The lace expansion on a tree with application to networks of self-avoiding walks

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July 8, 2002

Abstract

The lace expansion has been used successfully to study the critical behaviour in high dimensions of self-avoiding walks, lattice trees and lattice animals, and percolation. In each case, the lace expansion has been an expansion along a time interval. In this paper, we introduce the lace expansion on a tree, in which 'time' is generalised from an interval to a tree. We develop the expansion in the context of networks of mutually-avoiding self-avoiding walks joined together with the topology of a tree, in dimensions $d > 4$, and prove Gaussian behaviour for sufficiently spread-out networks consisting of long self-avoiding walks.

1 Introduction and results

1.1 Introduction

This paper has two distinct but related goals: (i) development of the lace expansion on a tree, and (ii) application of the lace expansion on a tree to the analysis of cycle-free networks of mutually-avoiding spread-out self-avoiding walks on $\mathbb{Z}^d$ in dimensions $d > 4$.

The lace expansion was introduced in [2] for the study of weakly self-avoiding walks in dimensions $d > 4$. It has since been applied to the strictly self-avoiding walk, to lattice trees and lattice animals, and to oriented and unoriented percolation. For reviews, see [10, 15, 17]. In previous work, the 'time' variable for the lace expansion has been indexed by an interval. Our first goal is to generalise the theory of laces to the case where the time variable is indexed by a tree.

Our second goal is to apply the lace expansion on a tree to cycle-free networks of self-avoiding walks. A single self-avoiding walk is often used as a model of a linear polymer in a good solution. Networks of self-avoiding walks can be used to model networks of polymers containing monomers

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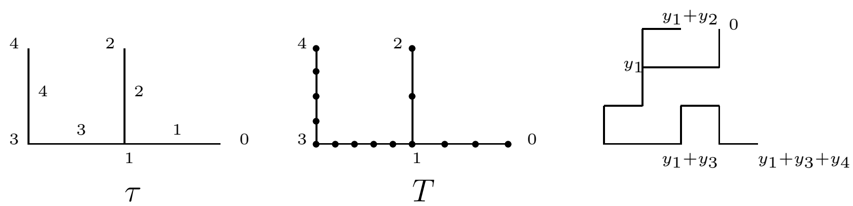

Figure 1: A shape τ with vertex and edge labels, the tree $T = (\tau, \vec{n})$ with $\vec{n} = (3, 2, 5, 4)$, and the range of a self-avoiding mapping $\omega \in \Omega_T(\vec{y})$ in $\mathbb{Z}^2$ with $y_1 = (-2, -1)$, $y_2 = (1, 1)$, $y_3 = (1, -2)$, $y_4 = (2, 0)$.

capable of making more than two chemical bonds, leading to branching. The rich critical behaviour expected for polymer networks when $d \leq 4$ has been studied in the physics literature [4, 5], but remains open as a mathematical problem. In this paper, we prove Gaussian behaviour for cycle-free networks in dimensions $d > 4$, for sufficiently "spread-out" models.

In [13], our methods are extended and applied to networks that are permitted to contain cycles. We expect that the lace expansion on a tree can also be applied to lattice trees, to simplify and extend the results of [3, 8]. In particular, the method should permit the double expansion used in [3, 8] to be performed instead in a simpler single step.

1.2 Trees and networks

For $r \geq 1$, let $\mathcal{T}_r$ denote the set of unlabelled trees with r edges. Given a tree, we denote the degree of a vertex i by d_i. Vertices of degree 1 will be referred to as *leaves* and vertices of degree 2 will be referred to as *path points*. Given $\tau \in \mathcal{T}_r$, we call one of its leaves the root and label the root by 0. The remaining vertices in τ are labelled in a fixed but arbitrary manner. We also label the edges of τ in a fixed but arbitrary manner. Edges of τ are regarded as directed away from the root. We refer to an element of $\mathcal{T}_r$, together with its vertex and edge labels, as a *shape*.

For $\vec{n} = (n_1, \ldots, n_r)$ with each n_i a positive integer, let $\mathcal{T}_r(\vec{n})$ denote the set of trees T that can be obtained by picking a shape $\tau \in \mathcal{T}_r$, and inserting $n_j - 1$ path points on edge j of τ ($j = 1, \ldots, r$). We refer to the vertices in T that correspond to the original vertices of τ in this procedure, i.e., the vertices that are not inserted as path points, as *branch points*. Note that a branch point i can have any degree $d_i \geq 1$. We identify T with the pair $(\tau, \vec{n})$.

Fix $T = (\tau, \vec{n}) \in \mathcal{T}_r(\vec{n})$. Let ω be a mapping from the vertex set of T into $\mathbb{Z}^d$, such that the root of T is mapped to the origin of $\mathbb{Z}^d$. For a directed edge $j = (j_1, j_2)$ in τ, let y_j denote the displacement $\omega(j_2) - \omega(j_1)$ of the embedded path in T corresponding to that edge. We write $\vec{y} = (y_1, \ldots, y_r)$ with each $y_j \in \mathbb{Z}^d$. Given $\vec{y}$ and T, let $\Omega_T(\vec{y})$ denote the set of ω such that $\omega(j_2) - \omega(j_1) = y_j$ for each edge j of τ. Thus $\Omega_T(\vec{y})$ consists of mappings such that the branch points in T corresponding to the vertices in edge j of τ are embedded in $\mathbb{Z}^d$ in such a way that they are separated by the displacement y_j. See Figure 1.

Given a function $D : \mathbb{Z}^d \to \mathbb{R}$, to each ω we associate a weight

$$W(\omega) = \prod_{i \in T : i \neq 0} D(\omega(i) - \omega(i^-)), \tag{1.1}$$

where the product is over the vertices of T and i^- denotes the neighbour of $i \in T$ that is closest to the root. Also, for each ω let

$$U_{ij}(\omega) = \begin{cases} -1 & \text{if } \omega(i) = \omega(j) \\ 0 & \text{if } \omega(i) \neq \omega(j). \end{cases} \tag{1.2}$$

The product $\prod_{i,j \in T: i \neq j}(1 + U_{ij}(\omega))$ is nonzero if and only if the mapping ω is one-to-one. In other words, this product is nonzero precisely when the image of T under ω is a network of mutually-avoiding self-avoiding walks with the topology of τ. For $T \in \mathcal{T}_r(\vec{n})$ and $\vec{y} \in \mathbb{Z}^{dr}$, we define

$$c_T(\vec{y}) = \sum_{\omega \in \Omega_T(\vec{y})} W(\omega) \prod_{i,j \in T: i \neq j} (1 + U_{ij}(\omega)). \tag{1.3}$$

The basic quantity of interest is $c_T(\vec{y})$. A *network* is an ω that gives rise to a nonzero contribution to (1.3), and we think of a network as identified with the image of ω in $\mathbb{Z}^d$. For example, with T and $\vec{y}$ as in Figure 1, and with $D(x) = 1$ if and only if x is one of the four unit vectors in $\mathbb{Z}^2$, $c_T(\vec{y})$ counts the number of network configurations in which the branches of T undergo nearest-neighbour mutually-avoiding self-avoiding walks on $\mathbb{Z}^2$ with the walk displacements specified by $\vec{y}$. The network depicted in Figure 1 gives one contribution to $c_T(\vec{y})$.

The special case in which the shape of the network is the graph τ_r consisting of r edges ($r \geq 1$) joined together at a single vertex of degree r is called a star-shaped network. Star-shaped networks will be central in our analysis.

Our results are in terms of the Fourier transform, which is defined for an absolutely summable function $f : \mathbb{Z}^{dr} \mapsto \mathbb{C}$ by

$$\hat{f}(\vec{k}) = \sum_{\vec{y} \in \mathbb{Z}^{dr}} f(\vec{y}) e^{i\vec{k}\cdot\vec{y}}, \qquad \vec{k} \in [-\pi, \pi]^{dr}, \tag{1.4}$$

with $\vec{k}\cdot\vec{y} = \sum_{j=1}^r k_j \cdot y_j$ ($k_j \in [-\pi,\pi]^d$, $y_j \in \mathbb{Z}^d$). We make the abbreviation $c_T = \hat{c}_T(\vec{0}) = \sum_{\vec{y}} c_T(\vec{y})$. When τ consists of a single edge, so that T can be identified with the interval $[0,n]$, we write simply $c_n(x) = c_T(x)$. We also write $n = \sum_j n_j = |T|$ to denote the number of edges in T, and, for $\vec{k} \in [-\pi,\pi]^{dr}$, we write $|k|^2 = \sum_{j=1}^r |k_j|^2$. Finally, we write $f_n \sim g_n$ to denote $\lim_{n\to\infty} f_n/g_n = 1$.

1.3 Main result

To state our main result, we must first specify the weight function D. The weight function D is required to be invariant under permutation of the components of x and under replacement of any component of x by its negative, and is assumed to obey the properties of Assumption D of [12]. These properties involve a positive parameter L, which serves to spread out the embeddings, and which we will take to be large. The properties are as follows. We require that $D(x) \geq 0$, $D(0) = 0$, $\sum_{x \in \mathbb{Z}^d} D(x) = 1$ and that there is an $\epsilon > 0$ such that $\sum_x |x|^{2+2\epsilon} D(x) < \infty$. In this paper, we strengthen the latter to require that

$$\sup_{x \in \mathbb{Z}^d} |x|^2 D(x) \leq CL^{2-d} \quad \text{and} \quad \sum_{x \in \mathbb{Z}^d} |x|^{2+2\epsilon} D(x) \leq CL^{2+2\epsilon}. \tag{1.5}$$

We require that $D(x) \leq CL^{-d}$ uniformly in x. Let

$$\sigma^2 = \sum_{x \in \mathbb{Z}^d} |x|^2 D(x), \tag{1.6}$$

where $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^d$. Denoting the supremum norm by $\|\cdot\|_\infty$, and writing $\hat{D}(k) = \sum_{x\in\mathbb{Z}^d} D(x)e^{ik\cdot x}$ ($k \in [-\pi, \pi]^d$), we also require that there are positive constants η, c_1, c_2 such that

$$c_1 L^2|k|^2 \leq 1 - \hat{D}(k) \leq c_2 L^2|k|^2 \quad (\|k\|_\infty \leq L^{-1}), \tag{1.7}$$

$$1 - \hat{D}(k) > \eta \quad (\|k\|_\infty \geq L^{-1}), \tag{1.8}$$

$$1 - \hat{D}(k) < 2 - \eta \quad (k \in [-\pi, \pi]^d). \tag{1.9}$$

It follows from (1.7) that σ is bounded above and below by multiples of L. Examples of functions D obeying the above assumptions are given in [12]. A simple example is

$$D(x) = \begin{cases} [(2L+1)^d - 1]^{-1} & 0 < \|x\|_\infty \leq L \\ 0 & \text{otherwise.} \end{cases} \tag{1.10}$$

With the above assumptions on D, we will show that the methods of [12] can be applied to the simplest network, for which $r = 1$ and τ is a single edge joining two vertices.

Our main result is the following theorem for sufficiently spread-out ($L \gg 1$) self-avoiding networks in dimensions $d > 4$. By definition, we take $V_1 = 1$. The constants V_{d_i} in the theorem depend only on the degree d_i of the branch point i, and will be referred to as *vertex factors*. The ϵ appearing in the theorem is the one appearing in (1.5). Our proof of the theorem is restricted to large L, although we expect the result to remain true for all $L \geq 1$.

Theorem 1.1. *Let $d > 4$, $\delta \in (0, \epsilon \wedge 1 \wedge \frac{d-4}{2})$, $r \geq 1$, and $\vec{n} = (n_1, \ldots, n_r)$ with $n = \sum_{j=1}^r n_j$, $n_j \sim nt_j$ and each $t_j \in (0, 1]$. Fix $T = (\tau, \vec{n}) \in \mathcal{T}_r(\vec{n})$. There is an L_0 such that for $L \geq L_0$ there exist positive constants v, μ, $V_2, V_3, \ldots$, $A = V_2^{-1}$ (all depending on d and L), and C_1, C_2 (depending on d but not L) such that the following statements hold as $n \to \infty$:*
(a) For all $\vec{k} \in \mathbb{R}^{dr}$ with $|k|^2$ bounded by a constant,

$$\hat{c}_T(\vec{k}(\sigma^2 vn)^{-1/2}) = A^r \Big(\prod_{i\in\tau} V_{d_i}\Big)\mu^n e^{-\sum_{j=1}^r |k_j|^2 t_j/2d}\big[1 + \mathcal{O}(n^{-(d-4)/2}) + \mathcal{O}(|k|^2 n^{-\delta})\big]. \tag{1.11}$$

(b)

$$\frac{1}{c_T}\sum_{\vec{y}\in\mathbb{Z}^{dr}} |y_j|^2\, c_T(\vec{y}) = \sigma^2 vn_j\big[1 + \mathcal{O}(n^{-\delta})\big] \quad (j = 1, \ldots, r). \tag{1.12}$$

(c)

$$C_1\mu^n L^{-dr} n^{-dr/2} \leq \sup_{\vec{x}\in\mathbb{Z}^{dr}} c_T(\vec{x}) \leq C_2\mu^n L^{-dr} n^{-dr/2}. \tag{1.13}$$

Constants implied by the $\mathcal{O}$ notation in error terms may depend on L. In addition, these constants are not uniform as $t_j \to 0$. Uniformity in this limit is not possible, since $t_j = 0$ effectively changes the underlying shape τ of T, which affects the vertex factors appearing in the leading behaviour of (1.11).

Since (1.11) gives $c_n = A\mu^n[1 + \mathcal{O}(n^{-(d-4)/2})]$, μ must be the "connective constant" given by $\mu = \lim_{n\to\infty} c_n^{1/n}$. We will see in (4.8) that $\mu = 1 + \mathcal{O}(L^{-d})$. Also, (1.12) gives $c_n^{-1}\sum_{x\in\mathbb{Z}^d}|x|^2 c_n(x) = \sigma^2 vn[1 + \mathcal{O}(n^{-\delta})]$, so $\sigma^2 v$ is the diffusion constant. Setting $k = 0$ in (1.11) gives $c_T^{1/n} \to \mu$ as $n = |T| \to \infty$. This shows that the connective constant μ also serves as the growth constant for the networks treated in Theorem 1.1. For $d = 3$, a closely related result is proved in [19]; the proof extends to general $d \geq 2$ [18].

Theorem 1.1 states that the constants A and V_2 are related by $A = V_2^{-1}$. In fact, this is required for consistency of Theorem 1.1. To see this, consider the statement of Theorem 1.1(a) for $r = 1$, $k = 0$ and $T = [0, 2n]$. In this case, Theorem 1.1(a) states that $c_{2n} \sim A\mu^{2n}$. On the other hand, we may regard $[0, 2n]$ as the unique tree in $\mathcal{T}_2(n, n)$, in which case Theorem 1.1(a) with $(k_1, k_2) = (0, 0)$ gives $c_{2n} \sim A^2 V_2 \mu^{2n}$. Therefore it must be the case that $A = V_2^{-1}$.

Consider the special case of a star-shaped network with $\tau = \tau_r$, with D given by (1.10). In this case, the $k = 0$ case of Theorem 1.1(a) states that $c_T = \hat{c}_T(0) \sim A^r V_r \mu^n = V_r \prod_{j=1}^r (A\mu^{n_j})$. This means that the number $[(2L+1)^d - 1]^n c_T$ of configurations of the star-shaped network, with arbitrary spatial locations for its leaves, is asymptotically equal to the number $\prod_{j=1}^r (A\mu^{n_j}[(2L+1)^d - 1]^{n_j})$ of configurations of a network of r *independent* self-avoiding walks, multiplied by a vertex factor V_r which takes into account the mutual avoidance of the r branches. The scaling by $n^{-1/2}$ of $\vec{k}$ in Theorem 1.1(a) indicates that a network with n vertices has spatial extent of order $n^{1/2}$, and this is reiterated in Theorem 1.1(b). Similar remarks apply for general cycle-free networks.

For $r = 1$, the results of Theorem 1.1 were proved using generating functions in [15, Theorems 6.1.1], but with weaker error estimates and without the lower bound of (c). We will give a very different proof of this simplest case, by applying the general inductive method of [12]. The inductive method requires the verification of certain assumptions, which we will verify in this paper. As is explained in [12], the inductive method, when combined with the verification of the assumptions provided below, also gives a version of a local central limit theorem for $c_n(x)$, when $r = 1$. This local central limit theorem is explicitly stated in [12, Theorem 1.3].

Theorem 1.1 can be used to conclude that a network consisting of a single self-avoiding walk converges weakly to Brownian motion, for $d > 4$ and large L. This is the content of the following corollary, which was proved using generating functions in [15, Theorem 6.1.8]. To illustrate an application of Theorem 1.1, we give a proof of the corollary.

Corollary 1.2. *Let T be an interval $[0, n]$, and assign probability c_n^{-1} to each self-avoiding walk of length n. Under the hypotheses of Theorem 1.1, the rescaled process $\big(\omega(\lfloor nt \rfloor)/\sqrt{\sigma^2 v n}\big)_{t \in [0,1]}$, linearly interpolated to produce a continuous mapping from $[0, 1]$ into $\mathbb{R}^d$, converges weakly to Brownian motion as $n \to \infty$.*

Proof. Weak convergence as a process is equivalent to convergence of finite-dimensional distributions and tightness [1]. Convergence of finite-dimensional distributions follows from Theorem 1.1(a), as follows. Let $r \geq 1$, and choose τ to be the path π_r of length r. The left side of (1.11) divided by c_n is then the characteristic function of the increments of a self-avoiding walk. By (1.11), this converges to $e^{-\sum_{j=1}^r |k_j|^2 t_j / 2d}$, which is the characteristic function of the increments of Brownian motion. (Note that $A^r \prod_{i \in \pi_r} V_{d_i} = A$ for all r, so the constants cancel when we normalise by c_n.) This proves the required convergence of finite-dimensional distributions. Tightness follows from a moment condition together with Theorem 1.1(b) for $r = 1$, as in the proof of [15, Lemma 6.6.3]. ∎

Convergence of the lace expansion for self-avoiding walks requires both $d > 4$ and the presence of a small parameter. In Theorem 1.1, we are obtaining the small parameter by taking L sufficiently large. Alternately, for the nearest-neighbour model, a small parameter could be introduced by replacing -1 in (1.2) by $-\lambda$. For $d > 4$, $r = 1$ and sufficiently small λ, results very close to Theorem 1.1 were proven in [2, 6, 11, 14]. For $d \geq 5$ and $r = 1$, results close to those of Theorems 1.1 were obtained for the nearest-neighbour strictly self-avoiding walk in [7, 9], via a computer-assisted proof. The extension to $r > 1$ in Theorem 1.1 is new.

1.4 Organisation

The remainder of this paper is organised as follows. Our first goal, development of the lace expansion on a tree, is carried out in Sections 2–3. In Section 2, we derive the lace expansion on a tree, and in Section 3, we develop the theory of laces. Our second goal, application of the lace expansion on a tree to networks of self-avoiding walks, is carried out in Sections 4–6. In Section 4, we reduce the proof of Theorem 1.1 to two propositions, Proposition 4.1 and 4.2. Proposition 4.1 gives bounds on the lace expansion on an interval, and Proposition 4.2 generalises this to bounds on the lace expansion on a tree. The proof of Theorem 1.1 in Section 4 is by induction on r, and the case $r = 1$ is proved by combining the general results of [12] with Proposition 4.1. Finally, we prove Propositions 4.1 and 4.2 in Sections 5 and 6, respectively.

2 The lace expansion

2.1 Graphs

Fix $r \geq 1$, $\vec{n} = (n_1, \dots, n_r)$ with each n_i a positive integer, and $T = (\tau, \vec{n}) \in \mathcal{T}_r(\vec{n})$. Given a set of values $U_{ij} \in \mathbb{R}$ for all pairs of distinct vertices $i, j \in T$, such as (1.2), we define

$$K[T] = \prod_{i,j \in T : i \neq j} (1 + U_{ij}). \tag{2.1}$$

We also define $K[T] = 1$ if T consists of a single vertex.

We refer to a pair $\{i, j\}$, with $i, j \in T$ as an *edge*. For simplicitly, we often write ij in place of $\{i, j\}$. We refer to i and j as the *endpoints* of ij. Since T is a tree, there is a unique path in T joining i and j. We require several definitions related to graphs on trees.

Definition 2.1. (a) An edge ij naturally determines an open interval of vertices of T, corresponding to the vertices of T on the path between i and j in T, with the endpoints i, j excluded.
(b) A *graph* on T is a set of edges. Let $\mathcal{B}(T)$ denote the set of all graphs on T.
(c) An edge *covers* all sites in the open interval in T determined by the edge. In addition, an edge containing a leaf as an endpoint *covers* that leaf. Given a graph $\Gamma \in \mathcal{B}(T)$, we say that a leaf or path point of T is *fully covered* by Γ if it is covered by an edge in Γ. We say a branch point of degree 2 is *fully covered* by Γ if it is covered by some edge of Γ. We say that a vertex b of degree $d_b \geq 3$ is *fully covered* by Γ if there exists a set of edges $i_1 j_1, \dots, i_l j_l \in \Gamma$ for some $l \geq \lceil \frac{d_b}{2} \rceil$ such that each of these edges covers b, with $\{i_1, j_1, \dots, , i_l, j_l\}$ containing elements of each of the d_b distinct subtrees of T that meet at b.
(d) A graph $\Gamma \in \mathcal{B}(T)$ is *connected* if each vertex of T is fully covered by Γ. We denote the set of connected graphs on T by $\mathcal{G}(T)$.
(e) Let S be a subtree of T. Given $\Gamma \in \mathcal{B}(T)$, we define the restriction of Γ to S, denoted $\Gamma|_S$, by $\Gamma|_S = \{ij \in \Gamma : i, j \in S\}$.

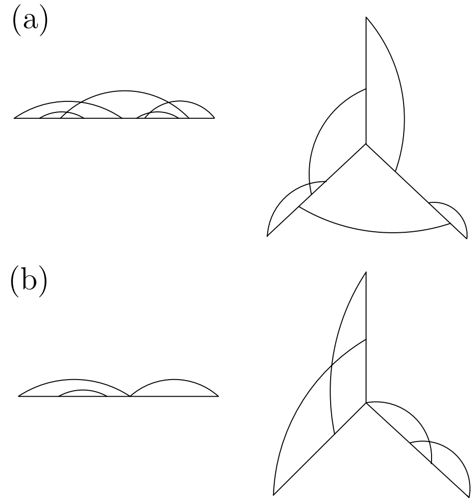

Figure 2: Examples of (a) connected graphs, (b) graphs that are not connected.

Expanding the product in (2.1) gives

$$K[T] = \sum_{\Gamma \in \mathcal{B}(T)} \prod_{ij \in \Gamma} U_{ij}. \tag{2.2}$$

We denote the restriction of the sum in (2.2) to connected graphs by

$$J[T] = \sum_{\Gamma \in \mathcal{G}(T)} \prod_{ij \in \Gamma} U_{ij}, \tag{2.3}$$

and set $J[T] = 1$ if T consists of a single vertex only.

Recall that branch points are defined to be the vertices of τ, identified with vertices of T, and that the root is a leaf. Let B denote the set of branch points that are not leaves of T. Suppose that T is such that $|B| \geq 2$, and let $b \in B$ denote the branch point adjacent to the root in τ. Define $\mathcal{E}(T) \subset \mathcal{B}(T)$ to be the set of all graphs Γ such that there is a subtree of T containing b and containing another vertex of B, both not as a leaf, and such that the restriction of Γ to that subtree is a connected graph on the subtree. We define

$$E[T] = \sum_{\Gamma \in \mathcal{E}(T)} \prod_{ij \in \Gamma} U_{ij}. \tag{2.4}$$

If we wish to emphasise the dependence of $E[T]$ or $\mathcal{E}(T)$ on b, then we will write $E_b[T]$ or $\mathcal{E}_b(T)$. If T is such that $|B| \leq 1$, then we set $\mathcal{E}(T) = \varnothing$ and $E[T] = 0$. In particular, $E[T] = 0$ if $r = 1, 2$, or if T has the star shape τ_r for any $r \geq 3$.

Let $r \geq 2$, and let b be the branch point of τ adjacent to the root. In our arbitary but fixed labelling of the edges of τ, we choose to label the edge joining the root to b by 1, and the remaining edges emanating from b by $2, \ldots, d_b$. Let $\bar{T}$ be the subtree of T consisting of the d_b paths of T corresponding to edges $1, \ldots, d_b$ of τ (including the branch points at which these paths terminate). Given a subtree $S \subset \bar{T}$ containing b, we denote the leaves of S by $s_1, \ldots, s_{d_b}$, where s_e lies in the path in T corresponding to edge e of τ. It is possible, for each e, that $s_e = b$. Thus S has the star shape τ_s for some $s \in \{0, 1, \ldots, d_b\}$, where τ_0 denotes the trivial graph consisting of a single vertex and no edges.

For $r = 1$, we may regard T as the interval $[0, n_1]$. In this case, we take $b = 0$ and $S = [0, s_1]$. We let $S_1 = [s_1, n_1]$ if $s_1 > 0$, and $S_1 = [1, n_1]$ if $s_1 = 0$.

For $r \geq 2$, we let S_1 be the path joining the root to s_1. For $e = 2, \ldots, d_b$, we let S_e denote the subtree of T consisting of s_e and all vertices and edges that can be reached by a path from the root after passing through s_e.

The quantities K, J and E are related by the following lemma.

Lemma 2.2. *For $r \geq 1$ and $T \in \mathcal{T}_r(\vec{n})$,*

$$K[T] = \sum_{S\colon b \in S \subset \bar{T}} J[S] \prod_{e=1}^{d_b} K[S_e] + E_b[T], \tag{2.5}$$

where the sum is over subtrees S of $\bar{T}$, and where b, $\bar{T}$ and S_e ($e = 1, \ldots, d_b$) are as defined above.

Proof. By (2.2), $K[T] = \sum_{\Gamma \in \mathcal{B}(T)} \prod_{ij \in \Gamma} U_{ij}$. We sum separately over $\mathcal{E}_b(T)$ and its complement. The contribution from $\mathcal{E}_b(T)$ gives the term $E_b[T]$ in (2.5). Given a graph $\Gamma \in \mathcal{B}(T) \backslash \mathcal{E}_b(T)$, let $S = S(\Gamma)$ denote the largest subgraph of T, containing the branch point b, such that $\Gamma|_S \in \mathcal{G}(S)$. Then $S \subset \bar{T}$, by definition of $\mathcal{E}_b(T)$. The first term on the right side of (2.5) corresponds to performing the sum over such graphs by first summing over subtrees S and then summing over all graphs such that $S(\Gamma) = S$. The latter sum factors to give the first term of (2.5).

The distinction made in the definition of S_1 for $r = 1$ and $s_1 = 0$ is due to the fact that, in this case, the connected component of Γ containing b is simply b itself. Since b is a leaf when $r = 1$, this means that the complement of this connected component in Γ is a graph on the interval obtained by removing b and its edge from T. ◻

2.2 Application to networks of self-avoiding walks

Let $T = (\tau, \vec{n}) \in \mathcal{T}_r(\vec{n})$, where $r \geq 1$. Combining (1.3), (2.1) and (2.5) gives

$$c_T(\vec{y}) = \sum_{\omega \in \Omega_T(\vec{y})} W(\omega) K[T] = \sum_{\omega \in \Omega_T(\vec{y})} W(\omega) \sum_{S\colon b \in S \subset \bar{T}} J[S] \prod_{e=1}^{d_b} K[S_e] + \sum_{\omega \in \Omega_T(\vec{y})} W(\omega) E[T]. \tag{2.6}$$

The first term on the right is the main term. To rewrite this term, we introduce some notation. We denote by $\bar{m} = (m_1, \ldots, m_{d_b})$ a vector having d_b components, in contrast to $\vec{n}$ which usually has r components. Given $\bar{m}$ with each component non-negative, we write $S_{\bar{m}}$ for the star-shaped tree with branches of length m_i. For $\bar{v} \in \mathbb{Z}^{d \cdot d_b}$, we define

$$\pi_{\bar{m}}^{(d_b)}(\bar{v}) = \sum_{\omega \in \Omega_{S_{\bar{m}}}(\bar{v})} W(\omega) J[S_{\bar{m}}]. \tag{2.7}$$

We also define

$$\varphi_T(\vec{y}) = \sum_{\omega \in \Omega_T(\vec{y})} W(\omega) E[T], \tag{2.8}$$

which vanishes for $r = 1, 2$ and which will constitute an error term for $r \geq 3$.

2.2.1 The case $r = 1$

Suppose $r = 1$. We identify T with the interval $[0, n]$, with $b = 0$. The case $r = 1$ is slightly different from $r \geq 2$, since in this case b is a leaf. In particular, the definition of S_1 above Lemma 2.2 is special when $S = \{b\}$, in which case $S_1 = [1, n]$.

In (2.6), we identify S with $[0, m]$, where $0 \leq m \leq n$. The sum over ω factors into a sum over independent walks on $[0, m]$ and walks on $[m, n]$. Extracting the $m = 0$ term explicitly, (2.6) becomes

$$
\begin{aligned}
c_n(y) &= \sum_{\omega \in \Omega_{[0,n]}(y)} W(\omega) K[1, n] + \sum_{m=1}^{n} \sum_{u \in \mathbb{Z}^d} \pi_m^{(1)}(u) c_{n-m}(y - u) \\
&= \sum_{u \in \mathbb{Z}^d} D(u) c_{n-1}(y - u) + \sum_{m=2}^{n} \sum_{u \in \mathbb{Z}^d} \pi_m^{(1)}(u) c_{n-m}(y - u). \qquad (2.9)
\end{aligned}
$$

In the first term on the right, we have factored the sum over walks into independent sums over the first step and the last $n - 1$ steps. In the second term, we have made the observation that $\pi_1^{(1)}(u) = \sum_{\omega \in \Omega_{[0,1]}(u)} D(u) U_{01}(\omega) = 0$ because $D(u) = 0$ if $u = 0$ and $U_{01}(\omega) = 0$ if $u \neq 0$. The identity (2.9) is the basic identity underlying the analysis of [2].

2.2.2 The case $r \geq 2$

Suppose $r \geq 2$. In the first term on the right side of (2.6), the sum over ω factors into $d_b + 1$ independent summations, corresponding to the portions of ω indexed by S and by $S_1, \ldots, S_{d_b}$. There are also implied sums over $\bar{m}$ and $\bar{v}$, which respectively represent the lengths of the branches of S and the spatial location of the leaves of S. The contribution from each S_e give rise to $c_{S_e}(\vec{y}_e - v_e)$, where $\vec{y}_e$ represents the subset of the components of $\vec{y}$ that label the edges of the shape of S_e, and $\vec{y}_e - v_e$ represents subtraction of v_e from the component of $\vec{y}_e$ corresponding to an edge of τ incident on b. Thus we obtain

$$
c_T(\vec{y}) = \sum_{\bar{0} \leq \bar{m} \leq \bar{n}} \sum_{\bar{v} \in \mathbb{Z}^{d \cdot d_b}} \pi_{\bar{m}}^{(d_b)}(\bar{v}) \prod_{e=1}^{d_b} c_{S_e}(\vec{y}_e - v_e) \; + \; \varphi_T(\vec{y}). \qquad (2.10)
$$

Here the notation $\bar{0} \leq \bar{m} \leq \bar{n}$ denotes summation over $0 \leq m_e \leq n_e$ for each $e = 1, \ldots, d_b$.

This is simplest when T is star shaped. In this case, T can be written as $T = (\tau_r, \bar{n})$ for some $\bar{n} = (n_1, \ldots, n_r)$ ($r = d_b \geq 2$), and we write $c_{\bar{n}}(\bar{y}) = c_T(\bar{y})$. Now $S_1, \ldots, S_{d_b}$ are all intervals, and (2.6) becomes

$$
c_{\bar{n}}(\bar{y}) = \sum_{\bar{0} \leq \bar{m} \leq \bar{n}} \sum_{\bar{v} \in \mathbb{Z}^{d \cdot d_b}} \pi_{\bar{m}}^{(d_b)}(\bar{v}) \prod_{e=1}^{d_b} c_{n_e - m_e}(y_e - v_e). \qquad (2.11)
$$

The special case of (2.11) with $r = 2$, corresponding to the shape τ_2 in which two leaves are joined by edges to a branch point of degree 2, was used in [16].

2.3 Laces

We use the notion of a *lace* to analyse $\pi_{\bar{m}}^{(d_b)}(\bar{v})$ of (2.7). This involves only the star shapes τ_r ($r \geq 1$). Accordingly, we restrict attention now to a tree T with the shape τ_r. We denote the

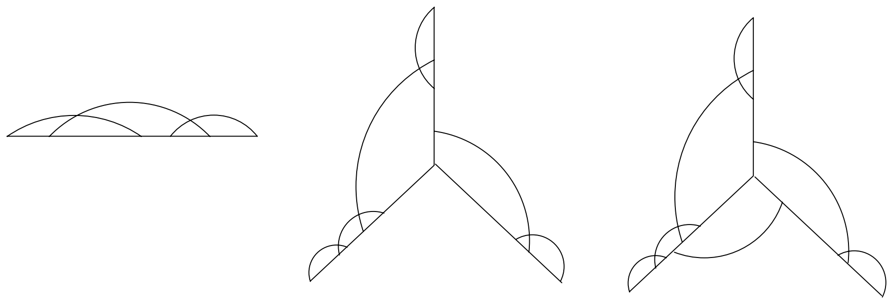

Figure 3: Examples of laces when $r = 1, 3$. The third lace is not minimally connected.

vertex of T corresponding to the vertex of τ_r of degree r by b, so that $d_b = r$. For $r = 1$, τ_1 is a single edge joining two vertices, and we take $b = 0$. In general, $T = (\tau_r, \bar{n})$ consists of a branch point b and r paths, of lengths $n_1, \ldots, n_r$, joining b to r leaves. We denote these paths (including their endpoints) as T_e ($e = 1, \ldots, r$). Throughout this section, we consider only such trees T. We begin with a preliminary definition.

Definition 2.3. Let $T = (\tau_r, \bar{n})$ with $r \geq 1$. Fix a connected graph $\Gamma \in \mathcal{G}(T)$ and a branch T_e, and let $\Gamma_e(\Gamma)$ denote the subgraph of Γ consisting of those edges that cover b and contain an endpoint in T_e. We can write $\Gamma_e(\Gamma) = \{i_1 j_1, \ldots, i_l j_l\}$, with each $j_m \in T_e$ and each $i_m \notin T_e$ (unless $r = 1$ in which case $i_m = b = 0$). We select from $\Gamma_e(\Gamma)$ the element or elements for which the distance from j_m to b is maximal. If there is a unique such edge, then we say it is the edge of Γ that is *associated to branch* T_e. If there is more than one such edge, then we select from those with j_m maximally distant from b the one with i_m furthest from b. If this still does not specify an edge uniquely then we choose the i_m that lies on the branch with highest label, and refer to $i_m j_m$ as the edge associated to branch T_e. We write $i^{(e)} j^{(e)}$ for the edge associated to T_e. For $r = 1$, $i^{(e)} = b = 0$, while for $r \geq 2$, $i^{(e)} \notin T_e$ and $j^{(e)} \in T_e$.

Definition 2.4. For $r \geq 1$, a *lace* on $T = (\tau_r, \bar{n})$ is a connected graph L such that: (a) if $ij \in L$ covers the branch point, then it is associated to a branch e for some e; and (b) if $ij \in L$ does not cover the branch point, then $L \backslash \{ij\}$ is not connected. We denote the set of laces on T by $\mathcal{L}(T)$.

Examples of laces are depicted in Figure 3. For $r = 1$, a lace L is minimally connected, i.e., no proper subset of L is connected. However, for $r \geq 2$, a lace may not be minimally connected.

We now define a prescription that associates to a connected graph $\Gamma \in \mathcal{G}(T)$ a corresponding lace $L \subset \Gamma$ on T. Given $\Gamma \in \mathcal{G}(T)$ and a branch T_e of T, we first define the T_e*-lace construction*. This construction produces a lace $\mathsf{L}_\Gamma(e)$ on a subinterval of T that contains T_e. For $r = 1$, the T_1-lace construction is exactly the prescription to obtain a lace from a connected graph that was first introduced by Brydges and Spencer [2]. For $r \geq 1$, $\mathsf{L}_\Gamma(e)$ consists of edges $i_1 j_1, i_2 j_2, \ldots, i_N j_N$, determined as follows. First, j_1, i_1 are given by

$$j_1 = \max\{j : \exists i : ij \in \Gamma_e(\Gamma)\}, \qquad i_1 = \min\{i : ij_1 \in \Gamma_e(\Gamma)\}, \tag{2.12}$$

where the order implied by the max and min is the order on T_e obtained by identifying T_e with $[0, n_e]$ with 0 identified with b, supplemented by identifying a vertex $i \notin T_e$ that lies a distance r_i from b with the integer $-r_i$. If i_1 is not unique, then we pick i_1 to lie on the branch with

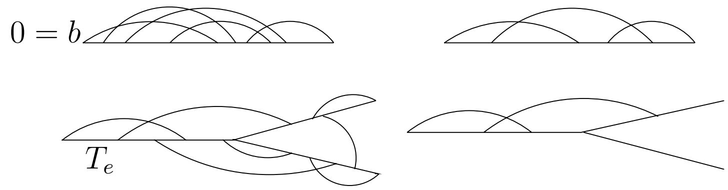

Figure 4: Examples of the T_e-lace construction, for $r = 1, 3$.

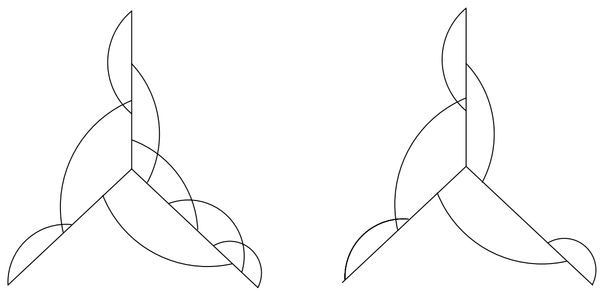

Figure 5: A connected graph Γ and its associated lace L_Γ.

highest possible label. The above is a reformulation of the definition of the edge associated to T_e in Definition 2.3, and $i_1 j_1 = i^{(e)} j^{(e)}$. If j_1 is not the leaf of T_e, then we define $j_2, i_2, \ldots$ by

$$j_{p+1} = \max\{j : \exists i < j_p \text{ such that } ij \in \Gamma\}, \qquad i_{p+1} = \min\{i : ij_{p+1} \in \Gamma\}. \tag{2.13}$$

The procedure terminates as soon as j_N is the leaf. It is clear from the construction that $\mathsf{L}_\Gamma(e)$ is a lace on an interval; see Figure 4. Our prescription that associates a lace L_Γ to a connected graph $\Gamma \in \mathcal{G}(T)$ is then given by

$$\mathsf{L}_\Gamma = \cup_{e=1}^{r} \mathsf{L}_\Gamma(e). \tag{2.14}$$

An example is depicted in Figure 5. The following proposition shows that L_Γ is indeed a lace.

Proposition 2.5. *Given $r \geq 1$, a tree $T = (\tau_r, \bar{n})$, and a connected graph $\Gamma \in \mathcal{G}(T)$, the graph L_Γ is a lace on T.*

Proof. By construction, the graph L_Γ is connected. In fact, for each branch T_e, every site $i \in T_e$ other than the branch point b is fully covered by an edge in the T_e-lace construction. Also, each T_e-lace construction produces an edge with an endpoint on T_e that covers b. Therefore b is fully covered.

To prove (a) in Definition 2.4, we note that an edge ij covering b can only be in L_Γ when it is obtained as the first edge $i_1 j_1$ in a T_e-lace construction. By definition, $i_1 j_1$ is associated to branch e. This proves (a).

To prove (b) in Definition 2.4, we note that when ij does not cover b, it can only be in L_Γ if $ij = i_p j_p$ is obtained in a T_e-lace construction with $p \geq 2$. If j_p is the leaf, then $\mathsf{L}_\Gamma \backslash \{i_p j_p\}$ does not cover the leaf and hence is not connected. If j_p is not the leaf, then there must be an edge $i_{p+1} j_{p+1} \in \mathsf{L}_\Gamma(e)$ with $i_{p+1} \geq j_{p-1}$. Then $\mathsf{L}_\Gamma \backslash \{i_p j_p\}$ does not cover the interval $[j_{p-1}, i_{p+1}]$ and hence is not connected. ☐

In the next proposition, we characterise the connected graphs Γ such that $\mathsf{L}_\Gamma = L$, using a notion of compatibility. Given a lace $L \in \mathcal{L}(T)$, we define $\mathcal{C}(L)$ to be the set of edges $ij \notin L$, $i, j \in T$, such that $\mathsf{L}_{L\cup\{ij\}} = L$. In other words, $\mathcal{C}(L)$ consists of those edges ij on T which are *compatible* with L in the sense that the connected graph Γ obtained by adding ij to L will yield L under the prescription L.

Proposition 2.6. $\mathsf{L}_\Gamma = L$ *if and only if* $L \subset \Gamma$ *is a lace and* $\Gamma \backslash L \subset \mathcal{C}(L)$.

Proof. Suppose that $\mathsf{L}_\Gamma = L$. By definition, $L = \mathsf{L}_\Gamma \subset \Gamma$, and L is a lace by Proposition 2.5. To prove the forward implication, it suffices to show that if $st \in \Gamma\backslash L$, then $st \in \mathcal{C}(L)$. The edges in L_Γ were chosen optimally with respect to the maxima and minima in (2.12)–(2.13), among all edges in Γ including st. Therefore the same edges will be selected in applying the prescription to $L \cup \{st\}$, i.e., $\mathsf{L}_{L\cup\{st\}} = L$ and hence $st \in \mathcal{C}(L)$.

Conversely, suppose that $L \subset \Gamma$ is a lace and $\Gamma\backslash L \subset \mathcal{C}(L)$. We show that $\mathsf{L}_\Gamma = L$ by showing that $\mathsf{L}_\Gamma \subset L$ and $L \subset \mathsf{L}_\Gamma$.

To prove that $\mathsf{L}_\Gamma \subset L$, we show that if $st \in \mathcal{C}(L) \cap \Gamma$ then $st \notin \mathsf{L}_\Gamma$. In fact, $st \in \mathcal{C}(L)$ means that $\mathsf{L}_{L\cup\{st\}} = L$, or, in other words, that edges in L are preferred over st in the max and min of (2.12)–(2.13). Therefore st will also not be selected in L_Γ.

Finally, we prove that $L \subset \mathsf{L}_\Gamma$. We first consider an edge $ij \in L$ that covers b. By Definition 2.4, $ij = i_L^{(e)} j_L^{(e)}$ for some e, where the subscript L indicates that ij is the edge in L associated to T_e. Since every $st \in \Gamma\backslash L$ is in $\mathcal{C}(L)$, it follows that $ij = i_L^{(e)} j_L^{(e)}$ is equal to the edge $i_\Gamma^{(e)} j_\Gamma^{(e)}$ in Γ associated to T_e. By definition of L_Γ, $i_\Gamma^{(e)} j_\Gamma^{(e)} \in \mathsf{L}_\Gamma$ and hence $ij \in \mathsf{L}_\Gamma$.

Next, suppose that $ij \in L$ does not cover b, so there is an e such that $i, j \in T_e$. If there is such an edge in L that is not in L_Γ, then there is a first such edge along T_e (starting from b). But then the T_e-lace construction applied to Γ selects an edge in $\Gamma\backslash L$ rather than ij. This selected edge cannot be compatible with L, contradicting $\Gamma\backslash L \subset \mathcal{C}(L)$. This completes the proof that $L \subset \mathsf{L}_\Gamma$. ∎

In particular, Proposition 2.6 implies that $\mathsf{L}_L = L$ if and only if L is a lace. Further development of the theory of laces is given in Section 3.

2.4 Resummation

We now use the prescription associating a lace L_Γ to a connected graph Γ, on a tree T with shape τ_r, to partially resum the sum over connected graphs in the definition (2.3) of $J[T]$. This leads to the resummation identity

$$\begin{aligned} J[S] &= \sum_{L\in\mathcal{L}(S)} \sum_{\Gamma:\mathsf{L}_\Gamma = L} \prod_{ij\in L} U_{ij} \prod_{i'j'\in\Gamma\backslash L} U_{i'j'} = \sum_{L\in\mathcal{L}(S)} \prod_{ij\in L} U_{ij} \sum_{C\subset\mathcal{C}(L)} \prod_{i'j'\in C} U_{i'j'} \\ &= \sum_{L\in\mathcal{L}(S)} \prod_{ij\in L} U_{ij} \prod_{i'j'\in\mathcal{C}(L)} (1 + U_{i'j'}), \end{aligned} \tag{2.15}$$

where the second equality follows from Proposition 2.6. This resummation achieves two results. First, the summation in (2.15) is over the set of laces, which is a much smaller set than the set of all connected graphs. Secondly, the self-avoidance interaction has been restored partially, on the compatible edges.

It will often be convenient to restrict attention to laces of a fixed size. For $N \geq 1$, let $\mathcal{L}^{(N)}(S)$ denote the set of laces on S consisting of exactly N edges. We define

$$J^{(N)}[S] = \sum_{L \in \mathcal{L}^{(N)}(S)} \prod_{ij \in L} (-U_{ij}) \prod_{i'j' \in \mathcal{C}(L)} (1 + U_{i'j'}) \tag{2.16}$$

and

$$\pi^{(d_b)}_{\bar{m},N}(\bar{y}) = \sum_{\omega \in \Omega_{S_{\bar{m}}}(\bar{y})} W(\omega) J^{(N)}[S_{\bar{m}}]. \tag{2.17}$$

By definition, $\pi^{(d_b)}_{\bar{m},N}(\bar{y}) \geq 0$, and

$$\pi^{(d_b)}_{\bar{m}}(\bar{y}) = \sum_{N=1}^{\infty} (-1)^N \pi^{(d_b)}_{\bar{m},N}(\bar{y}). \tag{2.18}$$

3 Classification of laces

In this section, we further develop the theory of laces. This theory will be needed in Section 6. Throughout this section, we fix $r = d_b \geq 2$ and a tree $T = (\tau_r, \bar{n})$ having the star shape τ_r. Thus b is the branch point in T of degree r. We always assume $n_e > 0$ for $e = 1, \ldots, r$.

3.1 Acyclic, cyclic and reducible laces

In this section, we derive some elementary consequences of the definition of a lace, and partition the set of laces into three distinct classes. Recall Definition 2.3, which defines the edge in a connected graph *associated* to a branch.

Lemma 3.1. *Let $T = (\tau_r, \bar{n})$ with $r = d_b \geq 2$. In a lace $L \in \mathcal{L}(T)$, the number of edges that cover the branch point b is at least $\lceil \frac{d_b}{2} \rceil$ and at most d_b.*

Proof. For the branch point to be fully covered, each branch of T must contain an endpoint of an edge that covers the branch point. Therefore L contains at least $\lceil \frac{d_b}{2} \rceil$ edges. On the other hand, each edge ij covering the branch point must be the edge associated to some branch by Definition 2.4(a). Hence, there are at most d_b edges covering the branch point. □

Let $L \in \mathcal{L}(T)$. Since $\mathsf{L}_L = L$, where L is given by the algorithm of Section 2.3, it follows that every edge in L that covers the branch point is the edge associated to some branch (possibly to two branches). Therefore a lace L will have fewer than d_b branches precisely when at least one edge is associated to both of the branches containing its endpoints.

Lemma 3.2. *Let $T = (\tau_r, \bar{n})$ with $r \geq 2$. Let $L \in \mathcal{L}(T)$ be a lace, and let S be a subtree of T (possibly with fewer branches than T) that contains b. If $L|_S$ is nonempty and $L|_S \in \mathcal{G}(S)$, then $L|_S \in \mathcal{L}(S)$.*

Proof. Suppose that $L \in \mathcal{L}(T)$ is such that $L|_S$ is nonempty and $L|_S \in \mathcal{G}(S)$. We will verify properties (a) and (b) of Definition 2.4.

For (a), if $ij \in L|_S \subset L$ covers the branch point, then it is associated in L to a branch e. However, recalling Definition 2.3, this implies that it is also associated in $L|_S$ to branch e.

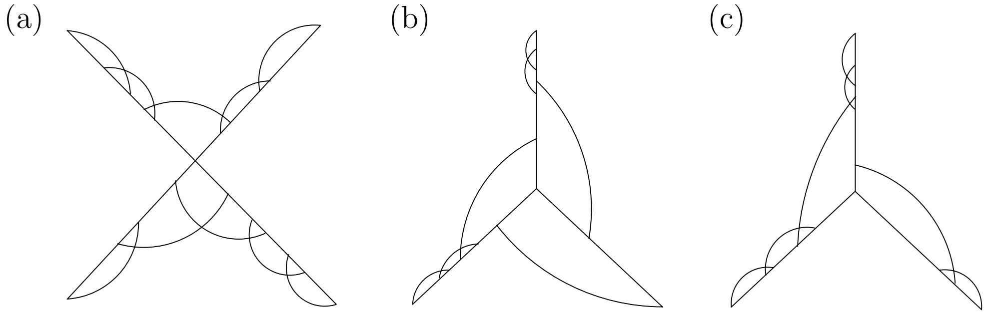

Figure 6: Examples of (a) reducible, (b) cyclic, (c) acyclic laces.

For (b), we must show that if $ij \in L|_S$ is such that $i, j \in T_e$ for some e, then $L|_S \backslash \{ij\}$ is not connected. Since $ij \in L$ and $L = \cup_{e=1}^{r} \mathsf{L}_L(e)$, it follows that $ij = i_p j_p$ for some $i_p j_p \in \mathsf{L}_L(e)$, with $p \geq 2$ (recall (2.13)). If j_p is a leaf of S, then j_p is not covered by $L|_S \backslash \{i_p j_p\}$. On the other hand, if j_p is not a leaf of S, then $i_p j_p$ is necessary in L to cover at least one vertex in the interval, and hence it is also necessary in $L|_S \subset L$. □

The following definition partitions $\mathcal{L}(T)$ into three disjoint classes.

Definition 3.3. Let $T = (\tau_r, \bar{n})$ with $r = d_b \geq 2$.
(a) A lace $L \in \mathcal{L}(T)$ is *reducible* if there is a proper subset $A \subset \{1, \ldots, d_b\}$ such that L can be written as a disjoint union of laces on each of $\cup_{e \in A} T_e$ and $\cup_{e \in A^c} T_e$. An *irreducible* lace is a lace that is not reducible.
(b) A lace is *cyclic* if it is irreducible and the edges covering the branch point b can be ordered as $\{i_k j_k : k = 1, \ldots, d_b\}$, with j_k and i_{k+1} on the same branch for $1 \leq k \leq d_b$. By convention, $i_{d_b+1} = i_1$.
(c) An irreducible lace that is not cyclic is called *acyclic*.

Note that when $d_b = 2, 3$, there are no reducible laces. Moreover, for $d_b \geq 4$, the set A in Definition 3.3(a) must have cardinality at least 2 and at most $d_b - 2$. Reducible laces do exist when $d_b \geq 4$. See Figure 6. For $d_b = 2$, a lace is cyclic if exactly two of its edges cover b, and is acyclic if exactly one of its edges covers b.

Definition 3.4. A connected graph (and, in particular, a lace) is called *minimal* if removal of any of its edges results in a graph that is not connected. A lace that is not minimal is called *non-minimal*.

Non-minimal laces do not exist for $r = 1$, but do exist for $r \geq 2$. Although it is possible for a lace to be non-minimal, any mimimal connected graph is a lace.

Definition 3.5. For $A \subset \{1, \ldots, r\}$, we write $T_A = \cup_{e \in A} T_e$ and $L_A = L|_{T_A}$. A lace L is said to have a *cyclic component* if there exists an $A \subset \{1, 2, \ldots, r\}$ such that L_A is a cyclic lace. A lace L is said to have a *non-minimal cyclic component* if it has a cyclic component that is non-minimal. We denote the set of all laces containing no non-minimal cyclic component by $\mathcal{M}(T)$, and let $\mathcal{N}(T) = \mathcal{L}(T) \backslash \mathcal{M}(T)$ denote the set of all laces containing a non-minimal cyclic component. (Note that laces in $\mathcal{M}$ need not be minimal.)

3.2 Properties of acyclic laces

The main result of this section is Lemma 3.7, which will be used in Section 6.2.3. We start with a preparatory lemma. Let $\Gamma(b) \subset L$ denote the set of edges in a lace L that cover the branch point b, and let M_e denote the number of edges in $\Gamma(b)$ that contain an endpoint in branch T_e.

Lemma 3.6. *Let $T = (\tau_r, \bar{n})$ with $r \geq 2$. If $L \in \mathcal{L}(T)$ is acyclic then there is at least one branch T_e with $M_e = 1$.*

Proof. There is nothing to prove if there is an e with $M_e = 1$, so assume that $M_e \geq 2$ for all e. Since $\Gamma(b)$ has at most d_b edges, $\sum_e M_e \leq 2d_b$. If $M_a \geq 3$ for some a, then since $M_e \geq 1$ for all e there must be a branch a' with $M_{a'} = 1$. Thus we may restrict attention to the case where $M_e = 2$ for all e.

Suppose $M_e = 2$ for all e. We will show that the lace is either reducible or cyclic. We refer to a pair of edges in $\Gamma(b)$ having an endpoint in branch e as e-partners. We pick one of the 1-partners, move to the branch e_0 (say) containing the other endpoint of that edge, identify the e_0-partner of that endpoint, move to the branch containing the other endpoint of the e_0-partner, and so on. We continue until we return to branch 1. Since $M_e = 2$ for all e, we always do return to branch 1. If all branches were visited in this operation, then the lace is cyclic by definition. If only a subset of the branches have been visited, then the lace is reducible. This follows, since the lace restricted to the set of visited branches will be a lace by Lemma 3.2, and similarly the restriction to the complement of the branches is a lace, since there are no edges in L that link the two sets of branches. $\square$

Given a lace $L \in \mathcal{L}(T)$, we denote the set of edges in L with both endpoints on branch T_e by $L^{(e)}$. We order the vertices on each branch from b (least vertex) to the leaf (greatest vertex).

Lemma 3.7. *Let $T = (\tau_r, \bar{n})$ with $r = d_b \geq 3$ and suppose $L \in \mathcal{L}(T)$ is acyclic. Then there is a branch T_a with $M_a = 1$ such that the restriction of L to $\cup_{e:e\neq a} T_e$ is a lace on $\cup_{e:e\neq a} T_e$.*

Proof. Let $O = \{e : M_e = 1\}$. Since L is acyclic, O is nonempty by Lemma 3.6. We will show that there exists an $a \in O$ such that T_a is as stated in the lemma.

Choose an $a_1 \in O$. Let $i_1 j_1$ denote the unique edge in $\Gamma(b)$ with $j_1 \in T_{a_1}$, and suppose i_1 is in branch $a_2 \neq a_1$. If $i_1 j_1$ is not the edge associated to branch a_2 (as in Definition 2.3), then the edge associated to branch a_2 will cover every vertex on T_{a_2} that is covered by $i_1 j_1$. In this case, by Lemma 3.2 the restriction of L to $\cup_{e:e\neq a_1} T_e$ is a lace on the restricted tree, and we are done.

So suppose, on the other hand, that $i_1 j_1$ is also associated to branch a_2. Then all vertices $i_1 \leq l \leq n_{a_2}$ of T_{a_2} are covered by the edges in $L^{(a_2)}$. Thus $L^{(a_2)} \cup \{i_1 j_1\} \cup L^{(a_1)}$ is a lace on $T_{a_1} \cup T_{a_2}$. Since L is irreducible, there must be an edge $i_2 j_2 \in L$ with $i_2 \in T_{a_2}$ and j_2 in a third branch T_{a_3}. We may choose $i_2 j_2$ (and a_3) such that i_2 is as close as possible to the branch point b. If $M_{a_3} = 1$, then branch T_{a_3} satisfies the requirement of the lemma, since $i_1 j_1$ covers the vertices on T_{a_2} that are covered by the edge $i_2 j_2$.

Therefore, we assume that there is another edge $i_3 j_3 \in \Gamma(b)$ with $i_3 \in T_{a_3}$. Since $i_2 j_2$ is not associated to branch T_{a_2}, it must be associated to branch T_{a_3}, and hence $i_3 \leq j_2$. We repeat the above procedure. At stage l, we denote the edge obtained by $i_l j_l$, with $i_l \in T_{a_l}$ and $j_l \in T_{a_{l+1}}$. As above, $T_{a_{l+1}}$ satisfies the claim if $M_{a_{l+1}} = 1$, since the edge $i_l j_l$ is not associated to branch T_{a_l}. On the other hand, if $M_{a_{l+1}} > 1$ then all other edges ij with precisely one endpoint i on T_{a_l} have $i \leq j_l$, since the edge $i_l j_l$ is associated to branch $T_{a_{l+1}}$.

Moreover, a new branch is visited at each stage of the construction. This can be seen from the fact that $i_p j_p$ is associated to branch $T_{a_{p+1}}$ for each p. Since a new branch is visited at each stage of the construction, the procedure must terminate on some branch. The terminal branch T_t must have $M_t = 1$. Since $i_{t-1} j_{t-1}$ is by construction associated to T_t but not to T_{t-1}, the restriction of L to the branches other than T_t is a lace. ☐

The above lemma has consequences not just for acyclic laces, as we show in the following two lemmas.

Lemma 3.8. *Let $r \geq 2$ and $T = (\tau_r, \bar{n})$. Let $L \in \mathcal{L}(T)$ be an irreducible lace. Then L has either $r-1$ or r edges covering the branch point.*

Proof. The proof is by induction on r. The statement of the lemma is true for $r = 2$. Let $r \geq 3$ and assume the statement for $r-1$. If L is cyclic, then L has r edges covering the branch point. If L is acyclic, then by Lemma 3.7 there is a branch T_a containing an endpoint of a unique edge in L covering the branch point, and such that the restriction of L to $\cup_{e:e\neq a} T_e$ is a lace. By the induction hypothesis, this restriction has either $r-1$ or $r-2$ edges covering the branch point. ☐

Lemma 3.9. *Let $r \geq 2$ and $T = (\tau_r, \bar{n})$. Let $L \in \mathcal{L}(T)$ be an irreducible lace. Then L has a cyclic component if and only if L has r edges covering the branch point.*

Proof. The proof is by induction on r. The statement of the lemma is true for $r = 2$. Let $r \geq 3$ and assume the statement for $r-1$. Suppose L has a cyclic component. If L is itself cyclic, then L has r edges covering the branch point. If L is acyclic, then by Lemma 3.7 there is a branch T_a containing an endpoint of a unique edge in L covering the branch point, and such that the restriction of L to $\cup_{e:e\neq a} T_e$ is a lace. The restriction of L to $\cup_{e:e\neq a} T_e$ must contain a cyclic component, and therefore must have $r-1$ edges covering the branch point. The converse follows similarly. ☐

3.3 Properties of minimal cyclic laces

The main result of this section is Lemma 3.11, which will be used in Section 6.2.4. The proof of Lemma 3.11 will make use of Lemma 3.10. Recall the definition of $L^{(e)}$ given above the statement of Lemma 3.7. We also recall that if $L \in \mathcal{L}(T)$ is cyclic then it contains precisely $r = d_b$ edges covering the branch point, and each branch of T contains endpoints of precisely two edges covering the branch point. Given a graph Γ, we denote its number of edges by $|\Gamma|$.

Lemma 3.10. *Let $T = (\tau_r, \bar{n})$ with $r \geq 2$, and let $L \in \mathcal{L}(T)$ be a minimal cyclic lace. Fix a branch T_e of T. Let $i_1^{(e)} j_1^{(e)} \in L$ denote the edge associated to branch e, and let $i_p^{(e)} j_p^{(e)}$ denote the edges of $\mathsf{L}_L(e)$ ordered from b to the leaf of e as in* (2.12)–(2.13). *Fix $ij \in L$ with $i \in T_e$, $j \notin T_e$ and $ij \neq i_1^{(e)} j_1^{(e)}$.*
(a) If $|L^{(e)}| = 0$ then $i < j_1^{(e)}$.
(b) If $|L^{(e)}| = 1$ then $i \leq i_2^{(e)} < j_1^{(e)}$.
(c) If $|L^{(e)}| \geq 2$ then $i \leq i_2^{(e)} < j_1^{(e)} \leq i_3^{(e)}$.

Proof. (a) If $|L^{(e)}| = 0$ then $j_1^{(e)}$ must be the leaf of branch e, which immediately implies that $j_1^{(e)} \geq i$. We need to rule out $j_1^{(e)} = i$. Since L is cyclic, the edge $i_1^{(e)} j_1^{(e)}$ is associated to T_e and to no other branch. Thus $i_1^{(e)} j_1^{(e)}$ is not required to cover any vertex on any other branch than T_e. Suppose $i_1^{(e)} \in T_{e'}$. If $j_1^{(e)} = i$, then each vertex covered by $i_1^{(e)} j_1^{(e)}$ is also covered by the union of ij

and $i_1^{(e')}j_1^{(e')}$, which contradicts the fact that L is a minimal lace.
(b) If $|L^{(e)}| = 1$ then $j_1^{(e)}$ cannot be the leaf of branch e, and $i_2^{(e)}j_2^{(e)}$ must cover $j_1^{(e)}$. Therefore $i_2^{(e)} < j_1^{(e)}$. If $i > i_2^{(e)}$, then each vertex covered by $i_1^{(e)}j_1^{(e)}$ is also covered by $\{ij, i_2^{(e)}j_2^{(e)}, i_1^{(e')}j_1^{(e')}\}$, where e' is such that $i_1^{(e)} \in T_{e'}$. Thus $i_1^{(e)}j_1^{(e)}$ can be removed without disconnecting the graph, which contradicts the fact that L is a minimal lace.
(c) The restriction of L to the union of T_e with the subset $[0, i_1^{(e)}]$ of $T_{e'}$ is $L^{(e)} \cup \{i_1^{(e)}j_1^{(e)}\}$, which is connected. By Lemma 3.2, it is therefore a lace. The last two inequalities follow from this. If $i > i_2^{(e)}$, then each vertex covered by $i_1^{(e)}j_1^{(e)}$ is also covered by $\{ij, i_2^{(e)}j_2^{(e)}, i_1^{(e')}j_1^{(e')}\}$. As in (b), this contradicts the fact that L is a minimal lace. □

Lemma 3.11. *Let $T = (\tau_r, \bar{n})$ with $r \geq 2$. Suppose $L \in \mathcal{L}(T)$ is a minimal cyclic lace. Let $i_1^{(1)}j_1^{(1)}$ denote the edge associated to branch 1, with $j_1^{(1)} \in T_1$. Fix $ij \in L$ with $i \in T_1 \backslash \{b\}$, $j \notin T_1$ and $ij \neq i_1^{(1)}j_1^{(1)}$. Let $T' = (\tau_r, \bar{n}')$, where n_1' is the distance from i to the branch point b and $n_e' = n_e$ for $e \neq 1$. Then*
(a) $\cup_{e=2}^r \mathsf{L}_L(e)$ is an acyclic lace on T'.
(b) If $L^{(e)}$ is empty then $\{ij_1^{(1)}\}$ is a lace on the interval $[i, n_1 = j_1^{(1)}]$. If $L^{(e)}$ is nonempty then either $L^{(e)} \cup \{ij_1^{(1)}\}$ is a lace on the interval $[i, n_1]$, or $L^{(e)}$ is a lace on the interval $[i, n_1]$, $i = i_2^{(e)}$, and $j_1^{(1)}$ is an element of the set $\mathcal{A}(L^{(e)})$ defined in (5.11).

Proof. (a) The graph $\Gamma = \cup_{e=2}^r \mathsf{L}_L(e)$ is nonempty and is connected on T'. Since $i_1^{(1)}j_1^{(1)}$ is associated to branch 1 and L is cyclic, Γ is the restriction of L to T'. By Lemma 3.2, Γ is therefore a lace on T'. Since Γ has only $r-1$ edges covering the branch point, it is either reducible or acyclic. However, since the edges in Γ that cover the branch point are obtained by removing exactly one edge from those of the cyclic lace L, Γ cannot be reducible.
(b) This follows from Lemma 3.10. □

3.4 Properties of laces containing a non-minimal cyclic component

In this section, we prove two lemmas that will be used in Section 6.3 to estimate the laces in $\mathcal{N}(T)$.

Lemma 3.12. *Let $T = (\tau_r, \bar{n})$ with $r \geq 2$. Let $L \in \mathcal{N}(T)$ be an irreducible lace containing a non-minimal cyclic component. Let $A \subset \{1, \ldots, r\}$ be such that L_A is a non-minimal cyclic lace on T_A, and let $i, j \in T_A$ be such that $L_A \backslash \{ij\} \in \mathcal{L}(T_A)$. Then $L \backslash \{ij\}$ is an acyclic lace on T.*

Proof. We begin by showing that $L\backslash\{ij\}$ is a lace on T. First, $L\backslash\{ij\}$ is connected. To see this, we note that every vertex of $T\backslash\{b\}$ remains fully covered by $L\backslash\{ij\}$, since $L_A\backslash\{ij\}$ is a lace by assumption. In addition, given any branch T_e, b is covered by an edge of $L\backslash\{ij\}$ containing an endpoint in T_e. To see that $L\backslash\{ij\}$ is a lace, we note that every edge not covering the branch point must be essential to maintain connectivity, since this was true already for L. In addition, every edge of L covering the branch point was associated to a branch of T, and the same must therefore be true for $L\backslash\{ij\}$. It remains to prove that $L\backslash\{ij\}$ is acyclic.

Since L is a lace, ij must cover the branch point or $L\backslash\{ij\}$ would not be connected. Therefore, $L\backslash\{ij\}$ has at most $r-1$ edges covering the branch point, so it is not cyclic and must be either acyclic or reducible. We prove by contradiction that $L\backslash\{ij\}$ is not reducible. Suppose, to the contrary, that there is a $B \subset \{1, \ldots, r\}$ with $2 \leq |B| \leq r-2$ such that $L\backslash\{ij\}$ is the disjoint union of L_B and L_{B^c}. Since L is irreducible by assumption, it must be the case that $i \in T_B$ and $j \in T_{B^c}$ (or vice versa), and that ij is the only edge in L that links T_B and T_{B^c}. However, since ij is in the cylic lace L_A, there must be another edge of L_A that links T_B and T_{B^c}. This contradiction proves that $L\backslash\{ij\}$ is not reducible, and hence it must be acyclic. □

The following lemma implies, in particular, that a lace containing at least two non-minimal cyclic components is reducible. In addition, repeated application of the lemma shows that a lace containing $i \geq 2$ non-minimal cyclic components can be decomposed into two laces with one containing a unique non-minimal cyclic component and the other containing $i-1$ non-minimal cyclic components. This is used in Section 6.3.3.

Lemma 3.13. *Let $T = (\tau_r, \bar{n})$ with $r \geq 4$, and let $L \in \mathcal{L}(T)$. Suppose that there are disjoint proper subsets $A_1, A_2 \subset \{1, \ldots, r\}$ such that L_{A_1} and L_{A_2} are cyclic laces. Then there is an $A \subset \{1, \ldots, r\}$ such that L_A and L_{A^c} are laces on T_A and T_{A^c} respectively, L is the disjoint union of L_A and L_{A^c}, and $L_{A_1} \subset L_A$, $L_{A_2} \subset L_{A^c}$.*

Proof. Let A_1, A_2 be such that L_{A_i} is a cyclic lace on T_{A_i} $(i = 1, 2)$. This implies that $A_1 \cap A_2 = \varnothing$. The proof is by contradiction. Suppose there is no $A \subset \{1, \ldots, r\}$ such that L is the disjoint union of L_A and L_{A^c} with L_A and L_{A^c} laces on T_A and T_{A^c} respectively, and $L_{A_1} \subset L_A$, $L_{A_2} \subset L_{A^c}$. Then there must be a set of edges $s_1t_1, \ldots, s_at_a$ such that each s_it_i covers the branch point, $s_1 \in T_{A_1}$, $t_a \in T_{A_2}$, and s_{i+1}, t_i are on the same branch for each i. The graph $L_{A_1} \cup L_{A_2} \cup \{s_1t_1, \ldots, s_at_a\}$ is a connected graph on a subtree $S \subset T$. By Lemma 3.2, it is therefore a lace on S. However, the degree of the branch point in S is $|A_1|+|A_2|+a-1$, whereas the number of edges covering the branch point is $|A_1| + |A_2| + a$. By Lemma 3.1, this contradicts the fact that $L_{A_1} \cup L_{A_2} \cup \{s_1t_1, \ldots, s_at_a\}$ is a lace. □

4 Proof of Theorem 1.1

In this section, we reduce the proof of Theorem 1.1 to two propositions, Propositions 4.1 and 4.2, whose proofs are deferred to Sections 5 and 6, respectively. We begin in Section 4.1 by reducing the case $r = 1$ of Theorem 1.1 to Proposition 4.1, which gives bounds on $\pi_n^{(1)}$. In Section 4.2, we state Proposition 4.2, which gives bounds on $\pi_{\bar{n}}^{(r)}$ $(r \geq 2)$ and $\varphi_T(\vec{y})$. In Section 4.3, we prove Theorem 1.1 for $r \geq 2$ by induction on r, assuming the two propositions.

4.1 The case $r = 1$

In this section, we reduce the $r = 1$ case of Theorem 1.1 to Proposition 4.1. Our analysis actually gives Theorem 1.1(a) uniformly for $|k|^2$ bounded by a small multiple of $\log n$, as in [12]. However, to simplify our analysis for $r \geq 2$, we have restricted the statement and proof of Theorem 1.1(a) to bounded $|k|^2$.

The results of the $r = 1$ case of Theorem 1.1 were shown in [12] to hold for solutions of a general recursion relation

$$f_0(k; z) = 1, \qquad f_n(k; z) = \sum_{m=1}^{n} g_m(k; z) f_{n-m}(k; z) + e_n(k; z) \qquad (n \geq 1), \tag{4.1}$$

subject to a certain set of assumptions. Here z is a non-negative parameter, and $k \in [-\pi, \pi]^d$ is a Fourier variable. For self-avoiding walk, the Fourier transform of (2.9) can be written in the form of (4.1), if we set

$$g_1(k;z) = z\hat{D}(k), \quad g_n(k;z) = \hat{\pi}^{(1)}_n(k)z^n \quad f_n(k;z) = \hat{c}_n(k)z^n, \quad e_n(k;z) = 0 \quad (n \geq 2). \tag{4.2}$$

The main result of [12] is that the $r = 1$ case of Theorem 1.1 holds provided certain assumptions apply. As described in [12, Section 1.4.1], the only substantial assumption to verify is Assumption G of [12]. Assumption G involves the parameter ϵ of (1.5) and the small parameter

$$\beta = L^{-d}. \tag{4.3}$$

In our present notation, the statement of Assumption G is as follows.

Assumption G. There is an L_0, an interval $I \subset [1-\alpha, 1+\alpha]$ with $\alpha \in (0,1)$, and a function $K_f \mapsto C_g(K_f)$, such that if the bounds

$$\int_{[-\pi,\pi]^d} \hat{D}(k)^2 |\hat{c}_m(k)| z^m \frac{d^dk}{(2\pi)^d} \leq K_f \beta m^{-d/2}, \quad \hat{c}_m(0) z^m \leq K_f, \quad |\nabla^2 \hat{c}_m(0)| z^m \leq K_f \sigma^2 m \tag{4.4}$$

hold for some $K_f > 1$, $L \geq L_0$, $z \in I$ and for all $1 \leq m \leq n$, then for that L and z, and for all $k \in [-\pi,\pi]^d$ and $2 \leq m \leq n+1$, the following bounds hold:

$$|\hat{\pi}^{(1)}_m(k)| z^m \leq C_g(K_f) \beta m^{-d/2}, \quad |\nabla^2 \hat{\pi}^{(1)}_m(0)| z^m \leq C_g(K_f) \sigma^2 \beta m^{-(d-2)/2}, \tag{4.5}$$

$$|\hat{\pi}^{(1)}_m(0)| m z^{m-1} \leq C_g(K_f) \beta m^{-(d-2)/2}, \tag{4.6}$$

$$|\hat{\pi}^{(1)}_m(k) - \hat{\pi}^{(1)}_m(0) - [1-\hat{D}(k)]\sigma^{-2}\nabla^2 \hat{\pi}^{(1)}_m(0)| z^m \leq C_g(K_f)[1-\hat{D}(k)]^{1+\epsilon'} \beta m^{-(d-2-2\epsilon')/2}, \tag{4.7}$$

with (4.7) valid for any $\epsilon' \in [0, \epsilon \wedge 1]$.

Note that Assumption G does *not* assume that (4.4) holds, but rather that (4.4) implies (4.5)–(4.7). We emphasise that (4.5)–(4.7) are to be concluded for all $2 \leq m \leq n+1$, whereas (4.4) is assumed only for $1 \leq m \leq n$. The inclusion of $m = n+1$ in (4.5)–(4.7) is crucial in [12] in the analysis of (4.1) by induction on n.

We will prove the $r = 1$ case of Theorem 1.1 by showing that Assumption G does hold. Once we have established Assumption G, it then follows from [12] that (4.4) and (4.5)–(4.7) hold respectively for *all* $m \geq 1$ and $m \geq 2$, for the choice $z = z_c = \mu^{-1}$. It will follow from this that the bounds in Proposition 4.1 hold for *all* $m \geq 1$, when $z = z_c = \mu^{-1}$. In addition, it follows from the results of [12] that

$$z_c = \mu^{-1} = 1 + \mathcal{O}(L^{-d}). \tag{4.8}$$

In Section 5, we will prove the following proposition, which verifies Assumption G by showing that (4.5)–(4.7) for $m \leq n+1$ follow from (4.4) for $m \leq n$. Note that the cases $q = 0, 2$ of (4.9) imply (4.5)–(4.6).

Proposition 4.1. *Assume that* (4.4) *holds for* $1 \leq m \leq n$ *and for fixed* $z \in I$ *with* α *fixed. Then there is a positive constant* C_g*, depending on* K_f*, such that for* L *sufficiently large*
(i)

$$\sum_{y \in \mathbb{Z}^d} |y|^q \sum_{N=1}^{\infty} \pi^{(1)}_{m,N}(y) z^m \leq \frac{C_g \beta \sigma^q}{m^{(d-q)/2}} \qquad (q = 0, 2, 4;\ 2 \leq m \leq n+1), \tag{4.9}$$

(ii) the bound (4.7) *holds for any* $\epsilon' \in [0, \epsilon]$ *and* $2 \leq m \leq n+1$.

4.2 Bounds for $r \geq 2$

The key estimates for the proof of Theorem 1.1 for $r \geq 2$ are contained in the following proposition, whose proof is deferred to Section 6. In its statement, we denote the set of permutations on $\{1, \ldots, s\}$ by Σ_s, and write $p \in \Sigma_s$ as $p = (p_1, \ldots, p_s)$. For $s \geq 2$, we also define

$$B_{\bar{m}}^{(s)} = \sum_{p \in \Sigma_s} \sum_{l=0}^{\lfloor \frac{s}{2} \rfloor} \prod_{i=1}^{l} (m_{p_{2i-1}} + m_{p_{2i}} + 1)^{-d/2} \prod_{j=2l+1}^{s} (m_{p_j} + 1)^{-(d-2)/2}. \tag{4.10}$$

Proposition 4.2. *(i) Let $d > 4$, $s \geq 2$, $q = 0, 2$, $m_e > 0$ ($e = 1, \ldots, s$), and $m = \sum_{e=1}^{s} m_e$. For L sufficiently large, there is a constant $K_s < \infty$ such that*

$$\sum_{\bar{x}} |x_e|^q \sum_{N=1}^{\infty} \pi_{\bar{m},N}^{(s)}(\bar{x}) \mu^{-m} \leq K_s \beta^{\lceil s/2 \rceil} \sigma^q m_e^{q/2} B_{\bar{m}}^{(s)}. \tag{4.11}$$

If some $m_e = 0$ but $\bar{m} \neq \bar{0}$, then (4.11) holds with the power of β reduced to $\lceil M/2 \rceil$, where M denotes the number of nonzero components of $\bar{m}$.
(ii) Let $d > 4$, $r \geq 2$, $q = 0, 2$, $T \in \mathcal{T}_r(\vec{n})$, and $\underline{n} = \min\{n_1, \ldots, n_r\}$. For L sufficiently large, there is a constant $K_r < \infty$ such that

$$\sum_{\vec{y}} |y_e|^q |\varphi_T(\vec{y})| \mu^{-n} \leq K_r \beta \sigma^q n_e^{q/2} (\underline{n} + 1)^{-(d-4)/2}. \tag{4.12}$$

The following elementary lemmas will be useful.

Lemma 4.3. *Let $d > 4$, $s \geq 2$, $\gamma \in [0, \frac{d-2}{2}]$, and $m = \sum_{e=1}^{s} m_e$. Assuming Proposition 4.2(i),*

$$\sum_{\bar{0} \leq \bar{m} \leq \bar{n} : \bar{m} \neq \bar{0}} \frac{1}{(n_e - m_e + 1)^\gamma} \sum_{\bar{x}} \sum_{N=1}^{\infty} \pi_{\bar{m},N}^{(s)}(\bar{x}) \mu^{-m} \leq \frac{C\beta}{(n_e + 1)^\gamma}, \tag{4.13}$$

$$\sum_{\bar{0} \leq \bar{m} \leq \bar{n}} \sum_{\bar{x}} |x_e|^2 \sum_{N=1}^{\infty} \pi_{\bar{m},N}^{(s)}(\bar{x}) \mu^{-m} \leq \begin{cases} C\beta\sigma^2 n_e^{0 \vee (6-d)/2} & (d \neq 6) \\ C\beta\sigma^2 \log n_e & (d = 6). \end{cases} \tag{4.14}$$

Proof. By Proposition 4.2(i), it suffices to prove that

$$\sum_{\bar{0} \leq \bar{m} \leq \bar{n}} \frac{1}{(n_e - m_e + 1)^\gamma} B_{\bar{m}}^{(s)} \leq \frac{C}{(n_e + 1)^\gamma}, \qquad \sum_{\bar{0} \leq \bar{m} \leq \bar{n}} m_e B_{\bar{m}}^{(s)} \leq \begin{cases} C n_e^{0 \vee (6-d)/2} & (d \neq 6) \\ C \log n_e & (d = 6). \end{cases} \tag{4.15}$$

In each case, we first interchange the sum over $\bar{m}$ with the sums over p and l in $B_{\bar{m}}^{(s)}$. Depending on the permutation p and the value of l, we associate the factors $(n_e - m_e + 1)^{-\gamma}$ and m_e either with $(m_e + 1)^{-(d-2)/2}$ or $(m_e + m_{e'} + 1)^{-d/2}$, for some e'. The remaining sums over m_i can be performed to yield constant factors, using $\sum_{m=0}^{n} (m+1)^{-(d-2)/2} \leq C$ and $\sum_{m=0}^{n} \sum_{m'=0}^{n'} (m + m' + 1)^{-d/2} \leq C$. There are four estimates remaining, corresponding to the above two pairs of possibilities. These will produce the desired decay.

Two of the inequalities are handled using

$$\sum_{m=0}^{n} (n - m + 1)^{-\gamma} (m + 1)^{-(d-2)/2} \leq C(n + 1)^{-\gamma}, \tag{4.16}$$

$$\sum_{m=0}^{n} m(m+1)^{-(d-2)/2} \le \begin{cases} Cn^{0\vee(6-d)/2} & (d \neq 6) \\ C\log n & (d=6), \end{cases} \tag{4.17}$$

while the other two are handled using

$$\sum_{m=0}^{n}\sum_{m'=0}^{n'} (n-m+1)^{-\gamma}(m+m'+1)^{-d/2} \le C\sum_{m=0}^{n}(n-m+1)^{-\gamma}(m+1)^{-(d-2)/2}, \tag{4.18}$$

$$\sum_{m=0}^{n}\sum_{m'=0}^{n'} m(m+m'+1)^{-d/2} \le C\sum_{m=0}^{n} m(m+1)^{-(d-2)/2} \tag{4.19}$$

in conjunction with (4.16)–(4.17). □

Lemma 4.4. *Let $d \ge 4$, $s \ge 2$, and write $\bar{n}' = (m, n_2, \ldots, n_s)$ and $\bar{n} = (n_1, n_2, \ldots, n_s)$. Then*

$$\sum_{m=0}^{n_1} B^{(s)}_{\bar{n}'}(n_1 - m + 1)^{-(d-2)/2} \le CB^{(s)}_{\bar{n}}, \tag{4.20}$$

$$\sum_{\bar{0}\le\bar{m}\le\bar{n}:\, m_1 > n_1/2} B^{(s)}_{\bar{m}} \le Cn_1^{-(d-4)/2}. \tag{4.21}$$

Proof. The first estimate follows immediately from (4.10) and the inequalities

$$\sum_{m=0}^{n_1}(n_1-m+1)^{-(d-2)/2}(m+1)^{-(d-2)/2} \le C(n_1+1)^{-(d-2)/2}, \tag{4.22}$$

$$\sum_{m=0}^{n_1}(n_1-m+1)^{-(d-2)/2}(m+n_e+1)^{-d/2} \le C(n_1+1)^{-(d-2)/2}(n_e+1)^{-(d-2)/2} + C(n_1+n_e+1)^{-d/2}. \tag{4.23}$$

The second estimate is elementary. □

4.3 Induction on r

In this section, we prove Theorem 1.1, assuming Propositions 4.1–4.2. The proof is by induction on the number r of edges in τ, where $T = (\tau, \vec{n}) \in \mathcal{T}_r(\vec{n})$. The case $r = 1$ was treated in Section 4.1 and this will initialise the induction, assuming Proposition 4.1.

Fix $T \in \mathcal{T}_r(\vec{n})$, with $r \ge 2$, and let $\kappa_e = k_e/\sqrt{\sigma^2 vn}$. Let $n = \sum_{e=1}^{r} n_e$. The starting point for the proof of Theorem 1.1, for $r \ge 2$, is the identity (2.10). Taking the Fourier transform of (2.10) gives

$$\hat{c}_T(\vec{\kappa}) = \sum_{\bar{0}\le\bar{m}\le\bar{n}} \hat{\pi}^{(d_b)}_{\bar{m}}(\bar{\kappa}) \prod_{e=1}^{d_b} \hat{c}_{S_e}(\vec{\kappa}^{(e)}) \;+\; \hat{\varphi}_T(\vec{\kappa}), \tag{4.24}$$

where $\bar{\kappa} = (\kappa_1, \ldots, \kappa_{d_b})$, and $\vec{\kappa}^{(e)}$ denotes the components of $\vec{\kappa}$ whose labels label edges of S_e. The bound of Proposition 4.2(ii) will be used to conclude that $\hat{\varphi}_T(\vec{\kappa})$ is an error term. Our proof of

Theorem 1.1 is based on the idea that the first term in (4.24) is approximately equal to

$$V_{d_b} \prod_{e=1}^{d_b} \hat{c}_{S_e^{(0)}}(\vec{\kappa}^{(e)}), \tag{4.25}$$

where $S_e^{(0)}$ denotes S_e when all $m_i = 0$, and

$$V_{d_b} = \sum_{\bar{0} \le \bar{m} < \bar{\infty}} \hat{\pi}_{\bar{m}}^{(d_b)}(\bar{0}) \mu^{-m} = \sum_{\bar{0} \le \bar{m} < \bar{\infty}} \sum_{\bar{x}} \pi_{\bar{m}}^{(d_b)}(\bar{x}) \mu^{-m}. \tag{4.26}$$

By definition, $\hat{\pi}_{\bar{0}}^{(d_b)}(\bar{x}) = \delta_{\bar{0},\bar{x}}$ and hence $\hat{\pi}_{\bar{0}}^{(d_b)}(\bar{0}) = 1$, which is the main contribution to V_{d_b}. The case $\gamma = 0$ of the first bound of Lemma 4.3 then implies that

$$V_{d_b} = 1 + \mathcal{O}(\beta) \qquad (d_b \ge 2). \tag{4.27}$$

Since $d_b \ge 2$ in (4.24), each component S_e has fewer than r edges and an inductive analysis is possible.

Proof of Theorem 1.1(a) assuming Proposition 4.2. Let $\underline{n} = \min\{n_1, \dots, n_r\}$. Define $E_T(\vec{\kappa}, \vec{n})$ by

$$\hat{c}_T(\vec{\kappa}) = A^r \Big(\prod_{i \in \tau} V_{d_i} \Big) \mu^n e^{-\sum_{j=1}^r |k_j|^2 n_j / 2dn} \left[1 + E_T(\vec{\kappa}, \vec{n}) \right]. \tag{4.28}$$

It suffices to prove that there is a finite constant C_r, for $r \ge 2$, such that $|E_T(\vec{0}, \vec{n})| \le C_r \underline{n}^{-(d-4)/2}$ and $|E_T(\vec{\kappa}, \vec{n}) - E_T(\vec{0}, \vec{n})| \le C_r |k|^2 \underline{n}^{-\delta}$, uniformly in $\vec{k}$ with $|k|^2$ bounded. For this, it suffices to show that

$$|E_T(\vec{0}, \vec{n})| \le C_r \underline{n}^{-(d-4)/2}, \qquad \big| \nabla_j^2 |_{\vec{k}=\vec{0}} E_T(\vec{\kappa}, \vec{n}) \big| \le C_r \underline{n}^{-\delta}, \tag{4.29}$$

where ∇_j denotes differentiation with respect to k_j. Constants in this proof, including C_r of (4.29), can depend on L. The case $r = 1$ of (4.29) has been established already in Section 4.1. We assume, as induction hypothesis, that E_T obeys (4.29) for T with fewer than r branches.

We first show that $\hat{\varphi}_T(\vec{\kappa})$ is an error term, which involves showing that $e^{\sum_{j=1}^r |k_j|^2 n_j / 2dn} \hat{\varphi}_T(\vec{\kappa}) \mu^{-n}$ obeys the bounds of (4.29). This does not require the induction hypothesis. In fact, Proposition 4.2(ii) gives

$$|\hat{\varphi}_T(\vec{\kappa})| \le K_r \beta \mu^n \underline{n}^{-(d-4)/2}, \tag{4.30}$$

which implies the first bound of (4.29) for this contribution. For the second bound of (4.29) for this contribution, we use

$$\big| \nabla_j^2 |_{\vec{k}=\vec{0}} \big(e^{\sum_{j=1}^r |k_j|^2 n_j / 2dn} \hat{\varphi}_T(\vec{\kappa}) \big) \big| \mu^{-n} \le C |\hat{\varphi}_T(\vec{\kappa})| \mu^{-n} + C \big| \nabla_j^2 |_{\vec{k}=\vec{0}} \hat{\varphi}_T(\vec{\kappa}) \big| \mu^{-n} \le C \underline{n}^{-(d-4)/2}, \tag{4.31}$$

where we have used Proposition 4.2(ii) and the fact that $|k|^2$ and n_j/n are bounded.

We can also easily dispense with the contribution to the first term on the right side of (4.24) due to terms where $m_e > n_e/2$ for some e. To show that this contribution is an error term, we will show that

$$W(\vec{\kappa}, \vec{n}) = e^{\sum_{j=1}^r |k_j|^2 n_j / 2dn} \mu^{-n} \sum_{\bar{0} \le \bar{m} \le \bar{n} : \exists e \, m_e > n_e/2} \hat{\pi}_{\bar{m}}^{(d_b)}(\bar{\kappa}) \prod_{e=1}^{d_b} \hat{c}_{S_e}(\vec{\kappa}^{(e)}) \tag{4.32}$$

obeys (4.29). For this, we will use the fact that $|\hat{c}_{S_e}(\vec{\kappa}^{(e)})|\mu^{-|S_e|} \leq c_{S_e}\mu^{-|S_e|}$ is uniformly bounded, which follows by neglecting the self-avoidance interaction between branches of S_e and applying the $r = 1$ result of Theorem 1.1(a). Therefore

$$|W(\vec{\kappa}, \vec{n})| \leq C \sum_{\bar{0} \leq \bar{m} \leq \bar{n}: \exists e \; m_e > n_e/2} |\hat{\pi}^{(d_b)}_{\bar{m}}(\bar{\kappa})|\mu^{-m}. \tag{4.33}$$

The right side of the above is bounded by $C\underline{n}^{-(d-4)/2}$ by Proposition 4.2(i) and (4.21), which proves the first bound of (4.29) for this contribution. For the second bound of (4.29), we apply $\nabla_j^2|_{\vec{k}=\vec{0}}$ to (4.32). When this operation is applied to the exponential factor or to the factor $\hat{\pi}^{(d_b)}_{\bar{m}}(\bar{\kappa})$ in (4.32), the above argument can be applied. When the operation is applied to a factor $\hat{c}_{S_e}(\vec{\kappa}^{(e)})$, then we can again neglect the self-avoidance interaction between branches of S_e and apply the $r = 1$ result of Theorem 1.1(b) to conclude that $|\nabla_j^2|_{\vec{k}=\vec{0}}\hat{c}_{S_e}(\vec{\kappa}^{(e)})| \leq C$. This leads to the desired bound.

Thus we need only concern ourselves with the main term of (4.24), with the summation restricted to $0 \leq \bar{m} \leq \bar{n}/2$. In the main term of (4.24), we write each S_e as $S_e = (\tau_e, \vec{n}^{(e)}) \in \mathcal{T}_{r_e}(\vec{n}^{(e)})$. The components of $\vec{n}^{(e)}$ are identical to those of $\vec{n}$ for the edges of T that are also in S_e, except for the edge f (say) incident to b in S_e, whose length is $n_f - m_f$. Note that each r_e is strictly less than r, since $\sum_{e=1}^{d_b} r_e = r$ and $r_1 = 1$ for the edge of τ joining b to the root. In addition, $n_e - m_e \geq n_e/2$ for each e, so every branch of every S_e has length of order n . By the induction hypothesis,

$$\hat{c}_{S_e}(\vec{\kappa}_e) = M_{S_e}(\vec{\kappa}_e, \vec{n}^{(e)}) \left[1 + E_{S_e}(\vec{\kappa}_e, \vec{n}^{(e)})\right], \tag{4.34}$$

where M_{S_e} represents the main term specified by the induction hypothesis and E_{S_e} obeys (4.29).

In the product over e in

$$\sum_{\bar{0} \leq \bar{m} \leq \bar{n}/2} \hat{\pi}^{(d_b)}_{\bar{m}}(\bar{\kappa}) \prod_{e=1}^{d_b} \hat{c}_{S_e}(\vec{\kappa}^{(e)}) = \sum_{\bar{0} \leq \bar{m} \leq \bar{n}/2} \hat{\pi}^{(d_b)}_{\bar{m}}(\bar{\kappa}) \prod_{e=1}^{d_b} \left(M_{S_e}(\vec{\kappa}_e, \vec{n}^{(e)}) \left[1 + E_{S_e}(\vec{\kappa}_e, \vec{n}^{(e)})\right]\right), \tag{4.35}$$

consider those terms which contain at least one factor E. We claim these terms are error terms. To see this, we need to show that

$$e^{\sum_{j=1}^r |k_j|^2 n_j/2dn} \sum_{\bar{0} \leq \bar{m} \leq \bar{n}/2} \hat{\pi}^{(d_b)}_{\bar{m}}(\bar{\kappa})\mu^{-m} \prod_{a \in F} E_{S_a}(\vec{\kappa}_a, \vec{n}^{(a)}) \prod_{e=1}^{d_b} M_{S_e}(\vec{\kappa}_e, \vec{n}^{(e)})\mu^{-n^{(e)}} \tag{4.36}$$

obeys the bounds of (4.29), where F is a nonempty subset of $\{1, \ldots, d_b\}$. Since $|k|^2$ is bounded, the first bound of (4.29) can be obtained by estimating all but one of the factors E_{S_a} by a constant and using the first bound of Lemma 4.3 with $\gamma = \delta$. The second bound of (4.29) can be obtained similarly, using (4.14).

Thus we need only show that

$$e^{\sum_{j=1}^r |k_j|^2 n_j/2dn} \sum_{\bar{0} \leq \bar{m} \leq \bar{n}/2} \hat{\pi}^{(d_b)}_{\bar{m}}(\bar{\kappa})\mu^{-m} \prod_{e=1}^{d_b} M_{S_e}(\vec{\kappa}_e, \vec{n}^{(e)})\mu^{-n^{(e)}} - A^r\Big(\prod_{i \in \tau} V_{d_i}\Big) \tag{4.37}$$

is an error term. Since

$$\prod_{e=1}^{d_b} M_{S_e}(\vec{\kappa}_e, \vec{n}^{(e)})\mu^{-n^{(e)}} = V_{d_b}^{-1} A^r\Big(\prod_{i \in \tau} V_{d_i}\Big) e^{-\sum_{j=1}^r |k_j|^2 n_j/2dn} e^{\sum_{j=1}^{d_b} m_j |k_j|^2/2dn}, \tag{4.38}$$

it suffices to show that the quantity

$$X(\vec{\kappa},\vec{n}) = \sum_{\bar{0}\leq\bar{m}\leq\bar{n}/2} \hat{\pi}^{(d_b)}_{\bar{m}}(\bar{\kappa})\mu^{-m} e^{\sum_{j=1}^{d_b} m_j|k_j|^2/2dn} \ - \ V_{d_b} \tag{4.39}$$

obeys the bounds of (4.29).

Recalling the definition of V_{d_b} in (4.26), we make the decomposition

$$X(\vec{\kappa},\vec{n}) = X_1(\vec{n}) + X_2(\vec{\kappa},\vec{n}) + X_3(\vec{\kappa},\vec{n}) \tag{4.40}$$

with

$$X_1(\vec{n}) = -\sum_{\bar{m}:\exists e: m_e > n_e/2} \hat{\pi}^{(d_b)}_{\bar{m}}(0)\mu^{-m}, \tag{4.41}$$

$$X_2(\vec{\kappa},\vec{n}) = \sum_{\bar{0}\leq\bar{m}\leq\bar{n}/2} \Big(\hat{\pi}^{(d_b)}_{\bar{m}}(\bar{\kappa}) - \hat{\pi}^{(d_b)}_{\bar{m}}(0)\Big)\mu^{-m}, \tag{4.42}$$

$$X_3(\vec{\kappa},\vec{n}) = \sum_{\bar{0}\leq\bar{m}\leq\bar{n}/2} \hat{\pi}^{(d_b)}_{\bar{m}}(\bar{\kappa})\mu^{-m}\Big(e^{\sum_{j=1}^{d_b} m_j|k_j|^2/2dn} - 1\Big). \tag{4.43}$$

It suffices to show that $X_1(\vec{n})$ obeys the first bound of (4.29), and, since $X_2(\vec{0},\vec{n}) = X_3(\vec{0},\vec{n}) = 0$, that $X_2(\vec{\kappa},\vec{n})$ and $X_3(\vec{\kappa},\vec{n})$ obey the second bound of (4.29).

By Proposition 4.2(i) and (4.21), $X_1(\vec{n})$ obeys $|X_1(\vec{n})| \leq C\underline{n}^{-(d-4)/2}$. For $X_2(\vec{\kappa},\vec{n})$, we use

$$|X_2(\vec{\kappa},\vec{n})| \leq C|\kappa|^2 \sum_{\bar{0}\leq\bar{m}\leq\bar{n}/2} \sum_x |x|^2 |\pi^{(d_b)}_{\bar{m}}(\bar{x})|\mu^{-m} \tag{4.44}$$

and the second estimate of Lemma 4.3. Finally, for $X_3(\vec{\kappa},\vec{n})$, we use $|e^t - 1| \leq Ct$ for t bounded to obtain

$$|X_3(\vec{\kappa},\vec{n})| \leq \frac{C}{n}\sum_{j=1}^{d_b} |k_j|^2 \sum_{\bar{0}\leq\bar{m}\leq\bar{n}/2} m_j \hat{\pi}^{(d_b)}_{\bar{m}}(\bar{\kappa})\mu^{-m}. \tag{4.45}$$

The right side of (4.45) obeys the same bound as the right side of (4.44). This advances the induction and completes the proof of Theorem 1.1(a). □

Proof of Theorem 1.1(b) assuming Proposition 4.2. Our goal is to prove that

$$\frac{\nabla_j^2 \hat{c}_T(\vec{0})}{c_T} = -\sigma^2 v n_j[1 + \mathcal{O}(\underline{n}^{-\delta})]. \tag{4.46}$$

The proof is by induction on r. We assume, as induction hypothesis, that (4.46) has been proven for trees with l branches, with $1 \leq l \leq r-1$. The $l = 1$ case was established in Section 4.1.

By (4.24),

$$\frac{1}{c_T}\nabla_j^2 \hat{c}_T(\vec{0}) = \sum_{\bar{0}\leq\bar{m}\leq\bar{n}} \hat{\pi}^{(d_b)}_{\bar{m}}(\bar{0}) \frac{\nabla_j^2 \prod_{e=1}^{d_b} \hat{c}_{S_e}(\vec{0})}{c_T} + \sum_{\bar{0}\leq\bar{m}\leq\bar{n}} \nabla_j^2 \hat{\pi}^{(d_b)}_{\bar{m}}(\bar{0}) \frac{\prod_{e=1}^{d_b} \hat{c}_{S_e}(\vec{0})}{c_T} + \frac{\nabla_j^2 \hat{\varphi}_T(\vec{0})}{c_T}. \tag{4.47}$$

The last two terms can be identified as error terms, using Theorem 1.1(a), Proposition 4.2 and Lemma 4.3. We handle the first term using the induction hypothesis. We consider separately the cases $0 \le \bar{m} \le \bar{n}/2$, and $m_e > n_e/2$ for some e. The contribution due to the latter case can be bounded using the $r = 1$ statement of Theorem 1.1, as in the proof of Theorem 1.1(a), by $\mathcal{O}(\underline{n}^{1-(d-4)/2})$. This is an error term.

The remaining contribution to the first term can be written using the induction hypothesis and Theorem 1.1(a) as

$$-\sum_{\bar{0} \le \bar{m} \le \bar{n}/2} \hat{\pi}^{(d_b)}_{\bar{m}}(\bar{0}) \mu^{-m} V_{d_b}^{-1} \sigma^2 v \tilde{n}_j [1 + \mathcal{O}(\underline{n}^{-\delta})], \tag{4.48}$$

where $\tilde{n}_j$ is equal to $n_j - m_j$ or n_j depending on whether j labels an edge adjacent to b or not. In the error term, we have used the fact that $n_j - m_j \ge n_j/2$. When edge j is not adjacent to b, the desired result follows using the bound on X_1 of (4.41). When edge j is adjacent to b, the desired result can be obtained using the second estimate of (4.15). □

Proof of Theorem 1.1(c) assuming Proposition 4.2. The lower bound follows from Theorem 1.1(a) exactly as in the proof of [12, Corollary 1.4]. The upper bound follows from the elementary inequality $c_T(\vec{y}) \le \prod_{j=1}^{r-1} c_{n_j}(y_j)$, together with the corresponding bound for $r = 1$ that has already been established in Section 4.1. □

5 Proof of Proposition 4.1

In this section we prove Proposition 4.1, which is repeated below for convenience as Proposition 5.1. The proposition is standard, but we will give a somewhat new proof based directly on laces rather than on Feynman diagrams representing walk trajectories.

Proposition 5.1. *Assume that* (4.4) *holds for* $1 \le m \le n$ *and for fixed* $z \in I$ *with* α *fixed. Then there is a positive constant* C_g*, depending on* K_f*, such that for* L *sufficiently large*
(i)

$$\sum_{y \in \mathbb{Z}^d} |y|^q \sum_{N=1}^{\infty} \pi^{(1)}_{m,N}(y) z^m \le \frac{C_g \beta \sigma^q}{m^{(d-q)/2}} \qquad (q = 0, 2, 4;\ 2 \le m \le n+1), \tag{5.1}$$

(ii) the bound (4.7) *holds for any* $\epsilon' \in [0, \epsilon]$ *and* $2 \le m \le n+1$.

In the course of the proof of Proposition 5.1, we will employ the estimates of the following lemma. For a function $f : \mathbb{Z}^d \to \mathbb{C}$, we write $\|f\|_\infty = \sup_{x \in \mathbb{Z}^d} |f(x)|$ and $\|f\|_1 = \sum_{x \in \mathbb{Z}^d} |f(x)|$, and for $\hat{f} : [-\pi, \pi]^d \to \mathbb{C}$ we write $\|\hat{f}\|_1 = (2\pi)^{-d} \int_{[-\pi,\pi]^d} |\hat{f}(k)| d^d k$. Also, for $f, g : \mathbb{Z}^d \to \mathbb{C}$, we denote their convolution by

$$(f * g)(x) = \sum_{y \in \mathbb{Z}^d} f(x-y) g(y). \tag{5.2}$$

Once we have proven Proposition 5.1, and thereby established Assumption G of [12], it then follows from [12] that (4.4) and the bounds of Lemma 5.2 hold for *all* $m \ge 0$, when $z = z_c = \mu^{-1}$.

Lemma 5.2. *Assume that* (4.4) *holds for* $1 \le m \le n$ *and for fixed* $z \in I$ *with* α *fixed. Then there is a* K*, depending on* K_f*, such that for* $0 \le m \le n$ *the following bounds hold:*

$$\hat{c}_m(0) z^m \le K, \quad \sum_x |x|^2 c_m(x) z^m \le K \sigma^2 m, \quad \|c_m\|_\infty z^m \le \begin{cases} K\beta(m+1)^{-d/2} & (m \ne 0) \\ 1 & (m = 0), \end{cases} \tag{5.3}$$

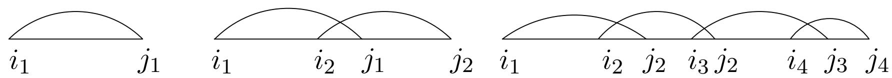

Figure 7: Laces in $\mathcal{L}^{(N)}[a,b]$ for $N = 1, 2, 4$.

$$\sup_x |x|^2 c_m(x) z^m \le K\sigma^2 \beta (m+1)^{-(d-2)/2}. \tag{5.4}$$

Proof. For $m \le n$, the first two bounds are immediate from (4.4). The third bound holds trivially for $m = 0$. For $m = 1$, we have $\|c_1\|_\infty z = z\|D\|_\infty \le (1+\alpha)CL^{-d}$, by (1.10). For $m \ge 2$, we use the elementary bound

$$c_l(x) \le (c_j * c_{l-j})(x) \quad (0 \le j \le l) \tag{5.5}$$

twice with $j = 1$ to conclude that

$$c_m(x) z^m \le (c_1 * c_1 * c_{m-2})(x) z^m \le z^2 \|\hat{D}^2 \hat{c}_{m-2}\|_1 z^{m-2}. \tag{5.6}$$

Thus (4.4) implies (5.3) for all $m \le n+2$, with $K = (1+\alpha)^2 K_f$.

For (5.4), when $m = 1$ we use (1.5) and the fact that $c_1(x) = D(x)$. For $m \ge 2$, we use (5.5) with $l = m$ and $j = \lfloor m/2 \rfloor$ to get

$$\begin{aligned} |x|^2 c_m(x) z^m &\le 2 \sum_y (|y|^2 + |x-y|^2) c_{\lfloor m/2 \rfloor}(y) c_{m-\lfloor m/2 \rfloor}(x-y) z^m \\ &\le 2\Big(\|c_{m-\lfloor m/2 \rfloor}\|_\infty \sum_y |y|^2 c_{\lfloor m/2 \rfloor}(y) z^m + \|c_{\lfloor m/2 \rfloor}\|_\infty \sum_w |w|^2 c_{m-\lfloor m/2 \rfloor}(w) z^m \Big). \end{aligned} \tag{5.7}$$

Then (5.4) follows from (5.3) for $m \le n$ with $K = CK_f^2$ (in fact, for m of the order of $2n$). ☐

Before starting the proof of Proposition 5.1, we introduce some definitions and new notation. We write $L \in \mathcal{L}^{(N)}[a,b]$ as $L = \{i_1 j_1, \ldots, i_N j_N\}$, with $i_l < j_l$ for each l. The fact that L is a lace is equivalent to a certain ordering of the i_p and j_p. For $N = 1$, we simply have $a = i_1 < j_1 = b$. For $N \ge 2$, $L \in \mathcal{L}^{(N)}[a,b]$ if and only if

$$a = i_1 < i_2, \quad i_{p+1} < j_p \le i_{p+2} \quad (p = 1, \ldots, N-2), \quad i_N < j_{N-1} < j_N = b \tag{5.8}$$

(for $N = 2$ the vacuous middle inequalities play no role). This can be seen from Figure 7. Thus L divides $[a,b]$ into $2N-1$ subintervals:

$$[i_1, i_2],\ [i_2, j_1],\ [j_1, i_3],\ [i_3, j_2],\ \ldots\ , [i_N, j_{N-1}],\ [j_{N-1}, j_N]. \tag{5.9}$$

Of these, intervals number 3, 5, ..., $(2N-3)$ can have zero length for $N \ge 3$, whereas all others have length at least 1. Each of the subintervals has length strictly less than $b-a$. In what follows, for $N \ge 2$ we write a lace $L \in \mathcal{L}^{(N-1)}[i,b]$ as $\{i_2 j_2, \ldots, i_N j_N\}$, with $i_2 = i$.

For $N = 1$, we define

$$J_x^{(1)}[a,b] = \delta_{x,\omega(b)} \prod_{ij \in \mathcal{C}(ab)} (1 + U_{ij}), \tag{5.10}$$

where $\mathcal{C}(ab)$ denotes the set of edges compatible with the lace $\{ab\}$ on $[a,b]$. Given a lace $L = \{i_2j_2,\ldots,i_Nj_N\} \in \mathcal{L}^{(N-1)}[i,b]$, with $N \geq 2$, we define

$$\mathcal{A}(L) = \begin{cases} [i+1, b-1] & (N=2) \\ [i+1, i_3] & (N \geq 3). \end{cases} \tag{5.11}$$

For $N \geq 2$, we define

$$J_x^{(N)}[a,b] = \sum_{i=a}^{b-2} K[a,i] \sum_{L\in\mathcal{L}^{(N-1)}[i,b]} \sum_{j\in\mathcal{A}(L)} \delta_{x,\omega(j)} \prod_{st\in L}(-U_{st}) \prod_{s't'\in\mathcal{C}(L)} (1+U_{s't'}). \tag{5.12}$$

Comparing with (2.16), it can then be seen that

$$J^{(N)}[a,b] \leq J^{(N)}_{\omega(a)}[a,b] \quad (N \geq 1). \tag{5.13}$$

In this comparison, L in (5.12) corresponds to $L\backslash\{i_1j_1\}$ in (2.16), i and j of (5.12) correspond to i_2 and j_1 of the lace L in (2.16), and the set of compatible edges in (2.16) has been reduced for an upper bound. (The inclusion of the term $i=a$ in (5.12) is unnecessary at this point, but will be useful in Section 6; see also Lemma 3.11(b).) Finally, for $N \geq 1$ we define

$$\pi^{(1)}_{m,N}(x,y) = \sum_{\omega\in\Omega_m(y)} W(\omega) J_x^{(N)}[0,m], \tag{5.14}$$

where $\Omega_m(y)$ denotes the set of m-step walks from 0 to y (not necessarily self-avoiding).

We first prove Proposition 5.1(i) for the case $q=0$.

Proof of Proposition 5.1(i) for $q=0$. By (2.18), it suffices to show that $\sum_y \pi^{(1)}_{m,N}(y) z^m \leq (C\beta)^N m^{-d/2}$, for all $N \geq 1$ and $2 \leq m \leq n+1$. By (2.17), (5.13) and (5.14), we have

$$\sum_{y\in\mathbb{Z}^d} \pi^{(1)}_{m,N}(y) \leq \sum_{y\in\mathbb{Z}^d} \pi^{(1)}_{m,N}(0,y) \leq \sup_{x\in\mathbb{Z}^d} \sum_{y\in\mathbb{Z}^d} \pi^{(1)}_{m,N}(x,y) \quad (N \geq 1). \tag{5.15}$$

It therefore suffices to show that

$$\sup_{x\in\mathbb{Z}^d} \sum_{y\in\mathbb{Z}^d} \pi^{(1)}_{m,N}(x,y) z^m \leq (C\beta)^N m^{-d/2} \quad (N \geq 1,\ 2 \leq m \leq n+1), \tag{5.16}$$

for some constant C depending on K_f. We will prove (5.16) jointly with the assertion that

$$\sum_{x\in\mathbb{Z}^d} \sum_{y\in\mathbb{Z}^d} \pi^{(1)}_{m,N}(x,y) z^m \leq (C\beta)^{N-1} \quad (N \geq 1,\ 2 \leq m \leq n+1), \tag{5.17}$$

by induction on N. The bounds (5.16)–(5.17) will also be useful in Section 6.2.

Since $\mathcal{C}(0m) \supset \mathcal{B}[1,m]$ for $m \geq 2$, the $N=1$ term obeys

$$\begin{aligned} \pi^{(1)}_{m,1}(x,y) &= \delta_{x,y} \sum_{\omega\in\Omega_m(y)} W(\omega) \prod_{ij\in\mathcal{C}(0m)} (1+U_{ij}) \leq \delta_{x,y} \sum_{\omega\in\Omega_m(y)} W(\omega) K[1,m] \\ &= \delta_{x,y}(D * c_{m-1})(y) \leq \delta_{x,y} \|c_{m-1}\|_\infty \quad (m \geq 2). \end{aligned} \tag{5.18}$$

By (4.4) and (5.3), this implies that

$$\sup_x \sum_y \pi^{(1)}_{m,1}(x,y)z^m \le z\|c_{m-1}\|_\infty z^{m-1} \le C\beta m^{-d/2} \quad (2 \le m \le n+1). \tag{5.19}$$

Similarly,

$$\sum_{x,y} \pi^{(1)}_{m,1}(x,y)z^m \le z\hat{c}_{m-1}(0)z^{m-1} \le C \quad (2 \le m \le n+1). \tag{5.20}$$

We begin the induction with the case $N = 2$. In this case, the sum over L in (5.12) consists of the single term $L = \{im\}$. Since $\mathcal{C}(im) \supset \mathcal{B}[i,j] \cup \mathcal{B}[j,m]$ for $j \in \mathcal{A}(L)$,

$$\begin{aligned}\sum_y \pi^{(1)}_{m,2}(x,y)z^m &\le \sum_y \sum_{0\le i<j<m} \sum_{\omega\in\Omega_m(y)} W(\omega)K[0,i]\delta_{\omega(i),y}K[i,j]\delta_{x,\omega(j)}K[j,m]z^m \\ &= \sum_{0\le i<j<m} \sum_y c_i(y)z^i c_{j-i}(x-y)z^{j-i}c_{m-j}(y-x)z^{m-j}. \end{aligned} \tag{5.21}$$

We will bound the above sum over y by using the first and third bounds of (5.3), together with the elementary estimate

$$\sup_x \Big| \sum_y f_1(y)f_2(x-y)f_3(y-x)\Big| \le \|f_j\|_1 \prod_{i:i\ne j} \|f_i\|_\infty \quad (j=1,2,3). \tag{5.22}$$

Note that when $m \le n+1$, the subscripts i, $j-i$, $m-j$ in (5.21) are all at most n.

We apply the l_∞ norm to the factors of $c_j(v)z^j$ with the largest two subscripts, which are necessarily positive. Thus these factors each provide $C\beta j^{-d/2}$. The remaining factor is bounded with the l_1 norm. Relabelling the summation indices, this leads to the bound

$$\sup_x \sum_y \pi^{(1)}_{m,2}(x,y)z^m \le C^3\beta^2 \sum_{\substack{m_1\ge m_2\ge m_3\\ m_1+m_2+m_3=m}} \frac{1}{(m_1+1)^{d/2}}\frac{1}{(m_2+1)^{d/2}}. \tag{5.23}$$

The summation constraints imply that $m_1 \ge m/3$, so that the sum on the right side is bounded, as required, by

$$\frac{C}{m^{d/2}} \sum_{m_2=0}^{m} \sum_{m_3=0}^{m_2} \frac{1}{(m_2+1)^{d/2}} \le \frac{C}{m^{d/2}} \quad (2 \le m \le n+1). \tag{5.24}$$

This proves the $N = 2$ case of (5.16).

To prove the $N = 2$ case of (5.17), we begin with

$$\sum_{x,y} \pi^{(1)}_{m,2}(x,y)z^m \le \sum_{0\le i<j<m} \sum_{x,y} c_i(y)z^i c_{j-i}(x-y)z^{j-i}c_{m-j}(y-x)z^{m-j}. \tag{5.25}$$

Again the subscripts on the c's are all less than n. We apply the l_∞ norm to the second or third factor, selecting the factor according to which of $j-i$ and $m-j$ is larger. The other two factors are bounded by the l_1 norm, to take care of the sums over x, y. Writing $\ell = m-i$, (5.25) can be bounded by

$$\sum_{x,y} \pi^{(1)}_{m,2}(x,y)z^m \le C^3\beta \sum_{\ell=1}^{m} \frac{\ell}{\ell^{d/2}} \le C^3\beta. \tag{5.26}$$

In the above bound, the factor $\ell^{d/2}$ arises since the larger time interval, referred to above, has length at least $\ell/2$, and the numerator ℓ accounts for the possible values of j between i and m.

To advance the induction, we fix $N \geq 3$ and assume that (5.16)–(5.17) hold for $N-1$. Writing $L \in \mathcal{L}^{(N-1)}[i,m]$ as $L = \{i_2 j_2, \ldots, i_N j_N\}$, we replace the factor $-U_{i_2 j_2}$ in the first product of (5.12) by

$$-U_{i_2 j_2} = \delta_{\omega(i_2),\omega(j_2)} = \sum_{v \in \mathbb{Z}^d} \delta_{\omega(i_2),v} \delta_{v,\omega(j_2)}. \tag{5.27}$$

Let $L' = L \backslash \{i_2, j_2\}$. For $j \in \mathcal{A}(L)$, we then have $\mathcal{C}(L) \supset \mathcal{B}[i_2, j] \cup \mathcal{B}[j, i_3] \cup \mathcal{C}(L')$. Using (5.27), we conclude from (5.12) that

$$J_x^{(N)}[0,m] \leq \sum_v \sum_{0 \leq i < j < m} K[0,i] \delta_{\omega(i),v} K[i,j] \delta_{x,\omega(j)} J_v^{(N-1)}[j,m]. \tag{5.28}$$

Therefore

$$\sum_y \pi_{m,N}^{(1)}(x,y) z^m \leq \sum_{0 \leq i < j < m} \sum_v c_i(v) z^i c_{j-i}(x-v) z^{j-i} \sum_y \pi_{m-j,N-1}^{(1)}(v-x, y-x) z^{m-j}. \tag{5.29}$$

This inequality is of the same form as (5.21), except $\sum_y \pi_{m-j,N-1}^{(1)}(v-x, y-x)$ appears instead of $c_{m-j}(v-x)$. For the case $N=2$, our bounds depended only on bounds on the l_∞ and l_1 norms of $c_s(v)$. If we knew that $\sum_y \pi_{m-j,N-1}^{(1)}(v-x, y-x)$ obeyed the same bounds, apart from the extra factors of $C\beta$ needed for general N, the proof would proceed exactly as before. But these required bounds are exactly given by the induction hypotheses (5.16)–(5.17). In fact, the only thing to check is that the powers of β work out as required.

When we take the supremum over x in (5.29), the factor with shortest time interval is bounded with the l_1 norm and the other two factors are bounded with the l_∞ norm. Note that there is no contribution when $m-j=0$, since $N-1>0$. Thus if $m-j$ is shortest, we obtain $\beta \cdot \beta \cdot \beta^{N-2} = \beta^N$, as required. If one of the other two factors is shortest, then we obtain $\beta \cdot \beta^{N-1} = \beta^N$, as required.

When we take the sum over x in (5.29), the shortest of the second two factors receives the l_1 norm and the other receives the l_∞ norm. If $m-j$ is shortest, then we obtain $\beta \cdot \beta^{N-2} = \beta^{N-1}$, as required. If the other factor is shortest, then we obtain $\beta^0 \cdot \beta^{N-1} = \beta^{N-1}$, as required.

This advances the induction and completes the proof for $q=0$. ∎

Before turning to the cases $q = 2, 4$, we pause to make an observation that will be used in those cases, and that will be crucial in Section 6. The observation concerns the fact that the upper bounds (5.16)–(5.17), which were at the heart of the proof of (5.1) for $q=0$, have a particular structure that we will exploit. This structure can be plainly seen in the bounds (5.21) and (5.25) for the case $N=2$. Each of these upper bounds involves sums over temporal and spatial variables of products of factors $c_{m_i}(y_i)$. These upper bounds were estimated in turn by applying the l_1 and l_∞ norms to these factors, depending on how the temporal variables were ordered. The structure of the upper bounds is the same for larger values of N, with the difference being a larger number of summations and factors of $c_{m_i}(y_i)$, as in the recursive estimate (5.29). For $N=1$, only a single factor of $c_{m_i}(y_i)$ was involved.

To be more explicit, we note that the proof of (5.16)–(5.17) shows that we may write

$$\pi_{m,N}^{(1)}(x,y) z^m \leq \sum_{\vec{y} \in \mathcal{Y}} \sum_{\vec{m} \in \mathcal{M}} \prod_i c_{m_i}(y_i) z^{m_i}, \tag{5.30}$$

where $\mathcal{M}$ restricts the m_i so that, in particular, $\sum_{i=1}^{2N-1} m_i = m$, and where $\mathcal{Y}$ is a certain subset of $\mathbb{Z}^{d(2N-1)}$ for $N \geq 3$, and where $\mathcal{Y}$ uniquely specifies the y_i in terms of x, y for $N = 1, 2$. Moreover, our bound on (5.30) proceeds by bounding each factor $c_{m_i}(y_i)$ with either the l_1 or the l_∞ norm.

We will rely on this observation by noting that if we change $\pi^{(1)}_{m,N}(y)$ or $\pi^{(1)}_{m,N}(x, y)$ by making a modification to the portion of a walk on a single subinterval (5.9), and if we can control the increase in the l_1 and the l_∞ norm of the portion of the walk on that subinterval, then we can control the size of the modification to $\pi^{(1)}_{m,N}(y)$ or $\pi^{(1)}_{m,N}(x, y)$. This is formalised in the following lemma, whose proof is a consequence of the above remarks.

Lemma 5.3. *Suppose that*

$$\sum_{\vec{y}\in\mathcal{Y}} \sum_{\vec{m}\in\mathcal{M}} \prod_i a^{(i)}_{m_i}(y_i) \leq B, \tag{5.31}$$

with the bound obtained by applying the inequalities $\|a^{(i)}_m\|_\infty \leq K\beta m^{-d/2}$ *or* $\|a^{(i)}_m\|_1 \leq K$ *to each factor* $a^{(i)}_{m_i}(y_i)$. *Suppose in addition that* $b^{(i)}$ *obeys*

$$\|b^{(i)}_m\|_\infty \leq \alpha_i K\beta m^{-d/2} \quad \text{and} \quad \|b^{(i)}_m\|_1 \leq \alpha_i K, \tag{5.32}$$

with α_i *independent of* m_i. *Then*

$$\sum_{\vec{y}\in\mathcal{Y}} \sum_{\vec{m}\in\mathcal{M}} \prod_i b^{(i)}_{m_i}(y_i) \leq \big(\prod_i \alpha_i\big) B. \tag{5.33}$$

Proof of Proposition 5.1(i) for $q = 2, 4$. Since $|y|^q \pi^{(1)}_{m,1}(y) = 0$ for all $y \in \mathbb{Z}^d$, we restrict attention to $N \geq 2$.

Fix $N \geq 2$. Because of the factor $\prod_{ij\in L}(-U_{ij})$ occurring in the definition of $\pi^{(1)}_{m,N}(y)$, a nonzero contribution occurs only for those ω for which $\omega(i) = \omega(j)$ for each edge $ij \in L$. Let I_j denote the j^{th} time interval listed in (5.9) ($j = 1, \ldots, 2N-1$), and let y_j denote the displacement performed on I_j by a walk ω contributing to $\pi^{(1)}_{m,N}(x)$. The constraints that $\omega(i) = \omega(j)$ for all $ij \in L$, together with the subinterval structure (5.9), impose the constraints

$$y_1 + y_2 = 0, \quad \sum_{j=2p}^{2p+2} y_j = 0 \quad (p = 1, \ldots, N-2), \quad y_{2N-2} + y_{2N-1} = 0. \tag{5.34}$$

It can also be seen from (5.9) that the total displacement y is given by

$$y = \sum_{i=1}^{\lceil N/2\rceil} y_{4i-3} = \sum_{i=1}^{\lfloor N/2\rfloor} y_{4i-1} = -\sum_{i=1}^{N-1} y_{2i}. \tag{5.35}$$

Using the first two identities of (5.35) (we will not need the third), together with the Cauchy–Schwarz inequality, we obtain

$$|y|^2 \leq \lceil N/2\rceil \sum_{i=1}^{\lceil N/2\rceil} |y_{4i-3}|^2, \qquad |y|^2 \leq \lfloor N/2\rfloor \sum_{i=1}^{\lfloor N/2\rfloor} |y_{4i-1}|^2. \tag{5.36}$$

We will prove bounds on $\sum_y |y|^q \pi^{(1)}_{m,N}(y)$ for $q = 2, 4$ by taking the factor $|y|^q$ inside the sum over laces defining $\pi^{(1)}_{m,N}(y)$, applying one or both of the estimates (5.36), and taking the factors involving $N/2$ and the sum or sums over i outside all the other sums. Then the walks on one (for $q=2$) or two (for $q=4$) of the subintervals carry an extra factor corresponding to the square of the displacement of ω on that subinterval.

In the proof for the $q=0$ case, we recursively bounded the contribution due to each subinterval either by $\|c_j\|_\infty z^j$ or by $\|c_j\|_1 z^j$, with $j \leq m \wedge n$ for each j. For $q=2,4$, using (5.36), similar bounds will apply except that $q/2$ of the subintervals will instead have an extra factor $|y_j|^2$ inside the norms. By (5.3)–(5.4), these norms are at most $\sigma^2 j \leq \sigma^2 m$ times larger than the norms without the $|y_j|^2$. Therefore, the presence of the factor $|y_j|^q$ increases the $q=0$ bound by $m\sigma^2$ for $q=2$ and by $m^2\sigma^4$ for $q=4$. Performing the summation(s) of (5.36) and using Lemma 5.3, we therefore obtain

$$\sum_{y\in\mathbb{Z}^d} |y|^q \pi^{(1)}_{m,N}(y) z^m \leq \frac{m^{q/2} N^q (C\beta)^N \sigma^q}{m^{d/2}} \quad (N\geq 2;\ 2\leq m\leq n+1;\ q=2,4). \tag{5.37}$$

This gives the desired result by taking L large and summing over N. □

For future reference, we note that the above proof also immediately yields the bounds

$$\sup_{x\in\mathbb{Z}^d} \sum_{y\in\mathbb{Z}^d} |y|^q \pi^{(1)}_{m,N}(x,y) z^m \leq \frac{N^q (C\beta)^N \sigma^q}{m^{(d-q)/2}} \quad (N\geq 1;\ 2\leq m\leq n+1;\ q=0,2), \tag{5.38}$$

$$\sum_{x,y\in\mathbb{Z}^d} |y|^q \pi^{(1)}_{m,N}(x,y) z^m \leq N^q C^N \beta^{N-1} \sigma^q m^{q/2} \quad (N\geq 1;\ 2\leq m\leq n+1;\ q=0,2). \tag{5.39}$$

Proof of Proposition 5.1(ii). We give separate arguments for $\|k\|_\infty \leq L^{-1}$ and $\|k\|_\infty \geq L^{-1}$. For $\|k\|_\infty \geq L^{-1}$, it follows from Proposition 5.1(i) and (1.8) that for $2\leq m\leq n+1$

$$\begin{aligned}\big|\hat{\pi}^{(1)}_m(k) - \hat{\pi}^{(1)}_m(0) - [1-\hat{D}(k)]\sigma^{-2}\nabla^2\hat{\pi}^{(1)}_m(0)\big| z^m &\leq \frac{C\beta}{m^{d/2}} + \frac{C\beta}{m^{d/2}} + \frac{C\beta[1-\hat{D}(k)]}{m^{(d-2)/2}} \\ &\leq \frac{C\beta[1-\hat{D}(k)]}{m^{(d-2)/2}} \leq \frac{C\beta[1-\hat{D}(k)]^2}{m^{(d-2)/2}}.\end{aligned} \tag{5.40}$$

This contribution satisfies (4.7).

Henceforth, we restrict attention to $\|k\|_\infty \leq L^{-1}$. By the triangle inequality,

$$\begin{aligned}&\big|\hat{\pi}^{(1)}_m(k) - \hat{\pi}^{(1)}_m(0) - [1-\hat{D}(k)]\sigma^{-2}\nabla^2\hat{\pi}^{(1)}_m(0)\big| z^m \\ &\quad\leq \Big|\hat{\pi}^{(1)}_m(k)z^m - \hat{\pi}^{(1)}_m(0)z^m - \frac{|k|^2}{2d}\nabla^2\hat{\pi}^{(1)}_m(0)z^m\Big| + \Big|[1-\hat{D}(k)]\sigma^{-2} - \frac{|k|^2}{2d}\Big|\ \big|\nabla^2\hat{\pi}^{(1)}_m(0)z^m\big|.\end{aligned} \tag{5.41}$$

By symmetry, the first term on the right side of (5.41) can be rewritten using

$$\hat{\pi}^{(1)}_m(k)z^m - \hat{\pi}^{(1)}_m(0)z^m - \frac{|k|^2}{2d}\nabla^2\hat{\pi}^{(1)}_m(0)z^m = \sum_x \Big(\cos(k\cdot x) - 1 + \frac{(k\cdot x)^2}{2}\Big)\pi^{(1)}_m(x)z^m. \tag{5.42}$$

For every $\epsilon' \in [0, \epsilon\wedge 1]$, there is a constant $c>0$ such that $|\cos t - 1 + \frac{1}{2}t^2| \leq ct^{2+2\epsilon'}$. Since $|k\cdot x|^{2+2\epsilon'} \leq |k|^{2+2\epsilon'}|x|^{2+2\epsilon'}$, it follows that

$$|\hat{\pi}^{(1)}_m(k)z^m - \hat{\pi}^{(1)}_m(0)z^m - \frac{|k|^2}{2d}\nabla^2\hat{\pi}^{(1)}_m(0)z^m| \leq c|k|^{2+2\epsilon'} \sum_x |x|^{2+2\epsilon'}|\pi^{(1)}_m(x)|. \tag{5.43}$$

By Hölder's inequality and Proposition 5.1(i) with $q = 4$,

$$\sum_x |x|^{2+2\epsilon'} |\pi_m^{(1)}(x)| z^m \le \left(\sum_x |\pi_m^{(1)}(x)| z^m \right)^{\frac{1-\epsilon'}{2}} \left(\sum_x |x|^4 |\pi_m^{(1)}(x)| z^m \right)^{\frac{1+\epsilon'}{2}} \le \frac{K\beta\sigma^{2+2\epsilon'}}{m^{(d-2-2\epsilon')/2}}. \quad (5.44)$$

The desired bound on the first term of (5.41) then follows by combining (5.43)–(5.44) with the lower bound of (1.7).

It follows from (1.5) and Hölder's inequality that $\sum_x |x|^{2+2\epsilon'} D(x) \le CL^{2+2\epsilon'}$ for each $\epsilon' \in [0, \epsilon]$. By Proposition 5.1(i) with $q = 2$, arguing as above and using (1.7), the second term of (5.41) is bounded by

$$\frac{K\beta}{m^{(d-2)/2}} \left| [1 - \hat{D}(k)] - \frac{|k|^2\sigma^2}{2d} \right| \le \frac{K\beta |k|^{2+2\epsilon'} L^{2+2\epsilon'}}{m^{(d-2)/2}} \le \frac{K\beta}{m^{(d-2)/2}} [1 - \hat{D}(k)]^{1+\epsilon'}, \quad (5.45)$$

which satisfies (4.7). □

6 Proof of Proposition 4.2

In this section, we prove Proposition 4.2. Proposition 4.2(i) is proved in Sections 6.1–6.3 and Proposition 4.2(ii) is proved in Section 6.4.

6.1 Overview of the proof of Proposition 4.2(i)

There are two statements in Proposition 4.2(i). Writing $z_c = \mu^{-1}$, the first statement is that the bound

$$\sum_{\bar{y}} |y_e|^q \sum_{N=1}^{\infty} \pi_{\bar{n},N}^{(s)}(\bar{y}) z_c^n \le C\beta^{\lceil s/2 \rceil} \sigma^q n_e^{q/2} B_{\bar{n}}^{(s)} \quad (q = 0, 2) \quad (6.1)$$

holds for $s \ge 2$ when $n_e > 0$ for all e. The second statement is that if $n'_e \ne 0$ for exactly M values of e', then (6.1) remains true if the power of β on the right side is reduced to $\lceil M/2 \rceil$. This second statement follows easily from (6.1) and the fact that in this case $\pi^{(s)}$ is equal to $\pi^{(M)}$ with strictly positive 'time' variables. So we may assume in Sections 6.2–6.3, where the proof is carried out, that $\bar{n}$ has strictly positive components.

By (2.16)–(2.17),

$$\sum_{N=1}^{\infty} \pi_{\bar{n},N}^{(s)}(\bar{y}) = \sum_{\omega \in \Omega_{S_{\bar{n}}}(\bar{y})} W(\omega) \sum_{L \in \mathcal{L}(S)} \prod_{ij \in L} (-U_{ij}) \prod_{i'j' \in \mathcal{C}(L)} (1 + U_{i'j'}). \quad (6.2)$$

Recall from Definition 3.5 that the set of laces on T is partitioned into the set $\mathcal{M}(T)$ of laces containing no non-minimal cyclic component and the set $\mathcal{N}(T)$ of laces that do contain a non-minimal cyclic component. We will consider the contributions to $\pi_{\bar{n},N}^{(s)}(\bar{y})$ due to $\mathcal{M}$ and $\mathcal{N}$ separately. In Section 6.2, we show that the contribution to (6.2) due to laces in $\mathcal{M}$ obeys the bound of (6.1), and in Section 6.3 we show that the contribution to (6.2) due to laces in $\mathcal{N}$ also obeys the bound of (6.1). The properties of laces provided in Section 3 play a crucial role.

6.2 Contribution from laces in $\mathcal{M}$

Suppose that $\bar{n}$ has all its components strictly positive. Let

$$\mu_{\bar{n}}^{(s)}(\bar{x}) = \sum_{\omega \in \Omega_{S_{\bar{n}}}(\bar{y})} W(\omega) \sum_{L \in \mathcal{M}(S)} \prod_{ij \in L} (-U_{ij}) \prod_{i'j' \in \mathcal{C}(L)} (1 + U_{i'j'}) \tag{6.3}$$

denote the contribution to (6.2) due to laces in $\mathcal{M}$. In this section, we prove that this contribution obeys the bound of (6.1), i.e., that

$$\sum_{\bar{y}} |y_e|^q \mu_{\bar{n}}^{(s)}(\bar{y}) z_c^n \le C\beta^{\lceil s/2 \rceil} \sigma^q n_e^{q/2} B_{\bar{n}}^{(s)} \quad (q = 0, 2). \tag{6.4}$$

The proof is by induction on the degree $s = d_b$ of the branch point b.

6.2.1 The induction on s for laces in $\mathcal{M}$

The induction hypothesis is twofold. First, we assume that the bounds (6.4) hold when s is replaced by $t = 2, \ldots, s-1$, with C depending only on t and on the dimension d.

For the second part of the induction hypothesis, given a lace $L \in \mathcal{L}(S)$, we define

$$\mathcal{V}(L) = \{b\} \bigcup \Big(\bigcup_{ij \in L} \{i, j\} \Big) \tag{6.5}$$

to be the union of the branch point b with the endpoints of edges in L. The set $\mathcal{V}(L)$ induces a division of the tree into intervals. We will refer to these intervals as the *subintervals* induced by L. When the tree is just an interval, the subintervals of L are listed in (5.9). The second part of the induction hypothesis is that the bound (6.4) is obtained by applying the l_1 or the l_∞ norms on the subintervals induced by L, as described before Lemma 5.3. In this case, we will say that the bound has the *subinterval property*.

We begin the induction by establishing (6.4) for $s = 2$. As we now explain, this case follows from estimates obtained already in Section 5. Note that for $s = 2$, if $L \in \mathcal{M}(S)$ then L must be minimal. Therefore, for n_1 and n_2 both positive, $\mu_{n_1,n_2}^{(2)}(y_1, y_2)$ is by definition the same as $\pi_{n_1+n_2}^{(1)}(y_2 - y_1)$, apart from an additional constraint $\delta_{\omega(n_1),-y_1}$ in the sum over ω that defines $\pi_{n_1+n_2}^{(1)}(y_2 - y_1)$ in (2.7). The sum over y_2 sums over the walk's position at time $n_1 + n_2$, while the sum over y_1 simply removes the constraint $\delta_{\omega(n_1),-y_1}$. Therefore, the $q = 0$ version of (6.4) for $s = 2$ follows immediately from Proposition 5.1(i). For $q = 2$, we may decompose y_e into subinterval displacements as in the proof of Proposition 5.1(i) for $q = 2$, and the bounds proceed as before except for the subinterval containing time n_1. This subinterval is special, since it carries a factor of the square of only part of its displacement, rather than its entire displacement. The desired bound will follow from Proposition 5.1(i) and Lemma 5.3, if we can show that the l_1 and l_∞ norms corresponding to this subinterval are bounded in the same way as if a factor of the square of the entire subinterval displacement were present. Thus we must show that the l_1 norm is bounded by Cmz_c^{-m} and the l_∞ norm by $Cz_c^{-m}m^{-(d-2)/2}$, where m is the length of the subinterval. To carry this out, we define $c_{m',m}(y', y)$ $(m' \le m)$ to be the number of m-step self-avoiding walks with $\omega(0) = 0$, $\omega(m') = y'$, and $\omega(m) = y$. Since $c_{m',m}(y', y) \le c_{m'}(y')c_{m-m'}(y - y')$, it follows immediately that $\sum_{y',y} |y'|^2 c_{m',m}(y', y) \le Cmz_c^{-m}$. Similarly, $\sup_y \sum_{y'} |y'|^2 c_{m',m}(y', y)$ can be seen to obey the bound stated above, by associating the supremum over y' to the part of the walk which achieves the maximum of m', $m-m'$, and associating the sum over y' to the other part of the walk. This proves the claim for $s=2$.

To advance the induction, we partition $\mathcal{M}(T)$ into the sets of reducible, acyclic and cyclic laces introduced in Definition 3.3. These sets will be denoted respectively by $\mathcal{M}_r(T)$, $\mathcal{M}_a(T)$, $\mathcal{M}_c(T)$, and the contribution to the right side of (6.3) from these sets will be denoted respectively by $\rho_{\bar{n}}^{(s)}(\bar{x})$, $\alpha_{\bar{n}}^{(s)}(\bar{x})$ and $\sigma_{\bar{n}}^{(s)}(\bar{x})$. This gives the decomposition

$$\mu_{\bar{n}}^{(s)}(\bar{x}) = \rho_{\bar{n}}^{(s)}(\bar{x}) + \alpha_{\bar{n}}^{(s)}(\bar{x}) + \sigma_{\bar{n}}^{(s)}(\bar{x}). \tag{6.6}$$

We will advance the induction by arguing separately for each of the three terms on the right side of (6.6).

6.2.2 Reducible laces in $\mathcal{M}$

This case arises only for $s \geq 4$. By definition, for any lace $L \in \mathcal{M}_r(S)$ there is a subset $A \subset \{1,\ldots,s\}$ with $2 \leq |A| \leq s-2$ such that $L = L_1 \cup L_2$ with $L_1 \in \mathcal{M}(\cup_{e\in A} T_e)$ and $L_2 \in \mathcal{M}(\cup_{e\notin A} T_e)$. Since $\mathcal{C}(L)$ consists of the disjoint union of $\mathcal{C}(L_1)$, $\mathcal{C}(L_2)$ and the compatible edges $\mathcal{C}_{1,2}$ linking $\cup_{e\in A} T_e$ and $\cup_{e\notin A} T_e$, we can replace the sum over $L \in \mathcal{M}_r(T)$ in the definition of $\rho_{\bar{n}}^{(s)}(\bar{x})$ by sums over L_1, L_2, via an upper bound using $\prod_{ij\in\mathcal{C}_{1,2}}(1+U_{ij}) \leq 1$. This gives

$$\rho_{\bar{n}}^{(s)}(\bar{x}) \leq \sum_{A\subset\{1,\ldots,s\}:2\leq|A|\leq s-2} \mu_{\bar{n}_1}^{(|A|)}(\bar{x}_1)\, \mu_{\bar{n}_2}^{(s-|A|)}(\bar{x}_2), \tag{6.7}$$

where $\bar{n}_i$ and $\bar{x}_i$ are the labels (which depend on A) appropriate to the laces L_i. The induction hypothesis then gives the desired bound, using the inequality

$$\beta^{\lceil |A|/2\rceil}\beta^{\lceil (s-|A|)/2\rceil} B_{\bar{n}_1}^{(|A|)} B_{\bar{n}_2}^{(s-|A|)} \leq \beta^{\lceil s/2\rceil} B_{\bar{n}}^{(s)}. \tag{6.8}$$

In addition, $\rho_{\bar{n}}^{(s)}(\bar{x})$ inherits the subinterval property from the induction hypothesis.

6.2.3 Acyclic laces in $\mathcal{M}$

In this section, we estimate $\alpha_{\bar{n}}^{(s)}(\bar{x})$. Recall that $\mathcal{A}(L)$ was defined in (5.11) for a lace L on an interval $[0,m]$. For a lace L on $S=(\tau_r,\bar{n})$ with $r\geq 2$, we define $\mathcal{A}(L)$ to be the union over the branches of the $r+2$ subintervals on each branch which are closest to the branch point. If there are fewer than $r+2$ subintervals on a branch, then $\mathcal{A}(L)$ is defined to include that branch entirely. We refer to elements of $\mathcal{A}(L)$ as *admissible* vertices. For $m\in\mathbb{N}$, let

$$\mathcal{A}_m(L) = \{i \in \mathcal{A}(L) : d(i,\mathcal{V}(L)) \geq m\} \tag{6.9}$$

denote the set of admissible vertices that are at least graph-distance m from the vertices of $\mathcal{V}(L)$.

For $s\geq 2$, we define

$$\mu_{\bar{n}}^{(s)}(v,\bar{x}) = \sum_{\omega\in\Omega_{S_{\bar{n}}}(\bar{y})} W(\omega) \sum_{L\in\mathcal{M}(S)} \sum_{i_1\in\mathcal{A}(L)} \delta_{v,\omega(i_1)} \prod_{ij\in L}(-U_{ij}) \prod_{i'j'\in\mathcal{C}(L)}(1+U_{i'j'}), \tag{6.10}$$

$$\mu_{\bar{n}}^{(s)}(v,\bar{x};m) = \sum_{\omega\in\Omega_{S_{\bar{n}}}(\bar{y})} W(\omega) \sum_{L\in\mathcal{M}(S)} \sum_{i_1\in\mathcal{A}_m(L)} \delta_{v,\omega(i_1)} \prod_{ij\in L}(-U_{ij}) \prod_{i'j'\in\mathcal{C}(L)}(1+U_{i'j'}). \tag{6.11}$$

Note that $\mu_{\bar{n}}^{(s)}(v,\bar{x}) = \mu_{\bar{n}}^{(s)}(v,\bar{x};0)$. For $s=1$, recalling the definition of $\pi_{n,N}^{(1)}(x,y)$ for $N \geq 1$ from (5.14), we define

$$\mu_n^{(1)}(x,y) = \sum_{N=1}^{\infty} \pi_{n,N}^{(1)}(x,y). \tag{6.12}$$

The bounds (5.38)–(5.39) then give

$$\sup_x \sum_y |y|^q \mu_n^{(1)}(x,y) z_c^n \leq C\beta\sigma^q (n+1)^{-(d-q)/2} \quad (n \geq 2,\ q=0,2), \tag{6.13}$$

$$\sum_{x,y} |y|^q \mu_n^{(1)}(x,y) z_c^n \leq C\sigma^q n^{q/2} \quad (n \geq 2,\ q=0,2). \tag{6.14}$$

Moreover, the analysis of Section 5 implies that these bounds have the subinterval property.

Lemma 6.1. *Let $s \geq 2$. If $\mu_{\bar{n}}^{(s)}(\bar{x})$ satisfies the bound (6.4), with the bound having the subinterval property, then*

$$\sup_v \sum_{\bar{x}} |x|^q \mu_{\bar{n}}^{(s)}(v,\bar{x};m) z_c^n \leq C\beta^{\lceil s/2 \rceil +1} \sigma^q m^{q/2} (m+1)^{-(d-2)/2} B_{\bar{n}}^{(s)} \quad (q=0,2), \tag{6.15}$$

with this bound also having the subinterval property (with the set $\mathcal{V}(L)$ augmented with the vertex $i_1 \in \mathcal{A}_m(L)$ occurring in (6.11)*).*

Proof. We consider only the case $q=0$ as $q=2$ is similar. Note that the bound (6.15) is $\beta(m+1)^{-(d-2)/2}$ times the right side of (6.1).

The difference between $\mu_{\bar{n}}^{(s)}(v,\bar{x};m)$ and $\mu_{\bar{n}}^{(s)}(\bar{x})$ resides only in the subinterval containing $i \in \mathcal{A}_m(L)$. A factor $c_j(y)$ in a bound on the latter is replaced by the factor $\sum_{i=m}^{j-m} c_{i,j}(v,y)$ in the former, where $c_{i,j}(v,y) = \sum_{\omega \in \Omega_j(y)} W(\omega) \delta_{v,\omega(i)} K[0,j]$ for $0 \leq i \leq j$. By Lemma 5.3, it suffices to show that

$$\sum_y \sum_{i=m}^{j-m} c_{i,j}(v,y) z_c^j \leq C\beta (m+1)^{-(d-2)/2}, \tag{6.16}$$

$$\sup_y \sum_{i=m}^{j-m} c_{i,j}(v,y) z_c^j \leq C\beta^2 (m+1)^{-(d-2)/2} (j+1)^{-d/2}, \tag{6.17}$$

with these bounds uniform in v. We prove (6.16)–(6.17) using $c_{i,j}(v,y) \leq c_i(v) c_{j-i}(y-v)$ in conjunction with the bounds of (5.3). For (6.16), we have

$$\sum_y \sum_{i=m}^{j-m} c_{i,j}(v,y) z_c^j \leq C\beta \sum_{i=m}^{j-m} (i+1)^{-d/2} \leq C\beta (m+1)^{-(d-2)/2}. \tag{6.18}$$

For (6.17), we consider only the case $i \geq j/2$, for which as required, we have

$$\begin{aligned} \sup_y \sum_{i=j/2}^{j-m} c_{i,j}(v,y) z_c^j &\leq C\beta^2 \sum_{i=j/2}^{j-m} (i+1)^{-d/2} (j-i+1)^{-d/2} \leq C\beta^2 (j+1)^{-d/2} \sum_{i=j/2}^{j-m} (j-i)^{-d/2} \\ &\leq C\beta^2 (j+1)^{-d/2} (m+1)^{-(d-2)/2}. \end{aligned} \tag{6.19}$$

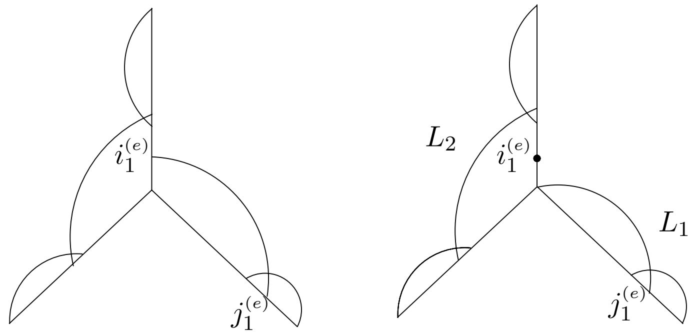

Figure 8: Reorganisation of an acyclic lace, in the case $i_2^{(e)} \neq b$.

The case $i \leq j/2$ is similar. By construction, the subinterval property is preserved. $\square$

We now advance our estimate on the contribution to $\mu_{\bar{n}}^{(s)}(\bar{y})$ due to acyclic laces, which is

$$\alpha_{\bar{n}}^{(s)}(\bar{x}) = \sum_{\omega \in \Omega_{S_{\bar{n}}}(\bar{y})} W(\omega) \sum_{L \in \mathcal{M}_a(T)} \prod_{ij \in L} (-U_{ij}) \prod_{i'j' \in \mathcal{C}(L)} (1 + U_{i'j'}). \tag{6.20}$$

Writing $\bar{x}_e$ to denote removal of the e^{th} component from $\bar{x}$, we have the following lemma.

Lemma 6.2. *For $s \geq 3$,*

$$\alpha_{\bar{n}}^{(s)}(\bar{x}) \leq \sum_{e=1}^{s} \sum_{v \in \mathbb{Z}^d} \mu_{n_e}^{(1)}(v, x_e) \mu_{\bar{n}_e}^{(s-1)}(v, \bar{x}_e). \tag{6.21}$$

Proof. By Lemma 3.7, for any acyclic lace $L \in \mathcal{L}_a(T)$ there is a branch T_e with $M_e = 1$ such that the restriction of L to $T_{e^c} = \cup_{a:a \neq e} T_a$ is a lace on T_{e^c}. We call this restriction L_2. Since we are considering $L \in \mathcal{M}_a(S) \subset \mathcal{M}(S)$, it follows that $L_2 \in \mathcal{M}(T_{e^c})$. Let $i_1^{(e)} j_1^{(e)}$ denote the edge in L associated to T_e, so $j_1^{(e)} \in T_e$ and $i_1^{(e)} \notin T_e$. Let $T_{e'}$ denote the branch containing $i_1^{(e)}$. Using the notation of Lemma 3.10, let $L_1 = L^{(e)}$ if $i_2^{(e)} = b$, and $L_1 = L^{(e)} \cup \{j_1^{(e)} b\}$ if $i_2^{(e)} \neq b$. See Figure 8. If $L^{(e)} \neq \varnothing$, then by construction $j_1^{(e)} \in \mathcal{A}(L_1 \backslash \{j_1^{(e)} b\})$ (note that the branch point b should not be confused with the b of (5.11)). To deal with the case $L^{(e)} = \varnothing$, which implies $L_1 \backslash \{j_1^{(e)} b\} = \varnothing$, for convenience of notation we will write $\mathcal{A}(L_1 \backslash \{j_1^{(e)} b\}) = \{n_e\}$ in this case.

In addition, we have defined the set of admissible vertices so that $\mathcal{A}(L_2)$ is large enough to ensure that $i_1^{(e)} \in \mathcal{A}(L_2)$. To see this, we argue as follows. Given L with $i_1^{(e)} \in T_{e'}$, it must be the case that $i_1^{(e)}$ lies in the interval $[b, j_1^{(e')}]$ of $T_{e'}$. The number of elements of $\mathcal{V}(L_2)$ that lie strictly between $j_1^{(e')}$ and b is at most s, including $i_2^{(e')}$ and one endpoint of each of the at most $s-1$ edges of L_2 that cover b. Thus there are at most $s+1$ subintervals of L_2 in which $i_1^{(e)}$ can lie, and by construction, these subintervals are the ones that are closest to b.

Given e, we use the fact that $\mathcal{C}(L_1) \cup \mathcal{C}(L_2) \subset \mathcal{C}(L)$ to estimate the final product of (6.20). This leads to

$$\begin{aligned} \alpha_{\bar{n}}^{(s)}(\bar{x}) \leq \sum_{e=1}^{s} \sum_{\omega \in \Omega_{S_{\bar{n}}}(\bar{x})} W(\omega) \sum_{L_2 \in \mathcal{M}(T_{e^c})} \sum_{i_1^{(e)} \in \mathcal{A}(L_2)} \prod_{ij \in L_2} (-U_{ij}) \prod_{i'j' \in \mathcal{C}(L_2)} (1 + U_{i'j'}) \\ \times \sum_{L_1 \in \mathcal{L}(T_e)} \sum_{j_1^{(e)} \in \mathcal{A}(L_1 \backslash \{j_1^{(e)} b\})} (-U_{i_1^{(e)} j_1^{(e)}}) \prod_{ij \in L_1 \backslash \{j_1^{(e)} b\}} (-U_{ij}) \prod_{i'j' \in \mathcal{C}(L_1)} (1 + U_{i'j'}). \end{aligned} \tag{6.22}$$

We denote the portion of ω corresponding to T_a by $\{\omega^{(a)}(i)\}_{i=0}^{n_a}$, for $a = 1, \ldots, s$, with $\omega^{(a)}(0) = b$ and $\omega^{(a)}(n_a) = x_a$. In (6.22), the walk $\omega^{(e)}$ is independent of the other $\omega^{(a)}$ apart from the factor

$$-U_{i_1^{(e)} j_1^{(e)}} = \delta_{\omega^{(e)}(j_1^{(e)}),\omega^{(e')}(i_1^{(e)})} = \sum_{v\in\mathbb{Z}^d} \delta_{\omega^{(e)}(j_1^{(e)}),v}\delta_{v,\omega^{(e')}(i_1^{(e)})}. \tag{6.23}$$

We insert (6.23) into (6.22) and move the sum over v next to the sum over e.

By (6.10), the sums over $i_1^{(e)}$ and $\omega^{(a)}$ ($a \neq e$) are equal to $\mu_{\bar{n}_e}^{(s-1)}(v, \bar{x}_e)$. We claim that the sums over $\omega^{(e)}$ and $j_1^{(e)}$ in (6.22) are bounded by $\mu_{n_e}^{(1)}(v, x_e)$ of (6.12). In fact, the case $|L_1| = 1$ gives $\pi_{\bar{n}_e,1}^{(1)}(v, \bar{x}_e)$. For $|L_1| > 1$, the case $i_2^{(e)} = b$ corresponds to the term $i = a$ in (5.12), while the case $i_2^{(e)} \neq b$ corresponds to $i > a$. The upper bound follows by using the estimate $1 + U_{ij} \leq 1$ for some of the edges compatible with L_1. This gives (6.21). □

We now proceed to bound $\sum_{\bar{x}} \alpha_{\bar{n}}(\bar{x})$, using (6.21). Recalling (6.11), we write

$$\mu_{\bar{n}_e}^{(s-1)}(v, \bar{x}_e) = \mu_{\bar{n}_e}^{(s-1)}(v, \bar{x}_e; n_e) + [\mu_{\bar{n}_e}^{(s-1)}(v, \bar{x}_e) - \mu_{\bar{n}_e}^{(s-1)}(v, \bar{x}_e; n_e)], \tag{6.24}$$

and deal with the two terms on the right side separately. By Lemma 6.2, the contribution of the first term to $\sum_{\bar{x}} \alpha_{\bar{n}}(\bar{x})$ is bounded by

$$\sum_e \Big(\sup_v \sum_{\bar{x}_e} \mu_{\bar{n}_e}^{(s-1)}(v, \bar{x}_e; n_e)\Big) \sum_{x_e,v} \mu_{n_e}^{(1)}(v, x_e). \tag{6.25}$$

Using the induction hypothesis and Lemma 6.1 to estimate the first factor, and using (6.14) for the second factor, this is at most

$$C\sum_e \beta^{\lceil \frac{s-1}{2}\rceil+1} z_c^{-n}(n_e+1)^{-(d-2)/2} B_{\bar{n}_e}^{(s-1)} \leq C\beta^{\lceil \frac{s}{2}\rceil} z_c^{-n} B_{\bar{n}}^{(s)}. \tag{6.26}$$

The corresponding contribution from the second term is bounded by

$$\sum_e \Big(\sup_v \sum_{x_e} \mu_{n_e}^{(1)}(v, x_e)\Big) \sum_{\bar{x}_e,v} [\mu_{\bar{n}_e}^{(s-1)}(v, \bar{x}_e) - \mu_{\bar{n}_e}^{(s-1)}(v, \bar{x}_e; n_e)]. \tag{6.27}$$

By (6.10)–(6.11),

$$\sum_v [\mu_{\bar{n}_e}^{(s-1)}(v, \bar{x}_e) - \mu_{\bar{n}_e}^{(s-1)}(v, \bar{x}_e; n_e)] = |\mathcal{A}(L_2)\backslash\mathcal{A}_{n_e}(L_2)|\; \mu_{\bar{n}_e}^{(s-1)}(\bar{x}_e). \tag{6.28}$$

Since there are at most $(s-1)(s+2)$ subintervals in $\mathcal{A}(L_2)$ and at most $2(n_e+1)$ vertices of $\mathcal{A}(L_2)\backslash\mathcal{A}_{n_e}(L_2)$ in each subinterval, it follows from the induction hypothesis that

$$\sum_{\bar{x}_e,v} [\mu_{\bar{n}_e}^{(s-1)}(v, \bar{x}_e) - \mu_{\bar{n}_e}^{(s-1)}(v, \bar{x}_e; n_e)] \leq 2(n_e+1)(s-1)(s+2)C\beta^{\lceil \frac{s-1}{2}\rceil} z_c^{n_e-n} B_{\bar{n}_e}^{(s-1)}, \tag{6.29}$$

By (6.13), (6.27) is therefore bounded by

$$C\beta^{\lceil \frac{s-1}{2}\rceil+1} z_c^{-n} \sum_e (n_e+1)^{-(d-2)/2} B_{\bar{n}_e}^{(s-1)} \leq C\beta^{\lceil \frac{s}{2}\rceil} z_c^{-n} B_{\bar{n}}^{(s)}. \tag{6.30}$$

Adding the bounds in (6.26) and (6.30) gives the required estimate for $\sum_{\bar{x}} \alpha_{\bar{n}}^{(s)}(\bar{x})$, for $q = 0$. By construction, the estimate possesses the subinterval property. The analysis is almost identical for $q = 2$, and we omit the details. This advances the induction for the contribution due to acyclic laces.

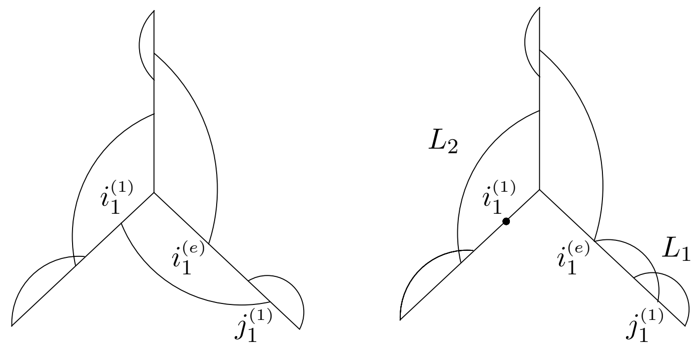

Figure 9: Reorganisation of a minimal cyclic lace.

6.2.4 Cyclic laces in $\mathcal{M}$

It remains to advance our estimate on the contribution to $\mu_{\bar{n}}^{(s)}(\bar{y})$ due to the cyclic laces, which is

$$\sigma_{\bar{n}}^{(s)}(\bar{x}) = \sum_{\omega \in \Omega_{S_{\bar{n}}}(\bar{y})} W(\omega) \sum_{L \in \mathcal{M}_c(S)} \prod_{ij \in L} (-U_{ij}) \prod_{i'j' \in \mathcal{C}(L)} (1 + U_{i'j'}). \tag{6.31}$$

By definition, laces in $\mathcal{M}_c(T)$ are minimal. By Lemma 3.11(a), given a minimal cyclic lace $L \in \mathcal{M}_c(T)$ there is an $n_1' \in [1, n_1]$ such that $L_2 = \cup_{e=2}^{s} \mathsf{L}_L(e) \in \mathcal{M}_a(S_{\bar{n}'})$ is an acyclic lace on $S_{\bar{n}'}$, where $n_e' = n_e$ for all $e \geq 2$. Let e be such that $i_1^{(e)} \in T_1$, and let e' be such that $i_1^{(1)} \in T_{e'}$. Then $i_1^{(e)}$ must be the point on T_1 labelled n_1'. By Lemma 3.10, $n_1' < n_1$.

If $L^{(1)} \neq \varnothing$, we conclude from Lemma 3.11(b) that $L_1 = L^{(1)} \cup \{j_1^{(1)} i_1^{(e)}\}$ is a lace if $i_2^{(1)} \neq i_1^{(e)}$, and $L_1 = L^{(1)}$ is a lace if $i_2^{(1)} = i_1^{(e)}$. In either case, $j_1^{(1)} \in \mathcal{A}(L_1 \backslash \{j_1^{(1)} i_1^{(e)}\})$. See Figure 9. In addition, if $L^{(e)} = \varnothing$, then $\{j_1^{(1)} i_1^{(e)}\} = \{n_1 i_1^{(e)}\}$ is a lace. For convenience of notation, in the latter case we write $\mathcal{A}(L_1 \backslash \{j_1^{(1)} i_1^{(e)}\}) = \{n_1\}$. In all cases, $i_1^{(1)} \in \mathcal{A}(L_2)$, and

$$L = L_1 \cup L_2 \cup \{i_1^{(1)} j_1^{(1)}\} \backslash \{j_1^{(1)} i_1^{(e)}\}. \tag{6.32}$$

Note that $\mathcal{C}(L_1) \cup \mathcal{C}(L_2) \subset \mathcal{C}(L)$. In (6.31), we use $1 + U_{i'j'} \leq 1$ for all $i'j' \in \mathcal{C}(L) \backslash [\mathcal{C}(L_1) \cup \mathcal{C}(L_2)]$. This leads to

$$\begin{aligned} \sigma_{\bar{n}}^{(s)}(\bar{x}) \;\; \leq \;\; & \sum_{n_1'=1}^{n_1-1} \sum_{\omega \in \Omega_{S_{\bar{n}}}(\bar{x})} W(\omega) \sum_{L_2 \in \mathcal{M}_a(S_{\bar{n}'})} \sum_{i_1^{(1)} \in \mathcal{A}(L_2)} \prod_{ij \in L_2} (-U_{ij}) \prod_{i'j' \in \mathcal{C}(L_2)} (1 + U_{i'j'}) \\ & \times \sum_{L_1 \in \mathcal{L}([n_1', n_1])} \sum_{j_1^{(1)} \in \mathcal{A}(L_1 \backslash \{j_1^{(1)} i_1^{(e)}\})} (-U_{i_1^{(1)} j_1^{(1)}}) \prod_{ij \in L_1 \backslash \{j_1^{(1)} i_1^{(e)}\}} (-U_{ij}) \prod_{i'j' \in \mathcal{C}(L_1)} (1 + U_{i'j'}). \end{aligned} \tag{6.33}$$

We denote the portion of ω corresponding to T_a by $\omega^{(a)}$, for $a = 2, \ldots, s$, with $\omega^{(a)}(0) = b$ and $\omega^{(a)}(n_a) = x_a$. We denote the portions of the walk corresponding to the intervals $[0, n_1'] \subset T_1$ and $[n_1', n_1] \subset T_1$ by $\omega_1^{(1)}$ and $\omega_2^{(1)}$ respectively. We set $x_1' = \omega_1^{(1)}(i_1^{(e)})$. The walk $\omega_2^{(1)}$ remains coupled to the other walks only by the factor $-U_{i_1^{(1)} j_1^{(1)}}$, which can be written as

$$-U_{i_1^{(1)} j_1^{(1)}} = \delta_{\omega^{(1)}(j_1^{(1)}), \omega^{(e')}(i_1^{(1)})} = \sum_{v \in \mathbb{Z}^d} \delta_{\omega^{(1)}(j_1^{(1)}), v} \delta_{v, \omega^{(e')}(i_1^{(1)})}. \tag{6.34}$$

The sum over $\omega_2^{(1)}$ in (6.33) is then bounded by $\pi^{(1)}_{n_1-n_1'}(v-x_1', x_1-x_1')$, as in the bound on (6.22). The sums over $\omega^{(a)}$ $(a \neq 1)$ and $(\omega_2^{(1)}(i))_{i=n_1'}^{n_1}$ are equal to $\alpha^{(s)}_{\bar{n}'}(v, \bar{x}')$, where $x_e' = x_e$ for $e \geq 2$. Writing $y_1 = x_1 - x_1'$ and $u = v - x_1'$, we have thus shown that

$$\sum_{\bar{x}} \sigma^{(s)}_{\bar{n}}(\bar{x}) \leq \sum_{n_1'=1}^{n_1} \sum_u \sum_{y_1} \mu^{(1)}_{n_1-n_1'}(u, y_1) \sum_{\bar{x}'} \alpha^{(s)}_{\bar{n}'}(u+x_1', \bar{x}'). \tag{6.35}$$

We now proceed as in Section 6.2.3, and write

$$\alpha^{(s)}_{\bar{n}'}(v, \bar{x}') = \alpha^{(s)}_{\bar{n}'}(v, \bar{x}'; n_1 - n_1') + [\alpha^{(s)}_{\bar{n}'}(v, \bar{x}') - \alpha^{(s)}_{\bar{n}'}(v, \bar{x}'; n_1 - n_1')] \tag{6.36}$$

where $\alpha^{(s)}_{\bar{n}}(v, \bar{x}; m)$ denotes the restriction of $\mu^{(s)}_{\bar{n}}(v, \bar{x}; m)$ in (6.11) to acyclic laces. The contribution to (6.35) from the first term in (6.36) is bounded above by

$$\sum_{n_1'=1}^{n_1} \Big(\sum_{v, y_1} \mu^{(1)}_{n_1-n_1'}(v, y_1) \Big) \Big(\sup_v \sum_{\bar{x}'} \alpha^{(s)}_{\bar{n}'}(v, \bar{x}'; n_1 - n_1') \Big). \tag{6.37}$$

The first factor in the above product is bounded by $Cz_c^{n_1'-n_1}$, by (6.14), with the subinterval property. For the second factor, we note that we have already shown in Section 6.2.3 that $\sum_{\bar{x}} \alpha^{(s)}_{\bar{n}}(\bar{x})$ is bounded by the right side of (6.1), with the bound having the subinterval property. The proof of Lemma 6.1 therefore implies that $\sup_v \sum_{\bar{x}} \alpha^{(s)}_{\bar{n}}(v, \bar{x}; m)$ is bounded by the right side of (6.15). Therefore, by Lemma 4.4, (6.37) is bounded above, as required, by

$$C\beta^{\lceil \frac{s}{2} \rceil + 1} z_c^{-n} \sum_{n_1'=1}^{n_1} (n_1 - n_1' + 1)^{-(d-2)/2} B^{(s)}_{\bar{n}'} \leq C\beta^{\lceil \frac{s}{2} \rceil + 1} z_c^{-n} B^{(s)}_{\bar{n}}. \tag{6.38}$$

By construction, this bound possesses the subinterval property.

The contribution to (6.35) from the second term in (6.36) is bounded above by

$$\sum_{n_1'=1}^{n_1} \Big(\sup_v \sum_{y_1} \mu^{(1)}_{n_1-n_1'}(v, y_1) \Big) \Big(\sum_{v, \bar{x}'} [\alpha^{(s)}_{\bar{n}'}(v, \bar{x}') - \alpha^{(s)}_{\bar{n}'}(v, \bar{x}'; n_1 - n_1')] \Big). \tag{6.39}$$

The first factor is bounded above by $C\beta z_c^{n_1'-n_1}(n_1 - n_1' + 1)^{-d/2}$, by (6.13). Arguing as in (6.28),

$$\Big| \sum_v [\alpha^{(s)}_{\bar{n}'}(v, \bar{x}') - \alpha^{(s)}_{\bar{n}'}(v, \bar{x}'; n_1 - n_1')] \Big| \leq 2(n_1 - n_1' + 1) s(s+2) \alpha^{(s)}_{\bar{n}'}(\bar{x}'). \tag{6.40}$$

Since we have shown in Section 6.2.3 that $\alpha^{(s)}_{\bar{n}'}(\bar{x}')$ obeys the bound of (6.4), (6.39) is therefore bounded above by

$$C\beta^{\lceil \frac{s}{2} \rceil + 1} z_c^{-n} \sum_{n_1'=1}^{n_1} (n_1 - n_1' + 1)^{-(d-2)/2} B^{(s)}_{\bar{n}'} \leq C\beta^{\lceil \frac{s}{2} \rceil + 1} z_c^{-n} B^{(s)}_{\bar{n}}, \tag{6.41}$$

using Lemma 4.4 for the last inequality. Again by construction, this bound possesses the subinterval property. The bounds (6.38) and (6.41) advance the induction for the minimal cyclic laces, for $q = 0$.

The case $q = 2$ can be handled similarly, using the inequality $|x_1|^2 \le 2(|x_1'|^2 + |y_1|^2)$ with $y_1 = x_1 - x_1'$. This gives rise to two terms. For the term with $|y_1|^2$, we use (6.13)–(6.14) with $q = 2$. For the term with $|x_1'|^2$, we use the fact established in Section 6.2.3 that $\alpha_{\bar{n}'}^{(s)}(\bar{x}')$ obeys the bound of (6.1) for $q = 2$.

The analysis of Sections 6.2.2–6.2.4 advances the induction on s, and completes the proof of (6.4).

6.3 Contribution from laces in $\mathcal{N}$

Suppose that $\bar{n}$ has all its components strictly positive. Let

$$\nu_{\bar{n}}^{(s)}(\bar{x}) = \sum_{\omega \in \Omega_{S_{\bar{n}}}(\bar{y})} W(\omega) \sum_{L \in \mathcal{N}(S)} \prod_{ij \in L} (-U_{ij}) \prod_{i'j' \in \mathcal{C}(L)} (1 + U_{i'j'}) \tag{6.42}$$

denote the contribution to (6.2) due to laces in $\mathcal{N}$. In this section, we complete the proof of Proposition 4.2(i) by showing that this contribution obeys the bound of (6.1), i.e., that

$$\sum_{\bar{y}} |y_e|^q \nu_{\bar{n}}^{(s)}(\bar{y}) z_c^n \le C\beta^{\lceil \frac{s}{2} \rceil} \sigma^q n_e^{q/2} B_{\bar{n}}^{(s)} \quad (q = 0, 2). \tag{6.43}$$

Let $\mathcal{N}_1(S)$ and $\mathcal{R}_1(S)$ respectively denote the sets of irreducible and reducible laces containing a unique non-minimal cyclic component. We will estimate the contributions to (6.42) due to $\mathcal{N}_1(S)$, $\mathcal{R}_1(S)$ and $\mathcal{N}(S) \backslash [\mathcal{N}_1(S) \cup \mathcal{R}_1(S)]$ in Sections 6.3.1, 6.3.2, and 6.3.3, respectively.

6.3.1 Laces in $\mathcal{N}_1$

Let $\mathcal{N}_1^{(e)}(S)$ denote the set of laces in $\mathcal{N}_1(S)$ such that the edge associated to T_e in the lace is in the unique non-minimal cyclic component and can be removed without disconnecting the lace. It follows that $\mathcal{N}_1(S) = \cup_{e=1}^s \mathcal{N}_1^{(e)}(S)$. Let

$$\mathcal{M}^{(e)}(S) = \{L \in \mathcal{M}(S) : \exists st \text{ with } L \cup \{st\} \in \mathcal{N}_1^{(e)}(S),\ st \text{ associated to } T_e \text{ for } L \cup \{st\}\}. \tag{6.44}$$

By Lemma 3.12, $\mathcal{M}^{(e)}(S) \subset \mathcal{M}_a(S)$ for each e. To simplify the notation, given $\mathcal{K}(S) \subset \mathcal{L}(S)$, we write

$$\pi_{\bar{n}}^{\mathcal{K}}(\bar{x}) = \sum_{\omega \in \Omega_{S_{\bar{n}}}(\bar{x})} W(\omega) \sum_{L \in \mathcal{K}(S)} \prod_{ij \in L} (-U_{ij}) \prod_{i'j' \in \mathcal{C}(L)} (1 + U_{i'j'}). \tag{6.45}$$

The following lemma will be used to estimate $\pi_{\bar{n}}^{\mathcal{N}_1}$.

Lemma 6.3. *Let $s \ge 2$. For each $e = 1, \ldots, s$,*

$$\pi_{\bar{n}}^{\mathcal{N}_1^{(e)}}(\bar{x}) \le \pi_{\bar{n}}^{\mathcal{M}^{(e)}}(\bar{x}). \tag{6.46}$$

Before proving Lemma 6.3, we give its application. By definition, $\pi_{\bar{n}}^{\mathcal{N}_1}(\bar{x}) \le \sum_{e=1}^s \pi_{\bar{n}}^{\mathcal{N}_1^{(e)}}(\bar{x})$. For $q = 0, 2$, it follows from Lemma 6.3 and the result of Section 6.2.3 that

$$\begin{aligned} \sum_{\bar{x}} |x_e|^q \pi_{\bar{n}}^{\mathcal{N}_1^{(e)}}(\bar{x}) z_c^n &\le \sum_{\bar{x}} |x_e|^q \pi_{\bar{n}}^{\mathcal{M}^{(e)}}(\bar{x}) z_c^n \le \sum_{\bar{x}} |x_e|^q \pi_{\bar{n}}^{\mathcal{M}_a}(\bar{x}) z_c^n \\ &= \sum_{\bar{x}} |x_e|^q \alpha_{\bar{n}}^{(s)}(\bar{x}) z_c^n \le C\beta^{\lceil \frac{s}{2} \rceil} \sigma^q n_e^{q/2} B_{\bar{n}}^{(s)}. \end{aligned} \tag{6.47}$$

This proves that the contribution to (6.42) due to $\mathcal{N}_1$ obeys (6.43).

It remains to prove Lemma 6.3. Given a lace $L \in \mathcal{M}^{(e)}(S)$, we define

$$\mathcal{P}^{(e)}(L) = \{st : L \cup \{st\} \in \mathcal{N}_1^{(e)}(S),\ st \text{ associated to } T_e \text{ for } L \cup \{st\}\}. \tag{6.48}$$

Each $st \in \mathcal{P}^{(e)}(L)$ has exactly one endpoint in T_e. We claim that

$$\mathcal{N}_1^{(e)}(S) = \dot{\bigcup}_{L \in \mathcal{M}^{(e)}(S)} \dot{\bigcup}_{st \in \mathcal{P}^{(e)}(L)} \{L \cup \{st\}\}. \tag{6.49}$$

First, the union on the right side of (6.49) is disjoint since the edge associated to T_e in $L \cup \{st\}$ must be st, by definition of $\mathcal{P}^{(e)}(L)$. The inclusion of the right side of (6.49) in the left side follows by definition. For the opposite inclusion, given $L \in \mathcal{N}_1(S)$, let st be the edge associated to T_e and let $L' = L \backslash \{st\}$. It suffices to show that $L' \in \mathcal{M}^{(e)}(S)$. For this, we first note that the removal of a redundant edge from a cyclic component cannot result in a reducible lace, so that L' is irreducible. By Lemma 3.8, L' has $r-1$ edges covering the branch point, and therefore by Lemma 3.9 it has no cyclic component. Thus $L' \in \mathcal{M}^{(e)}(S)$.

The set $\mathcal{P}^{(e)}(L)$ can be totally ordered, as follows. First, we order the vertices on T_e from the branch point to the leaf. Vertices on branches other than T_e are ordered by their distance from the branch point (a greater distance corresponds to a lesser vertex), with vertices of equal distance from the branch point ordered by their branch numbers. Using this order, we always take $s < t$ for $st \in \mathcal{P}^{(e)}(L)$, and set $st > ij$ if $t > j$ or if $t = j$ and $s > i$. The proof of Lemma 6.3 uses a resummation argument based on the following lemma.

Lemma 6.4. *Given a lace $L \in \mathcal{M}^{(e)}(S)$ and $st \in \mathcal{P}^{(e)}(L)$,*

$$\mathcal{C}(L \cup \{st\}) = \mathcal{C}(L) \,\dot{\cup}\, \{ij \in \mathcal{P}^{(e)}(L) : ij < st\}. \tag{6.50}$$

Proof of Lemma 6.3 subject to Lemma 6.4. By (6.49),

$$\pi_{\bar{n}}^{\mathcal{N}_1^{(e)}}(\bar{x}) = \sum_{\omega \in \Omega_{S_{\bar{n}}}(\bar{x})} W(\omega) \sum_{L \in \mathcal{M}^{(e)}(S)} \prod_{ij \in L} (-U_{ij}) \sum_{st \in \mathcal{P}^{(e)}(L)} (-U_{st}) \prod_{i'j' \in \mathcal{C}(L \cup \{st\})} (1 + U_{i'j'}). \tag{6.51}$$

Using (6.50), we can resum to obtain that

$$\begin{aligned}\sum_{st \in \mathcal{P}^{(e)}(L)} (-U_{st}) \prod_{i'j' \in \mathcal{C}(L \cup \{st\})} (1 + U_{i'j'}) &= \prod_{i'j' \in \mathcal{C}(L)} (1 + U_{i'j'}) \sum_{st \in \mathcal{P}^{(e)}(L)} (-U_{st}) \prod_{ij \in \mathcal{P}^{(e)}(L): ij < st} (1 + U_{ij}) \\ &= \prod_{i'j' \in \mathcal{C}(L)} (1 + U_{i'j'}) \big[1 - \prod_{st \in \mathcal{P}^{(e)}(L)} (1 + U_{st})\big]. \end{aligned} \tag{6.52}$$

The claim then follows by applying the estimate $0 \le 1 - \prod_{st \in \mathcal{P}^{(e)}(L)} (1 + U_{st}) \le 1$. $\square$

Proof of Lemma 6.4. Let

$$\mathcal{C}^{(e)}(L) = \{st \notin L : \mathsf{L}_{L \cup \{st\}}(e) = \mathsf{L}_L(e)\} \tag{6.53}$$

denote the edges that are compatible with the T_e-lace $\mathsf{L}_L(e)$. Recall that for any connected graph Γ, $\mathsf{L}_\Gamma = \cup_{e=1}^r \mathsf{L}_\Gamma(e)$. It follows that

$$\mathcal{C}(L) = \bigcap_{e'=1}^{r} \mathcal{C}^{(e')}(L). \tag{6.54}$$

We claim that if $L \cup \{st\} \in \mathcal{N}_1^{(e)}(S)$ and $st \in \mathcal{P}^{(e)}(L)$, then

$$\mathcal{C}^{(e)}(L \cup \{st\}) = \mathcal{C}^{(e)}(L) \cup \{ij \in \mathcal{P}^{(e)}(L) : ij < st\}, \tag{6.55}$$

$$\mathcal{P}^{(e)}(L) \subset \mathcal{C}^{(e')}(L) = \mathcal{C}^{(e')}(L \cup \{st\}) \quad \text{(for all } e' \neq e\text{)}. \tag{6.56}$$

From (6.54)–(6.56), it follows that

$$\begin{aligned}
\mathcal{C}(L \cup \{st\}) = \bigcap_{e'=1}^{r} \mathcal{C}^{(e')}(L \cup \{st\}) &= \big(\bigcap_{e' \neq e} \mathcal{C}^{(e')}(L)\big) \cap (\mathcal{C}^{(e)}(L) \cup \{ij \in \mathcal{P}^{(e)}(L) : ij < st\}) \\
&= \big(\bigcap_{e'=1}^{r} \mathcal{C}^{(e')}(L)\big) \cup \Big(\big(\bigcap_{e' \neq e} \mathcal{C}^{(e')}(L)\big) \cap \{ij \in \mathcal{P}^{(e)}(L) : ij < st\}\Big) \\
&= \mathcal{C}(L) \cup \{ij \in \mathcal{P}^{(e)}(L) : ij < st\}.
\end{aligned} \tag{6.57}$$

By definition of $\mathcal{P}^{(e)}(L)$, no element of $\mathcal{P}^{(e)}(L)$ is compatible with L. Thus the union on the right side of (6.57) is disjoint, which gives (6.50). It remains to prove (6.55)–(6.56).

We begin with the observation that if $ij \in \mathcal{P}^{(e)}(L)$, then ij is associated only to branch T_e for $L \cup \{ij\}$. In fact, if ij were also associated to a second branch, then L would have at most $r-2$ edges covering the branch point and hence would be reducible by Lemma 3.8.

Turning now to (6.55), we prove both inclusions. Suppose that $ij \in \mathcal{C}^{(e)}(L \cup \{st\})$. We are done if $ij \in \mathcal{C}^{(e)}(L)$, so assume that $ij \notin \mathcal{C}^{(e)}(L)$. We need only consider the case in which ij covers the branch point, and we may assume that $j, t \in T_e$. Since $st \in \mathcal{P}^{(e)}(L)$, the edge st is associated to branch T_e for $L \cup \{st\}$. Since $ij \in \mathcal{C}^{(e)}(L \cup \{st\})$, it follows that $ij < st$. Since $ij \notin \mathcal{C}^{(e)}(L)$, it must be the case that ij is associated to T_e for $L \cup \{ij\}$. Moreover, ij cannot also be associated to a second branch for $L \cup \{ij\}$, or L would be reducible for the reason indicated in the previous paragraph. Thus every edge in L that covers the branch point is associated to a branch other than T_e for $L \cup \{ij\}$, and hence $L \cup \{ij\}$ is a lace. Since L has no non-minimal cyclic component, it suffices to show that $L \cup \{ij\}$ has a cyclic component (which will necessarily be the unique non-minimal cyclic component). However, since L is irreducible, $L \cup \{ij\}$ is also irreducible. By Lemma 3.8, $L \cup \{ij\}$ has r edges covering the branch point, and hence contains a cyclic component by Lemma 3.9. We have shown that $ij \in \{i'j' \in \mathcal{P}^{(e)}(L) : i'j' < st\}$.

Suppose, on the other hand, that $ij \in \mathcal{C}^{(e)}(L) \cup \{i'j' \in \mathcal{P}^{(e)}(L) : i'j' < st\}$. If $ij \in \mathcal{C}^{(e)}(L)$, then ij is not selected in the prescription $\mathsf{L}_{L \cup \{ij\}}(e)$. It therefore is not selected in $\mathsf{L}_{L \cup \{ij\} \cup \{st\}}(e)$, and hence $ij \in \mathcal{C}^{(e)}(L \cup \{st\})$. This shows that $\mathcal{C}^{(e)}(L) \subset \mathcal{C}^{(e)}(L \cup \{st\})$. If $ij \in \{i'j' \in \mathcal{P}^{(e)}(L) : i'j' < st\}$, then $ij \in \mathcal{C}^{(e)}(L \cup \{st\})$ because $ij < st$.

Next, we prove that $\mathcal{P}^{(e)}(L) \subset \mathcal{C}^{(e')}(L)$. Suppose that $ij \in \mathcal{P}^{(e)}(L)$, with $j \in T_e$ and $i \in T_f$. Then ij is associated in $L \cup \{ij\}$ to T_e and not to T_f. Therefore $ij \in \mathcal{C}^{(e')}(L)$ for all $e' \neq e$.

Finally, we prove that $\mathcal{C}^{(e')}(L) = \mathcal{C}^{(e')}(L \cup \{st\})$. The argument two paragraphs above showing $\mathcal{C}^{(e)}(L) \subset \mathcal{C}^{(e)}(L \cup \{st\})$ applies also to give $\mathcal{C}^{(e')}(L) \subset \mathcal{C}^{(e')}(L \cup \{st\})$. For the opposite inclusion, we argue as follows. Suppose that $ij \in \mathcal{C}^{(e')}(L \cup \{st\})$, i.e., that $\mathsf{L}_{L \cup \{st\} \cup \{ij\}}(e') = \mathsf{L}_{L \cup \{st\}}(e')$. Since st is associated only to branch T_e in $L \cup \{st\}$, it is not associated to branch $T_{e'}$ and therefore $\mathsf{L}_{L \cup \{st\}}(e') = \mathsf{L}_L(e')$. Since st is associated only to branch T_e in $L \cup \{st\} \cup \{ij\}$, it follows similarly that $\mathsf{L}_{L \cup \{st\} \cup \{ij\}}(e') = \mathsf{L}_{L \cup \{ij\}}(e')$. Therefore, $\mathsf{L}_{L \cup \{ij\}}(e') = \mathsf{L}_L(e')$, and hence $ij \in \mathcal{C}^{(e')}(L)$. This completes the proof of (6.56). ∎

6.3.2 Laces in $\mathcal{R}_1$

To estimate $\pi_{\bar{n}}^{\mathcal{R}_1}(\bar{x})$, we write $L \in \mathcal{R}_1(S)$ as a disjoint union of laces L_A and L_{A^c} on T_A and T_{A^c}, with L_A containing no non-minimal cyclic component, and with L_{A^c} irreducible and containing a unique non-minimal cyclic component. Then

$$\pi_{\bar{n}}^{\mathcal{R}_1}(\bar{x}) \leq \sum_{A\subset\{1,\ldots,s\}:2\leq|A|\leq s-2} \pi_{\bar{n}_A}^{\mathcal{M}}(\bar{x}_A)\, \pi_{\bar{n}_{A^c}}^{\mathcal{N}_1}(\bar{x}_{A^c}) = \sum_{A\subset\{1,\ldots,s\}:2\leq|A|\leq s-2} \mu_{\bar{n}_A}^{(|A|)}(\bar{x}_A)\, \pi_{\bar{n}_{A^c}}^{\mathcal{N}_1}(\bar{x}_{A^c}), \tag{6.58}$$

where we make the abbreviation $\bar{n}_I = (n_i)_{i\in I}$. By the estimates on $\mu_{\bar{n}_A}^{(|A|)}(\bar{x}_A)$ and $\pi_{\bar{n}_{A^c}}^{\mathcal{N}_1}(\bar{x}_{A^c})$ obtained in Sections 6.2 and 6.3.1 respectively, the desired estimate on $\pi_{\bar{n}}^{\mathcal{R}_1}(\bar{x})$ follows as in (6.7)–(6.8).

6.3.3 Laces in $\mathcal{N}$ containing at least two non-minimal cyclic components

For $i \geq 1$, let $\mathcal{K}_i(S)$ be the set of laces containing precisely i non-minimal cyclic components. In particular, $\mathcal{K}_1(S) = \mathcal{N}_1(S) \cup \mathcal{R}_1(S)$. We wish to bound $\sum_{i=2}^{s} \pi_{\bar{n}}^{\mathcal{K}_i}(\bar{x})$. We use induction on i, with the induction initialised using the bounds on $\pi_{\bar{n}}^{\mathcal{K}_1}$ of Sections 6.3.1–6.3.2. By Lemma 3.13 (see also the discussion preceding Lemma 3.13), any $L \in \cup_{i=2}^{s}\mathcal{K}_i(S)$ is reducible and can be reduced into two laces, one with a unique non-minimal cyclic component and the other with $i-1$ non-minimal cyclic components. Therefore,

$$\pi_{\bar{n}}^{\mathcal{K}_i}(\bar{x}) \leq \sum_{A\subset\{1,\ldots,s\}:2\leq|A|\leq s-2} \pi_{\bar{n}_A}^{\mathcal{K}_1}(\bar{x}_A)\, \pi_{\bar{n}_{A^c}}^{\mathcal{K}_{i-1}}(\bar{x}_{A^c}). \tag{6.59}$$

The right side can then be estimated using the induction hypothesis. □

6.4 Proof of Proposition 4.2(ii)

This section contains the proof of Proposition 4.2(ii). We discuss only the $q = 0$ case, as the extension to $q = 2$ is straightforward. By (2.4) and (2.8),

$$\varphi_T(\vec{y}) = \sum_{\omega\in\Omega_T(\vec{y})} W(\omega) \sum_{\Gamma\in\mathcal{E}_b(T)} \prod_{ij\in\Gamma} U_{ij}. \tag{6.60}$$

The set $\mathcal{E}_b(T)$ was defined in Section 2.1. We subdivide $\mathcal{E}_b(T)$ into a disjoint union of $\mathcal{E}_b^{(1)}(T)$ and $\mathcal{E}_b^{(2)}(T)$, as follows. A *branch* of T consists of a path in T whose endpoints are two branch points in T that are adjacent in the shape τ of T. Let $\mathcal{F}^{(1)}$ denote the set of edges ij that cover b and whose endpoints lie on distinct non-adjacent branches of T. Let $\mathcal{F}^{(2)}$ denote the set of edges ij whose endpoints i and j lie either on the same branch or on adjacent branches. We define $\mathcal{E}_b^{(1)}$ to be the set of graphs in $\mathcal{E}_b$ that contain at least one edge from $\mathcal{F}^{(1)}$, and define $\mathcal{E}_b^{(2)}$ to be the set of graphs in $\mathcal{E}_b$ consisting only of edges in $\mathcal{F}^{(2)}$. Then we define $\varphi_T^{(i)}$ by replacing $\mathcal{E}_b$ by $\mathcal{E}_b^{(i)}$ $(i = 1, 2)$ in (6.60).

We begin with a bound on $\varphi_T^{(1)}$, using an approach similar to the method of [13, Proposition 2.4]. A graph in $\mathcal{E}_b^{(1)}$ can be uniquely decomposed into a subgraph consisting of edges in $\mathcal{F}^{(2)}$ and a nonempty subgraph consisting of edges in $\mathcal{F}^{(1)}$. The sum defining $\varphi_T^{(1)}$ can thus be factored and resummed to give

$$\sum_{\Gamma\in\mathcal{E}_b^{(1)}(T)} \prod_{ij\in\Gamma} U_{ij} = \Big[\prod_{ij\in\mathcal{F}^{(2)}} (1 + U_{ij}) \Big] \Big[\prod_{ij\in\mathcal{F}^{(1)}} (1 + U_{ij}) - 1 \Big]. \tag{6.61}$$

We insert (6.61) into the definition of $\varphi_T^{(1)}(\vec{y})$, and sum over $\vec{y}$. Writing T_e for the e^{th} branch of T, the first factor on the right side of (6.61) can be bounded above by $\prod_{e=1}^r K[T_e]$. The factor $K[T_e]$ ensures that the embedding of T_e is a self-avoiding walk.

The second factor on the right side of (6.61) is nonzero only if there is a branch e' adjacent to b and containing a leaf of T, and a branch e'' that is not adjacent to b, such that the embeddings of $T_{e'}$ and $T_{e''}$ intersect each other. Thus, as an upper bound, we obtain a network of possibly mutually-intersecting self-avoiding walks, two of which *must* intersect each other. This is bounded above by the sum over $x \in \mathbb{Z}^d$ of such a network, where an intersection as above is required to occur at x. We obtain a further upper bound by neglecting the mutual avoidance of the two parts of $T_{e'}$ on either side of x, and similarly for $T_{e''}$. This gives an upper bound by a network consisting of a polygon containing $\omega(b)$ and x, with sides consisting of at least three self-avoiding walks (because $T_{e'}$ and $T_{e''}$ are not adjacent), and with branches emerging from the polygon. Removal of the polygon from T leaves a number of connected components $\{R_j\}_{j=1}^q$ (the branches emerging from the polygon). We include in R_j the polygon vertex at which R_j is attached (such a vertex is either a branch point or a point in T corresponding to the intersection point x, and is included also in the polygon.) A branch R_j consists of a tree of self-avoiding walks that are possibly mutually intersecting, and by the $r = 1$ case of Theorem 1.1(a) the contribution due to these branches is bounded above by $C\mu^{\sum_{j=1}^q |R_j|}$. Using translation invariance to treat $\omega(b)$ as the origin, this leads to the upper bound

$$\sum_{\vec{y}} \varphi_T(\vec{y}) z_c^{|T|} \leq C \max_{e',e'',e_i} \sum_{x\in\mathbb{Z}^d} \sum_{m_{e'}=1}^{n_{e'}} \sum_{m_{e''}=1}^{n_{e''}} c_{m_{e'}}(x) z_c^{m_{e'}} \sum_j (c_{n_{e_1}} * \cdots * c_{n_{e_j}} * c_{m_{e''}})(x) z_c^{n_{e_1}+\cdots+n_{e_j}+m_{e''}}. \tag{6.62}$$

The number of terms in the sum over j depends on the relative positions of $T_{e'}$ and $T_{e''}$, and is at least one and at most $r - 1$.

The sum over x of the convolution factor in (6.62) is bounded by a constant, while its sup over x is easily seen from the third bound of (5.3) to be bounded above by a constant multiple of $\beta(n_{e_1}+\cdots+n_{e_j}+m_{e''})^{-d/2}$ (constants here depend on j and hence on r). Thus, we may regard the convolution as a single self-avoiding walk, as far as estimates are concerned. The desired bound on $\sum_{\vec{y}} \varphi_T^{(1)}(\vec{y}) z_c^{|T|}$ then follows

$$\sum_x \sum_{m_1=\underline{n}}^{\infty} \sum_{m_2=0}^{m_1} c_{m_1}(x) z_c^{m_1} c_{m_2}(x) z_c^{m_2} \leq C \sum_{m_1=\underline{n}}^{\infty} m_1 \beta m_1^{-d/2} \leq C\beta \underline{n}^{-(d-4)/2}, \tag{6.63}$$

where m_1 corresponds to the larger of $m_{e'}$ and $n_{e_1} + \cdots + n_{e_j} + m_{e''}$ in (6.62). Here, we are also using Lemma 5.3.

It remains to estimate $\varphi_T^{(2)}$. We follow the basic method of [13, Proposition 2.8]. Let B denote the set of branch points of T that are not leaves, and assume that $|B| \geq 2$. Let $B_b = \{b_1, \ldots, b_k\}$ denote the subset of B consisting of branch points in B that are adjacent to b in τ. For $b_l \in B_b$, let $\mathcal{E}_{b,b_l}(T)$ denote the set of all graphs Γ, consisting of bonds in $\mathcal{F}^{(2)}$, for which there is a subtree $S(\Gamma)$ of T containing b and b_l (both not as a leaf) such that the restriction of Γ to $S(\Gamma)$ is a connected graph on $S(\Gamma)$. Then $\mathcal{E}_b^{(2)}(T) = \cup_{l=1}^k \mathcal{E}_{b,b_l}(T)$. For $A \subset B_b$, let $\mathcal{E}_{b,A}(T) = \cap_{b_l\in A} \mathcal{E}_{b,b_l}(T)$. By the

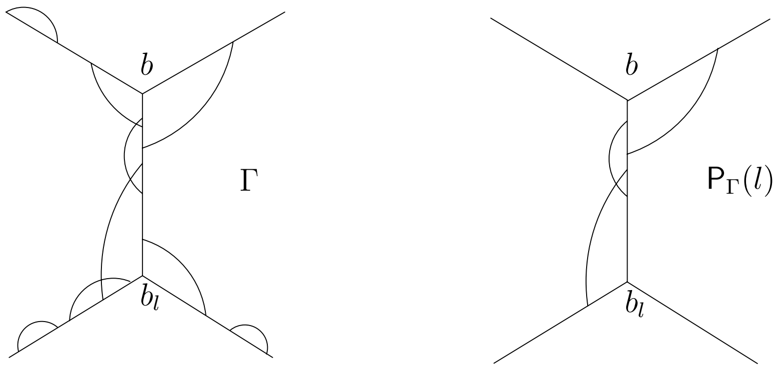

Figure 10: Example of the lace $\mathsf{P}_\Gamma(l)$ associated to a graph Γ.

inclusion-exclusion relation,

$$\Big| \sum_{\Gamma \in \mathcal{E}_b^{(2)}(T)} \prod_{ij \in \Gamma} U_{ij} \Big| \leq \sum_{A \subset B_b : A \neq \varnothing} \Big| \sum_{\Gamma \in \mathcal{E}_{b,A}(T)} \prod_{ij \in \Gamma} U_{ij} \Big|. \tag{6.64}$$

Fix $\Gamma \in \mathcal{E}_{b,b_l}(T)$, with $b_l \in B_b$. Let the interval joining b and b_l in T be labelled by e_l. We will associate to Γ a lace $L_l \subset \Gamma$ on an interval $I_l \subset T$ that contains b and b_l as interior points. This involves a natural generalisation of the T_{e_l}-lace construction to trees that are not star shaped. Since we are now dealing only with edges in $\mathcal{F}^{(2)}$, edges that cover b have endpoints in branches adjacent to b. We may therefore extend the definition of the edge associated to branch T_{e_l}, given in Definition 2.3, to apply also to Γ. We may also extend the definition of the T_{e_l}-lace construction, where now the construction ends when an edge is selected that covers b_l. If there is a tie when attempting to select this final edge, we choose the edge having an endpoint on the branch with smallest label, in some labelling of the branches. This generalised T_{e_l}-lace construction produces a lace $L_l = \mathsf{P}_\Gamma(l) \subset \Gamma$ on an interval I_l that contains b and b_l as interior points; see Figure 10. Note that a lace produced in this manner has at least two edges, that the first and last edges in the lace cover b and b_l respectively, and that no other edge covers b or b_l.

Given $A \subset B_b$ and $\Gamma \in \mathcal{E}_{b,A}(T)$, let $L_A = \cup_{b_l \in A} \mathsf{P}_\Gamma(l) = \mathsf{P}_\Gamma(A)$ and $S_A = \cup_{b_l \in A} I_l$. The fact that we are working only with edges in $\mathcal{F}^{(2)}$ implies that S_A is a star-shaped tree with branch point b. By definition, L_A is a lace on S_A, since the construction has been tailored to select the edge associated to each branch from the edges covering the branch point. We denote by $\mathcal{Q}(L)$ the set of edges $ij \in \mathcal{F}^{(2)}$ that are *compatible* with L in the sense that the lace associated to the graph $L \cup \{ij\}$ is L. It can be shown, as in the proof of Proposition 2.6, that $\mathsf{P}_\Gamma(A) = L$ if and only if $L \subset \Gamma$ is a lace on S_A and $\Gamma \backslash L \subset \mathcal{Q}(L)$. We may therefore resum, as in (2.15), to obtain

$$\sum_{\Gamma \in \mathcal{E}_{b,A}(T)} \prod_{ij \in \Gamma} U_{ij} = \sum_{S_A} \sum_{L \in \tilde{\mathcal{L}}(S_A)} \prod_{ij \in L} U_{ij} \prod_{i'j' \in \mathcal{Q}(L)} (1 + U_{i'j'}), \tag{6.65}$$

where the sum over S_A is the sum over subtrees of T that contain b and the elements of A, not as a leaf. The tilde on $\tilde{\mathcal{L}}(S_A)$ denotes the subset of $\mathcal{L}(S_A)$ consisting of laces that can arise from the prescription $\mathsf{P}_\Gamma(A)$.

Removal of S_A from T leaves a number of connected components $\{R_j\}_{j=1}^q$. We include in R_j the vertex of S_A to which R_j is attached. (Such vertices are either branch points covered by the lace L, or endpoints of S_A, and are included also in S_A.) For $L \in \tilde{\mathcal{L}}(S_A)$, it follows from the definition of $\mathcal{Q}(L)$ that

$$\mathcal{C}^{(2)}(L) \cup \big(\cup_j \mathcal{B}^{(2)}(R_j)\big) \subset \mathcal{Q}(L), \tag{6.66}$$

where $\mathcal{C}^{(2)}(L)$ denotes the edges in $\mathcal{B}(S_A)$ that are compatible with the lace $L \in \tilde{\mathcal{L}}(S_A)$, and $\mathcal{B}^{(2)}(R_j) = \mathcal{B}(R_j) \cap \mathcal{F}^{(2)}$. We denote the branches of R_j by R_{jl}, and note that $\cup_l \mathcal{B}(R_{jl}) \subset \mathcal{B}^{(2)}(R_j)$. Therefore,

$$\prod_{i'j' \in \mathcal{Q}(L)} (1 + U_{i'j'}) \leq \prod_{i'j' \in \mathcal{C}^{(2)}(L)} (1 + U_{i'j'}) \prod_{j,l} K[R_{jl}], \tag{6.67}$$

and hence, by (6.65) and (6.67),

$$\Big| \sum_{\Gamma \in \mathcal{E}_{b,A}(T)} \prod_{ij \in \Gamma} U_{ij} \Big| \leq \sum_{S_A} \sum_{L \in \tilde{\mathcal{L}}(S_A)} \prod_{ij \in L} (-U_{ij}) \prod_{i'j' \in \mathcal{C}^{(2)}(L)} (1 + U_{i'j'}) \prod_{j,l} K[R_{jl}]. \tag{6.68}$$

We insert (6.68) into (6.64), and insert the result into (6.60) with $\mathcal{E}_b$ replaced by $\mathcal{E}_b^{(2)}$. The contribution due to the branches R_{jl} is bounded by a constant, by the $r = 1$ version of Theorem 1.1(a).

By (2.16)–(2.18), the remaining factors obey an upper bound

$$\sum_{\vec{y}} |\varphi_T^{(2)}(\vec{y})| z_c^{|T|} \leq C \sum_{A \subset B_b : A \neq \varnothing} \sum_{S_A} \sum_{\bar{x}} \sum_{N=2}^{\infty} \tilde{\pi}_{\bar{m}(S_A),N}^{(d_b(S_A))}(\bar{x}) z_c^{|S_A|}, \tag{6.69}$$

where $d_b(S_A)$ denotes the degree of b in S_A, the components of $\bar{m}(S_A)$ give the lengths of the branches of S_A, at least one component of $\bar{m}(S_A)$, say m_1, exceeds $\underline{n}$, and $\tilde{\pi}$ is related to π by replacement of $L \in \mathcal{L}(S_A)$ by $L \in \tilde{\mathcal{L}}(S_A)$ (note that $\tilde{\mathcal{L}}(S_A) \subset \mathcal{L}(S_A)$) and replacement of $\mathcal{C}(L)$ by the slightly smaller set $\mathcal{C}^{(2)}(L)$ in (2.16)–(2.18). Since compatible edges linking distinct subintervals are neglected in our bounds on π (consistent with the subinterval property), our bounds on π apply also to $\tilde{\pi}$. We may therefore use the result of Proposition 4.2(i) to bound the sum over x to obtain

$$\sum_{\vec{y}} |\varphi_T^{(2)}(\vec{y})| z_c^{|T|} \leq C\beta \sum_{s=2}^{d_b} \sum_{\bar{0} \leq \bar{m} \leq \bar{\infty} : m_1 \geq \underline{n}} B_{\bar{m}}, \tag{6.70}$$

where $\bar{m}$ has s components. An application of (4.21) (extended slightly to replace $\bar{n}$ by $\bar{\infty}$) completes the proof. With a bit more care, the factor β on the right side of (6.70) can be replaced by β^2.

Acknowledgements

This work was supported in part by NSERC of Canada. It was carried out in part during extensive visits by both authors to Microsoft Research and the Fields Institute. The work of GS was also carried out in part at McMaster University. We thank Eric Derbez, Takashi Hara, Antal Járai and Akira Sakai for useful discussions. We are grateful to Antal Járai for drawing our attention to an important error in an earlier version of this paper.

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