Asymptotic expansions in n^{-1} for percolation critical values on the n-cube and $\mathbb{Z}^n$

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Abstract

We use the lace expansion to prove that the critical values for nearest-neighbour bond percolation on the n-cube $\{0,1\}^n$ and on the integer lattice $\mathbb{Z}^n$ have asymptotic expansions, with rational coefficients, to all orders in powers of n^{-1}.

1 Main results

1.1 Main result for $\mathbb{Z}^n$

We consider bond percolation on $\mathbb{Z}^n$ with edge set consisting of pairs $\{x, y\}$ of vertices in $\mathbb{Z}^n$ with $\|x-y\|_1 = 1$, where $\|w\|_1 = \sum_{j=1}^n |w_j|$ for $w \in \mathbb{Z}^n$. Bonds (edges) are independently occupied with probability p and vacant with probability $1-p$. The critical value is defined by

$$p_c(\mathbb{Z}^n) = \inf\{p : \exists \text{ an infinite connected cluster of occupied bonds a.s.}\}. \tag{1.1}$$

Given a vertex x of $\mathbb{Z}^n$, let $C(x)$ denote the connected cluster of x, i.e., the set of vertices y such that y is connected to x by a path consisting of occupied bonds. Let $|C(x)|$ denote the cardinality of $C(x)$, and let $\chi(p) = \mathbb{E}_p|C(0)|$ denote the expected cluster size of the origin. Results of [1, 21] imply that

$$p_c(\mathbb{Z}^n) = \sup\{p : \chi(p) < \infty\}. \tag{1.2}$$

is an equivalent definition of the critical value. Our main result for $\mathbb{Z}^n$ is the following theorem.

Theorem 1.1. *Consider bond percolation on $\mathbb{Z}^n$. There are rational numbers $a_i(\mathbb{Z})$ such that for all $M \geq 1$,*

$$p_c(\mathbb{Z}^n) = \sum_{i=1}^{M} a_i(\mathbb{Z})(2n)^{-i} + O((2n)^{-M-1}) \qquad \text{as } n \to \infty. \tag{1.3}$$

The constant in the error term depends on M.

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The convergence of $p_c(\mathbb{Z}^n)$ to the leading term $(2n)^{-1}$ has been studied in [4, 6, 14, 16, 20], with various error estimates. The formula (1.3) for $M = 3$ was obtained in [17, 18], and with a simplified proof in [19], with $a_1(\mathbb{Z}) = 1$, $a_2(\mathbb{Z}) = 1$, $a_3(\mathbb{Z}) = \frac{7}{2}$. Thus, $p_c(\mathbb{Z}^n) = (2n)^{-1} + (2n)^{-2} + \frac{7}{2}(2n)^{-3} + O((2n)^{-4})$. The expansion

$$p_c(\mathbb{Z}^n) = \frac{1}{2n} + \frac{1}{(2n)^2} + \frac{7}{2(2n)^3} + \frac{16}{(2n)^4} + \frac{103}{(2n)^5} + \cdots \tag{1.4}$$

was reported in [12], but with no rigorous bound on the remainder.

We expect that the full asymptotic expansion $\sum_{n=1}^{\infty} a_i(\mathbb{Z})(2n)^{-i}$ does *not* converge for any n (i.e., the formal power series $\sum_{i=1}^{\infty} a_i(\mathbb{Z})t^i$ has radius of convergence zero), although we have no serious evidence for this conjecture. On a very naive level, the coefficients $1, 1, \frac{7}{2}, 16, 103$ stated in (1.4) are going in the uncomfortable direction. Also, there is an example where divergence has been proven. In [11], Fisher and Singh review expansions in the inverse dimension for the critical temperature of spin systems, and in particular for the N-vector model [13]. It has not been proven that such expansions are asymptotic for general N. However, for the spherical model [5], which corresponds to the limit $N \to \infty$ of the N-vector model, the critical temperature $T_c(\mathbb{Z}^n)$ is exactly equal to

$$T_c(\mathbb{Z}^n) = \left[\int_{[-\pi,\pi]^n} \frac{1}{2(n - \sum_{j=1}^n \cos k_j)} \frac{dk_1}{2\pi} \cdots \frac{dk_n}{2\pi} \right]^{-1}. \tag{1.5}$$

The integral (1.5) has an asymptotic expansion to all orders in powers of $1/n$, but this expansion is divergent [13]. There is no reason to expect that such divergence is limited to the spherical model.

As a side remark, we note that it is pointed out in [13] that there are sign changes in the expansion of $T_c(\mathbb{Z}^n)$ at orders $(2n)^{-12}$ and $(2n)^{-20}$. From (1.4), one might guess that $a_i(\mathbb{Z}) \geq 0$ for all i, but the spherical model shows that negative coefficients can occur relatively late in the series. Indeed, although we do not determine their overall signs except for $i = 1, 2, 3$, in our proof there are contributions to the $a_i(\mathbb{Z})$ of both signs.

For the connective constant for self-avoiding walks on $\mathbb{Z}^n$, existence of an asymptotic expansion to all orders in $(2n)^{-1}$ was proved in [17], but the corresponding result for percolation was not obtained. Our method is based on the lace expansion and follows the same general approach as that used for the connective constant in [17], but the details here are significantly different and substantially more difficult.

1.2 Main result for $\mathbb{Q}_n$

We also consider bond percolation on the n-cube $\mathbb{Q}_n$, which has vertex set $\{0,1\}^n$ and edge set consisting of pairs $\{x, y\}$ of vertices in $\{0,1\}^n$ with $\|x - y\|_1 = 1$, where we regard $\mathbb{Q}_n$ as an additive group with addition component-wise modulo 2. Again bonds are independently occupied with probability p and vacant with probability $1 - p$.

For percolation on a finite graph, such as $\mathbb{Q}_n$, the characterizations of the critical value used for $\mathbb{Z}^n$ are inapplicable. In [8, 9, 10] (in particular, see [10]), it was shown that there is a small positive constant λ_0 such that the critical value $p_c(\mathbb{Q}_n) = p_c(\mathbb{Q}_n; \lambda_0)$ for the n-cube can be defined implicitly by

$$\chi(p_c(\mathbb{Q}_n)) = \lambda_0 2^{n/3}. \tag{1.6}$$

Given λ_0, (1.6) uniquely specifies $p_c(\mathbb{Q}_n)$, since $\chi(p)$ is a polynomial in p that increases from $\chi(0) = 1$ to $\chi(1) = 2^n$. As discussed in [10], $p_c(\mathbb{Q}_n; \lambda_0)$ depends only weakly on the choice of λ_0. This point is also reflected in the Remark below. Our main result for $\mathbb{Q}_n$ is the following theorem.

Theorem 1.2. *Consider bond percolation on $\mathbb{Q}_n$. Let $M \geq 1$, fix constants c, c' (independent of n but possibly depending on M), and choose p such that $\chi(p) \in [cn^M, c'n^{-2M}2^n]$. Then there are rational numbers $a_i(\mathbb{Q})$, independent of p, c, c', such that for all $M \geq 1$,*

$$p = \sum_{i=1}^{M} a_i(\mathbb{Q}) n^{-i} + O(n^{-M-1}) \qquad \text{as } n \to \infty. \tag{1.7}$$

The constant in the error term depends on M, c, c', but does not depend otherwise on p.

Remark. Theorem 1.2 states that, for $\mathbb{Q}_n$, any p for which $\chi(p) \in [cn^M, c'n^{-2M}2^n]$ will have the *same* expansion up to an error $O(n^{-M-1})$, with the error independent of p. Note that the expansion (1.7) is valid simultaneously to all orders M if we choose p to lie eventually in all intervals $[cn^M, c'n^{-2M}2^n]$. Thus, (1.7) holds simultaneously for all M for any p for which $\chi(p)$ is in the interval $[f_n, f_n^{-1}2^n]$, where f_n is a sequence that grows faster than any power and more slowly than $e^{\alpha n}$ for all $\alpha > 0$ (e.g., $f_n = 2^{\sqrt{n}}$). In particular, (1.7) holds simultaneously for all M when $p = p_c(\mathbb{Q}_n; \lambda_0)$ for *any* constant $\lambda_0 > 0$, large or small, with the $a_i(\mathbb{Q})$ independent of λ_0.

The leading term for the critical value $p_c(\mathbb{Q}_n)$ was identified as n^{-1} in [3], and this was refined in [7]. It was subsequently proved (see [10, (1.10)]) that

$$1 - \lambda_0^{-1} 2^{-n/3} \leq n p_c(\mathbb{Q}_n) \leq 1 + O(n^{-1}). \tag{1.8}$$

This gives (1.7) for $M = 1$ for $p = p_c(\mathbb{Q}_n)$. In [19], the formula (1.7) for $M = 3$ was obtained, with $a_1(\mathbb{Q}) = 1$, $a_2(\mathbb{Q}) = 1$, $a_3(\mathbb{Q}) = \frac{7}{2}$, and it was predicted (but not proved) that $a_4(\mathbb{Q})$ and $a_4(\mathbb{Z})$ are different. Thus, $p_c(\mathbb{Q}_n) = n^{-1} + n^{-2} + \frac{7}{2}n^{-3} + O(n^{-4})$. We again expect that the full expansion $\sum_{i=1}^{\infty} a_i(\mathbb{Q}) n^{-i}$ is divergent for every n, but we have no proof for this.

In [8, 9, 10], it was conjectured that the phase transition for $\mathbb{Q}_n$ takes place within a scaling window of width $2^{-n/3}$ centred at $p_c(\mathbb{Q}_n)$, and it was proved that the scaling window has width at most $e^{-cn^{1/3}}$. Except in the unlikely circumstance that the expansion (1.7) eventually terminates as a polynomial in n^{-1}, any truncation of the expansion gives a result that lies outside this exponentially small scaling window for large enough n. The non-perturbative definition (1.6) of $p_c(\mathbb{Q}_n)$ therefore tracks the scaling window more accurately than any polynomial in n^{-1} can ever do.

On the other hand, the asymptotic expansion gives information on the phase transition on $\mathbb{Q}_n$, at every polynomial scale. To explain this, we first recall some results from [10]. Let $p = p_c(\mathbb{Q}_n) + \epsilon n^{-1}$, where ϵ may depend on n, and let $\mathcal{C}_{\max}$ denote a cluster of maximal size. By [10, Theorem 1.1], if $\epsilon < 0$ and $\lim_{n\to\infty} |\epsilon| 2^{n/3} = \infty$, then

$$\chi(p) = \frac{1}{|\epsilon|}[1 + o(1)], \tag{1.9}$$

$$\frac{c_1}{\epsilon^2} \leq |\mathcal{C}_{\max}| \leq \frac{2(\log 2)n}{\epsilon^2}[1 + o(1)] \quad \text{a.a.s.}, \tag{1.10}$$

where c_1 is a universal constant, and where we say that a sequence E_n of events holds a.a.s. (asymptotically almost surely) if $\lim_{n\to\infty} \mathbb{P}(E_n) = 1$. On the other hand, by [10, Theorems 1.4, 1.5], there are universal constants c, c_2, c_3, c_4 such that if $e^{-cn^{1/3}} \le \epsilon \le 1$, then

$$c_2\epsilon^2 2^n \le \chi(p) \le c_3\epsilon^2 2^n, \tag{1.11}$$

$$|\mathcal{C}_{\max}| \ge c_4\epsilon 2^n \qquad \text{a.a.s.} \tag{1.12}$$

For $M \ge 1$, let

$$p_c^{(M)}(\mathbb{Q}_n) = \sum_{i=1}^{M} a_i(\mathbb{Q})n^{-i} \tag{1.13}$$

denote an approximate critical value. Let $p = p_c^{(M)}(\mathbb{Q}_n) + \delta n^{-M}$, so that by (1.7) $p = p_c(\mathbb{Q}_n) + \epsilon n^{-1}$ with $\epsilon = \delta n^{1-M}[1 + O(\delta^{-1}n^{-1})]$. It follows from (1.9)–(1.10) that for fixed $\delta < 0$, as $n \to \infty$, we have

$$\chi(p) = |\delta|^{-1}n^{M-1}[1 + o(1)], \tag{1.14}$$

$$\frac{1}{2}c_1\delta^{-2}n^{2M-2} \le |\mathcal{C}_{\max}| \le 2(\log 2)\delta^{-2}n^{2M-1}[1 + o(1)] \qquad \text{a.a.s.} \tag{1.15}$$

In addition, it follows from (1.11)–(1.12) that for fixed $\delta > 0$,

$$\frac{1}{2}c_2\delta^2 n^{2(1-M)}2^n \le \chi(p) \le 2c_3\delta^2 n^{2(1-M)}2^n, \tag{1.16}$$

$$|\mathcal{C}_{\max}| \ge \frac{1}{2}c_4\delta n^{1-M}2^n \qquad \text{a.a.s.} \tag{1.17}$$

This shows that the largest cluster jumps from poly-logarithmic in the volume for $\delta < 0$, to proportional to the volume up to poly-logarithmic factors for $\delta > 0$. Thus there is a phase transition on scale n^{-M} at $p_c^{(M)}(\mathbb{Q}_n)$, for each $M \ge 1$. A similar transition takes place if we instead consider $p = p_c(\mathbb{Q}_n) + \delta n^{-M}$, where the critical point $p_c(\mathbb{Q}_n)$ is of course *independent* of M. Although $p_c(\mathbb{Q}_n)$ locates the phase transition simultaneously for perturbations on scale n^{-M} for all M, there is nevertheless a satisfying concreteness to the polynomial $p_c^{(M)}(\mathbb{Q}_n)$ in n^{-1}, compared to the implicitly defined $p_c(\mathbb{Q}_n)$ of (1.6). Moreover, $p_c^{(M)}(\mathbb{Q}_n)$ is independent of the parameter λ_0.

In the next proposition, we conclude from (1.14) and (1.16) that Theorem 1.2 follows if we can prove (1.7) for any one particular choice of p such that $\chi(p)$ lies in the interval $[cn^M, c'n^{-2M}2^n]$. We will make a convenient choice in (2.4) below.

Proposition 1.3. *For percolation on $\mathbb{Q}_n$, if p obeys* (1.7) *for some fixed sequence p (depending on n) such that $\chi(p) \in [cn^M, c'n^{-2M}2^n]$, then* (1.7) *is valid for every such p.*

Proof. Define p_1 and p_2 by $\chi(p_1) = cn^M$ and $\chi(p_2) = c'n^{-2M}2^n$. Then for large n, $p_1 \le p_2$. We will prove that $p_1 \ge p_c(\mathbb{Q}_n) - O(n^{-M-1})$ and $p_2 \le p_c(\mathbb{Q}_n) + O(n^{-M-1})$, where the constants in the error bounds depend only on c, c'. This suffices, since it implies that if $\chi(p), \chi(p') \in [cn^M, c'n^{-2M}2^n]$, then $|p' - p| = O(n^{-M-1})$.

To prove the bound on p_1, we let $\epsilon_1 = 2c^{-1}n^{-M}$. By (1.9), for n sufficiently large,

$$\chi(p_1) = \frac{2}{\epsilon_1} \geq \chi(p_c(\mathbb{Q}_n) - \epsilon_1 n^{-1}), \tag{1.18}$$

so by the monotonicity of χ,

$$p_1 \geq p_c(\mathbb{Q}_n) - \frac{\epsilon_1}{n} = p_c(\mathbb{Q}_n) - \frac{2}{cn^{M+1}}. \tag{1.19}$$

For the bound on p_2, we let $\epsilon_2 = \sqrt{c'/c_2}n^{-M}$. By (1.11),

$$\chi(p_2) = c_2\epsilon_2^2 2^n \leq \chi(p_c(\mathbb{Q}_n) + \epsilon_2 n^{-1}), \tag{1.20}$$

so by the monotonicity of χ,

$$p_2 \leq p_c(\mathbb{Q}_n) + \frac{\epsilon_2}{n} = p_c(\mathbb{Q}_n) + \sqrt{\frac{c'}{c_2}}\frac{1}{n^{M+1}}. \tag{1.21}$$

□

2 Application of the lace expansion

We prove Theorems 1.1–1.2 simultaneously, using the lace expansion. Our method relies on convergence of the lace expansion but is otherwise largely insensitive to the fine details of how the model is defined. In particular, we expect that the proof could be extended from the n-cube $\{0,1\}^n$ to graphs with vertex set $\{0,1,\ldots,r\}^n$ with $r > 1$ fixed (see [9]), though we do not pursue this further here.

We write $\mathbb{G}$ in place of $\mathbb{Q}_n$ and $\mathbb{Z}^n$ when we wish to refer to both models simultaneously. We write Ω for the degree of $\mathbb{G}$, so that $\Omega = 2n$ for $\mathbb{Z}^n$ and $\Omega = n$ for $\mathbb{Q}_n$.

For $\mathbb{Q}_n$ or $\mathbb{Z}^n$ with n large, the lace expansion [16] gives rise to an identity

$$\chi(p) = \frac{1 + \hat{\Pi}_p}{1 - \Omega p[1 + \hat{\Pi}_p]}, \tag{2.1}$$

valid for $p \leq p_c(\mathbb{G})$. The function $\hat{\Pi}_p$ is finite for this range of p. Although we do not display the dependence explicitly in the notation, $\hat{\Pi}_p$ does depend on the graph $\mathbb{Q}_n$ or $\mathbb{Z}^n$. For a derivation of the lace expansion, see, e.g., [9, Section 3]. It follows from (2.1) that

$$\Omega p = \frac{1}{1 + \hat{\Pi}_p} - \chi(p)^{-1}. \tag{2.2}$$

For $\mathbb{Z}^n$, (2.1) follows from results in [16, Section 4.3.2]. (Note the notational difference that in [16] what we are calling here $\hat{\Pi}_p$ is equal to $\sum_{n=0}^{\infty}(-1)^n \hat{g}_n(0)$.) Since $\chi(p_c(\mathbb{Z}^n)) = \infty$, (2.2) gives

$$2np_c(\mathbb{Z}^n) = \frac{1}{1 + \hat{\Pi}_{p_c(\mathbb{Z}^n)}}. \tag{2.3}$$

Bounds of [16] imply that $|\hat{\Pi}_{p_c(\mathbb{Z}^n)}| = O(n^{-1})$. This gives $p_c(\mathbb{Z}^n) = (2n)^{-1} + O(n^{-2})$, which is (1.3) for $M = 1$.

For $\mathbb{Q}_n$, the identity (2.1) is established in [9, (6.1)] (in the notation of [9], $\chi(p) = \hat{\tau}_p(0)$). We fix a sequence f_n as in the remark below Theorem 1.2. That is, we require that $\lim_{n\to\infty} f_n n^{-M} = \infty$ for every positive integer M and that $\lim_{n\to\infty} f_n e^{-\alpha n} = 0$ for every $\alpha > 0$. In view of Proposition 1.3, it suffices to prove (1.7) for $p = \bar{p}$, where $\bar{p}$ is defined by

$$\chi(\bar{p}) = f_n. \tag{2.4}$$

For $p = \bar{p}$, (2.2) gives

$$n\bar{p} = \frac{1}{1 + \hat{\Pi}_{\bar{p}}} + O(f_n^{-1}). \tag{2.5}$$

The second term on the right hand side is an error term for (1.7), and can be neglected in the proof of Theorem 1.2. It follows from results in [9] that $|\hat{\Pi}_{\bar{p}}| \leq O(n^{-1})$. In more detail, it follows from [9, Proposition 5.2] that $|\hat{\Pi}_p| \leq \text{const}(\lambda^3 \vee \beta)$, where $\lambda = \chi(p)2^{-n/3} \leq f_n 2^{-n/3}$ for $p \leq \bar{p}_c(\mathbb{Q}_n)$, and β is proportional to n^{-1} by [9, Proposition 2.1]. With (2.5), this implies that $\bar{p} = n^{-1} + O(n^{-2})$, which is (1.7) for $M = 1$.

Henceforth, we will write

$$\bar{p}_c = \bar{p}_c(\mathbb{G}) = \begin{cases} \bar{p} & (\mathbb{G} = \mathbb{Q}_n) \\ p_c(\mathbb{Z}^n) & (\mathbb{G} = \mathbb{Z}^n). \end{cases} \tag{2.6}$$

The identities (2.3) and (2.5) give recursive equations for $\bar{p}_c$, in which an input for $\bar{p}_c$ on the right hand side gives rise to an improved value of $\bar{p}_c$ on the left hand side. To prove Theorems 1.1–1.2 using this recursion, we will apply the following proposition.

Proposition 2.1. *Fix $M \geq 1$. For $\mathbb{G} = \mathbb{Q}_n$ and $\mathbb{G} = \mathbb{Z}^n$, there are rational numbers $\alpha_{j,i,M} = \alpha_{j,i,M}(\mathbb{G})$ and a positive integer L_M, all independent of n, such that for $p \leq \bar{p}_c(\mathbb{G})$,*

$$\hat{\Pi}_p = \sum_{i=1}^{L_M} \sum_{j=0}^{i-1} \alpha_{j,i,M} \Omega^j p^i + O(\Omega^{-M-1}). \tag{2.7}$$

The constant in the error term depends on M.

Proof of Theorems 1.1–1.2 assuming Proposition 2.1. We restrict attention to $p = \bar{p}_c$, which is sufficient by Proposition 1.3. The proof is by induction on M. As discussed above, we know that (1.3) and (1.7) hold for $M = 1$. We assume that (1.3) and (1.7) hold for $1, \ldots, M$, and prove the corresponding result for $M + 1$.

By (2.3) and (2.5),

$$\bar{p}_c = \frac{1}{\Omega} \frac{1}{1 + \hat{\Pi}_{\bar{p}_c}} + O(\Omega^{-M-2}) \tag{2.8}$$

(in fact the error term is zero for $\mathbb{Z}^n$ and is $O(n^{-1}f_n^{-1})$ for $\mathbb{Q}_n$). By the induction hypothesis, there are rational $\beta_{k,i} = \beta_{k,i}(\mathbb{G})$ such that for $i = 1, \ldots, L_M$,

$$\bar{p}_c^i = \left[\sum_{j=1}^{M} a_j \Omega^{-j} + O(\Omega^{-M-1}) \right]^i = \Omega^{-i} \left[\sum_{k=0}^{M-1} \beta_{k,i} \Omega^{-k} + O(\Omega^{-M}) \right]. \tag{2.9}$$

By Proposition 2.1, this implies that

$$\hat{\Pi}_{\bar{p}_c} = \sum_{i=1}^{L_M} \sum_{j=0}^{i-1} \alpha_{j,i,M} \Omega^j \Omega^{-i} \left[\sum_{k=0}^{M-1} \beta_{k,i} \Omega^{-k} + O(\Omega^{-M}) \right] + O(\Omega^{-M-1})$$
$$= \sum_{l=1}^{M} \gamma_{l,M} \Omega^{-l} + O(\Omega^{-M-1}), \tag{2.10}$$

for some rational coefficients $\gamma_{l,M} = \gamma_{l,M}(\mathbb{G})$. Substitution of (2.10) into (2.8) shows that there are rational coefficients $a_{j,M}$ such that

$$\bar{p}_c = \sum_{j=1}^{M+1} a_{j,M} \Omega^{-j} + O(\Omega^{-M-2}). \tag{2.11}$$

The consistency of (2.11) with the induction hypothesis implies that for $j = 1, \ldots, M$ the coefficients $a_{j,M}$ in fact depend only on j. This completes the proof. □

Note that the precise value of L_M is not needed in the above proof. It remains to prove Proposition 2.1. For this, we will use the description of $\hat{\Pi}_p$ given in the next section.

3 The function $\hat{\Pi}_p$

The function $\hat{\Pi}_p$ has the form

$$\hat{\Pi}_p = \sum_{N=0}^{\infty} (-1)^N \hat{\Pi}_p^{(N)}. \tag{3.1}$$

To define $\hat{\Pi}_p^{(N)}$ (see [9] or [16]), we need the following definition.

Definition 3.1. (i) Given a bond configuration, we say that x is *connected to* y, and write $x \leftrightarrow y$, if $x = y$ or if there is a path from x to y consisting of occupied bonds. Given a set A of vertices of $\mathbb{G}$, we write $x \leftrightarrow A$ for the event that $x \leftrightarrow y$ for some $y \in A$. Also, we say that x is *doubly connected to* y, and write $x \Leftrightarrow y$, if $x = y$ or if there are at least two bond-disjoint paths from x to y consisting of occupied bonds.
(ii) Given a bond configuration, vertices x, y, and a set A of vertices of $\mathbb{G}$, we say x and y are *connected through* A, and write $x \overset{A}{\leftrightarrow} y$, if every occupied path connecting x to y has at least one bond with an endpoint in A.
(iii) Given a bond configuration, and a bond b, we define $\tilde{C}^b(x)$ to be the set of vertices connected to x in the new configuration obtained by setting b to be vacant.
(iv) Given a bond configuration and vertices x, y, we say that the directed bond (u, v) is *pivotal* for $x \leftrightarrow y$ if (a) $x \leftrightarrow y$ occurs when the bond $\{u, v\}$ is set occupied, and (b) $x \leftrightarrow y$ does not occur when $\{u, v\}$ is set vacant, but $x \leftrightarrow u$ and $v \leftrightarrow y$ do occur. (Note that there is a distinction between the events $\{(u, v)$ is pivotal for $x \leftrightarrow y\}$ and $\{(u, v)$ is pivotal for $y \leftrightarrow x\} = \{(v, u)$ is pivotal for $x \leftrightarrow y\}$.)
(v) Given vertices v, x and a set of vertices A, let

$$E'(v, x; A) = \{v \overset{A}{\leftrightarrow} x\} \cap \{\nexists \text{ occupied pivotal } (u', v') \text{ for } v \leftrightarrow x \text{ s.t. } v \overset{A}{\leftrightarrow} u'\}. \tag{3.2}$$

We refer to the second event on the right hand side of (3.2) as the "NP" (no pivotal) condition.

Henceforth, we drop the subscript p in $\mathbb{P}_p$, $\mathbb{E}_p$ and $\hat{\Pi}_p$, and write, e.g., $\hat{\Pi}$ in place of $\hat{\Pi}_p$. By definition,

$$\hat{\Pi}^{(0)} = \sum_{x \neq 0} \mathbb{P}(0 \Leftrightarrow x), \tag{3.3}$$

where $\mathbb{P}$ denotes probability for bond percolation with bond density p, and

$$\hat{\Pi}^{(1)} = \sum_{x} \sum_{(u_0,v_0)} p\mathbb{E}_0 \left[I[0 \Leftrightarrow u_0] \mathbb{E}_1 I[E'(v_0, x; \tilde{C}_0^{(u_0,v_0)}(0))] \right]. \tag{3.4}$$

On the right hand side of (3.4), the cluster $\tilde{C}_0^{(u_0,v_0)}(0)$ is *random* with respect to the expectation $\mathbb{E}_0$, but $\tilde{C}_0^{(u_0,v_0)}(0)$ should be regarded as a *fixed* set inside the probability $\mathbb{P}_1$. The latter introduces a second percolation model which is independent of the original percolation model. However, the law of the indicator $I[E'(v_0, x; \tilde{C}_0^{(u_0,v_0)}(0))]$ is given by the joint probabilities $\mathbb{P}_0$ and $\mathbb{P}_1$.

In general, for $N \geq 1$,

$$\begin{aligned} \hat{\Pi}^{(N)} = \sum_{x} \sum_{(u_0,v_0)} \cdots \sum_{(u_{N-1},v_{N-1})} & p^N \mathbb{E}_0 I[0 \Leftrightarrow u_0] \\ & \times \mathbb{E}_1 I[E'(v_0, u_1; \tilde{C}_0)] \cdots \mathbb{E}_{N-1} I[E'(v_{N-2}, u_{N-1}; \tilde{C}_{N-2})] \mathbb{E}_N I[E'(v_{N-1}, x; \tilde{C}_{N-1})], \end{aligned} \tag{3.5}$$

where we have used the abbreviation $\tilde{C}_j = \tilde{C}_j^{(u_j,v_j)}(v_{j-1})$ (with $v_{-1} = 0$), and where each sum over (u, v) is a sum over all directed bonds. The expectations in (3.5) are mutually related through the $\tilde{C}$ clusters. We use subscripts for $\tilde{C}$ and the expectations, to indicate to which expectation $\tilde{C}$ belongs, and refer to the bond configuration corresponding to expectation j as the "level-j" configuration. We also write F_j to indicate an event F at level-j.

We will rewrite (3.5) as follows. We set $u_N = x$ and $v_{-1} = 0$, and write

$$E_0 = \{0 \Leftrightarrow u_0\}_0, \tag{3.6}$$

$$E_j = E'(v_{j-1}, u_j, \tilde{C}_{j-1})_j \qquad (j = 1, \ldots, N), \tag{3.7}$$

and

$$E^{(N)} = E_0 \cap E_1 \cap \cdots \cap E_N. \tag{3.8}$$

We write $\mathbb{E}^{(N)}$ for the joint expectation $\mathbb{E}_0 \mathbb{E}_1 \cdots \mathbb{E}_N$. We write the sum over x and all (u_j, v_j) $(j = 0, \ldots, N-1)$ as $\sum_{x,(u_j,v_j)}$. With this notation, and applying Fubini's theorem, (3.5) takes the more compact form

$$\hat{\Pi}^{(N)} = \sum_{x,(u_j,v_j)} p^N \mathbb{E}^{(N)} \left[I[E^{(N)}] \right]. \tag{3.9}$$

It is known that there is a constant C independent of N and n such that for all $N \geq 0$,

$$0 \leq \hat{\Pi}^{(N)} \leq \left(\frac{C}{\Omega} \right)^{N \vee 1} \quad \text{uniformly in } p \leq \bar{p}_c(\mathbb{G}). \tag{3.10}$$

For $\mathbb{Q}_n$, (3.10) is established in [9, Lemma 5.4] (where $\hat{\Pi}^{(N)}$ is written as $\sum_x \Pi^{(N)}(x)$). In more detail, [9, Lemma 5.4] states that $\hat{\Pi}^{(N)} \leq [\text{const}(\lambda^3 \vee \beta)]^{N \vee 1}$, where $\lambda = \chi(p) 2^{-n/3} \leq f_n 2^{-n/3}$ for $p \leq \bar{p}_c(\mathbb{Q}_n)$. In addition, it is shown in [9, Proposition 2.1] that β can be chosen proportional to n^{-1}. This gives (3.10) for $\mathbb{Q}_n$. For $\mathbb{Z}^n$, (3.10) follows from results in [16, Section 4.3.2] (we emphasize again that there are notational differences in [16]).

It follows immediately from (3.10) that to prove (2.7) it is sufficient to prove that there are rational numbers $\alpha_{j,i,M}$ such that

$$\sum_{N=0}^{M}(-1)^N\hat{\Pi}^{(N)} = \sum_{i=1}^{L_M}\sum_{j=0}^{i-1}\alpha_{j,i,M}\Omega^j p^i + O(\Omega^{-M-1}). \tag{3.11}$$

Thus it is sufficient to show that for each $N \leq M$ there are rational numbers $\alpha_{j,i,M}^{(N)}$ such that

$$\hat{\Pi}^{(N)} = \sum_{i=1}^{L_M}\sum_{j=0}^{i-1}\alpha_{j,i,M}^{(N)}\Omega^j p^i + O(\Omega^{-M-1}). \tag{3.12}$$

Suppose that we could show instead that

$$\hat{\Pi}^{(N)} = \sum_{i=0}^{L_M}\sum_{j=0}^{L_M}\alpha_{j,i,M}^{(N)}\Omega^j p^i + O(\Omega^{-M-1}). \tag{3.13}$$

The sum on the right hand side can be rewritten as

$$\sum_{k=-L_M}^{L_M}\Omega^k\left(\sum_{i=-k\vee 0}^{(L_M-k)\wedge L_M}\alpha_{i+k,i,M}^{(N)}(\Omega p)^i\right), \tag{3.14}$$

so the coefficient of Ω^k in $\hat{\Pi}^{(N)}$ of (3.13) is

$$\sum_{i=-k\vee 0}^{(L_M-k)\wedge L_M}\alpha_{i+k,i,M}^{(N)}(\Omega p)^i. \tag{3.15}$$

Suppose that $k > -(N \vee 1)$. Then for every p in the interval $(\frac{1}{2}\bar{p}_c, \bar{p}_c)$ (say), Ωp is bounded away from zero, and hence the above coefficient must be zero since otherwise (3.10) would be violated for large Ω. This then implies, in particular, that $\alpha_{i+k,i,M}^{(N)} = 0$ for $k \geq 0$. Thus, (3.13) implies (3.12). The remainder of the paper is devoted to the proof of (3.13), for fixed $N \leq M$.

In estimating error terms, it is convenient to work with an upper bound for the event $E^{(N)}$ defined in (3.8). Let $E \circ F$ denote disjoint occurrence of the events E and F (see [15, Theorem 2.12] for the very important BK inequality which applies for events occurring disjointly). We define

$$F_0(0,u_0,w_0,z_1) = \{0 \leftrightarrow u_0\} \circ \{0 \leftrightarrow w_0\} \circ \{w_0 \leftrightarrow u_0\} \circ \{w_0 \leftrightarrow z_1\}, \tag{3.16}$$

$$F'(v_{i-1},t_i,z_i,u_i,w_i,z_{i+1}) = \{v_{i-1} \leftrightarrow t_i\} \circ \{t_i \leftrightarrow z_i\} \circ \{t_i \leftrightarrow w_i\} \circ \{z_i \leftrightarrow u_i\} \circ \{w_i \leftrightarrow u_i\} \circ \{w_i \leftrightarrow z_{i+1}\}, \tag{3.17}$$

$$F''(v_{i-1},t_i,z_i,u_i,w_i,z_{i+1}) = \{v_{i-1} \leftrightarrow w_i\} \circ \{w_i \leftrightarrow t_i\} \circ \{t_i \leftrightarrow z_i\} \circ \{t_i \leftrightarrow u_i\} \circ \{z_i \leftrightarrow u_i\} \circ \{w_i \leftrightarrow z_{i+1}\}, \tag{3.18}$$

$$F(v_{i-1},t_i,z_i,u_i,w_i,z_{i+1}) = F'(v_{i-1},t_i,z_i,u_i,w_i,z_{i+1}) \cup F''(v_{i-1},t_i,z_i,u_i,w_i,z_{i+1}), \tag{3.19}$$

$$F_N(v_{N-1},t_N,z_N,x) = \{v_{N-1} \leftrightarrow t_N\} \circ \{t_N \leftrightarrow z_N\} \circ \{t_N \leftrightarrow x\} \circ \{z_N \leftrightarrow x\}. \tag{3.20}$$

We also define $F^{(0)} = \{0 \Leftrightarrow u_0\}$, and, for $N \geq 1$, let

$$F^{(N)} = \bigcup_{\vec{t},\vec{w},\vec{z}} \Big(F_0(0, u_0, w_0, z_1)_0 \cap \Big(\bigcap_{i=1}^{N-1} F(v_{i-1}, t_i, z_i, u_i, w_i, z_{i+1})_i \Big) \cap F_N(v_{N-1}, t_N, z_N, x)_N \Big), \quad (3.21)$$

where $\vec{t} = (t_1, \ldots, t_N)$, $\vec{w} = (w_0, \ldots, w_{N-1})$ and $\vec{z} = (z_1, \ldots, z_N)$, and where the subscripts on the F events indicate the level to which the event belongs. It is then a standard estimate (see [9, Section 4.1] for a discussion using our present notation) that for $N \geq 0$,

$$E^{(N)} \subset F^{(N)}. \quad (3.22)$$

4 The polynomial

In this section, we prove (3.13) (and hence Proposition 2.1) apart from the estimation of several error terms that are identified below. These error terms are bounded in Section 5, to complete the proof of (3.13). We begin with some definitions.

4.1 Basic concepts

For $N \geq 0$, let $\mathbb{G}^{(N)} = (\mathbb{G}(0), \ldots, \mathbb{G}(N))$ denote $(N+1)$ copies of $\mathbb{G}$. This graph contains the edge set of $\mathbb{G}$ within each component $\mathbb{G}(i)$, and no edge links any vertex of $\mathbb{G}(i)$ with any vertex of $\mathbb{G}(j)$ when $i \neq j$. We consider percolation on $\mathbb{G}^{(N)}$, which is the natural setting for the event $E^{(N)}$ of (3.8). We denote the set of vertices of $\mathbb{G}^{(N)}$ by $(\mathbb{V}(0), \ldots, \mathbb{V}(N))$, where $\mathbb{V}(j)$ is the vertex set of $\mathbb{G}(j)$. In the event $E_j = E'(v_{j-1}, u_j; \tilde{C}_{j-1})_j$, the random set $\tilde{C}_{j-1} \subset \mathbb{V}(j-1)$ is identified with the corresponding subset of $\mathbb{V}(j)$, in (3.2).

Definition 4.1. Let $N \geq 0$. Given $A = (A(0), \ldots, A(N))$ with each $A(j) \subset \mathbb{V}(j)$, and an event E on $\mathbb{G}^{(N)}$, the event $\{E \textit{ on } A\}$ is the set of configurations for which E occurs in the possibly modified configuration in which each bond with one or more endpoints not in A is set vacant.

We write $A^c = (A(0)^c, \ldots, A(N)^c)$, and, for $B = (B(0), \ldots, B(N))$, we define $A \cap B = (A(0) \cap B(0), \ldots, A(N) \cap B(N))$. It is a direct consequence of Definition 4.1 that

$$\{E^c \text{ on } A\} = \{E \text{ on } A\}^c, \quad (4.1)$$

$$\{E \cap F \text{ on } A\} = \{E \text{ on } A\} \cap \{F \text{ on } A\}, \quad (4.2)$$

$$\{\{E \text{ on } A\} \text{ on } B\} = \{E \text{ on } A \cap B\}, \quad (4.3)$$

so the notion "on A" is well behaved with respect to the operations of set theory.

We will use the fact that for A as in Definition 4.1,

$$\{E^{(N)} \text{ on } A\} \subset \{F^{(N)} \text{ on } A\} \subset F^{(N)}, \quad (4.4)$$

where the first inclusion is a consequence of the proof of (3.22), and the second inclusion holds since $F^{(N)}$ is increasing.

Definition 4.2. Fix a bond configuration on $\mathbb{G}^{(N)}$, fix the summation variables x, (u_j, v_j) $(j = 0, \ldots, N-1)$ of (3.9), and let $u_N = x$, $v_{-1} = 0$.
(i) The *backbone* at level-j is the random set of vertices defined by

$$B_j = \{y \in \mathbb{V}(j) : (v_{j-1} \leftrightarrow y)_j \circ (y \leftrightarrow u_j)_j\} \qquad (j = 0, \ldots, N). \tag{4.5}$$

The *extended backbone* B_j^+ at level-j is

$$B_j^+ = \begin{cases} B_j \cup \{y \in \tilde{C}_j : (v_{j-1} \leftrightarrow y)_j \circ (y \leftrightarrow B_{j+1})_j\} & (j = 0, \ldots, N-1) \\ B_N & (j = N), \end{cases} \tag{4.6}$$

where B_{j+1} is identified with the corresponding subset of $V(j)$.
(ii) The *dimension* $\vec{D} = (D_1, \ldots, D_n) \in \{0,1\}^n$ is defined by setting $D_i = 1$ if there is a $j \in \{0, \ldots, N\}$ and a $y \in B_j^+$ with i^{th} coordinate $y_i \neq 0$, and otherwise $D_i = 0$. Let $\mathcal{P}_j$ denote the collection of paths consisting of level-j paths from v_{j-1} to u_j, and level-j paths in $\tilde{C}_j$ from v_{j-1} to B_{j+1}. Then $\vec{D}$ indicates all directions in $\mathbb{G}$ that are explored by occupied paths in $\mathcal{P}_j$, for all j.

Given $R \geq 1$, let $\mathbb{B}_R = \{x \in \mathbb{V} : \|x\|_\infty \leq R\}$ (note that for $\mathbb{Q}_n$, $\mathbb{B}_R = \mathbb{Q}_n$ for all R). Let $\mathbb{B}_R^{(N)} = (\mathbb{B}_R(0), \ldots, \mathbb{B}_R(N))$ denote $(N+1)$ copies of $\mathbb{B}_R$. Given $\vec{d} \in \{0,1\}^n$, let

$$\mathbb{V}_{\vec{d}} = \{x \in \mathbb{V} : x_i = 0 \text{ if } d_i = 0\}, \tag{4.7}$$

$$\mathbb{V}_{\vec{d}}^{(N)} = (\mathbb{V}_{\vec{d}}(0), \ldots, \mathbb{V}_{\vec{d}}(N)) = (N+1) \text{ copies of } \mathbb{V}_{\vec{d}}, \tag{4.8}$$

$$\mathbb{V}_{\vec{d},R}^{(N)} = \mathbb{V}_{\vec{d}}^{(N)} \cap \mathbb{B}_R^{(N)}. \tag{4.9}$$

Taking $\vec{d}$ to be the random vector $\vec{D}$, we extend Definition 4.1 to the case where A is the *random* set $\mathbb{V}_{\vec{D},R}^{(N)}$ by the disjoint union

$$\{E \text{ on } \mathbb{V}_{\vec{D},R}^{(N)}\} = \bigcup_{\vec{d} \in \{0,1\}^n} \Big(\{E \text{ on } \mathbb{V}_{\vec{d},R}^{(N)}\} \cap \{\vec{D} = \vec{d}\}\Big). \tag{4.10}$$

We will use the following lemma.

Lemma 4.3. *For $N \geq 0$ and $R > 0$,*

$$\{E^{(N)} \textit{ on } \mathbb{B}_R^{(N)}\} = \{E^{(N)} \textit{ on } \mathbb{V}_{\vec{D},R}^{(N)}\}. \tag{4.11}$$

Proof. The basic step in the proof is to observe that the event $E^{(N)}$ is determined by occupied paths in the extended backbones B_j^+ $(j = 0, 1, \ldots, N)$, and by the definition of $\vec{D}$, these paths lie in $\mathbb{V}_{\vec{D}}^{(N)}$. From this, we see that

$$E^{(N)} \cap \{\vec{D} = \vec{d}\} = \{E^{(N)} \text{ on } \mathbb{V}_{\vec{d}}^{(N)}\} \cap \{\vec{D} = \vec{d}\}. \tag{4.12}$$

Therefore,

$$\begin{aligned} \{E^{(N)} \text{ on } \mathbb{B}_R^{(N)}\} &= \bigcup_{\vec{d} \in \{0,1\}^n} \{E^{(N)} \text{ on } \mathbb{B}_R^{(N)}\} \cap \{\vec{D} = \vec{d}\} \\ &= \bigcup_{\vec{d} \in \{0,1\}^n} \{\{E^{(N)} \text{ on } \mathbb{V}_{\vec{d}}^{(N)}\} \text{ on } \mathbb{B}_R^{(N)}\} \cap \{\vec{D} = \vec{d}\} \\ &= \bigcup_{\vec{d} \in \{0,1\}^n} \{E^{(N)} \text{ on } \mathbb{V}_{\vec{d},R}^{(N)}\} \cap \{\vec{D} = \vec{d}\} \\ &= \{E^{(N)} \text{ on } \mathbb{V}_{\vec{D},R}^{(N)}\}, \end{aligned} \tag{4.13}$$

where we used (4.12) for the second equality, (4.3) for the third, and (4.10) for the fourth. □

4.2 The iteration

Recall the definition of $\hat{\Pi}^{(N)}$ in (3.9). For $R_0 > 0$, we write

$$\hat{\Pi}^{(N)} = \hat{\Pi}^{(N)}_{R_0} + \mathcal{E}^{(N)}_{1,R_0}, \tag{4.14}$$

where

$$\hat{\Pi}^{(N)}_{R_0} = \sum_{x,(u_j,v_j)} p^N \mathbb{E}^{(N)} \left[I[E^{(N)} \text{ on } \mathbb{B}^{(N)}_{R_0}] \right], \tag{4.15}$$

$$\mathcal{E}^{(N)}_{1,R_0} = \sum_{x,(u_j,v_j)} p^N \mathbb{E}^{(N)} \left[I[E^{(N)}] - I[E^{(N)} \text{ on } \mathbb{B}^{(N)}_{R_0}] \right]. \tag{4.16}$$

In Lemma 4.5 below, we will choose $R_0 = R_0(M)$ in such a manner that

$$\mathcal{E}^{(N)}_{1,R_0} = O(\Omega^{-M-1}), \tag{4.17}$$

so this term is an error term. The bound (4.17) is a reflection of the fact that configurations that exit $\mathbb{B}^{(N)}_{R_0}$, with R_0 large depending on M, must contain a long extended backbone path, and this gives rise to an error term. This notion will be formalized in Proposition 5.1 below.

By Lemma 4.3,

$$\hat{\Pi}^{(N)}_{R_0} = \sum_{x,(u_j,v_j)} p^N \mathbb{E}^{(N)} \left[I[E^{(N)} \text{ on } \mathbb{V}^{(N)}_{\vec{D},R_0}] \right]. \tag{4.18}$$

Let $\|\vec{d}\| = \sum_{i=1}^n |d_i|$ denote the ℓ^1 norm. Given $R_0, R_1 > 0$, we write

$$\hat{\Pi}^{(N)}_{R_0} = \hat{\Pi}^{(N)}_{R_0,R_1} + \mathcal{E}^{(N)}_{2,R_0,R_1}, \tag{4.19}$$

where

$$\hat{\Pi}^{(N)}_{R_0,R_1} = \sum_{x,(u_j,v_j)} p^N \mathbb{E}^{(N)} \left[I[E^{(N)} \text{ on } \mathbb{V}^{(N)}_{\vec{D},R_0}] I[\|\vec{D}\| \le R_1] \right], \tag{4.20}$$

$$\mathcal{E}^{(N)}_{2,R_0,R_1} = \sum_{x,(u_j,v_j)} p^N \mathbb{E}^{(N)} \left[I[E^{(N)} \text{ on } \mathbb{V}^{(N)}_{\vec{D},R_0}] I[\|\vec{D}\| > R_1] \right]. \tag{4.21}$$

We show below in Corollary 4.9 that for $R_0 = R_0(M)$ chosen as above, and for suitably chosen $R_1 = R_1(M)$,

$$\mathcal{E}^{(N)}_{2,R_0,R_1} = O(\Omega^{-M-1}), \tag{4.22}$$

so this term is an error term. The bound (4.22) will follow from the fact that large $\|\vec{D}\|$ implies the existence either of a long extended backbone path, or of many distinct extended backbone paths.

By definition,

$$\hat{\Pi}^{(N)} = \hat{\Pi}^{(N)}_{R_0,R_1} + \mathcal{E}^{(N)}_{1,R_0} + \mathcal{E}^{(N)}_{2,R_0,R_1}, \tag{4.23}$$

with

$$\hat{\Pi}^{(N)}_{R_0,R_1} = \sum_{\vec{d}_1 : \|\vec{d}_1\| \le R_1} \sum_{x,(u_j,v_j)} p^N \mathbb{E}^{(N)} \left[I[E^{(N)} \text{ on } \mathbb{V}^{(N)}_{\vec{d}_1,R_0}] I[\vec{D} = \vec{d}_1] \right]. \tag{4.24}$$

Our goal is to rewrite the expectation in (4.24), up to error terms, as an expectation of an event that occurs on some $\mathbb{V}_{\vec{d},R}$ with $\|\vec{d}\|$ and R bounded depending only on M. The factor $I[\vec{D} = \vec{d}_1]$ is not yet of this form, and needs to be rewritten.

We write $\vec{d} > \vec{d}'$ if $d_i \geq d'_i$ for all $i = 1, \ldots, n$ and $d_j > d'_j$ for some j. Given $\bar{Q}$ (large), $\vec{R} = (R_0, \ldots, R_{\bar{Q}+1})$ with $R_0 < \cdots < R_{\bar{Q}+1}$, $q \in \{1, \ldots, \bar{Q}\}$, and $\|\vec{d}_q\| \leq R_q$, we proceed as follows. First, we make the decomposition

$$I[\vec{D} = \vec{d}_q] = I[\{\vec{D} = \vec{d}_q\} \text{ on } \mathbb{V}^{(N)}_{\vec{d}_q, R_{q-1}}] I[\vec{D} = \vec{d}_q] + I[\{\vec{D} < \vec{d}_q\} \text{ on } \mathbb{V}^{(N)}_{\vec{d}_q, R_{q-1}}] I[\vec{D} = \vec{d}_q], \tag{4.25}$$

where we have used the fact that it is impossible for $\vec{D} > \vec{d}_q$ to hold on $\mathbb{V}^{(N)}_{\vec{d}_q, R_{q-1}}$. In the first term on the right hand side of (4.25), we make the replacement

$$I[\vec{D} = \vec{d}_q] = I[\vec{D} \geq \vec{d}_q] - I[\vec{D} > \vec{d}_q]. \tag{4.26}$$

Since $\{\{\vec{D} = \vec{d}_q\} \text{ on } \mathbb{V}^{(N)}_{\vec{d}_q, R_{q-1}}\} \subset \{\vec{D} \geq \vec{d}_q\}$, we obtain

$$\begin{aligned} I[\vec{D} = \vec{d}_q] = {} & I[\{\vec{D} = \vec{d}_q\} \text{ on } \mathbb{V}^{(N)}_{\vec{d}_q, R_{q-1}}] \\ & + I[\{\vec{D} < \vec{d}_q\} \text{ on } \mathbb{V}^{(N)}_{\vec{d}_q, R_{q-1}}] I[\vec{D} = \vec{d}_q] \\ & - I[\{\vec{D} = \vec{d}_q\} \text{ on } \mathbb{V}^{(N)}_{\vec{d}_q, R_{q-1}}] I[\vec{D} > \vec{d}_q]. \end{aligned} \tag{4.27}$$

In the last term on the right hand side, we insert the factor

$$1 = I[\|\vec{D}\| > R_{q+1}] + I[\|\vec{D}\| \leq R_{q+1}]. \tag{4.28}$$

This gives

$$\begin{aligned} I[\vec{D} = \vec{d}_q] = {} & I[\{\vec{D} = \vec{d}_q\} \text{ on } \mathbb{V}^{(N)}_{\vec{d}_q, R_{q-1}}] \\ & + I[\{\vec{D} < \vec{d}_q\} \text{ on } \mathbb{V}^{(N)}_{\vec{d}_q, R_{q-1}}] I[\vec{D} = \vec{d}_q] \\ & - I[\{\vec{D} = \vec{d}_q\} \text{ on } \mathbb{V}^{(N)}_{\vec{d}_q, R_{q-1}}] I[\|\vec{D}\| > R_{q+1}] \\ & - \sum_{\substack{\vec{d}_{q+1} : \vec{d}_{q+1} > \vec{d}_q \\ \|\vec{d}_{q+1}\| \leq R_{q+1}}} I[\{\vec{D} = \vec{d}_q\} \text{ on } \mathbb{V}^{(N)}_{\vec{d}_q, R_{q-1}}] I[\vec{D} = \vec{d}_{q+1}], \end{aligned} \tag{4.29}$$

since in the third term $\vec{D} > \vec{d}_q$ follows from the facts that $\|\vec{D}\| > R_{q+1} > R_q \geq \|\vec{d}_q\|$ and $\vec{D} \geq \vec{d}_q$.

We define $\mathcal{S}_Q(\vec{R})$ to be the set of $(\vec{d}_1, \ldots, \vec{d}_Q)$ such that $\vec{d}_1 < \cdots < \vec{d}_Q$ and $\|\vec{d}_i\| \leq R_i$ for each

i, and we make the abbreviation $\vec{\Delta}_Q = (\vec{d}_1, \ldots, \vec{d}_Q)$. We also use the abbreviations

$$T = \prod_{q=1}^{Q} I[\{\vec{D} = \vec{d}_q\} \text{ on } \mathbb{V}^{(N)}_{\vec{d}_q, R_{q-1}}], \tag{4.30}$$

$$T_3 = \left(\prod_{q=1}^{Q-1} I[\{\vec{D} = \vec{d}_q\} \text{ on } \mathbb{V}^{(N)}_{\vec{d}_q, R_{q-1}}] \right) I[\{\vec{D} < \vec{d}_Q\} \text{ on } \mathbb{V}^{(N)}_{\vec{d}_Q, R_{Q-1}}] I[\vec{D} = \vec{d}_Q], \tag{4.31}$$

$$T_4 = \left(\prod_{q=1}^{Q} I[\{\vec{D} = \vec{d}_q\} \text{ on } \mathbb{V}^{(N)}_{\vec{d}_q, R_{q-1}}] \right) I[\|\vec{D}\| > R_{Q+1}], \tag{4.32}$$

$$T_5 = \left(\prod_{q=1}^{\bar{Q}} I[\{\vec{D} = \vec{d}_q\} \text{ on } \mathbb{V}^{(N)}_{\vec{d}_q, R_{q-1}}] \right) I[\vec{D} = \vec{d}_{\bar{Q}+1}]. \tag{4.33}$$

Iteration of (4.29) leads to

$$\begin{aligned} \sum_{\vec{d}_1 : \|\vec{d}_1\| \le R_1} & I[E^{(N)} \text{ on } \mathbb{V}^{(N)}_{\vec{d}_1, R_0}] I[\vec{D} = \vec{d}_1] \\ &= \sum_{Q=1}^{\bar{Q}} (-1)^{Q-1} \sum_{\vec{\Delta}_Q \in \mathcal{S}_Q(\vec{R})} I[E^{(N)} \text{ on } \mathbb{V}^{(N)}_{\vec{d}_1, R_0}] \big(T + T_3 - T_4 \big) \\ &\qquad + (-1)^{\bar{Q}} \sum_{\vec{\Delta}_{\bar{Q}+1} \in \mathcal{S}_{\bar{Q}+1}(\vec{R})} I[E^{(N)} \text{ on } \mathbb{V}^{(N)}_{\vec{d}_1, R_0}] T_5. \end{aligned} \tag{4.34}$$

We insert the identity (4.34) into the right hand side of (4.24). With (4.23), this gives

$$\hat{\Pi}^{(N)} = \mathcal{M}^{(N)}_{\vec{R}, \bar{Q}} + \mathcal{E}^{(N)}_{\vec{R}, \bar{Q}}, \tag{4.35}$$

where the right hand side is defined as follows. First, the main term is

$$\mathcal{M}^{(N)}_{\vec{R}, \bar{Q}} = \sum_{Q=1}^{\bar{Q}} (-1)^{Q-1} \sum_{\vec{\Delta}_Q \in \mathcal{S}_Q(\vec{R})} \sum_{x, (u_j, v_j)} p^N \mathbb{E}^{(N)} \Big[I[E^{(N)} \text{ on } \mathbb{V}^{(N)}_{\vec{d}_1, R_0}] T \Big]. \tag{4.36}$$

The error term is

$$\mathcal{E}^{(N)}_{\vec{R}, \bar{Q}} = \mathcal{E}^{(N)}_{1, R_0} + \mathcal{E}^{(N)}_{2, R_0, R_1} + \sum_{j=3}^{5} \mathcal{E}^{(N)}_{j, \vec{R}, \bar{Q}}, \tag{4.37}$$

where the first two terms on the right hand side are given by (4.16) and (4.21), and

$$\mathcal{E}^{(N)}_{3, \vec{R}, \bar{Q}} = \sum_{Q=1}^{\bar{Q}} (-1)^{Q-1} \sum_{\vec{\Delta}_Q \in \mathcal{S}_Q(\vec{R})} \sum_{x, (u_j, v_j)} p^N \mathbb{E}^{(N)} \Big[I[E^{(N)} \text{ on } \mathbb{V}^{(N)}_{\vec{d}_1, R_0}] T_3 \Big], \tag{4.38}$$

$$\mathcal{E}^{(N)}_{4, \vec{R}, \bar{Q}} = \sum_{Q=1}^{\bar{Q}} (-1)^{Q} \sum_{\vec{\Delta}_Q \in \mathcal{S}_Q(\vec{R})} \sum_{x, (u_j, v_j)} p^N \mathbb{E}^{(N)} \Big[I[E^{(N)} \text{ on } \mathbb{V}^{(N)}_{\vec{d}_1, R_0}] T_4 \Big], \tag{4.39}$$

$$\mathcal{E}^{(N)}_{5, \vec{R}, \bar{Q}} = (-1)^{\bar{Q}} \sum_{\vec{\Delta}_{\bar{Q}+1} \in \mathcal{S}_{\bar{Q}+1}(\vec{R})} \sum_{x, (u_j, v_j)} p^N \mathbb{E}^{(N)} \Big[I[E^{(N)} \text{ on } \mathbb{V}^{(N)}_{\vec{d}_1, R_0}] T_5 \Big]. \tag{4.40}$$

We will show in Section 4.3 that $\bar{Q}(M)$ and $\vec{R}(M)$ can be chosen such that $\mathcal{E}^{(N)}_{\vec{R},\bar{Q}} = O(\Omega^{-M-1})$. Given this bound on the error term, to complete the proof of (3.13) it suffices to prove the following proposition.

Proposition 4.4. *Let $\bar{Q}(M)$ and $\vec{R}(M)$ be given. The main term $\mathcal{M}^{(N)}_{\vec{R},\bar{Q}}$ is a polynomial in Ω and p of the form $\sum_{i,j=0}^{L_M} \alpha^{(N)}_{j,i,M}\Omega^j p^i$, as in (3.13), with rational coefficients and with degree depending only on M.*

Proof. Since $\bar{Q}$ depends only on M, it suffices to show that each term in the sum over Q in (4.36) is a polynomial of the desired form. Thus, for fixed $Q \in \{1, \ldots, \bar{Q}\}$, we will show that

$$\sum_{\vec{\Delta}_Q \in \mathcal{S}_Q(\vec{R})} \sum_{x,(u_j,v_j)} p^N \mathbb{E}^{(N)}\Big[I[E^{(N)} \text{ on } \mathbb{V}^{(N)}_{\vec{d}_1,R_0}] \prod_{q=1}^{Q} I[\{\vec{D} = \vec{d}_q\} \text{ on } \mathbb{V}^{(N)}_{\vec{d}_q,R_{q-1}}]\Big] \tag{4.41}$$

is a polynomial of the desired form.

We perform the sum over $\vec{\Delta}_Q$ in the order

$$\sum_{\vec{d}_Q : \|\vec{d}_Q\| \le R_Q} \quad \sum_{\substack{\vec{\Delta}_{Q-1} \in \mathcal{S}_{Q-1}(\vec{R}) : \\ \vec{d}_{Q-1} < \vec{d}_Q}} , \tag{4.42}$$

which puts the sum over $\vec{d}_Q$ last. When $Q = 1$, the second sum is absent. We define an equivalence relation on $\{0,1\}^n$ by regarding $\vec{d}$ and $\vec{d}'$ as equivalent if $\|\vec{d}\| = \|\vec{d}'\|$, i.e., if $\vec{d}$ and $\vec{d}'$ have the same number of components taking the value 1. If $\vec{d}$ and $\vec{d}'$ are equivalent, then by symmetry they give rise to equal contributions to the sum over $\vec{d}_Q$ in (4.41). Let $[\vec{d}]_m$ denote the element of $\{0,1\}^n$ that consists of m ones followed by $n-m$ zeros. Then (4.41) is equal to

$$\sum_{m=1}^{R_Q} \binom{n}{m} \sum_{\substack{\vec{\Delta}_{Q-1} \in \mathcal{S}_{Q-1}(\vec{R}) : \\ \vec{d}_{Q-1} < [\vec{d}]_m}} \sum_{x,(u_j,v_j)} p^N \mathbb{E}^{(N)}\Big[I[E^{(N)} \text{ on } \mathbb{V}^{(N)}_{\vec{d}_1,R_0}] \prod_{q=1}^{Q} I[\{\vec{D} = \vec{d}_q\} \text{ on } \mathbb{V}^{(N)}_{\vec{d}_q,R_{q-1}}]\Big], \tag{4.43}$$

where $\vec{d}_Q$ is equal to $[\vec{d}]_m$.

The number of terms in the sum over $\vec{\Delta}_{Q-1}$ depends only on $\vec{R}$, and hence depends only on M. Also, the cardinality of the set $\mathbb{V}^{(N)}_{[\vec{d}]_m,R_{Q-1}}$ is bounded by a constant depending only on M, uniformly in $m \le R_Q$, and this set contains the sets $\mathbb{V}^{(N)}_{\vec{d}_q,R_{q-1}}$ for $q < Q$. The event $\{E^{(N)} \text{ on } \mathbb{V}^{(N)}_{\vec{d}_1,R_0}\}$ implies that $x, u_j, v_j \in \mathbb{V}^{(N)}_{\vec{d}_1,R_0}$, and hence the number of terms in the sum over $x, (u_j, v_j)$ is bounded by a constant depending only on M. Thus, it suffices to prove that the expectation in (4.43) is a polynomial of the desired form.

The expectation in (4.43) is the probability of an event that only depends on the occupation status of bonds in $\mathbb{V}^{(N)}_{[\vec{d}]_m,R_{Q-1}}$. Explicitly, this probability is the sum, over the *finitely* many configurations on $\mathbb{V}^{(N)}_{[\vec{d}]_m,R_{Q-1}}$ for which the product of indicators is 1, of $p^x(1-p)^y$, where x and y are the number of occupied and vacant bonds, respectively, in the configuration. Thus, this probability is a polynomial in p with integer coefficients. Therefore, taking into account the binomial coefficient in (4.43), the coefficients of the polynomial (4.43) in n and p must all be rational numbers. (We note that rational coefficients arise here rather than the integer coefficients found for the self-avoiding walk in [17], because in counting dimensions under symmetry, combinations arise here whereas for the self-avoiding walk it was permutations.) □

4.3 Error estimates

We now formulate three lemmas which show that we can choose $\bar{Q}$ and $R_0 < R_1 < \cdots < R_{\bar{Q}+1}$, all depending on M, such that each of $\mathcal{E}^{(N)}_{1,R_0}$, $\mathcal{E}^{(N)}_{2,R_0,R_1}$, and $\mathcal{E}^{(N)}_{j,\vec{R},\bar{Q}}$ $(j = 3, 4, 5)$ is $O(\Omega^{-M-1})$. As noted above, these estimates imply (3.13) and thus complete the proof of Proposition 2.1. Proofs of the three lemmas will be given in Section 5.

The first lemma gives the desired bound on $\mathcal{E}^{(N)}_{1,R_0}$.

Lemma 4.5. *Let $M \geq 1$. There exists a $K_1 = K_1(M)$ and an $r_0(M)$ such that if $R_0 \geq r_0(M)$ then for $N \leq M$ and $p \leq \bar{p}_c$,*

$$|\mathcal{E}^{(N)}_{1,R_0}| \leq K_1 \Omega^{-M-1}. \tag{4.44}$$

The corollary to the second lemma gives the desired bound on $\mathcal{E}^{(N)}_{3,\vec{R},\bar{Q}}$.

Lemma 4.6. *Let $M \geq 1$. There exists a $K_2 = K_2(M)$ and an $r(M)$ such that if $R \geq r(M)$ then for $N \leq M$ and $p \leq \bar{p}_c$,*

$$\sum_{\vec{d} \in \{0,1\}^n} \sum_{x,(u_j,v_j)} p^N \mathbb{E}^{(N)} \left[I[F^{(N)}] I[\{\vec{D} < \vec{d}\} \text{ on } \mathbb{V}^{(N)}_{\vec{d},R}] I[\vec{D} = \vec{d}] \right] \leq K_2 \Omega^{-M-1}. \tag{4.45}$$

Corollary 4.7. *Let $M \geq 1$ and $\bar{Q} \geq 2$. If $r(M) \leq R_Q \leq R_{\bar{Q}}$ for $Q = 0, \ldots, \bar{Q}-1$, then for $N \leq M$ and $p \leq \bar{p}_c$,*

$$|\mathcal{E}^{(N)}_{3,\vec{R},\bar{Q}}| \leq \bar{Q} 2^{(\bar{Q}-1)R_{\bar{Q}}} K_2 \Omega^{-M-1}. \tag{4.46}$$

Proof. By (4.4) and (4.31), the summand in the sum over Q in (4.38) is bounded above by

$$\sum_{\vec{\Delta}_Q \in \mathcal{S}_Q(\vec{R})} \sum_{x,(u_j,v_j)} p^N \mathbb{E}^{(N)} \left[I[F^{(N)}] I[\{\vec{D} < \vec{d}_Q\} \text{ on } \mathbb{V}^{(N)}_{\vec{d}_Q, R_{Q-1}}] I[\vec{D} = \vec{d}_Q] \right]. \tag{4.47}$$

Given $\vec{d}_Q$ with $\|\vec{d}_Q\| \leq R_Q \leq R_{\bar{Q}}$, the number of $\vec{d}_1, \ldots, \vec{d}_{Q-1}$ with $\vec{d}_1 < \cdots < \vec{d}_{Q-1} < \vec{d}_Q$ and $\|\vec{d}_i\| \leq R_i$ is bounded above by $2^{(\bar{Q}-1)R_{\bar{Q}}}$. Thus (4.47) is bounded above by

$$2^{(\bar{Q}-1)R_{\bar{Q}}} \sum_{\vec{d}_Q \in \{0,1\}^n} \sum_{x,(u_j,v_j)} p^N \mathbb{E}^{(N)} \left[I[F^{(N)}] I[\{\vec{D} < \vec{d}_Q\} \text{ on } \mathbb{V}^{(N)}_{\vec{d}_Q, R_{Q-1}}] I[\vec{D} = \vec{d}_Q] \right], \tag{4.48}$$

and now (4.45) can be applied. □

The corollary to the third lemma gives the desired bounds on $\mathcal{E}^{(N)}_{2,R_0,R_1}$, and $\mathcal{E}^{(N)}_{j,\vec{R},\bar{Q}}$ $(j = 4, 5)$.

Lemma 4.8. *Let $M \geq 1$ and $R \geq 1$. There exists a constant $C(R, M)$, and a sequence g_R with $\lim_{R\to\infty} g_R = \infty$, such that for $N \leq M$ and $p \leq \bar{p}_c$,*

$$\sum_{x,(u_j,v_j)} p^N \mathbb{E}^{(N)} \left[I[F^{(N)}] I[\|\vec{D}\| > R] \right] \leq C(R, M) \Omega^{-g_R}. \tag{4.49}$$

Corollary 4.9. *For $M \geq 1$ and for any R_0, we can choose $\bar{Q} = \bar{Q}(M)$ and $R_i = R_i(M)$ $(i = 1, \ldots, \bar{Q}+1)$, with $R_0 < R_1 < R_2 < \cdots < R_{\bar{Q}+1}$, such that for $N \leq M$ and $p \leq \bar{p}_c$,*

$$\left|\mathcal{E}^{(N)}_{2,R_0,R_1}\right| \leq C(R_1, M) \Omega^{-M-1}, \quad \left|\mathcal{E}^{(N)}_{4,\vec{R},\bar{Q}}\right| \leq C'(\bar{Q}, \vec{R}) \Omega^{-M-1}, \tag{4.50}$$

$$\left|\mathcal{E}^{(N)}_{5,\vec{R},\bar{Q}}\right| \leq 2^{\bar{Q} R_{\bar{Q}+1}} C(\bar{Q}-1, M) \Omega^{-M-1}, \tag{4.51}$$

where $C'(\bar{Q}, \vec{R})$ is a constant depending on $\bar{Q}$ and $\vec{R}$, and hence only on M.

Proof. The bound on $\mathcal{E}^{(N)}_{2,R_0,R_1}$ is an immediate consequence of (4.21), (4.4) and (4.49).

For the bound on $\mathcal{E}^{(N)}_{5,\vec{R},\bar{Q}}$, we note from (4.40), (4.4) and (4.33) that

$$\left|\mathcal{E}^{(N)}_{5,\vec{R},\bar{Q}}\right| \leq \sum_{\vec{\Delta}_{\bar{Q}+1}\in\mathcal{S}_{\bar{Q}+1}(\vec{R})} \sum_{x,(u_j,v_j)} p^N \mathbb{E}^{(N)}\left[I[F^{(N)}]I[\vec{D}=\vec{d}_{\bar{Q}+1}]\right]. \tag{4.52}$$

For fixed $\vec{d}_{\bar{Q}+1}$, the number of $\vec{d}_1,\dots,\vec{d}_{\bar{Q}}$ with $\vec{d}_1 < \cdots < \vec{d}_{\bar{Q}} < \vec{d}_{\bar{Q}+1}$ and $\|\vec{d}_i\| \leq R_i \leq R_{\bar{Q}+1}$ is bounded above by $2^{\bar{Q}R_{\bar{Q}+1}}$. On the other hand, since the $\vec{d}_i$ are strictly increasing, it must be the case that $\|\vec{d}_{\bar{Q}+1}\| \geq \bar{Q}$, and hence

$$\left|\mathcal{E}^{(N)}_{5,\vec{R},\bar{Q}}\right| \leq 2^{\bar{Q}R_{\bar{Q}+1}} \sum_{x,(u_j,v_j)} p^N \mathbb{E}^{(N)}\left[I[F^{(N)}]I[\|\vec{D}\| \geq \bar{Q}]\right]. \tag{4.53}$$

By Lemma 4.8, we can choose $\bar{Q}=\bar{Q}(M)$ such that

$$\left|\mathcal{E}^{(N)}_{5,\vec{R},\bar{Q}}\right| \leq 2^{\bar{Q}R_{\bar{Q}+1}} C(\bar{Q}-1,M)\Omega^{-M-1}. \tag{4.54}$$

Finally, we prove the bound on $\mathcal{E}^{(N)}_{4,\vec{R},\bar{Q}}$. The number of $\vec{d}_Q$ with $\|\vec{d}_Q\| \leq R_Q$ is at most $n^{\|\vec{d}_Q\|} \leq n^{R_Q}$. Given such a $\vec{d}_Q$, the number of $(\vec{d}_1,\dots,\vec{d}_{Q-1})$ with $\vec{d}_1 < \cdots < \vec{d}_{Q-1} < \vec{d}_Q$ is at most $2^{(Q-1)R_Q}$. Therefore, by (4.39), (4.32) and (4.4),

$$\left|\mathcal{E}^{(N)}_{4,\vec{R},\bar{Q}}\right| \leq \sum_{Q=1}^{\bar{Q}} 2^{(Q-1)R_Q} n^{R_Q} \sum_{x,(u_j,v_j)} p^N \mathbb{E}^{(N)}\left[I[F^{(N)}]I[\|\vec{D}\| > R_{Q+1}]\right]. \tag{4.55}$$

By Lemma 4.8, given any choice of R_0 and any choice of R_1, we can choose $R_{Q+1}=R_{Q+1}(R_Q,M) > R_Q$ sequentially and increasing for $Q=1,\dots,\bar{Q}$, so that

$$\begin{aligned}\left|\mathcal{E}^{(N)}_{4,\vec{R},\bar{Q}}\right| &\leq \sum_{Q=1}^{\bar{Q}} 2^{(Q-1)R_Q} n^{R_Q} C(R_Q,M)\Omega^{-M-1-R_Q} \\ &\leq \Omega^{-M-1} \sum_{Q=1}^{\bar{Q}} 2^{(Q-1)R_Q} C(R_Q,M)\Omega^{-M-1}.\end{aligned} \tag{4.56}$$

This is the desired estimate. □

This gives the desired bounds on the error terms. The value of $\bar{Q}(M)$ is fixed by Corollary 4.9, we take $R_0 = r_0(M) \vee r(M)$, and we fix $R_{\bar{Q}+1} > \cdots > R_1 > R_0$ according to Corollary 4.9. Then the restrictions of Lemma 4.5 and Corollaries 4.7 and 4.9 are all obeyed.

It remains to prove Lemmas 4.5, 4.6 and 4.8. This will be done in Section 5.

5 Proof of error estimates

In this section, we complete the proof of Proposition 2.1 by proving Lemmas 4.5, 4.6 and 4.8. We begin by recalling some basic facts.

5.1 Preliminaries

Let $D(y-x) = \Omega^{-1}$ if x and y are neighbours, and $D(y-x) = 0$ otherwise. Thus $D(y-x)$ is the transition probability for simple random walk on $\mathbb{G}$ to make a step from x to y. Let $\tau_p(y-x) = \mathbb{P}(x \leftrightarrow y)$ denote the percolation two-point function, and let $\tau_p^{(i)}(x)$ denote the probability that there is an occupied self-avoiding path from 0 to x of length at least i.

We define the Fourier transform of an absolutely summable function f on the vertex set $\mathbb{V}$ of $\mathbb{G}$ by

$$\hat{f}(k) = \sum_{x \in \mathbb{V}} f(x) e^{ik \cdot x} \qquad (k \in \mathbb{V}^*), \tag{5.1}$$

where $\mathbb{V}^* = \{0, \pi\}^n$ for $\mathbb{Q}_n$ and $\mathbb{V}^* = [-\pi, \pi]^n$ for $\mathbb{Z}^n$. Let

$$(f * g)(x) = \sum_{y \in \mathbb{V}} f(y) g(x-y) \tag{5.2}$$

denote convolution. Recall from [2] that $\hat{\tau}_p(k) \geq 0$ for all k.

For i, j non-negative integers, let

$$T_p^{(i,j)} = \begin{cases} 2^{-n} \sum_{k \in \{0,\pi\}^n} |\hat{D}(k)|^i \hat{\tau}_p(k)^j & (\mathbb{G} = \mathbb{Q}_n) \\ \int_{[-\pi,\pi]^n} |\hat{D}(k)|^i \hat{\tau}_p(k)^j \frac{d^n k}{(2\pi)^n} & (\mathbb{G} = \mathbb{Z}^n), \end{cases} \tag{5.3}$$

$$T_p = \sup_x (p\Omega)(D * \tau_p * \tau_p * \tau_p)(x). \tag{5.4}$$

Recall from [19, Section 3] that for $\mathbb{G} = \mathbb{Z}^n$ and $\mathbb{G} = \mathbb{Q}_n$, there are constants $K_{i,j}$ and K such that for all $p \leq \bar{p}_c(\mathbb{G})$,

$$T_p^{(i,j)} \leq K_{i,j} \Omega^{-i/2} \quad (i, j \geq 0), \tag{5.5}$$

$$T_p \leq K \Omega^{-1}, \tag{5.6}$$

$$\sup_x \tau_p^{(i)}(x) \leq \begin{cases} K\Omega^{-1} & (i = 1) \\ 2^i K_{i,1} \Omega^{-i/2} & (i \geq 2). \end{cases} \tag{5.7}$$

The above bounds are valid for $n \geq 1$ for $\mathbb{Q}_n$, and for n larger than an absolute constant for $\mathbb{Z}^n$, except that (5.5) also requires $n \geq 2j+1$ for $\mathbb{Z}^n$.

5.2 Proof of Lemmas 4.5, 4.6 and 4.8

Now we prove Lemmas 4.5, 4.6 and 4.8. The proofs use the following proposition. Recall the definition of $\mathcal{P}_j$ in Definition 4.2(ii), and let $\mathcal{P}_{l,L}$ denote the subset of $\mathcal{P}_l$ consisting of paths of length less than L.

Proposition 5.1. *Let $M \geq 1$. There is a constant $K_3 = K_3(M)$ such that for $l \leq N \leq M$ and $p \leq \bar{p}_c$,*

$$\sum_{x,(u_j,v_j)} p^N \mathbb{E}^{(N)} \Big[I[F^{(N)}] I[\{\exists \text{ occupied } \omega \in \mathcal{P}_l \backslash \mathcal{P}_{l,10(M+1)}\}] \Big] \leq K_3 \Omega^{-M-1}. \tag{5.8}$$

Proof. We assume some familiarity with diagrammatic estimates, as in [9, Section 4].

We begin by rewriting (3.21) as

$$F^{(N)} = \bigcup_{\vec{t},\vec{w},\vec{z}} \bigcap_{j=0}^{N} F_j, \tag{5.9}$$

where we have made the abbreviations $F_0 = F_0(0, u_0, w_0, z_0)_0$, $F_i = F(v_{i-1}, t_i, z_i, u_i, w_i, z_{i+1})_i$, and $F_N = F_N(v_{N-1}, t_N, z_N, x)_N$. We will use the estimate

$$\mathbb{P}^{(N)}\Big(G \cap \bigcup_{\vec{t},\vec{w},\vec{z}} \bigcap_{j=0}^{N} F_j\Big) \leq \sum_{\vec{t},\vec{w},\vec{z}} \mathbb{P}^{(N)}\Big(G \cap \bigcap_{j=0}^{N} F_j\Big), \tag{5.10}$$

with $G = \{\exists \text{ occupied } \omega \in \mathcal{P}_l \backslash \mathcal{P}_{l,10(M+1)}\}$ (for fixed l, M). The standard bounds on $\hat{\Pi}^{(N)}$ use (5.10) with G equal to the whole probability space. In the standard bounds, after applying the BK inequality, each of the disjoint connections in (3.16)–(3.20), say $\{y_1 \leftrightarrow y_2\}$, gives rise to a two-point function $\tau_p(y_2 - y_1)$. The overall effect is to bound $\hat{\Pi}^{(N)}$ by a Feynman diagram, which is then bounded by products of $T^{(i,3)}$ as in [9, Section 4]. We will modify this standard procedure to prove the proposition.

We use $G \subset G_1 \cup G_2$, where G_1 is the event that one of the disjoint connections in (3.16)–(3.20), say $\{y_1 \leftrightarrow y_2\}$, is replaced by the disjoint occurrence of $\{y_1 \leftrightarrow y_2$ via a path consisting of at least $2(M+1)$ occupied bonds$\}$, and $G_2 = G \backslash G_1$.

For the contribution due to G_1, the standard diagrammatic bounds give an upper bound identical to that for $\hat{\Pi}^{(N)}$, except that the factor $\tau_p(y_2 - y_1)$ is replaced by $\tau_p^{(2M+2)}(y_2 - y_1)$. This replaces the bound

$$\hat{\Pi}^{(N)} \leq \begin{cases} T_p & (N = 0) \\ T_p^{(0,3)}(2T_p^{(0,3)}T_p)^N & (N \geq 1) \end{cases} \tag{5.11}$$

of [9, Proposition 4.1] by a sum of terms in which one factor $T_p^{(0,3)}$ or T_p is replaced by $T_p^{(2M+2,3)}$. The number of terms in the sum depends only on N, and $N \leq M$. By (5.5), this new upper bound is at most $O(\Omega^{-M-1})$, as required. Thus, we are left to deal with G_2.

To estimate the right hand side of (5.10) with $G = G_2$, we may assume that each of the connections in $F_0, \ldots, F_N$ is achieved by a path consisting of at most $2(M+1)$ bonds. On the other hand, there must exist an occupied path in $\mathcal{P}_l$ consisting of at least $10(M+1)$ bonds. The coexistence of this path with the disjoint occupied paths required by the event F_l implies that we can find vertices a, b and disjoint paths such that two of the disjoint connections in F_l and/or F_{l+1}, say $\{y_1 \leftrightarrow y_2\}$ and $\{y_3 \leftrightarrow y_4\}$, become replaced by

$$\{y_1 \leftrightarrow a\} \circ \{a \leftrightarrow y_2\} \circ \{a \leftrightarrow b\} \circ \{y_3 \leftrightarrow b\} \circ \{b \leftrightarrow y_4\}. \tag{5.12}$$

Moreover, the connection from a to b must be achieved by a path of length at least $2(M+1)$. This follows from the fact that the long occupied path in $\mathcal{P}_l$ can partially coincide with at most four of the paths realizing the connections of F_l, and these paths have total length at most $8(M+1)$. Such a worst-case scenario is depicted in the first example in Figure 1, where the long path in $\mathcal{P}_1$ emerges at a after coinciding along four paths realizing connections in F_1. Therefore, we can in fact replace (5.12) by

$$\{y_1 \leftrightarrow a\} \circ \{a \leftrightarrow y_2\} \circ \{a \leftrightarrow b \text{ by a path of length at least } 2(M+1)\} \circ \{y_3 \leftrightarrow b\} \circ \{b \leftrightarrow y_4\}. \tag{5.13}$$

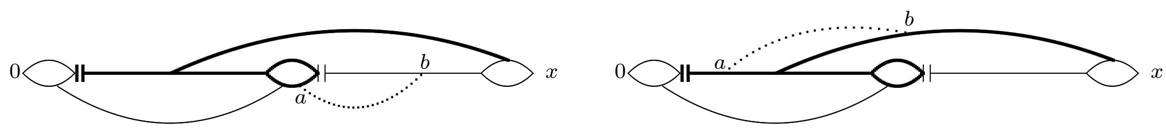

Figure 1: Examples of disjoint connections satisfying G_2. Each solid path joining a pair of vertices has length at most $2(M+1)$, and the first and second dotted paths have lengths at least $2(M+1)$ and $6(M+1)$, respectively.

In an upper bound achieved via the BK inequality, the two factors $\tau_p(y_1-y_2)\tau_p(y_3-y_4)$ normally present in the upper bound on $\hat{\Pi}^{(N)}$ are replaced by

$$\begin{aligned}
&\sum_{a,b} \tau_p(y_1-a)\tau_p(a-y_2)\tau_p^{(2M+2)}(b-a)\tau_p(y_3-b)\tau_p(b-y_4) \\
&\qquad \le \left(\sup_{a,b} \tau_p^{(2M+2)}(b-a)\right) \sum_a \tau_p(y_1-a)\tau_p(a-y_2) \sum_b \tau_p(y_3-b)\tau_p(b-y_4) \\
&\qquad \le 2^{2M+2} K_{2M+2,1} \Omega^{-M-1} (\tau_p * \tau_p)(y_1-y_2)(\tau_p * \tau_p)(y_3-y_4), \qquad (5.14)
\end{aligned}$$

using (5.7) to estimate $\sup_{a,b} \tau_p^{(2M+2)}(b-a)$. It remains to show that the result of replacing the factors $\tau_p(y_1-y_2)\tau_p(y_3-y_4)$ by $(\tau_p * \tau_p)(y_1-y_2)(\tau_p * \tau_p)(y_3-y_4)$, in the standard bound on $\hat{\Pi}^{(N)}$, gives rise to a bounded quantity.

This replacement has the effect of adding a vertex to each of the lines joining y_1 to y_2 and y_3 to y_4 in the standard diagrammatic bound on $\hat{\Pi}^{(N)}$. This leads to a bound in which one factor of $T_p^{(0,3)}$ or T_p in (5.11) is replaced by $T_p^{(0,5)}$ or $T_p^{(1,5)}$, or a product of two factors of $T_p^{(0,3)}$ and/or T_p is replaced by $T_p^{(0,4)} T_p^{(0,4)}$ (taking an upper bound), depending on whether the two vertices are added to the same triangle or not. The result is finite by (5.5). □

Proof of Lemma 4.5. Fix $M \ge 1$ and $0 \le N \le M$. We write simply R in place of R_0, and define

$$\tilde{C}_{j,R} = \{y : \{v_{j-1} \leftrightarrow y \text{ without using } (u_j, v_j)\} \text{ on } \mathbb{B}_R^{(N)}\} \quad (j = 0, \ldots, N-1), \qquad (5.15)$$

$$E_{j,R} = \begin{cases} E_0 & (j=0) \\ E'(v_{j-1}, u_j; \tilde{C}_{j-1,R})_j & (j = 1, \ldots, N), \end{cases} \qquad (5.16)$$

$$E_R^{(N)} = \bigcap_{j=0}^{N} E_{j,R}. \qquad (5.17)$$

By definition,

$$I[E^{(N)} \text{ on } \mathbb{B}_R^{(N)}] = I[E_R^{(N)} \text{ on } \mathbb{B}_R^{(N)}], \qquad (5.18)$$

and we may therefore rewrite the difference occurring in the definition of $\mathcal{E}_{1,R}^{(N)}$ in (4.16) as

$$I[E^{(N)}] - I[E^{(N)} \text{ on } \mathbb{B}_R^{(N)}] = \Big(I[E_R^{(N)}] - I[E_R^{(N)} \text{ on } \mathbb{B}_R^{(N)}]\Big) + \Big(I[E^{(N)}] - I[E_R^{(N)}]\Big). \qquad (5.19)$$

We will prove that

$$\left|I[E_R^{(N)}] - I[E_R^{(N)} \text{ on } \mathbb{B}_R^{(N)}]\right| \le 2I[F^{(N)}] \sum_{j=0}^{N} I[B_j \cap (\mathbb{B}_R^{(N)})^c \neq \varnothing], \tag{5.20}$$

$$\left|I[E^{(N)}] - I[E_R^{(N)}]\right| \le I[F^{(N)}] \sum_{j=1}^{N} I[B_j \cap \tilde{C}_{j-1} \backslash \tilde{C}_{j-1,R} \neq \varnothing]. \tag{5.21}$$

Assuming (5.20)–(5.21), the proof is completed as follows. Consider first a configuration contributing to the summand on the right hand side of (5.20). In such a configuration, there must be a backbone path of length $R/(N+1) \ge R/(M+1)$, since otherwise no backbone path could exit $\mathbb{B}_R^{(N)}$. We take $R = 10(M+1)^2$ and apply Proposition 5.1. Similarly, in a configuration contributing to the summand in the right hand side of (5.21), if there is an element of $B_j \cap \tilde{C}_{j-1} \backslash \tilde{C}_{j-1,R}$ that lies outside of $\mathbb{B}_R^{(N)}$, then there must be a path in $\mathcal{P}_j$ that exits $\mathbb{B}_R^{(N)}$. This implies that there is an occupied path in some $\mathcal{P}_i$ of length at least $R/(M+1)$, and we again take $R = 10(M+1)^2$ and apply Proposition 5.1. On the other hand, if $B_j \cap \tilde{C}_{j-1} \backslash \tilde{C}_{j-1,R} \subset \mathbb{B}_R^{(N)}$, then a path in $\tilde{C}_{j-1}$ must travel from v_{j-2} outside of $\mathbb{B}_R^{(N)}$ before intersecting B_j, so there must then be a path in $\mathcal{P}_{j-1}$ that exits $\mathbb{B}_R^{(N)}$, and again the previous argument applies. It remains to prove (5.20)–(5.21).

Proof of (5.20). By (4.1),

$$I[E_R^{(N)}] - I[E_R^{(N)} \text{ on } \mathbb{B}_R^{(N)}] = I[E_R^{(N)} \cap \{(E_R^{(N)})^c \text{ on } \mathbb{B}_R^{(N)}\}] - I[(E_R^{(N)})^c \cap \{E_R^{(N)} \text{ on } \mathbb{B}_R^{(N)}\}]. \tag{5.22}$$

By (4.4), $\{E^{(N)} \text{ on } \mathbb{B}_R^{(N)}\} \subset F^{(N)}$, and the proof of (3.22) also easily extends to yield $E_R^{(N)} \subset F^{(N)}$. By (4.1)–(4.2), it follows that

$$\begin{aligned} I[E_R^{(N)} \cap \{(E_R^{(N)})^c \text{ on } \mathbb{B}_R^{(N)}\}] &\le I[E_R^{(N)}] \sum_{j=0}^{N} I[E_{j,R} \cap \{E_{j,R}^c \text{ on } \mathbb{B}_R^{(N)}\}] \\ &\le I[F^{(N)}] \sum_{j=0}^{N} I[E_{j,R} \cap \{E_{j,R}^c \text{ on } \mathbb{B}_R^{(N)}\}], \end{aligned} \tag{5.23}$$

$$\begin{aligned} I[(E_R^{(N)})^c \cap \{E_R^{(N)} \text{ on } \mathbb{B}_R^{(N)}\}] &\le I[\{E_R^{(N)} \text{ on } \mathbb{B}_R^{(N)}\}] \sum_{j=0}^{N} I[E_{j,R}^c \cap \{E_{j,R} \text{ on } \mathbb{B}_R^{(N)}\}] \\ &\le I[F^{(N)}] \sum_{j=0}^{N} I[E_{R,j}^c \cap \{E_{j,R} \text{ on } \mathbb{B}_R^{(N)}\}]. \end{aligned} \tag{5.24}$$

Thus, it suffices to show that (i) if $E_{j,R} \cap \{E_{j,R}^c \text{ on } \mathbb{B}_R^{(N)}\}$ occurs, or (ii) if $E_{j,R}^c \cap \{E_{j,R} \text{ on } \mathbb{B}_R^{(N)}\}$ occurs, then there is a path in B_j that exits $\mathbb{B}_R^{(N)}$.

We first consider case (i). It is clear that if $E_0 \cap \{E_0^c \text{ on } \mathbb{B}_R^{(N)}\}$ occurs, then there is a path in B_0 that exits $\mathbb{B}_R^{(N)}$. So we consider $j \ge 1$. Given sets A, B of vertices, it suffices to show that if $E'(v,x;A) \cap \{E'(v,x;A)^c \text{ on } B\}$ occurs, then there must be an occupied path from v to x that exits B. Recall from (3.2) that the event $E'(v,x;A)$ is the intersection of the event $\{v \overset{A}{\leftrightarrow} x\}$ with the NP condition. Therefore,

$$\begin{aligned} I[E'(v,x;A) \cap \{E'(v,x;A)^c \text{ on } B\}] = I[E'(v,x;A)]I[\{v \overset{A}{\leftrightarrow} x\}^c \text{ on } B] \\ + I[E'(v,x;A)]I[\{v \overset{A}{\leftrightarrow} x\} \text{ on } B]I[\text{NP}^c \text{ on } B]. \end{aligned} \tag{5.25}$$

In the first term on the right hand side, the event $E'(v,x;A)$ requires that $\{v \overset{A}{\leftrightarrow} x\}$. If every path from v to x stays inside B, then also $\{v \overset{A}{\leftrightarrow} x\}$ on B. The second factor therefore ensures that there must be a connection from v to x that exits B, as required. In the second term on the right hand side, NP holds, but not on B. This means that, on B, there is a pivotal bond (u',v') for the connection from v to x such that $v \overset{A}{\leftrightarrow} u'$, but that there is no such bond when the entire configuration on $\mathbb{G}$ is used. This can only happen if there is an occupied path from v to x that exits B, with this path either making (u',v') no longer pivotal, or providing a path from v to u' that does not intersect A.

Next, we consider case (ii). The case $j=0$ cannot occur. We consider $j \geq 1$, and proceed as in the proof of case (i). By (3.2),

$$I[E'(v,x;A)^c \cap \{E'(v,x;A) \text{ on } B\}] = I[E'(v,x;A) \text{ on } B] I[\{v \overset{A}{\leftrightarrow} x\}^c \cap \text{NP}] + I[E'(v,x;A) \text{ on } B] I[\text{NP}^c]. \tag{5.26}$$

In the first term on the right hand side, the event $\{v \overset{A}{\leftrightarrow} x\}$ occurs on B but does not occur on $\mathbb{G}$. This implies that there is an occupied path from v to x that exits B (and does not contain a vertex in A), as required. The second term on the right hand side is zero. To see this, we first observe that the event NP^c implies that (on $\mathbb{G}$) there is an occupied pivotal bond (u',v') for $v \leftrightarrow x$ such that $v \overset{A}{\leftrightarrow} u'$. The bond (u',v') must also be pivotal for the connection from v to x on B. Moreover, since $E'(v,x;A)$ occurs on B, it must be that v is connected to u' on B and $\{v \overset{A}{\leftrightarrow} u'\}^c$ occurs on B. This contradicts $v \overset{A}{\leftrightarrow} u'$ on $\mathbb{G}$, and hence the second term is indeed zero. This completes the proof of (5.20).

Proof of (5.21). We begin with the identity

$$I[E^{(N)}] - I[E_R^{(N)}] = \sum_{j=1}^{N} \prod_{i=0}^{j-1} I[E_{i,R}] \Big(I[E_j] - I[E_{j,R}] \Big) \prod_{i=j+1}^{N} I[E_i], \tag{5.27}$$

in which the absent term with $j=0$ is equal to zero. It suffices to show that $|I[E_j] - I[E_{j,R}]|$ is bounded above by the indicator that B_j intersects $\tilde{C}_{j-1} \setminus \tilde{C}_{j,R}$, multiplied by either $I[E_j]$ or $I[E_{j,R}]$. The former gives rise to the summand on the right hand side of (5.21), while the latter, in combination with the products over i in (5.27), ensures that all connections necessary to imply $F^{(N)}$ are present. We proceed to obtain this estimate for $|I[E_j] - I[E_{j,R}]|$.

For $A \subset A'$, we write

$$I[E'(v,x;A')] - I[E'(v,x;A)] = I[E'(v,x;A)^c \cap E'(v,x;A')] - I[E'(v,x;A) \cap E'(v,x;A')^c], \tag{5.28}$$

and we consider the two terms on the right hand side separately. For the first term, we use the fact that if NP occurs for A' then it also occurs for A. Therefore, the event in the first term implies that $E'(v,x;A') \cap \{v \overset{A'}{\leftrightarrow} x\} \cap \{v \overset{A}{\leftrightarrow} x\}^c$ occurs. In particular, there must be an occupied path from v to x that intersects $A' \setminus A$. Similarly, for the second term in (5.28), we have $\{v \overset{A}{\leftrightarrow} x\} \subset \{v \overset{A'}{\leftrightarrow} x\}$, so the event of the second term implies that $E'(v,x;A)$ occurs, and that the NP condition holds for A, but not for A'. The latter implies that, as required, there is an occupied path from v to

x containing an element in $A' \setminus A$. This completes the proof of (5.21), and thus the proof of the lemma. □

Proof of Lemma 4.6. We first argue that if $\vec{D} = \vec{d}$, but $\{\vec{D} < \vec{d}\}$ on $\mathbb{V}^{(N)}_{\vec{d},R}$, then there must be an occupied path of length at least $10(M+1)$ in some $\mathcal{P}_l$, and hence,

$$I[\{\vec{D} < \vec{d}\} \text{ on } \mathbb{V}^{(N)}_{\vec{d},R}]I[\vec{D} = \vec{d}] \leq \sum_l I[\exists \text{ occupied } \omega \in \mathcal{P}_l \setminus \mathcal{P}_{l,10(M+1)}]I[\vec{D} = \vec{d}]. \tag{5.29}$$

Suppose, to the contrary, that all occupied paths in each $\mathcal{P}_l$ have length at most $10(M+1)$. Let $R \geq r(M) = 10(M+1)^2$. Then if $\vec{D} = \vec{d}$, it must also be the case that $\{\vec{D} = \vec{d}\}$ on $\mathbb{V}^{(N)}_{\vec{d},R}$, since the paths that determine $\vec{D}$ travel at most a distance $10(M+1)$ in each of the $N+1$ expectations, and hence travel a total distance at most $10(M+1)(N+1) \leq R$. This proves (5.29).

We substitute (5.29) into the left hand side of (4.45), perform the sum over $\vec{d}$ using the indicator $I[\vec{D} = \vec{d}]$, and apply Proposition 5.1, to obtain the desired estimate. □

Proof of Lemma 4.8. We again assume some familiarity with the methods of [9, Section 4]. Fix M and $N \leq M$, and fix a positive number $a < 1$.

Suppose that $F^{(N)}$ occurs and that $\|\vec{D}\| > R$. By Proposition 5.1, we need only consider the case in which all occupied paths in each $\mathcal{P}_j$ have length at most $10(R^a+1)$, since the complement obeys the desired estimate with $g_R = R^a + 1$. We make this assumption throughout the proof.

Given a bond configuration, we can select a sequence of occupied level-j paths $\eta^j_i \in \mathcal{P}_j$ (a path may consist of a single vertex and the paths need not be disjoint), for $i = 1,2,3$ and $j = 0,\ldots,N$, which together ensure that $F^{(N)} = \cup_{\vec{t},\vec{w},\vec{z}} \cap_{j=0}^N F_j$ occurs (see (5.9)). In Figure 2, the paths $\eta^j_1, \eta^j_2, \eta^j_3$ are the two paths joining v_{j-1} to u_j and the path joining v_{j-1} to B_{j+1}. Denote the union of the vertices in $\eta^j_1, \eta^j_2, \eta^j_3$ by A_j, and let $A = \cup_{j=0}^N A_j$. By our assumption, the set A explores at most $(N+1)30(R^a+1) \leq 30(M+1)(R^a+1)$ dimensions. We consider the case where R is large (depending on M), so that in particular $R > 30(M+1)(R^a+1)$. This implies that there must be additional occupied paths in $\cup_{j=0}^N \mathcal{P}_j$ that explore additional dimensions. In fact, there must be some j for which the number of dimensions explored at level-j exceeds $R' = (M+1)^{-1}[R - 30(M+1)(R^a+1)]$, and hence the number of these paths exceeds $R'' = R'/10(R^a+1)$. We fix such a j. More precisely, given a bond configuration for which $\|\vec{D}\| > R$, we can find a j and a sequence of occupied paths $\omega_1, \ldots, \omega_{R''} \in \mathcal{P}_j$, such that ω_1 enters a dimension not entered by A, and, for $l \geq 2$, ω_l enters a dimension not entered by $A \cup (\cup_{k<l}\omega_k)$, where the union refers to a union of vertices. Note that since $a < 1$, $R'' \to \infty$ as $R \to \infty$ with M fixed.

The paths η^j_i ($j = 0,\ldots,N$, $i = 1,2,3$) ensure that the disjoint connections required by the event $F^{(N)}$ occur. If we were to neglect the fact that the paths ω_l are occupied, an application of the BK inequality would lead to the standard diagrammatic estimates for $\hat{\Pi}^{(N)}$, as described, e.g., in [9, Section 4]. With this in mind, we take the paths ω_l into account sequentially, as follows.

First, since ω_1 enters a dimension not yet entered by A_j (where j is the special level fixed above), there is a vertex in ω_1 that is not in any of the η^j_i ($i = 1,2,3$). By following ω_1 forward and backwards until it hits A_j or B_{j+1} for the first time, we obtain a portion ω'_1 of ω_1 that begins in A_j and ends in either A_j or B_{j+1} and that is disjoint from the paths η^j_i.

If ω'_1 both begins and ends in A_j, then it has the effect of connecting two vertices on the paths $\eta^j_1, \eta^j_2, \eta^j_3$ by an occupied path which is disjoint from these paths. If we apply the BK inequality in

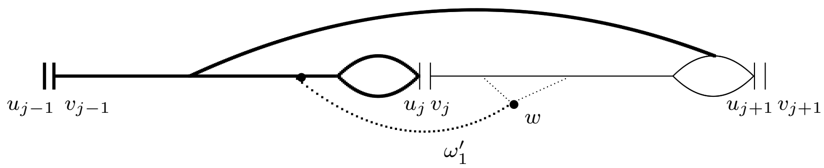

Figure 2: Examples of disjoint connections required by ω_1'. The path ω_1' is a level-j path, whereas the other two dotted paths are part of the level-$(j+1)$ backbone.

this situation, we produce Feynman diagrams that are constructed from those bounding $\hat{\Pi}^{(N)}$ by adding two vertices on diagram lines and joining them by a new line. In other words, we replace the product (say) $\tau_p(y_2 - y_1)\tau_p(y_4 - y_3)$ by

$$\sum_{a,b} \tau_p(y_2 - a)\tau_p(a - y_1)\tau_p(y_4 - b)\tau_p(b - y_3)\tau_p^{(1)}(b - a) \tag{5.30}$$

as in (5.12). As in the proof of Proposition 5.1, we bound the factor $\sup_{a,b} \tau_p^{(1)}(b-a)$ by $K\Omega^{-1}$ using (5.7), and we are left with a diagram with two additional vertices a, b. This diagram was bounded by a constant in the proof of Proposition 5.1.

If ω_1' begins in A_j and ends in B_{j+1}, then this has the effect of augmenting $\eta_1^j, \eta_2^j, \eta_3^j$ with a disjoint path from a vertex on one of these paths to the end of ω_1' in B_{j+1}. Call this latter endpoint w. Since w is in the first entry of ω_1' into B_{j+1}, we can augment the level-$(j+1)$ paths $\eta_1^{j+1}, \eta_2^{j+1}, \eta_3^{j+1}$ by a disjoint level-$(j+1)$ path passing through w, as indicated in Figure 2 (this is a worst-case scenario, and possibly w lies on one of the paths η_i^{j+1}). If we apply the BK inequality to this configuration, the result is a sum of Feynman diagrams, with extra lines due to these additional disjoint connections. We can begin to bound this diagram by extracting a factor $\sup_{a,b} \tau_p^{(1)}(b-a) \leq K\Omega^{-1}$ from the connection due to ω_1', and then a factor $T_p^{(0,2)} \leq K_{0,2}$ due to the level-$(j+1)$ connections that contain w and are bond-disjoint from the level-$(j+1)$ connections η_i^{j+1}. This leaves a standard $\hat{\Pi}^{(N)}$ diagram with at most three extra vertices, and this is bounded by a constant by standard bounds, using (5.5) (provided we take the dimension sufficiently large).

The above explains the procedure if there were only one path ω_1. However, we are interested in the situation where there is a large number R'' of paths ω_l. In this case, we first find the path ω_1' as above. Because ω_2 explores a dimension not entered by $A \cup \omega_1$, we can find a subpath ω_2' of ω_2 that starts at a vertex in $\eta_1^j, \eta_2^j, \eta_3^j$ or ω_1' and ends either in this set or in B_{j+1}, and that is disjoint from $\eta_1^j, \eta_2^j, \eta_3^j$ and ω_1'. If ω_2' ends in B_{j+1}, then we can find disjoint level-$(j+1)$ paths as explained above. If we had only these two paths ω_1', ω_2', we would apply the BK inequality as usual to produce a Feynman diagram, and then estimate this diagram by first bounding the lines created by ω_2', obtaining a factor Ω^{-1} from the fact that ω_2' takes at least one step, then bounding the lines created by ω_1', obtaining a second factor Ω^{-1}. This leaves a standard $\hat{\Pi}^{(N)}$ diagram with at most six additional vertices, and this can be bounded by a constant.

The general case is handled similarly. Each ω_l' gives rise to a diagram line that produces a factor Ω^{-1}, creating an overall factor $\Omega^{-R''}$. In bounding diagram lines iteratively, we may encounter lines with extra vertices (where lines already bounded were previously attached). However, the number of these vertices on any one line is less than $4R''$, since each ω_l' adds in total at most four vertices to the diagram, as in Figure 2. After all the lines due to the ω_l' have been bounded using suprema, we are left with a standard $\hat{\Pi}^{(N)}$ diagram, again with at most $4R''$ extra vertices. This is bounded by a constant depending on R'' and M, for n sufficiently large, using (5.5). The number of diagrams produced depends only on M and R''. Since $R'' < R$, we end up with an overall bound that is a constant multiple of $\Omega^{-R''}$, where the constant depends on R and M.

This completes the proof. □

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