and on this direct sum we define

$$\mathrm{Loc}_X = \overline{\mathrm{Loc}}_X \oplus \overline{\mathrm{Loc}}_{X \cap \{a,b\}}, \tag{50}$$

where $\overline{\mathrm{Loc}}_\varnothing$ is interpreted as zero. Symmetry considerations for the integral representation (22) restrict the domain of $\overline{\mathrm{Loc}}$ in our applications so that its range reduces to polynomials of the form V as in (43).

6. Gaussian integration

6.1. The super-expectation. For a $\Lambda \times \Lambda$ matrix A, we define

$$S_A(\Lambda) = \sum_{x,y \in \Lambda} \Big(\varphi_x A_{xy} \bar{\varphi}_x + \psi_x A_{xy} \bar{\psi}_y \Big). \tag{51}$$

When $A = m^2 - \Delta$ this is the same as $S(\Lambda)$ which was defined in (27). Let C be a positive-definite $\Lambda \times \Lambda$ matrix. Then $A = C^{-1}$ exists. We introduce the notation

$$\mathbb{E}_C F = \int_{\mathbb{C}^\Lambda} e^{-S_A(\Lambda)} F, \tag{52}$$

for F a form in $\mathcal{N}$. The integral is defined as described under Proposition 3.1. We call C the covariance because $\mathbb{E}_C \bar{\phi}_a \phi_b = C_{ab}$. More generally, if F is a form of degree zero, i.e., a function of ϕ, then $\mathbb{E}_C F$ is a standard Gaussian expectation for a complex valued random variable ϕ with covariance C [10].

We define a space $\mathcal{N}^\times$ in the same way as $\mathcal{N}$ is defined, but with ϕ doubled to (ϕ, ξ) so that (ϕ, ψ) doubles to the pair $(\phi, \psi), (\xi, \eta)$ with $\eta = (2\pi i)^{-1/2} d\xi$. The external field σ is not doubled. We define $\theta : \mathcal{N} \to \mathcal{N}^\times$ by

$$(\theta F)(\phi, \xi) = \sum_{y \in \Lambda^*} \frac{1}{y!} F_y(\phi + \xi)(\psi + \eta)^y. \tag{53}$$

We write $\mathbb{E}_C \theta F$ for the element of $\mathcal{N}$ obtained when the integral over $\mathbb{C}^\Lambda$ in $\mathbb{E}_C$ applies *only* to (ξ, η). In the general case where F is a form this is not standard probability theory, because $\mathbb{E}_C \theta F$ takes values in $\mathcal{N}$. To keep this in mind we call this a *super-expectation*. The variables and forms (ξ, η) that are integrated out are called *fluctuation fields*.

6.2. Finite-range decomposition of covariance. Suppose C and C_j, $j = 1, \dots, N$, are positive-definite $\Lambda \times \Lambda$ matrices such that

$$C = \sum_{j=1}^{N} C_j. \tag{54}$$

Let $C' = \sum_{k=2}^{N} C_k$. Then, as in the standard theory of Gaussian random variables, the $\mathbb{E}_C$ expectation can be performed progressively:

$$\mathbb{E}_C F = \mathbb{E}_{C' + C_1} F = \mathbb{E}_{C'} \big(\mathbb{E}_{C_1} \theta F \big). \tag{55}$$

For further details, see [12].

From now on we work with $C = (m^2 - \Delta)^{-1}$, where Δ is the finite difference Laplacian on the periodic lattice Λ. Given any sufficiently large dyadic integer L, there exists a decomposition $C = \sum_{j=1}^{N} C_j$ such that C_j is positive-definite and

$$C_j(x, y) = 0 \quad \text{if} \quad |x - y| \geq L^j/2. \tag{56}$$

This is called the *finite range* property. The existence of such a decomposition is established in [7] for the case where Λ is replaced by $\mathbb{Z}^d$. In [6, Lecture 2] it is briefly explained how the decomposition for the periodic Λ case is obtained from the $\mathbb{Z}^d$ case, for Λ a torus of side L^N. To accommodate this restriction on the side of Λ the infinite volume limit in Theorem 4.1 is taken with a sequence of tori with sides L^N, $N \in \mathbb{N}$.

We conclude this section with an informal discussion of scaling estimates that guide the proof. Equation (55) says that F, which depends on a field with covariance C, can be replaced by $\mathbb{E}_{C_1}\theta F$, which depends on a field characterised by the covariance C'. Repeating this operation j times will replace F by a new F that depends on a *field at scale* j characterised by the covariance $\sum_{k=j+1}^{N} C_k$. According to estimates in [7], this sum is dominated by the first term which satisfies

$$|\nabla_x^\alpha \nabla_y^\beta C_{j+1}(x,y)| \le \text{const}\, L^{-2j[\phi]-|\alpha|_1 j-|\beta|_1 j}, \tag{57}$$

where the symbol $[\phi]$, which is called the *dimension* of the field, was defined in (46). The typical field at scale j behaves like "half a covariance," and in particular the standard deviation of φ_x is $\approx L^{-j[\phi]}$. Furthermore, the estimate on derivatives in (57) says that typical fields at scale j are roughly constant over distances of order L^j.

We can now explain why the terms in V as defined by (43) play a pre-eminent role. For a cube B of side L^j, which contains L^{dj} points,

$$\sum_{x\in B} \varphi_{j,x}^p \approx L^{(d-p[\phi])j}. \tag{58}$$

In the case of $d = 4$, for which $[\phi] = 1$, this scales down when $p > 4$ and φ^p is said to be *irrelevant*. The power $p = 4$ neither decays nor grows, and is called *marginal*. Powers $p < 4$ grow with the scale, and are called *relevant*. Since the derivatives in (57) provide powers of L, the monomial $\varphi(-\Delta)\bar\varphi$ is marginal. Thus $\tau, \tau_\Delta, \tau^2$ are the supersymmetric marginal and relevant monomials.

6.3. Progressive integration. To prove Theorem 4.1 using (32) we have to calculate

$$\int_{\mathbb{C}^\Lambda} e^{-S(\Lambda)-V_0(\Lambda)} = \mathbb{E}_C e^{-V_0(\Lambda)}, \tag{59}$$

where V_0 is given by (31). This V_0 equals V as defined in (43), with (g, ν, z, λ, q) replaced by $(g_0, \nu_0, z_0, \lambda_0, q_0)$ with

$$q_0 = 0, \qquad \lambda_0 = 1. \tag{60}$$

Sections 6.1 and 6.2 have taught us that we can evaluate $\mathbb{E}_C e^{-V_0(\Lambda)}$ by the following iteration: let

$$Z_0 = e^{-V_0(\Lambda)}. \tag{61}$$

Inductively define Z_j, $j = 0, \dots, N$, by

$$Z_{j+1} = \mathbb{E}_{C_{j+1}}\theta Z_j. \tag{62}$$

Then

$$\mathbb{E}_C e^{-V_0(\Lambda)} = Z_N. \tag{63}$$

Therefore the proof of Theorem 4.1 now depends on the analysis of the sequence Z_j. Our proof will depend on showing that the Z_j simplify as j increases. In fact, in the next section we will see that they become more Gaussian, in the sense that the $g\tau^2$ term becomes smaller. The index j will be called a *scale*.

7. Perturbation theory and flow equations

In this section we start to prove that Z_j becomes more Gaussian as j increases. To do this we adapt to our particular setting a perturbative calculation of the kind that appears in [30].

For $X \subset \Lambda$ and V as defined in (43), define

$$I_{j,X}(V) = e^{-V(X)}\big(1 + \tfrac{1}{2}W_j(V,X)\big), \tag{64}$$

where

$$W_j(V,X) = (1 - \mathrm{Loc}_X)F_{w_j}\big(V(X), V(\Lambda)\big) \tag{65}$$

with

$$w_j = \sum_{i=1}^{j} C_i, \tag{66}$$

$$F_{w_j}\big(V(X), V(\Lambda)\big) = \sum_{n\geq 1} \frac{1}{n!}\big(D_R^n V(X)\big) w_j^n \big(D_L^n V(\Lambda)\big), \qquad X \subset \Lambda \setminus \{a,b\}; \tag{67}$$

the latter sum truncates at $n = 4$ due to our quartic interaction. The symbols D_R and D_L denotes right and left differentiation with respect to fields. The "left/right" is to specify signs, but this and the precise definition are not of immediate importance, so we just give an example. If X contains a or b there is an additional combinatorial factor of 2 multiplying terms in $F_{w_j}\big(V(X), V(\Lambda \setminus X)\big)$ that are linear in $\sigma, \bar{\sigma}$.

Example 2. For $V = \psi\bar{\psi}$ and $X = \{x\}$, $\big(D_R^n V(X)\big) w_j^n \big(D_L^n V(\Lambda)\big)$ equals

$$\begin{cases} \sum_{y\in\Lambda} \Big(\psi_x w_j(x,y)\bar{\psi}_y + \bar{\psi}_x w_j(x,y)\psi_y\Big) & n = 1 \\ -\sum_{y\in\Lambda} w_j^2(x,y) & n = 2. \end{cases} \tag{68}$$

When $j = 0$, $I_{j,X}(V) = e^{-V(\Lambda)}$ because $w_0 = 0$. Therefore we can choose the coupling constants to make it equal to Z_0. Furthermore, $I_{j,X}(V)$ has the martingale-like property exhibited in Proposition 7.1, which says that integrating out the fluctuation field ξ_{j+1} is approximately the same as changing the coupling constants in V to new coupling constants called $(g_{\mathrm{pt}}, \nu_{\mathrm{pt}}, z_{\mathrm{pt}}, \lambda_{\mathrm{pt}}, q_{\mathrm{pt}})$. The formulas for the new coupling constants are called *perturbative flow equations*.

Proposition 7.1. *As a formal power series in* (g, ν, z, λ, q),

$$\mathbb{E}_{C_{j+1}} I_{j,\Lambda}(V) = I_{j+1,\Lambda}(V_{\rm pt}) \mod (g, \nu, z, \lambda, q)^3, \tag{69}$$

where

$$V_{\rm pt} = V_{\rm pt}(V) \tag{70}$$

has the same form (43) *as* V*, with* (g, ν, z, λ, q) *replaced by*

$$g_{\rm pt} = g - c_g g^2 + r^{\rm pt}_{g,j}, \tag{71}$$

$$\nu_{\rm pt} = \nu_+ + r^{\rm pt}_{\nu,j}, \tag{72}$$

$$z_{\rm pt} = z + r^{\rm pt}_{z,j}, \tag{73}$$

$$\lambda_{\rm pt} = \left(1 + \sum_{y \in \Lambda} \big(\nu_+ w_{j+1}(0,y) - \nu w_j(0,y) \big) \right) \lambda, \tag{74}$$

$$q_{\rm pt} = q + \lambda^2 \, C_{j+1}(a,b), \tag{75}$$

where $c_g > 0$, $\nu_+ = \nu + 2gC_{j+1}(0,0)$, *and* $r^{\rm pt}_{g,j}$, $L^{2j} r^{\rm pt}_{\nu,j}$, $r^{\rm pt}_{z,j}$ *are computable uniformly bounded homogeneous polynomials of degree* 2 *in* (g, ν, z). *There are* g^2 *terms in* $r^{\rm pt}_{g,j}$, *but they are summable in* j *and therefore do not overpower* $c_g g^2$.

The β function. The right hand side of (71) is known as the *β function*. The simpler recursion obtained by setting $r^{\rm pt}_{\nu,j} = 0$, namely

$$\bar{g}_{j+1} = \bar{g}_j - c_g \bar{g}_j^2, \qquad \bar{g}_0 = g_0, \tag{76}$$

creates a sequence $\bar{g}_j$ that tends to zero like j^{-1} as $j \to \infty$. The sequence Z_j becomes more Gaussian due to the famous observation, known as *infra-red asymptotic freedom*, that (76) controls the behaviour of the more complex recursion of Proposition 7.1 and drives the τ^2 term to zero.

8. The renormalisation group map

The problem with the second order perturbative calculation in Section 7 is that the error is not only of order 3 in the coupling constants, but it also fails to be uniform in the volume Λ. The remedy is not to work with $I_{j,\Lambda}$, but with $\prod_{B \subset \Lambda} I_{j,B}$ where B is a cube and the allowed cubes pave Λ. The idea is that by choosing the side of B to be bigger than the range of C_{j+1}, we can take advantage of independence of cubes that do not touch to more or less use our perturbation theory with Λ replaced by individual cubes. This idea requires a systematic organisation which we describe in this section.

8.1. Scales and the circle product. Let $L \geq 3$ be an integer. Let $R = L^N$, and let $\Lambda = \mathbb{Z}^d/(R\mathbb{Z}^d)$.

Definition 1. (a) *Blocks.* For each $j = 0, 1, \ldots, N$, the torus Λ is paved in a natural way by L^{N-j} disjoint d-dimensional cubes of side L^j. The cube that contains the origin has the form (for L odd)

$$\{x \in \Lambda : |x| \leq \frac{1}{2}(L^j - 1)\}, \tag{77}$$

and all the other cubes are translates of this one by vectors in $L^j\mathbb{Z}^d$. We call these cubes *j-blocks*, or *blocks* for short, and denote the set of j-blocks by $\mathcal{B}_j = \mathcal{B}_j(\Lambda)$.
(b) *Polymers.* A union of j-blocks is called a *polymer* or *j-polymer*, and the set of j-polymers is denoted $\mathcal{P}_j = \mathcal{P}_j(\Lambda)$. The size $|X|_j$ of $X \in \mathcal{P}_j$ is the number of j-blocks in X.
(c) *Connectivity.* A subset $X \subset \Lambda$ is said to be *connected* if for any two points $x_a, x_b \in X$ there exists a path $(x_i, i = 0, 1, \ldots n) \in X$ with $\|x_{i+1} - x_i\|_\infty = 1$, $x_0 = x_a$ and $x_n = x_b$. According to this definition, a polymer can be decomposed into connected components; we write $\mathcal{C}(X)$ for the set of connected components of X. We say that two polymers X, Y *do not touch* if $\min\{\|x - y\|_\infty : x \in X, y \in Y\} > 1$.
(d) *Small sets.* A polymer $X \in \mathcal{P}_j$ is said to be a *small set* if $|X|_j \leq 2^d$ and X is connected. Let $\mathcal{S}_j$ be the set of all small sets in $\mathcal{P}_j$.
(e) *Small set neighbourhood.* For $X \subset \Lambda$ let

$$X^* = \bigcup_{Y \in \mathcal{S}_j : X \cap Y \neq \varnothing} Y. \tag{78}$$

The *polymers* of Definition 1 have nothing to do with long chain molecules. This concept has a long history in statistical mechanics going back to the important paper [17].

Proposition 8.1. *Suppose that $X_1, \ldots, X_n \in \mathcal{P}_{j+1}$ do not touch each other and let $F_i(X_i) \in \mathcal{N}(X_i)$. The expectation $\mathbb{E}_{C_{j+1}}$ has the* factorisation property*:*

$$\mathbb{E}_{C_{j+1}} \prod_{i=1}^{n} F_i(X_i) = \prod_{m=1}^{n} \mathbb{E}_{C_{j+1}} F_i(X_i). \tag{79}$$

Proof. Gaussian random variables are independent if and only if the off-diagonal part of their covariance matrix vanishes. This generalises to our forms setting, and so the proposition follows from the finite range property of C_{j+1}. □

Given forms F, G defined on $\mathcal{P}_j$, let

$$(F \circ G)(\Lambda) = \sum_{X \in \mathcal{P}_j} F(X) G(\Lambda \setminus X). \tag{80}$$

This defines an associative product, which is also commutative provided F and G both have even degree.

8.2. The renormalisation group map. Recall that we have defined $I_{j,X}(V)$ in (64). Given a yet-to-be-constructed sequence V_j, for $X \in \mathcal{P}_j$, let

$$I_j(X) = \prod_{B \in \mathcal{B}_j} I_{j,B}(V_j). \tag{81}$$

We have defined V_0 in (31). Let $K_0(X) = \mathbb{1}_{X=\varnothing}$. Then the Z_0 defined in (61) is also given by

$$Z_0 = I_0(\Lambda) = (I_0 \circ K_0)(\Lambda), \tag{82}$$

because $I_{0,\Lambda}(\Lambda) = e^{-V_0(\Lambda)}$ since $w_0 = 0$.

Definition 2. We say that $K : \mathcal{P}_j \to \mathcal{N}$ has the *component factorisation property* if

$$K(X) = \prod_{Y \in \mathcal{C}(X)} K(Y). \tag{83}$$

Suppose, inductively, that we have constructed (V_j, K_j) where $K_j : \mathcal{P}_j \to \mathcal{N}$ is such that

$$\begin{aligned} &(i) \quad Z_j = (I_j \circ K_j)(\Lambda), \\ &(ii) \quad K_j \text{ has the component factorisation property,} \\ &(iii) \quad \text{For } X \in \mathcal{P}_j,\ K_j(X) \in \mathcal{N}(X^*). \end{aligned} \tag{84}$$

Our objective is to define (V_{j+1}, K_{j+1}), where $K_{j+1} : \mathcal{P}_{j+1} \to \mathcal{N}$ has the same properties at scale $j+1$. Then the action of $\mathbb{E}_{C_{j+1}}\theta$ on Z_j has been expressed as the map:

$$(V_j, K_j) \mapsto (V_{j+1}, K_{j+1}). \tag{85}$$

This map will be constructed next. We call it the *renormalisation group map*. Unlike $Z_j \mapsto \mathbb{E}\theta Z_j$ it is not linear, so this looks like a poor trade, but in fact it is a good trade because the data (V_j, K_j) is local, unlike creatures such as $\exp(-V_j(\Lambda))$ in Z_j. The component factorisation property and Proposition 8.1 allow us to work with K_j on the domain of all connected sets in $\mathcal{P}_j$. We can prove that $K_j(X)$ is very small when the number of blocks in X is large; in fact, only the restriction of K_j to the small sets $\mathcal{S}_j$ plays an important role.

9. The inductive step: construction of V_{j+1}

In accordance with the program set out in Section 8.2 we describe how V_{j+1} is constructed, given (V_j, K_j). Our definition of V_{j+1} will be shown to have an additional property that there is an associated K_{j+1}, which, as a function of K_j, is contractive in norms described in Section 10.

Recall that the set $\mathcal{S}$ of small sets was given in Definition 1. For $B \in \mathcal{B}_j$ not containing a, b define V_{j+1} to be the local interaction determined by:

$$\begin{aligned} \hat{V}_j(B) &= V_j(B) + \mathrm{Loc}_B \sum_{Y \in \mathcal{S}, Y \supset B} \frac{1}{|Y|} I_j(Y)^{-1} K_j(Y), \\ V_{j+1} &= V_{\mathrm{pt}}(\hat{V}_j), \end{aligned} \tag{86}$$

where $V_{\mathrm{pt}} = V_{\mathrm{pt}}(V)$ with generic argument V is defined in (70). Recalling the discussion of "relevant terms" just after (58), in (86) V_{j+1} has been defined so that relevant and marginal terms inside K_j are absorbed into V_{j+1} in such a manner that they will not contribute to K_{j+1}. If B contains a or b the combinatorial factor $\frac{1}{|Y|}$ is modified for terms in $\mathrm{Loc}_B K_j$ which are divisible by σ or $\bar{\sigma}$.

We have completed the V part of the inductive construction of the sequence (V_j, K_j). Before discussing the K induction we have to define some norms so that we can state the contractive property.

10. Norms for K

Let $\mathfrak{h}_j > 0$ and $\mathfrak{s}_j > 0$. For a test function f as defined in Section 5.2 we introduce a norm

$$\|f\|_{\Phi_j} = \sup_{x,y \in \Lambda^*, z \in \{1,2\}^*} \sup_{|\alpha|_\infty \le 3} \mathfrak{h}_j^{-p-q} \mathfrak{s}_j^{-r} L^{j|\alpha|_1} |\nabla^\alpha f_{x,y,z}|. \tag{87}$$

Multiple derivatives up to order 3 on each argument are specified by the multi-index α. The gradient ∇ represents the finite-difference gradient, and the supremum is taken componentwise over both the forward and backward gradients. A test function f is required to have the property that $f_{x,y,z} = 0$ whenever the sequence x has length $p > 9$ or the sequence z has length $r > 2$; there is no restriction on the length of y. By the definition of the norm, test functions satisfy

$$|\nabla^\alpha f_{x,y,z}| \le \mathfrak{h}_j^{p+q} \mathfrak{s}_j^r L^{-j|\alpha|_1} \|f\|_{\Phi_j}. \tag{88}$$

We discuss the choice of $\mathfrak{s}_j$ in Section 12 when it first plays a role, and here we focus on $\mathfrak{h}_j$. An important choice is

$$\mathfrak{h}_j = \ell_j = \ell_0 L^{-j[\phi]}, \tag{89}$$

for a given ℓ_0. The $L^{-j[\phi]}$ is there because unit norm test functions of one variable should obey the same estimates as a typical field, and test functions of more than one variable should obey the estimates that a product of typical fields obeys.

Recall the pairing defined in (41) and, for $F \in \mathcal{N}$ and $\phi \in \mathbb{C}^\Lambda$, let

$$\|F\|_{T_{\phi,j}} = \sup_{g : \|g\|_{\Phi_j} \le 1} |\langle F, g \rangle_\phi| \,. \tag{90}$$

The following proposition provides properties of this seminorm that are well adapted to the control of K.

Proposition 10.1. *Let $F, F_1, F_2 \in \mathcal{N}$. The T_ϕ norm obeys the product property*

$$\|F_1 F_2\|_{T_{\phi,j}} \le \|F_1\|_{T_{\phi,j}} \|F_2\|_{T_{\phi,j}}, \tag{91}$$

and, if ℓ_0 is chosen large enough, the integration property

$$\|\mathbb{E}_{C_{j+1}} F\|_{T_{\phi,j}(\mathfrak{h}_j)} \le \mathbb{E}_{C_{j+1}} \|F\|_{T_{\phi+\xi,j}(2\mathfrak{h}_j)}. \tag{92}$$

For further details, see [12]. The second conclusion shows that the norm controls the forms when a fluctuation field is integrated out: on the right hand side the norm is a zero degree form, and hence the expectation is a standard Gaussian expectation.

The most important case of the T_ϕ seminorm is the case $\phi = 0$, but knowing that $\|K(X)\|_{T_0} < \infty$ cannot tell us whether $K(X)$ is integrable. For this we must limit the growth of $K(X)$ as $\phi \to \infty$, and the resolution of this issue will be obtained using Definition 3 below.

Our intuitive picture of $K_j(X)$, where $X \in \mathcal{P}_j$, is that it is dominated by a local version of the remainder $(g, \nu, z, \lambda, q)^3$ in (69). To estimate such remainders we must, in particular, estimate $I_{j,X}$ which contains $\exp(-g_j \sum_{x\in X} |\varphi_x|^4)$. By (57) the typical field φ at scale j is roughly constant on scale L^j, and X contains $O(L^{jd})$ points. Therefore this factor looks like $\exp(-g_j L^{dj} |\varphi|^4)$. This is a function of $\varphi/\mathfrak{h}_j$ with $\mathfrak{h}_j \approx g_j^{-1/4} L^{-jd/4}$, which in four dimensions can be rewritten as $g_j^{-1/4} L^{-j[\phi]}$ because $[\phi] = 1$. We want to prove that g_j decays in the same way as does $\bar{g}_j$ in (76), and with this in mind we replace g_j by the known sequence $\bar{g}_j$. This leads us to our second choice

$$\mathfrak{h}_j = h_j = k_0 \bar{g}_j^{-1/4} L^{-j[\phi]},$$

where the constant k_0 is determined so that $\exp(-V_j(B))$ will, uniformly in j, have a $T_\phi(h_j)$ norm close to one.

In the previous discussion we made the assumption that the typical φ at scale j is roughly constant on scales L^j. Our norm recognises this; it is a weighted L_∞ norm, where the weight permits growth as fields become atypical. The weight is called a large field regulator and is defined next.

Consider a test function f that is an ersatz field φ, namely a complex-valued function $f = f_x$ for $x \in \Lambda$. For $X \subset \Lambda$, we write $f \in \Pi(X)$ if f restricted to X is a polynomial of degree three or less. We define a seminorm on $\phi = (\varphi, \bar{\varphi})$ by

$$\|\phi\|_{\tilde{\Phi}_j(X)} = \inf\{\|\varphi - f\|_{\Phi_j(\ell_j)} : f \in \Pi(X)\}; \tag{93}$$

note that we are setting $\mathfrak{h}_j = \ell_j$ in the above equation.

Definition 3. Let $j \in \mathbb{N}_0$, $X \in \mathcal{P}_j$, and $\phi \in \mathbb{C}^\Lambda$. The *large-field regulator* is given by

$$\tilde{G}_j(X, \phi) = \prod_{B \in \mathcal{B}_j(X)} \exp \|\phi\|^2_{\tilde{\Phi}_j(B^*)}, \tag{94}$$

where B^* is the small set neighbourhood of B defined in (78). For each $X \in \mathcal{P}_j$, we define a seminorm on $\mathcal{N}(X^*)$ as follows. For $K(X) \in \mathcal{N}(X^*)$, we define $\|K(X)\|_{\tilde{G}_j,h_j}$ to be the best constant C in

$$\|K(X)\|_{T_{\phi,j}(h_j)} \le C\tilde{G}_j(X,\phi), \tag{95}$$

where we have made explicit in the notation the fact that the norm on the left hand side is based on the choice $\mathfrak{h}_j = h_j$.

11. The inductive step completed: existence of K_{j+1}

We have already specified V_{j+1} in (86). Now we complete the inductive step by constructing K_{j+1} such that (84) holds. The following theorem is at the heart of our method [12]. It provides K_{j+1} and says that we can continue to prolong the sequence (V_j, K_j) for as long as the coupling constants (g_j, ν_j, z_j) remain small. Moreover, in this prolongation, the T_0 norm of K_{j+1} remains third order in the coupling constants and is therefore much smaller than the perturbative (K-independent) part of V_{j+1}.

For $a \ge 0$, set $f_j(a, \varnothing) = 0$, and define

$$f_j(a,X) = 3 + a(|X|_j - 2^d)_+, \qquad X \in \mathcal{P}_j \text{ with } X \neq \varnothing. \tag{96}$$

Note that $f_j(a,X) = 3$ when $X \in \mathcal{S}_j$, but that $f_j(a,X)$ is larger than 3 and increases with the size of $|X|_j$ if $X \notin \mathcal{S}_j$. We fix a to have a sufficiently small positive value.

The following theorem is proved for two different choices of the norm pairs $\|\cdot\|_j$ and $\|\cdot\|_{j+1}$, in (97) and (98), and for two corresponding choices of the small parameter $\epsilon_{\delta I}$, as follows:

- $\|\cdot\|_j = \|\cdot\|_{\tilde{G}_j,h_j}$ with $h_j = k_0 \bar{g}_j^{-1/4} L^{-j[\phi]}$, and $\|\cdot\|_{j+1} = \|\cdot\|_{\tilde{G}_{j+1},h_{j+1}}$ with $h_{j+1} = k_0 \bar{g}_{j+1}^{-1/4} L^{-(j+1)[\phi]}$. The small parameter $\epsilon_{\delta I}$ is proportional to $g_j^{1/4}$.
- $\|\cdot\|_j = \|\cdot\|_{T_0,\ell_j}$ with $\ell_j = \ell_0 L^{-j[\phi]}$, and $\|\cdot\|_{j+1} = \|\cdot\|_{T_0,\ell_{j+1}}$. The small parameter $\epsilon_{\delta I}$ is proportional to g_j.

Define a cone $C = \{(g_j,\nu_j,z_j) | g > 0, |\nu| \vee |z| \le bg, g_j \le c(b,L)\}$. The constant b is determined in Section 12, and $c(b,L)$ is a function of b, L constructed in the proof of the next theorem.

Theorem 11.1. *Let $(g_j,\nu_j,z_j) \in C$. Let a be sufficiently small, and let M be any (large) positive constant that is independent of d, L. There is a constant $c_{\rm pt}$ (depending on d, L) such that the following holds. Suppose that $K_j : \mathcal{P}_j \to \mathcal{N}_j$ has properties* (84) *and satisfies*

$$\|K_j(X)\|_j \le M c_{\rm pt} \epsilon_{\delta I}^{f_j(a,X)}, \qquad X \in \mathcal{P}_j \text{ connected}, \tag{97}$$

Then, if L is sufficiently large (depending on M), there exists $K_{j+1} : \mathcal{P}_{j+1} \to \mathcal{N}_{j+1}$ with properties (84) *at scale $j+1$ and*

$$\|K_{j+1}(U)\|_{j+1} \le 2c_{\rm pt}\epsilon_{\delta I}^{f_{j+1}(a,U)}, \qquad U \in \mathcal{P}_{j+1} \text{ connected.} \tag{98}$$

12. Decay of the two-point function

Finally, we combine the machinery we have developed, to outline the proof of Theorem 4.1. As we have already noted, Theorem 1.2 is a consequence of Theorem 4.1.

We must study the coupling constant flow. The linear map $\mathrm{Loc}_B : \mathcal{N} \to \mathcal{V}$ is bounded in T_0 norm [12], so according to the inductive assumption (97) on the T_0 norm of K_j, the coupling constants in $\hat{V}_j$ of (86) are small (third order) adjustments to the coupling constants in V_j. Theorem 11.1 ensures that this smallness is preserved as the scale advances.

We first consider the case $(\lambda_0, q_0) = (0,0)$. In this case, $(\lambda_j, q_j) = (0,0)$ for all j. The definition of V_{j+1} in (86) then gives rise to a non-perturbative version of the flow equations of Proposition 7.1, in which the effect of K is now taken into account. When $V_j \mapsto V_{j+1}$ is expressed as

$$(g_j, \nu_j, z_j) \mapsto (g_{j+1}, \nu_{j+1}, z_{j+1}) \tag{99}$$

we find that

$$g_{j+1} = g_j - c_g g_j^2 + r_{g,j}, \tag{100}$$
$$\nu_{j+1} = \nu_j + 2gC_{j+1}(0,0) + r_{\nu,j}, \tag{101}$$
$$z_{j+1} = z_j + r_{z,j}, \tag{102}$$
$$K_{j+1} = r_{K,j}(g_j, \nu_j, z_j, K_j), \tag{103}$$

where the r's now depend also on K_j, and where we have added the map $r_{K,j} : (g_j, \nu_j, z_j, K_j) \mapsto K_{j+1}$ defined by Theorem 11.1. Furthermore, we prove that the r's are Lipschitz functions of (g_j, ν_j, z_j, K_j), where K belongs to a Banach space normed by a combination of the norms in Section 11. These are the properties needed to prove that K only causes a small deformation of the perturbative flow $V \mapsto V_{\rm pt}$.

The main theorem now reduces to an exercise in dynamical systems. We prove that with a suitable choice of the constant b defining the cone C there is a *Lipschitz stable manifold* of initial conditions $(z_0, \nu_0) = h(m^2, g_0)$ for which the sequence (V_j, K_j), $j = 0, \dots, N$, has a limit as $N \to \infty$ and $m^2 \downarrow 0$. We call this the *global trajectory*. For $m^2 = 0$, the global trajectory tends to the fixed point $(V, K) = (0,0)$. In particular, $g_j \to 0$, which is infra-red asymptotic freedom. Referring to (29), we have four unknown parameters g_0, ν_0, z_0, m^2 related by three equations, and now there is a fourth equation $(z_0, \nu_0) = h(m^2, g_0)$. By the implicit function theorem we solve for the unknowns as functions of (g, ν). As $\nu \downarrow \nu_c(g)$, $m^2 \downarrow 0$ and vice-versa.

Now we consider the flow for (λ_j, q_j). According to (60), $\lambda_0 = 1$ and $q_0 = 0$. Using (50), we prove that the terms $r_{g,j}, r_{\nu,j}, r_{z,j}$ do not depend on λ_j, q_j and thus the coupling constants g, ν, z have no dependence on λ, q. From (86) we find

$$\lambda_{j+1} = \left(1 + \sum_{y\in\Lambda} \big(\nu_{j+1} w_{j+1}(0,y) - \nu_j w_j(0,y)\big)\right) \lambda_j + r_{\lambda,j}, \tag{104}$$

$$q_{j+1} = q_j + \lambda^2\, C_{j+1}(a,b) + r_{q,j}, \tag{105}$$

where $r_{\lambda,j}, r_{q,j}$ are corrections that include contributions from K_j.

Recall that $\mathcal{S}_j$ was defined in Definition 1. Let $s_{a,b}$ be the first scale j such that there exists a polymer in $\mathcal{S}_j$ that contains $\{a,b\}$. The correction $r_{q,j}$ is zero for all scales $j < s_{a,b}$: according to (50) and the definition of $\hat{V}$ in (86) there can be no $\sigma\bar{\sigma}$ contribution from K_j until the first scale where there is a set $X \in \mathcal{S}_j$ that covers $\{a,b\}$. Also, by the finite range property, $C_{j+1}(a,b) = 0$ for $j < s_{a,b}$. Thus (105) gives

$$q_N = \sum_{j=s_{a,b}}^{N} \big(\lambda_j^2\, C_{j+1}(a,b) + r_{q,j}\big)\,. \tag{106}$$

At scale N, Λ is a single block in $\mathcal{B}_N$, so by the definition of the circle product, Z_N is simply given by

$$Z_N = (I_N \circ K_N)(\Lambda) = I_N(\Lambda) + K_N(\Lambda). \tag{107}$$

The final renormalisation group map is the action of $\mathbb{E}_{C_N}$, not $\mathbb{E}_{C_N}\theta$. This means that the fields ϕ, ψ are to be set to zero in I_N, K_N, and only dependence on σ remains. By (64) we compute two σ derivatives of I_N and find

$$-\left.\frac{\partial^2}{\partial\sigma\partial\bar{\sigma}}\right|_0 Z_N = q_N - K_{\bar{\sigma}\sigma}, \qquad \text{where} \quad K_{\bar{\sigma}\sigma} = \left.\frac{\partial^2 K_N(\Lambda)}{\partial\sigma\partial\bar{\sigma}}\right|_0. \tag{108}$$

The $\bar{\sigma}\sigma$ derivative is a coefficient in the pairing (41), and the T_0 norm bounds this pairing, so Theorem 11.1 gives

$$|K_{\bar{\sigma}\sigma}| \le \|K\|_{T_0,N} \mathfrak{s}_N^{-2} \le O(g_N^3)\mathfrak{s}_N^{-2}. \tag{109}$$

We are able to prove Theorem 11.1 with

$$\mathfrak{s}_j = \mathfrak{s}_0 \ell_{j\wedge s_{a,b}}^{-1} \approx O(L^{j\wedge s_{a,b}}), \tag{110}$$

where $\mathfrak{s}_0$ is a constant, so that, when $N > s_{a,b}$,

$$|K_{\bar{\sigma}\sigma}| \le O(g_N^3) L^{-2(N\wedge s_{a,b})} = O(g_N^3 |a-b|^{-2}). \tag{111}$$

This tends to zero as $N \to \infty$.

By a similar estimate we can control the $r_{\lambda,j}$, $r_{q,j}$ terms in (104), (106). These contain σ derivatives of the K_j terms in (86). The conclusion is that $\lambda_\infty = \lim_{N\to\infty} \lambda_N$ and $q_\infty = \lim_{N\to\infty} q_N$ exist and are bounded away from zero.

By (32), the left hand side of (30) is given by

$$\lim_{\nu\downarrow\nu_c}(1+z_0)\lim_{\Lambda\uparrow\mathbb{Z}^d}\int_{\mathbb{C}^\Lambda} e^{-S(\Lambda)-\tilde{V}_0(\Lambda)}\bar{\varphi}_a\varphi_b = \lim_{m^2\downarrow 0}(1+z_0)q_\infty. \tag{112}$$

From (104) and (106) we find that

$$\lim_{m^2\downarrow 0} q_\infty \sim \lambda_\infty^2 \sum_{j=s_{a,b}}^{\infty} C_{j+1}(a,b), \tag{113}$$

where $m^2 = 0$ in C_{j+1}, and $\sim$ means that the ratio of the left hand side and the right hand side tends to one as $a-b\to\infty$. Next, we use the finite range property to restore the scales $j < s_{a,b}$ to the sum, which then becomes the complete finite range decomposition for the infinite volume simple random walk two-point function $(-\Delta)^{-1}(a,b)$,

$$\lim_{m^2\downarrow 0} q_\infty \sim \lambda_\infty^2(-\Delta)^{-1}(a,b). \tag{114}$$

The right hand side of (114), and hence of (112), is thus asymptotic to a multiple of $|a-b|^{-2}$ as $|a-b|\to\infty$, as desired, since the inverse Laplacian has this behaviour.

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