

Critical two-point function of the 4-dimensional weakly self-avoiding walk

Roland Bauerschmidt*, David C. Brydges† and Gordon Slade†

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Abstract

We prove $|x|^{-2}$ decay of the critical two-point function for the continuous-time weakly self-avoiding walk on $\mathbb{Z}^d$, in the upper critical dimension $d = 4$. This is a statement that the critical exponent η exists and is equal to zero. Results of this nature have been proved previously for dimensions $d \geq 5$ using the lace expansion, but the lace expansion does not apply when $d = 4$. The proof is based on a rigorous renormalisation group analysis of an exact representation of the continuous-time weakly self-avoiding walk as a supersymmetric field theory. Much of the analysis applies more widely and has been carried out in a previous paper, where an asymptotic formula for the susceptibility is obtained. Here, we show how observables can be incorporated into the analysis to obtain a pointwise asymptotic formula for the critical two-point function. This involves perturbative calculations similar to those familiar in the physics literature, but with error terms controlled rigorously.

1 Main result

1.1 Introduction

The critical behaviour of the self-avoiding walk depends on the spatial dimension d . The upper critical dimension is 4, and for $d \geq 5$ the lace expansion has been used to prove that the asymptotic behaviour is Gaussian [23, 32–34, 45]. In particular, for the *strictly* self-avoiding walk in dimensions $d \geq 5$, the critical two-point function has $|x|^{-(d-2+\eta)}$ decay with critical exponent $\eta = 0$, both for spread-out walks [24, 33] and for the nearest-neighbour walk [32]. For $d = 3$, the problem remains completely unsolved from a mathematical point of view, but numerical and other evidence provides convincing evidence that the behaviour is not Gaussian. In particular, numerical values of the critical exponents γ and ν [25, 44], together with Fisher’s relation $\gamma = (2 - \eta)\nu$, indicate that the critical two-point function has approximate decay $|x|^{-1.031}$ for $d = 3$. For $d = 2$, the critical two-point function is predicted to decay as $|x|^{-5/24}$ [42], and recent work suggests that the scaling behaviour is described by $\text{SLE}_{8/3}$ [39], but neither has been proved. The case of $d = 1$ is of interest

*School of Mathematics, Institute for Advanced Study, Princeton, NJ 08540 USA. E-mail: brt@math.ias.edu.

†Department of Mathematics, University of British Columbia, Vancouver, BC, Canada V6T 1Z2. E-mail: db5d@math.ubc.ca, slade@math.ubc.ca.

for weakly self-avoiding walk, where a fairly complete understanding has been obtained [36]. More about mathematical results for self-avoiding walk can be found in [10, 40].

In the present paper, we prove that the critical two-point function of the continuous-time weakly self-avoiding walk is asymptotic to a multiple of $|x|^{-2}$ as $|x| \rightarrow \infty$, in dimension $d = 4$. This is a statement that the critical exponent η exists and is equal to zero. The proof is based on a rigorous renormalisation group method; a summary of the method and proof is given in [15]. Early indications of the critical nature of the dimension $d = 4$ were given in [3, 12], following proofs of triviality of ϕ^4 field theory above dimension 4 [2, 28].

Logarithmic corrections to scaling are common in statistical mechanical models at the upper critical dimension, and are predicted for the susceptibility and correlation length and several other interesting quantities [13, 38, 47], but not for the leading decay of the critical two-point function of the 4-dimensional self-avoiding walk. In [6], it is proved that the susceptibility of the 4-dimensional weakly self-avoiding walk does have a logarithmic correction to scaling, with exponent $\frac{1}{4}$. We now extend the methods of [6] to study the critical two-point function.

We use an integral representation to rewrite the two-point function of the continuous-time weakly self-avoiding walk as the two-point function of a supersymmetric field theory, and apply a rigorous renormalisation group argument to analyse the field theory. In fact, our proof involves an extension of the ideas and structure developed in [6], and to avoid repetition we refer below frequently to [6] for ideas and notation that apply without modification to our present purpose. A feature present here but not in [6] is the use of a complex observable field σ ; this requires aspects of [7, 20–22] concerning observables that were not used in [6]. A similar extension was used to study correlations of the dipole gas in [26].

We expect our general approach to apply more widely. In particular, work is in progress to extend our methods and use observable fields to study the critical behaviour of the 4-dimensional weakly self-avoiding walk with nearest-neighbour contact attraction [5], to prove existence of logarithmic corrections to scaling for certain critical 4-dimensional polymer networks [11], and to analyse critical correlation functions of the 4-dimensional n -component $|\varphi|^4$ spin model [11].

1.2 Main result

We now define the two-point function for continuous time weakly self-avoiding walk, and state our main result. Fix a dimension $d > 0$. Let X be the stochastic process on $\mathbb{Z}^d$ with right-continuous sample paths, that takes its steps at the times of the events of a rate- $2d$ Poisson process. Steps are independent both of the Poisson process and of all other steps, and are taken uniformly at random to one of the $2d$ nearest neighbours of the current position. Let E_a denote the corresponding expectation for the process started at $X(0) = a$. The *local time* at x up to time T is defined by $L_{x,T} = \int_0^T \mathbb{1}_{X(s)=x} ds$, and the *self-intersection local time* up to time T is $I(T) = \sum_{x \in \mathbb{Z}^d} L_{x,T}^2$. The continuous-time weakly self-avoiding walk *two-point function* is then defined by

$$G_{g,\nu}(a, b) = \int_0^\infty E_a \left(e^{-gI(T)} \mathbb{1}_{X(T)=b} \right) e^{-\nu T} dT, \quad (1.1)$$

where $g > 0$, and ν is a parameter (possibly negative) chosen so that the integral converges. By translation invariance, $G_{g,\nu}(a, b)$ only depends on a, b via $a - b$. For $d = 4$, the continuous-time weakly self-avoiding walk is identical to the lattice Edwards model; see [40, Section 10.1].

In (1.1), self-intersections are suppressed by the factor $e^{-gI(T)}$. In the limit $g \rightarrow \infty$, if ν is simultaneously sent to $-\infty$ in a suitable g -dependent manner, it is known that the limit of the two-point function (1.1) is a multiple of the two-point function of the standard discrete-time strictly self-avoiding walk [16]. The model defined by (1.1) is predicted to be in the same universality class as the strictly self-avoiding walk for all $g > 0$. Our analysis is restricted to small $g > 0$.

The *susceptibility* is defined by

$$\chi_g(\nu) = \sum_{b \in \mathbb{Z}^d} G_{g,\nu}(a, b), \quad (1.2)$$

and the critical value $\nu_c(g)$ is defined by $\nu_c(g) = \inf\{\nu \in \mathbb{R} : \chi_g(\nu) < \infty\}$. It is proved in [6, Lemma A.1] that $\nu_c = \nu_c(g, d) \in (-\infty, 0]$ for all $g > 0$ and $d > 0$, and that moreover

$$\chi_g(\nu) < \infty \quad \text{if and only if} \quad \nu > \nu_c. \quad (1.3)$$

Our main result is the following theorem which gives the decay of the critical two-point function in dimension 4, for sufficiently small g . In fact, in (4.51) we produce an exact formula for $G_{g,\nu_c(g)}(a, b)$, which in principle could be used to compute higher order asymptotic behaviour.

Theorem 1.1. *Let $d = 4$. There exists $\delta > 0$ such that for each $g \in (0, \delta)$ there exists $c(g) > 0$ such that as $|a - b| \rightarrow \infty$,*

$$G_{g,\nu_c(g)}(a, b) = \frac{c(g)}{|a - b|^2} \left(1 + O\left(\frac{1}{\log|a - b|}\right) \right). \quad (1.4)$$

In [15], an extension of Theorem 1.1 states that the critical two-point function has decay $|a - b|^{2-d}$ for all dimensions $d \geq 4$, but [15] provides only a sketch of proof. Our principal interest is the critical dimension $d = 4$, and we provide the details of the proof for $d = 4$ here. The restriction to $d = 4$ is to avoid additional complications required to handle general high dimensions. We will return to the general case in a future publication.

We define the Laplacian Δ on $\mathbb{Z}^d$ by $(\Delta f)_x = \sum_{e:|e|=1} (f_{x+e} - f_x)$. For $g = 0$ and $\nu \geq 0$, the two-point function is given by $G_{0,\nu}(a, b) = (-\Delta + \nu)_{ab}^{-1}$, and $\nu_c(0) = 0$. Theorem 1.1 proves that for $d = 4$ and small positive g , the critical two-point function has the same $|a - b|^{-2}$ decay as the lattice Green function $-\Delta_{ab}^{-1}$. In contrast, for $\nu > \nu_c(g)$, $G_{g,\nu}(a, b)$ decays exponentially as $|a - b| \rightarrow \infty$; an elementary proof is sketched below in Proposition 2.1.

It is shown in [6, Theorem 1.2] that for $d = 4$, as $g \downarrow 0$,

$$\nu_c(g) = -ag(1 + O(g)), \quad (1.5)$$

where the positive constant a is given by $a = -2\Delta_{00}^{-1}$. In particular, (1.5) implies that $\nu_c(g) < 0$ for small positive g .

A variant of the 4-dimensional Edwards model was analysed in [37] using a renormalisation group method. Although this variant is not a model of walks taking steps in a lattice, it is presumably in the same universality class as the 4-dimensional self-avoiding walk, and the results of [37] are of a similar nature to ours. For the 4-dimensional φ^4 model, related results were obtained via block spin renormalisation in [29–31, 35], and via partially renormalised phase space expansion in [27]. Our method is applied to the n -component $|\varphi|^4$ model in [8, 11].

2 Integral representation for the two-point function

The proof of Theorem 1.1 is based on an integral representation for a finite volume approximation of the two-point function. To discuss this, we first show how the two-point function can be approximated by a two-point function on a finite torus.

2.1 Finite volume approximation

Let $L \geq 3$ and $N \geq 1$ be integers, and let $\Lambda = \Lambda_N = \mathbb{Z}^d / L^N \mathbb{Z}^d$ denote the discrete torus of side L^N . We are ultimately interested in the limit $N \rightarrow \infty$, and regard Λ_N as a finite volume approximation to $\mathbb{Z}^d$. It is convenient at times to consider Λ_N to be a box (approximately) centred at the origin in $\mathbb{Z}^d$, without identifying opposite sides to create the torus. For fixed $a, b \in \mathbb{Z}^d$, we can then regard a, b as points in Λ_N provided N is large enough, and we make this identification throughout the paper. In particular, we tacitly assume that N is sufficiently large to contain given a, b .

For $a, b \in \Lambda_N$, let

$$G_{N,g,\nu}(a, b) = \int_0^\infty E_a^{\Lambda_N} (e^{-gI(T)} \mathbb{1}_{X(T)=b}) e^{-\nu T} dT, \quad (2.1)$$

where $E_a^{\Lambda_N}$ denotes the continuous-time simple random walk on the torus Λ_N , started from the point a . By the Cauchy–Schwarz inequality, $T = \sum_{x \in \Lambda} L_T^x \leq (|\Lambda| I(T))^{1/2}$ and hence $I(T) \geq T^2/|\Lambda|$, from which we conclude that the integral (2.1) is finite for all $g > 0$ and $\nu \in \mathbb{R}$. The following proposition shows that the infinite volume two-point function (1.1) can be approximated by the finite volume two-point function (2.1), and that it is possible to study the critical two-point function on $\mathbb{Z}^d$ in the double limit in which first $N \rightarrow \infty$ and then $\nu \downarrow \nu_c$.

Proposition 2.1. *Let $d > 0$, $g > 0$, and $\nu > \nu_c(g)$. Then $G_\nu(a, b)$ decays exponentially in $|a - b|$, and*

$$G_{g,\nu}(a, b) = \lim_{N \rightarrow \infty} G_{N,g,\nu}(a, b). \quad (2.2)$$

Also, for all $\nu \geq \nu_c(g)$,

$$G_{g,\nu}(a, b) = \lim_{\nu' \downarrow \nu} \lim_{N \rightarrow \infty} G_{N,g,\nu'}(a, b). \quad (2.3)$$

Proof. We fix $g > 0$ and drop it from the notation. Once we prove (2.2), (2.3) then follows because, by monotone convergence, $G_{g,\nu}(a, b)$ is right continuous for $\nu \geq \nu_c(g)$. Fix $\nu > \nu' > \nu_c$ and $S > 0$. Let $c_T(a, b) = E_a(e^{-gI(T)} \mathbb{1}_{X(T)=b})$ and $c_{N,T}(a, b) = E_a^{\Lambda_N}(e^{-gI(T)} \mathbb{1}_{X(T)=b})$. By the triangle inequality,

$$|G_\nu - G_{N,\nu}| \leq \int_0^S |c_T(a, b) - c_{N,T}(a, b)| e^{-\nu T} dT + \int_S^\infty c_T(a, b) e^{-\nu T} dT + \int_S^\infty c_{N,T}(a, b) e^{-\nu T} dT, \quad (2.4)$$

where to simplify the notation we write $G_\nu = G_\nu(a, b)$ and $G_{N,\nu} = G_{N,\nu}(a, b)$. For the analysis of the right-hand side of (2.4), we define $\chi_N(\nu) = \sum_{b \in \Lambda} G_{N,\nu}(a, b)$, and recall from [6, Lemma 2.1] that $\chi_N(\nu) \leq \chi(\nu)$. From this it follows that

$$\limsup_{N \rightarrow \infty} G_{N,\nu'} \leq \limsup_{N \rightarrow \infty} \chi_N(\nu') \leq \chi(\nu') < \infty. \quad (2.5)$$

Let $\delta = \nu - \nu' > 0$. Then

$$\int_S^\infty c_T(a, b) e^{-\nu T} dT \leq e^{-\delta S} G_{\nu'}, \quad \limsup_{N \rightarrow \infty} \int_S^\infty c_{N,T}(a, b) e^{-\nu T} dT \leq e^{-\delta S} \limsup_{N \rightarrow \infty} G_{N, \nu'}. \quad (2.6)$$

This shows that the last two terms in (2.4) can be made as small as desired, uniformly in N , by choosing S large.

To estimate the first contribution, let $(Y_t)_{t \geq 0}$ be a rate- $2d$ Poisson process with corresponding probability measure P . Since contributions to the difference $|c_T - c_{N,T}|$ only arise from walks that reach the inner boundary $\partial\Lambda$ of the torus (identified with a subset of $\mathbb{Z}^d$ so that it does have a boundary), for any $0 \leq T \leq S$ we have

$$\begin{aligned} |c_T(a, b) - c_{N,T}(a, b)| &\leq E_a \left(e^{-gI(T)} \mathbb{1}_{\{X([0,T]) \cap \partial\Lambda \neq \emptyset\}} \right) + E_a^\Lambda \left(e^{-gI(T)} \mathbb{1}_{\{X([0,T]) \cap \partial\Lambda \neq \emptyset\}} \right) \\ &\leq P_a \{X([0, T]) \cap \partial\Lambda \neq \emptyset\} + P_a^\Lambda \{X([0, T]) \cap \partial\Lambda \neq \emptyset\} \\ &\leq 2P \{Y_T \geq \text{diam}(\Lambda)\} \leq 2P \{Y_S \geq \text{diam}(\Lambda)\}, \end{aligned} \quad (2.7)$$

and the right-hand side goes to zero as $N \rightarrow \infty$ with S fixed. By this estimate, the first integral in (2.4) converges to 0 as $N \rightarrow \infty$, for any fixed S . This completes the proof of (2.2) and hence of (2.3).

We do not use the exponential decay in this paper, and its proof is standard, so we only sketch the argument. Given any $\alpha > 0$, we write $x = b - a$ and make the division

$$G_\nu(a, b) = \int_0^{\alpha|x|} c_T(a, b) e^{-\nu T} dT + \int_{\alpha|x|}^\infty c_T(a, b) e^{-\nu T} dT. \quad (2.8)$$

For the second integral on the right-hand side, we set $c_T = \sum_{b \in \mathbb{Z}^d} c_T(a, b)$ and use $c_T(a, b) \leq c_T$. It can be shown that c_T obeys $c_{S+T} \leq c_S c_T$ and that this implies $c_T^{1/T} \rightarrow e^{\nu_c}$ as $T \rightarrow \infty$, from which we see that $c_T(a, b) \leq C_\epsilon e^{(\nu_c + \epsilon)T}$ for any $\epsilon > 0$. This gives exponential decay in x for the second integral. For the first integral, we recall the Chernoff estimate for the Poisson distribution, in the form that if X is Poisson with mean λ and $k > \lambda$, then $P(X > k) \leq e^{-\lambda} (e\lambda/k)^k$. Since a walk can travel from a to b in time T only if the number of steps taken is at least x , it follows from the Chernoff bound that if $2d\alpha < 1$ and $T \leq \alpha|x|$ then

$$c_T(a, b) \leq P(Y_T \geq |x|) \leq e^{-2dT} (2dT e)^{|x|} |x|^{-|x|} \leq (2de\alpha)^{|x|}. \quad (2.9)$$

By choosing α sufficiently small depending on ν (recall that $\nu < 0$ is possible), we see that the first term on the right-hand side of (2.8) also exhibits exponential decay in x . $\blacksquare$

2.2 Integral representation

We use the supersymmetric integral representation for the two-point function discussed in detail in [6, Section 3]. We refer to that discussion for further details, notation, and definitions, and here we recall the minimum needed for our present purposes.

In terms of the complex boson field $\phi, \bar{\phi}$ and conjugate fermion fields $\psi, \bar{\psi}$ introduced in [6, Section 3], and using the same notation, for $x \in \Lambda$ we define the differential forms

$$\tau_x = \phi_x \bar{\phi}_x + \psi_x \wedge \bar{\psi}_x, \quad (2.10)$$

$$\tau_{\Delta,x} = \frac{1}{2} \left(\phi_x (-\Delta \bar{\phi})_x + (-\Delta \phi)_x \bar{\phi}_x + \psi_x \wedge (-\Delta \bar{\psi})_x + (-\Delta \psi)_x \wedge \bar{\psi}_x \right), \quad (2.11)$$

where $\Delta = \Delta_\Lambda$ is the lattice Laplacian on Λ given by $\Delta \phi_x = \sum_{y:|y-x|=1} (\phi_y - \phi_x)$. Here $\wedge$ denotes the wedge product; we drop the wedge from the notation subsequently with the understanding that forms are always multiplied using this anti-commutative product.

Let $\mathbb{E}_C$ denote the Gaussian super-expectation with covariance matrix C , as defined in [6, Definition 3.2]. In [6, (4.7)–(4.8)], it is shown that for $N < \infty$, $g > 0$, $\nu \in \mathbb{R}$, $m^2 > 0$, and $z_0 > -1$,

$$G_{N,g,\nu}(a, b) = (1 + z_0) \mathbb{E}_C \left(e^{-U_0(\Lambda)} \bar{\phi}_a \phi_b \right), \quad (2.12)$$

where $C = (-\Delta + m^2)^{-1}$,

$$U_0(\Lambda) = \sum_{x \in \Lambda} (g_0 \tau_x^2 + \nu_0 \tau_x + z_0 \tau_{\Delta,x}), \quad (2.13)$$

and

$$g_0 = g(1 + z_0)^2, \quad \nu_0 = (1 + z_0)\nu - m^2. \quad (2.14)$$

The identity (2.12) is a rewriting of an identity from [14, 18] that was inspired by [41, 43]; see also [19, Theorem 5.1] for a self-contained proof.

In [6], we also use (2.12), but there write V instead of U . In the present paper, we use V for an extension of U that incorporates also an *observable field*, discussed next.

2.3 Observable field

We introduce an external field $\sigma \in \mathbb{R}$ and define

$$V_0(\Lambda) = U_0(\Lambda) - \sigma \bar{\phi}_a - \bar{\sigma} \phi_b. \quad (2.15)$$

We refer to σ as the *observable field*. Then we can compute the two-point function using the identity

$$G_{N,g,\nu}(a, b) = (1 + z_0) \frac{\partial^2}{\partial \sigma \partial \bar{\sigma}} \Big|_0 \mathbb{E}_C e^{-V_0(\Lambda)}, \quad (2.16)$$

which follows from (2.12). To prove Theorem 1.1, we analyse the derivative of the Gaussian super-expectation on the right-hand side of (2.16), without making further reference to its connection with self-avoiding walks.

An external field is also employed to analyse the susceptibility in [6, Section 4.1], but in a different way. There the external field is a test function $J : \Lambda \rightarrow \mathbb{R}$, and $U_0(\Lambda)$ becomes replaced by $U_0(\Lambda) - \sum_{x \in \Lambda} (J_x \bar{\phi}_x + \bar{J}_x \phi_x)$. In [6] the interest is in the *constant* external field $J_x = 1$ for all $x \in \Lambda$, and the macroscopic regularity of this test function is important. Here, in contrast, (2.15) corresponds to setting $J_x = \sigma \mathbb{1}_{x=a}$ and $\bar{J}_x = \bar{\sigma} \mathbb{1}_{x=b}$ (so the two are not precisely complex conjugates). To work with such a *singular* external field, we use a different analysis based on ideas prepared in [20–22]. It would be desirable to allow all coupling constants to be spatially varying, not just the external field. This extension has been achieved for hierarchical models in [1].

Our attention to the dependence on the external field is quite limited: we only wish to compute the derivative (2.16), and as such we make no use of the dependence of any functional dependence

on $\sigma, \bar{\sigma}$ beyond expansion to second order, i.e., including terms of order 1, $\sigma, \bar{\sigma}, \sigma\bar{\sigma}$. We formalise this notion by identifying quantities with the same expansion to second order, as follows. Recall the space $\mathcal{N}$ of even differential forms introduced in [6, Section 3.1], which we now denote instead by $\mathcal{N}^\varnothing$. As in [6, (3.5)], an element of $\mathcal{N}^\varnothing$ has the form

$$\sum_{x,y} F_{x,y}(\phi, \bar{\phi}) \psi^x \bar{\psi}^y. \quad (2.17)$$

We extend this notion by now allowing the coefficients $F_{x,y}$ to be functions of the external field $\sigma, \bar{\sigma}$ as well as of the boson field $\phi, \bar{\phi}$. Let $\mathcal{N}$ be the resulting algebra of differential forms. Let $\mathcal{I}$ denote the ideal in $\mathcal{N}$ consisting of those elements of $\mathcal{N}$ whose expansion to second order in the external field is zero. The quotient algebra $\mathcal{N}/\mathcal{I}$ then has the direct sum decomposition

$$\mathcal{N}/\mathcal{I} = \mathcal{N}^\varnothing \oplus \mathcal{N}^a \oplus \mathcal{N}^b \oplus \mathcal{N}^{ab}, \quad (2.18)$$

where elements of $\mathcal{N}^a, \mathcal{N}^b, \mathcal{N}^{ab}$ are respectively given by elements of $\mathcal{N}^\varnothing$ multiplied by σ , by $\bar{\sigma}$, and by $\sigma\bar{\sigma}$. For example, $\phi_x \bar{\phi}_y \psi_x \bar{\psi}_x \in \mathcal{N}^\varnothing$, and $\sigma \bar{\phi}_x \in \mathcal{N}^a$. There are canonical projections $\pi_\alpha : \mathcal{N} \rightarrow \mathcal{N}^\alpha$ for $\alpha \in \{\varnothing, a, b, ab\}$. We use the abbreviation $\pi_* = 1 - \pi_\varnothing = \pi_a + \pi_b + \pi_{ab}$. This topic is discussed further around [20, (1.60)]. Since we have no further use of $\mathcal{N}$, to simplify the notation we henceforth write simply $\mathcal{N}$ instead of $\mathcal{N}/\mathcal{I}$. As functions of the external field, elements of $\mathcal{N}$ are then polynomials of degree at most 2, by definition. For example, we identify $e^{\sigma \bar{\phi}_a + \bar{\sigma} \phi_b}$ and $1 + \sigma \bar{\phi}_a + \bar{\sigma} \phi_b + \sigma \bar{\sigma} \bar{\phi}_a \phi_b$, as both are elements of the same equivalence class in the quotient space.

3 Renormalisation group map

In this section, we sketch only the most important ingredients of our renormalisation group method from [6, 7, 21, 22]. A more detailed introduction is given in [6] (see also [8]), to which we refer.

3.1 Progressive Gaussian integration

We use decompositions of the torus and $\mathbb{Z}^4$ covariances $C = (-\Delta_{\Lambda_N} + m^2)^{-1}$ and $(-\Delta_{\mathbb{Z}^4} + m^2)^{-1}$, as discussed in [6, Section 5.1], and we use the same notation as in [6]. These decompositions take the form

$$(-\Delta_{\mathbb{Z}^4} + m^2)^{-1} = \sum_{j=1}^{\infty} C_j \quad (m^2 \in [0, \delta)), \quad (3.1)$$

$$C = (-\Delta_{\Lambda_N} + m^2)^{-1} = \sum_{j=1}^{N-1} C_j + C_{N,N} \quad (m^2 \in (0, \delta)), \quad (3.2)$$

where the covariance $C_{N,N}$ is special because of the effect of the torus. The particular finite-range decomposition we use is developed in [4, 17], with properties given in [7]. The finite-range condition is the statement that $C_{j;x,y} = 0$ when $|x - y| \geq \frac{1}{2}L^j$; this condition is important for results we use

from [21, 22]. As discussed in [6, Section 5.1], the Gaussian super-expectation of $F \in \mathcal{N}$ can be carried out progressively, via the identity

$$\mathbb{E}_C F = (\mathbb{E}_{C_{N,N}} \circ \mathbb{E}_{C_{N-1}} \theta \circ \cdots \circ \mathbb{E}_{C_1} \theta) F. \quad (3.3)$$

The external field $\sigma, \bar{\sigma}$ is treated as a constant by the super-expectation. To compute $\mathbb{E}_C e^{-V_0(\Lambda)}$ of (2.16), we use (3.3), and define

$$Z_0 = e^{-V_0(\Lambda)}, \quad Z_{j+1} = \mathbb{E}_{C_{j+1}} \theta Z_j \quad (j < N). \quad (3.4)$$

For $j+1 = N$, we interpret the convolution $\mathbb{E}_{C_{j+1}} \theta$ as the expectation $\mathbb{E}_{C_{N,N}}$, i.e., the last covariance is taken to be the one appropriate for the torus Λ_N , and the convolution is projected to the degree-0 part of the differential form (i.e., the fermion field is set to 0) and has the boson field evaluated at $\phi = 0$. Then the desired expectation is given by

$$\mathbb{E}_C e^{-V_0(\Lambda)} = Z_N. \quad (3.5)$$

Thus we are led to study the recursion $Z_j \mapsto Z_{j+1}$. By (2.16), the two-point function is given by

$$G_{N,g,\nu}(a, b) = (1 + z_0) Z_{N;\sigma\bar{\sigma}}, \quad (3.6)$$

where $Z_{N;\sigma\bar{\sigma}} \in \mathcal{N}^\varnothing$ denotes the coefficient of $\sigma\bar{\sigma}$ in $Z_N \in \mathcal{N}$, i.e., $\pi_{ab} Z_N = \sigma\bar{\sigma} Z_{N;\sigma\bar{\sigma}}$.

3.2 The interaction functional

Let $\mathcal{Q}^{(0)}$ and $\mathcal{Q}^{(1)}$ respectively denote the vector space of local polynomials of the form

$$V^{(0)} = g\tau^2 + \nu\tau + z\tau_\Delta + \lambda_a \sigma\bar{\phi} + \lambda_b \bar{\sigma}\phi, \quad (3.7)$$

$$V^{(1)} = g\tau^2 + \nu\tau + z\tau_\Delta + \lambda_a \sigma\bar{\phi} + \lambda_b \bar{\sigma}\phi + \lambda_q \sigma\bar{\sigma}, \quad (3.8)$$

where $g, \nu, z, \lambda, q \in \mathbb{R}$, $\lambda_a = -\lambda \mathbb{1}_a$, $\lambda_b = -\lambda \mathbb{1}_b$, and the indicator functions are defined by the Kronecker delta $\mathbb{1}_{a,x} = \delta_{a,x}$. The terms involving σ are referred to as *observables*, while the terms involving τ^2 , τ , and τ_Δ are *bulk terms*. We will frequently use the identifications $\mathcal{Q}^{(0)} \cong \mathbb{R}^4$ and $\mathcal{Q}^{(1)} \cong \mathbb{R}^5$, and write $V = V^{(0)} = (g, \nu, z, \lambda)$, $V^{(1)} = (g, \nu, z, \lambda, q)$, and $U = \pi_\varnothing V = (g, \nu, z) \in \mathbb{R}^3$.

Recall from [6, Section 5.3] the set $\mathcal{B}_j$ of scale- j blocks, and the set $\mathcal{P}_j$ of scale- j polymers in Λ . We also recall from [6, Section 5.4] the interaction functional $I_j : \mathcal{Q}^{(0)} \times \mathcal{P}_j \rightarrow \mathcal{N}$ defined for $B \in \mathcal{B}_j$, $X \in \mathcal{P}_j$, and $V \in \mathcal{Q}^{(0)}$ by

$$I_j(V, B) = e^{-V(B)}(1 + W_j(V, B)), \quad I_j(V, X) = \prod_{B \in \mathcal{B}_j} I_j(V, B), \quad (3.9)$$

where W_j is an explicit quadratic function of V defined in [7]. In particular, $W_0 = 0$. We often write simply $I_j(X) = I_j(V, X)$. By (3.9), $I_0(V, X) = e^{-V(X)}$ for all $X \subset \Lambda$, with $V(X) = \sum_{x \in X} V_x$.

Motivation for the definition (3.9) is given in [7, Section 2]. In the present paper, we do not give the details of the definitions of W_j and I_j since we do not need them here. They are, however, important in [7, 21, 22] and we rely on results from those references. The V domain of I_j is larger here than in [6], due to the presence of observables, but the larger domain is permitted and present in the analysis of [7, 21, 22].

3.3 Renormalisation group coordinates

Given $F_1, F_2 : \mathcal{P}_j \rightarrow \mathcal{N}$, we define the *circle product* $F_1 \circ F_2 : \mathcal{P}_j \rightarrow \mathcal{N}$ by

$$(F_1 \circ F_2)(Y) = \sum_{X \in \mathcal{P}_j : X \subset Y} F_1(X) F_2(Y \setminus X) \quad (Y \in \mathcal{P}_j). \quad (3.10)$$

The terms $X = \emptyset$ and $X = \Lambda$ are included in the summation on the right-hand side, and we demand that all functions $F : \mathcal{P}_j \rightarrow \mathcal{N}$ obey $F(\emptyset) = 1$. The circle product depends on the scale j , is associative, and is also commutative due to our restriction in $\mathcal{N}$ to forms of even degree. Its identity element is $\mathbb{1}_\emptyset$, defined by $\mathbb{1}_\emptyset(X) = 1$ if X is empty, and otherwise $\mathbb{1}_\emptyset(X) = 0$.

In the definition of I_0 we set $V = V_0$, with V_0 defined in (2.15), so that $I_0(X) = I_0(V_0, X) = e^{-V_0(X)}$ for all $X \subset \Lambda$. Let $K_0 : \mathcal{P}_0 \rightarrow \mathcal{N}$ be defined by $K_0 = \mathbb{1}_\emptyset$, and set $q_0 = 0$. Then $Z_0 = I_0(V_0, \Lambda)$ of (3.4) is also given by

$$Z_0 = I_0(\Lambda) = e^{q_0 \sigma \bar{\sigma}} (I_0 \circ K_0)(\Lambda). \quad (3.11)$$

Our strategy is to define $q_j \in \mathbb{R}$, $V_j \in \mathcal{Q}^{(0)}$, $K_j : \mathcal{P}_j \rightarrow \mathcal{N}$, and set $I_j = I_j(V_j)$, so as to maintain this form as

$$Z_j = e^{q_j \sigma \bar{\sigma}} (I_j \circ K_j)(\Lambda) \quad (0 \leq j \leq N) \quad (3.12)$$

in the recursion $Z_j \mapsto Z_{j+1} = \mathbb{E}_{C_{j+1}} \theta Z_j$ of (3.4), with the initial condition given by (3.11). At the final scale $j = N$, the only two polymers are the single block $\Lambda = \Lambda_N$ and the empty set $\emptyset$, and since $I_j(\emptyset) = K_j(\emptyset) = 1$, by assumption, (3.12) simply reads

$$Z_N = e^{q_N \sigma \bar{\sigma}} (I_N \circ K_N)(\Lambda) = e^{q_N \sigma \bar{\sigma}} (I_N(\Lambda) + K_N(\Lambda)). \quad (3.13)$$

If we set $\delta q_{j+1} = q_{j+1} - q_j$, then (3.12) can equivalently be written as

$$\mathbb{E}_{C_{j+1}} \theta (I_j \circ K_j)(\Lambda) = e^{\delta q_{j+1} \sigma \bar{\sigma}} (I_{j+1} \circ K_{j+1})(\Lambda). \quad (3.14)$$

In view of (3.14), and since I_j is determined by V_j , we are led to study the *renormalisation group map*

$$(V_j, K_j) \mapsto (\delta q_{j+1}, V_{j+1}, K_{j+1}). \quad (3.15)$$

The coupling constants of $V_j \in \mathcal{Q}^{(0)}$ are written as $g_j, \nu_j, z_j, \lambda_j$. Ultimately we express the two-point function in terms of the sequence (q_j) , so this sequence is fundamentally important in the proof of Theorem 1.1.

3.4 Renormalisation group map

To implement the above strategy, given suitable $V_j \in \mathcal{Q}^{(0)}$ and $K_j : \mathcal{P}_j \rightarrow \mathcal{N}$, we define $\delta q_{j+1} \in \mathbb{R}$, $V_{j+1} \in \mathcal{Q}^{(0)}$ and $K_{j+1} : \mathcal{P}_{j+1} \rightarrow \mathcal{N}$ in such a way that

$$Z_{j+1} = \mathbb{E}_{C_{j+1}} \theta Z_j = e^{q_j \sigma \bar{\sigma}} \mathbb{E}_{C_{j+1}} \theta (I_j \circ K_j)(\Lambda) = e^{q_{j+1} \sigma \bar{\sigma}} (I_{j+1} \circ K_{j+1})(\Lambda) \quad (j < N). \quad (3.16)$$

Thus (3.12) does retain its form under progressive integration. We use the explicit choice for the renormalisation group map (3.15) that is given in [22], from now on. This choice achieves (3.16) for fixed $j < N$, assuming that (V_j, K_j) is in an appropriate domain, and it provides good estimates for $(\delta q_{j+1}, V_{j+1}, K_{j+1})$.

To simplify the notation, we write (3.15) as $(V, K) \mapsto (V_+, K_+)$ with $V_+ = (\delta q_+, V_+^{(0)}) \in \mathcal{Q}^{(1)}$. We typically drop subscripts j and write $+$ in place of $j+1$, also leave the dependence of the maps on the mass parameter m^2 of the covariance $(-\Delta + m^2)^{-1}$ implicit.

3.5 Bulk flow

In [22, (1.68)], it is shown that the renormalisation group map has the property

$$\pi_{\emptyset} V_+(V, K) = V_+(\pi_{\emptyset} V, \pi_{\emptyset} K), \quad \pi_{\emptyset} K_+(V, K) = K_+(\pi_{\emptyset} V, \pi_{\emptyset} K). \quad (3.17)$$

Thus, under (3.15), the *bulk coordinates* $(\pi_{\emptyset} V_j, \pi_{\emptyset} K_j)$ satisfy a closed evolution equation of their own. We denote its evolution map by $(V_+^{\emptyset}, K_+^{\emptyset})$ and write $U = \pi_{\emptyset} V$. Then (3.15) reduces to

$$(U_{j+1}, \pi_{\emptyset} K_{j+1}) = (V_+^{\emptyset}(U_j, \pi_{\emptyset} K_j), K_+^{\emptyset}(U_j, \pi_{\emptyset} K_j)). \quad (3.18)$$

The construction of a critical global renormalisation group flow of the bulk coordinates (3.18) is achieved in [6]. Namely, there is a construction of $(U_j, \pi_{\emptyset} K_j)$ for $0 \leq j \leq N$ such that (3.18) holds for all $0 \leq j \leq N$. This construction provides detailed information about the sequence U_j , and good estimates on $\pi_{\emptyset} K_j$, sufficient for studying the infinite volume limit at the critical point. In Section 4, we use this bulk flow to study observables.

It is convenient to change perspective on which variables are independent. The weakly self-avoiding walk has parameters g, ν . In (2.13), additional parameters m^2, g_0, ν_0, z_0 were introduced. For the moment we consider these as independent variables and do not consider g, ν directly. The relation between m^2, g_0, ν_0, z_0 and the original parameters g, ν is addressed in Section 3.6.

To state the result about the bulk flow, let $\bar{g}_j$ be the (m^2, g_0) -dependent sequence determined by $\bar{g}_{j+1} = \bar{g}_j - \beta_j \bar{g}_j^2$ with $\bar{g}_0 = g_0$ and with $\beta_j = \beta_j(m^2)$ defined in [6, (6.5)]. We also recall the sequence χ_j defined in [6, (6.7)], but its precise definition is not important for our present needs. It obeys $0 \leq \chi_j \leq 1$, eventually decays exponentially when $m^2 > 0$, and is identically equal to 1 when $m^2 = 0$. Also, by [6, Proposition 6.1] and [6, (8.22)] respectively,

$$\chi_j \bar{g}_j \leq O\left(\frac{g_0}{1 + g_0 j}\right) \quad \text{uniformly in } (m^2, g_0) \in [0, \delta)^2, \quad (3.19)$$

$$\sum_{k=j}^{\infty} \chi_k \bar{g}_k^2 = O(\chi_j \bar{g}_j). \quad (3.20)$$

Without multiplication by χ_j , the sequence $\bar{g}_j$ converges to 0 when $m^2 = 0$ but not when $m^2 > 0$. (To apply (2.3), in which the limit $\nu \downarrow \nu_c$ follows the limit $N \rightarrow \infty$, we do consider limits $j \rightarrow \infty$ with $m^2 > 0$, corresponding to $\nu > \nu_c$, to prove Theorem 1.1.)

The following theorem is a reduced version of [6, Proposition 8.1]. Some of its notation is explained after the statement.

Theorem 3.1. *Let $d = 4$ and let $\delta > 0$ be sufficiently small. There exist $M > 0$ and an infinite sequence of functions $U_j = (g_j^c, \nu_j^c, z_j^c)$ of $(m^2, g_0) \in [0, \delta)^2$, independent of $N \in \mathbb{N}$, such that:*

(i) *assuming $\sigma = 0$, given $N \in \mathbb{N}$, for initial conditions $U_0 = (g_0, \nu_0^c, z_0^c)$ with $g_0 \in (0, \delta)$, $K_0 = \mathbb{1}_{\emptyset}$, and with mass $m^2 \in [0, \delta)$, a flow $(U_j, K_j) \in \mathbb{D}_j^{\emptyset}$ exists such that (3.18) holds for all $j + 1 < N$, and given $m^2 \in [\delta L^{-2(N-1)}, \delta)$, also for $j + 1 = N$. Then, in particular,*

$$\|K_j\|_{\mathcal{W}_j} = \|\pi_{\emptyset} K_j\|_{\mathcal{W}_j} \leq M \chi_j \bar{g}_j^3 \quad (j \leq N) \quad (3.21)$$

and $g_j^c = O(\bar{g}_j)$. In addition, $z_j^c = O(\chi_j \bar{g}_j)$ and $\nu_j = O(\chi_j L^{-2j} \bar{g}_j)$.

(ii) *z_0^c, ν_0^c are continuous in $(m^2, g_0) \in [0, \delta)^2$.*

The definition of the $\mathcal{W}_j$ norm in (3.21) is discussed at length in [22], and we do not repeat the details here, as we now only need the fact that (3.21) with $j = N$ implies that

$$|\pi_\emptyset K_N(\Lambda)| \prec \chi_N \bar{g}_N^3, \quad (3.22)$$

uniformly in $m^2 \in [\delta L^{-2(N-1)}, \delta)$, as a consequence of [22, (1.64)].

The $\mathcal{W}_j = \mathcal{W}_j(\tilde{s})$ norm depends on a parameter $\tilde{s} = (\tilde{m}^2, \tilde{g}) \in [0, \delta) \times (0, \delta)$. Its significance is discussed in [6, Section 6.3]. In particular, it is discussed there that useful choices of this parameter depend on the scale j , as well as on approximate values of the mass parameter m^2 of the covariance and the coupling constant g_j . Throughout the paper, we use the convenient convention that when the parameter $\tilde{s}$ is omitted, it is given by $\tilde{s} = s_j = (m^2, \tilde{g}_j(m^2, g_0))$. Here $\tilde{m}^2 = m^2$ is the mass parameter of the covariance, and $\tilde{g} = \tilde{g}_j$ is defined in terms of the initial condition g_0 by

$$\tilde{g}_j = \tilde{g}_j(m^2, g_0) = \bar{g}_j(0, g_0) \mathbb{1}_{j \leq j_m} + \bar{g}_{j_m}(0, g_0) \mathbb{1}_{j > j_m}, \quad (3.23)$$

where the *mass scale* j_m is the smallest integer j such that $L^{2j} m^2 \geq 1$. By [6, Lemma 7.4],

$$\tilde{g}_j = \bar{g}_j + O(\bar{g}_j^2), \quad (3.24)$$

so the two sequences are the same to leading order. However, $\tilde{g}_j$ is more convenient for aspects of the analysis in [6].

The domain $\mathbb{D}_j^\emptyset = \mathbb{D}_j^\emptyset(\tilde{s})$ also depends on $\tilde{s}$ (with the same convention when the parameter is omitted) and is defined as follows. For the universal constant $C_{\mathcal{D}} \geq 2$ determined in [6], for $j < N$,

$$\mathbb{D}_j^\emptyset(\tilde{s}) = \{(g, \nu, z) \in \mathbb{R}^3 : C_{\mathcal{D}}^{-1} \tilde{g} < g < C_{\mathcal{D}} \tilde{g}, L^{2j} |\nu|, |z| \leq C_{\mathcal{D}} \tilde{g}\} \times B_{\mathcal{W}_j^\emptyset}(\alpha \tilde{\chi}_j \tilde{g}^3). \quad (3.25)$$

The first factor is the important stability domain defined in [21, (1.55)], restricted to the bulk coordinates and real scalars. In the second factor, $B_X(a)$ denotes the open ball of radius a centred at the origin of the Banach space X , and where α is fixed in [6]; it can be taken to be $10M$ where M is the constant of Theorem 3.1. The space $\mathcal{W}^\emptyset$ is the restriction of $\mathcal{W}$ to elements $K \in \mathcal{W}$ with $\pi_* K(X) = 0$ for all polymers X . Since the renormalisation group acts triangularly, by (3.17), the distinction between $\mathcal{W}$ and $\mathcal{W}^\emptyset$ is unimportant for the bulk flow, and $\mathcal{W}^\emptyset$ is denoted by $\mathcal{W}$ in [6].

3.6 Change of variables

Theorem 3.1 is stated in terms of the parameters m^2, g_0 , rather than the parameters g, ν of the weakly self-avoiding walk. The following proposition, proved in [6, Proposition 4.2(ii)], relates these sets of parameters via the functions z_0^c, ν_0^c of Theorem 3.1 and (2.14).

Proposition 3.2. *Let $d = 4$ and let $\delta_1 > 0$ be sufficiently small. There exists a function $[0, \delta_1]^2 \rightarrow [0, \delta]^2$, written $(g, \varepsilon) \mapsto (\tilde{m}^2(g, \varepsilon), \tilde{g}_0(g, \varepsilon))$, such that (2.14) holds with $\nu = \nu_c(g) + \varepsilon$, if $z_0 = z_0^c(\tilde{m}^2, \tilde{g}_0)$ and $\nu_0 = \nu_0^c(\tilde{m}^2, \tilde{g}_0)$. The functions $\tilde{m}, \tilde{g}_0$ are right-continuous as $\varepsilon \downarrow 0$, with $\tilde{m}^2(g, 0) = 0$, and $\tilde{m}^2(g, \varepsilon) > 0$ if $\varepsilon > 0$.*

We also write

$$\tilde{z}_0(g, \varepsilon) = z_0^c(\tilde{m}^2(g, \varepsilon), \tilde{g}_0(g, \varepsilon)), \quad \tilde{\nu}_0(g, \varepsilon) = \nu_0^c(\tilde{m}^2(g, \varepsilon), \tilde{g}_0(g, \varepsilon)). \quad (3.26)$$

The functions $\tilde{z}_0, \tilde{\nu}_0$ are right-continuous as $\varepsilon \downarrow 0$. For the problem without observables, considered in [6], we analysed the sequence Z_j by choosing variables as follows. First, starting from (g, ν) , Proposition 3.2 gives us $(\tilde{m}^2, \tilde{g}_0)$, and then Theorem 3.1 gives us an initial condition $U_0 = (\tilde{g}_0, \tilde{z}_0, \tilde{\nu}_0)$ for which there exists a global bulk flow of the renormalisation group map. In the next section, we extend this to include observables.

4 Observable flow

It follows from Proposition 2.1 and (3.6) that

$$G_{g, \nu_c}(a, b) = \lim_{\varepsilon \downarrow 0} \lim_{N \rightarrow \infty} G_{N, g, \nu_c + \varepsilon}(a, b) = \lim_{\varepsilon \downarrow 0} \left((1 + z_0) \lim_{N \rightarrow \infty} Z_{N; \sigma \bar{\sigma}} \right), \quad (4.1)$$

provided the parameters (m^2, g_0, ν_0, z_0) implicit on the right-hand side obey (2.14) with $\nu = \nu_c(g) + \varepsilon$. To analyse (4.1) via the renormalisation group flow, our remaining task is to supplement the bulk flow of Theorem 3.1 with the flow of the observable coupling constants λ_j, q_j and of the observable part $\pi_* K_j$ of K_j . In other words, we extend Theorem 3.1 to the case of nonzero σ . This is truly an extension, in the sense that the bulk flow needs no modification because the equations for $\lambda_j, q_j, \pi_* K_j$ depend on but do not appear in the flow of $(g_j, z_j, \nu_j, \pi_{\emptyset} K_j)$ which corresponds to $\sigma = 0$, by (3.17). With the estimates provided by [22], we will prove Theorem 1.1 using the kind of perturbative calculations familiar in the physics literature, but now in a mathematically rigorous manner.

In the remainder of the paper, we write

$$f \prec g \text{ when there is a } C > 0 \text{ such that } f \leq Cg; \quad (4.2)$$

the constant C is always uniform in g, ε and the scale j but may depend on L .

4.1 Perturbative flow of observables

Definition 4.1. Given $a, b \in \Lambda$, the *coalescence scale* j_{ab} is defined by

$$j_{ab} = \lfloor \log_L(2|a - b|) \rfloor. \quad (4.3)$$

The coalescence scale is related to the finite-range property of the covariance decomposition mentioned in Section 3.1, namely that $C_{j; x, y} = 0$ if $|x - y| \geq \frac{1}{2}L^j$. Thus j_{ab} is such that $C_{j_{ab}; a, b} = 0$, but $C_{j_{ab}+1; a, b}$ need not be zero. By definition, $L^{-2j_{ab}}$ is bounded above and below by multiples of $|a - b|^{-2}$, in fact $L^{j_{ab}} \leq 2|a - b|$.

In [7], the flow of the coupling constants in V is computed at a perturbative level. The perturbative flow is without control of errors uniformly in the volume, and we address the uniform control below. The perturbative flow is given by a map $V = (g, \nu, z, \lambda, q) \mapsto V_{\text{pt}} = (g_{\text{pt}}, \nu_{\text{pt}}, z_{\text{pt}}, \lambda_{\text{pt}}, q_{\text{pt}})$; here we are only interested in λ, q . The perturbative flow of λ, q is reported in [7, (3.34)–(3.35)] as the scale-dependent map $V \mapsto (\lambda_{\text{pt}}, q_{\text{pt}})$ given by

$$\lambda_{\text{pt}} = \begin{cases} (1 - \delta[\nu w^{(1)}])\lambda & (j + 1 < j_{ab}) \\ \lambda & (j + 1 \geq j_{ab}), \end{cases} \quad (4.4)$$

$$q_{\text{pt}} = q + \lambda^2 C_{j+1; a, b}. \quad (4.5)$$

In (4.4)–(4.5), j refers to the scale of the initial V , with $(\lambda_{\text{pt}}, q_{\text{pt}})$ being scale- $(j+1)$ objects. Also, $w^{(1)} = w_j^{(1)} = \sum_{x \in \Lambda} \sum_{i=1}^j C_{i;0x}$, and

$$\delta[\nu w^{(1)}] = \nu^+ w_{j+1}^{(1)} - \nu w_j^{(1)} \quad \text{with } \nu^+ = \nu + 2gC_{j+1;0,0}. \quad (4.6)$$

The coalescence scale j_{ab} has the property that $q_{\text{pt}} = 0$ if $q = 0$ for $j \leq j_{ab}$ because the factor $C_{j+1;a,b}$ on the right-hand side of (4.5) is zero when $j+1 \leq j_{ab}$. The considerations that lead to the stopping of the flow of λ at the coalescence scale in (4.4) are discussed in [7, Section 3.2].

As discussed above (3.14), it is convenient to express the renormalisation group map in terms of δq rather than q . For this, we identify elements $V \in \mathcal{Q}^{(0)}$ with elements of $\mathcal{Q}^{(1)}$ having $q = 0$, and, when $V \in \mathcal{Q}^{(0)}$ we write δq_{pt} instead of q_{pt} .

4.2 A single renormalisation group step

Now we consider the renormalisation group map

$$(V, K) \mapsto (V_+^{(1)}, K_+) = (\delta q_+, V_+, K_+), \quad (4.7)$$

which pertains not only to the bulk, but also to $\lambda, \delta q$, as well as $\pi_* K = (\pi_a K, \pi_b K, \pi_{ab} K)$.

To state the estimates we require for this map from [22], recall (3.25), and define similarly

$$\mathbb{D}_j(\tilde{s}) = \{(g, \nu, z, \lambda) \in \mathbb{R}^4 : C_{\mathcal{D}}^{-1} \tilde{g} < g < C_{\mathcal{D}} \tilde{g}, L^{2j} |\nu|, |z| \leq C_{\mathcal{D}} \tilde{g}, |\lambda| \leq C_{\mathcal{D}}\} \times B_{\mathcal{W}_j}(\alpha \tilde{\chi}_j \tilde{g}^3). \quad (4.8)$$

The first factor is the same as [22, (1.55)], but restricted to real values. Compared to $\mathbb{D}^\varnothing$ of (3.25), the coupling constant λ is included in $\mathbb{D}$ of (4.8), and, as in [22], the Banach spaces $\mathcal{W}_j = \mathcal{W}_j(\tilde{s})$ now also pertain to K with components in $\mathcal{N}^a, \mathcal{N}^b, \mathcal{N}^{ab}$. The domain $\mathbb{D}^\varnothing$ is obtained by projecting both factors in the definition (4.8) by the appropriate definitions of $\pi_\varnothing$ on $\mathcal{Q}^{(0)}$ and $\mathcal{W}_j$ separately. As in (3.25), compared to [22], we have replaced occurrences of $\chi^{3/2}$ and $\chi^{1/2}$ by χ in (4.8).

A j -dependent norm on $\mathcal{Q}^{(1)}$ is defined by

$$\|V\|_{\mathcal{Q}} = \max\{|g|, L^{2j} |\nu_j|, |z_j|, \ell_j \ell_{\sigma,j} |\lambda|, \ell_{\sigma,j}^2 |q|\} \quad (4.9)$$

where

$$\ell_j = \ell_0 L^{-j}, \quad \ell_{\sigma,j} = 2^{(j-j_{ab})+} L^{(j \wedge j_{ab})} \tilde{g}. \quad (4.10)$$

The significance of the weights $\ell_j, \ell_{\sigma,j}$ is explained in [21, Remark 3.3]; the constant $\ell_0 > 0$ is determined in [21, (1.72)] and is of no direct importance here.

The following theorem concerns a single renormalisation group step (3.15), with observables. It is a reduced version of the main result of [22], combining the relevant parts of [22, Theorems 1.10–1.11, 1.13] into a single statement. Such a result was used in [6, Theorem 6.3], but now observables are included in V and K . In fact, only the observable part of the statement is of interest here—the bulk flow is independent and has already been analysed—but it is convenient to state the theorem in its general form, applying to both bulk and observables simultaneously. The bounds on derivatives provided by [22] are not stated in Theorem 4.2 as they are not needed here.

The map $V_+^{(1)} = (\delta q_+, V_+)$ is a perturbation of the map V_{pt} discussed in Section 4.1, and it is convenient to describe it in terms of the difference

$$R_+(V, K) = V_+^{(1)}(V, K) - V_{\text{pt}}(V). \quad (4.11)$$

Thus R_+ is an element of $\mathcal{Q}^{(1)}$ with components for all coupling constants $(g, \nu, z, \lambda, \delta q)$. As in [6], considerable care is required to express the continuity of the maps R_+, K_+ in the mass parameter m^2 , and we define the intervals

$$\mathbb{I}_j = \begin{cases} [0, \delta] & (j < N) \\ [\delta L^{-2(N-1)}, \delta] & (j = N), \end{cases} \quad (4.12)$$

and, for $\tilde{m}^2 \in \mathbb{I}_j$,

$$\tilde{\mathbb{I}}_j = \tilde{\mathbb{I}}_j(\tilde{m}^2) = \begin{cases} [\frac{1}{2}\tilde{m}^2, 2\tilde{m}^2] \cap \mathbb{I}_j & (\tilde{m}^2 \neq 0) \\ [0, L^{-2(j-1)}] \cap \mathbb{I}_j & (\tilde{m}^2 = 0). \end{cases} \quad (4.13)$$

For the statement of the theorem, we write $\tilde{s} = (\tilde{m}^2, \tilde{g})$ and $\tilde{s}_+ = (\tilde{m}^2, \tilde{g}_+)$. We assume $\frac{1}{2}\tilde{g}_+ \leq \tilde{g} \leq 2\tilde{g}_+$ and write $\tilde{\chi} = \chi_j(\tilde{m}^2)$. We subsequently use the explicit choice $\tilde{s} = s_j$ and $\tilde{s}_+ = s_{j+1}$, discussed in Section 3.5, and the choice of α mentioned below (3.25). Then in particular $\tilde{\chi} = \chi_j$.

Theorem 4.2. *Let $d = 4$. Let $C_{\mathcal{D}}$ and L be sufficiently large. There exist $M > 0$ and $\delta > 0$ such that for $\tilde{g} \in (0, \delta)$ and $\tilde{m}^2 \in \mathbb{I}_+$, and with the domain $\mathbb{D}$ defined using any $\alpha > M$, the maps*

$$R_+ : \mathbb{D}(\tilde{s}) \times \tilde{\mathbb{I}}_+(\tilde{m}^2) \rightarrow \mathcal{Q}^{(1)}, \quad K_+ : \mathbb{D}(\tilde{s}) \times \tilde{\mathbb{I}}_+(\tilde{m}^2) \rightarrow \mathcal{W}_+(\tilde{s}_+) \quad (4.14)$$

are analytic in (V, K) , and satisfy the estimates

$$\|R_+\|_{\mathcal{Q}} \leq M\tilde{\chi}\tilde{g}^3, \quad \|K_+\|_{\mathcal{W}_+} \leq M\tilde{\chi}\tilde{g}^3. \quad (4.15)$$

In addition, R_+, K_+ are jointly continuous in all arguments m^2, V, K .

In a precise but non-trivial sense, Theorem 4.2 shows that the error to the perturbative calculation of Section 4.1 is of third-order in the coupling constants. However, unlike the bulk coupling constants, which remain small, the observables are not small, e.g., $\lambda_0 = 1$, and this is compensated by the weights in (4.10). In [22, Proposition 1.14], it is shown that

$$R_+^\lambda = \pi_a R_+ = \pi_b R_+ = 0 \text{ for } j \geq j_{ab}, \quad R_+^q = \pi_{ab} R_+ = 0 \text{ for } j < j_{ab}. \quad (4.16)$$

It is further shown in [22, Proposition 1.14] that, with (4.16) and definition of the $\mathcal{Q}$ norm (4.9), Theorem 4.2 implies that, for $(V, K) \in \mathbb{D}_j$,

$$|R_+^\lambda| \prec \chi_j \tilde{g}_j^2 \mathbb{1}_{j < j_{ab}}, \quad (4.17)$$

$$|R_+^q| \prec |a - b|^{-2} \chi_j 4^{-(j-j_{ab})} \tilde{g}_j \mathbb{1}_{j \geq j_{ab}}. \quad (4.18)$$

4.3 Observable flow

The achievement of Theorem 4.2 is to show that if (V_j, K_j) lies in the domain $\mathbb{D}_j$, then we have good control of λ_{j+1}, q_{j+1} and also the observable part of K_{j+1} (whose bulk part has been controlled along with the bulk coupling constants already in Theorem 3.1). The following proposition links scales together via an inductive argument to conclude that (V_j, K_j) remains in $\mathbb{D}_j$ for *all* $j \leq N$.

In particular, this requires that the bulk flow is well-defined for all $j \leq N$. For this, we recall that, given the parameters $(m^2, g_0) \in [0, \delta)^2$, the critical initial conditions for the global existence of the bulk renormalisation group flow are given by

$$U_0 = U_0^c = (g_0, z_0^c(m^2, g_0), \nu_0^c(m^2, g_0)), \quad (4.19)$$

by Theorem 3.1. We also recall the corresponding sequence $U_j(m^2, g_0)$. As a consequence, using (3.17) and Theorem 4.2, the next proposition shows that the flow with observables, and with initial conditions

$$\pi_\emptyset V_0 = U_0^c, \quad \lambda_0 = 1, \quad q_0 = 0, \quad (4.20)$$

exists for all $j \leq N$.

Proposition 4.3. (i) For $(m^2, g_0) \in [\delta L^{-2(N-1)}, \delta) \times (0, \delta)$, there is a choice of (q_j, V_j, K_j) such that (3.16) holds for $0 \leq j \leq N$. This choice is such that $\pi_\emptyset V_j = U_j(m^2, g_0)$, and

$$\lambda_j = \begin{cases} (1 + \nu_j w_j^{(1)})^{-1} \left(1 + \sum_{k=0}^{j-1} \check{v}_{\lambda,k} \right) & (j+1 < j_{ab}) \\ \lambda_{j_{ab}-1} & (j+1 \geq j_{ab}), \end{cases} \quad (4.21)$$

$$q_j = \sum_{i=j_{ab}-1}^{j-1} (\lambda_{j_{ab}-1}^2 C_{i+1;a,b} + v_{q,i}), \quad (4.22)$$

$$\|K_j\|_{W_j} \leq M \chi_j \tilde{g}_j^3, \quad (4.23)$$

with M as in (4.15), and with $\check{v}_{\lambda,j}, v_{q,j} \in \mathbb{R}$ obeying, uniformly in $(m^2, g_0) \in [0, \delta)^2$,

$$|\check{v}_{\lambda,j}| \prec \chi_j \tilde{g}_j^2 \mathbb{1}_{j < j_{ab}}, \quad |v_{q,j}| \prec |a-b|^{-2} \chi_j 4^{-(j-j_{ab})} \tilde{g}_j \mathbb{1}_{j \geq j_{ab}}. \quad (4.24)$$

(ii) Each of λ_j, q_j is independent of N , and is defined as a continuous function of $(m^2, g_0) \in [0, \delta)^2$.

Proof. (i) As a preliminary step, we introduce a change to variables that diagonalises the evolution of λ to linear order in V . For (V_j, K_j) , we write $\lambda_{\text{pt}} = \lambda_{\text{pt}}(V_j)$ and $v_{\lambda,j} = R_{j+1}^\lambda(V_j, K_j)$. Then the λ -component of (4.7) can be written as $\lambda_{j+1} = \lambda_{\text{pt}} + v_{\lambda,j}$. We define

$$\check{\lambda}_j = \lambda_j (1 + \nu_j w_j^{(1)}). \quad (4.25)$$

By (4.4), the recursion for $\check{\lambda}_j$ can then be written as

$$\check{\lambda}_{j+1} = \check{\lambda}_j + \check{v}_{\lambda,j} \quad (4.26)$$

with

$$\check{v}_{\lambda,j} = (\nu_{j+1} - \nu_j^+) \lambda_j w_{j+1}^{(1)} + v_{\lambda,j} (1 + \nu_{j+1} w_{j+1}^{(1)}) - \delta_j [\nu w^{(1)}] \lambda_j \nu_{j+1} w_{j+1}^{(1)}. \quad (4.27)$$

The solution to (4.26) with initial condition $\lambda_0 = 1$ is $\check{\lambda}_j = 1 + \sum_{k=0}^{j-1} \check{v}_{\lambda,k}$, and hence

$$\lambda_j = (1 + \nu_j w_j^{(1)})^{-1} \left(1 + \sum_{k=0}^{j-1} \check{v}_{\lambda,k} \right). \quad (4.28)$$

By (4.5) and (4.11), and with $v_{q,j} = R_{j+1}^q(V_j, K_j)$, δq_j is simply given by

$$\delta q_{j+1} = \delta q_{\text{pt}} + v_{q,j} = \lambda_j^2 C_{j+1;a,b} + v_{q,j}. \quad (4.29)$$

Now we can prove (4.21)–(4.24) by induction on j , with induction hypothesis:

IH_j : for all $k \leq j$, $(V_k, K_k) \in \mathbb{D}_k$, (4.21)–(4.24) hold with j replaced by k .

By direct verification, IH_0 holds (with $\check{v}_{\lambda,-1} = v_{q,-1} = 0$).

We assume IH_j and show that it implies IH_{j+1} . By IH_j and the bound (4.15) of Theorem 4.2, K_{j+1} obeys (4.23). In particular, this estimate implies $K_{j+1} \in B_{\mathcal{W}_j}(\alpha \chi_j \tilde{g}_j)$.

By (3.17)–(3.18), $\pi_{\emptyset} V_j = U_j$ for all j , and by Theorem 3.1, U satisfies the bounds required for $\pi_{\emptyset} V$ in the definition of $\mathbb{D}$. Therefore, to verify $(V_{j+1}, K_{j+1}) \in \mathbb{D}_{j+1}$, it suffices to show $\lambda_{j+1} \leq C_{\mathcal{D}}$.

By (4.11), (4.4), and (4.17), $\lambda_j = \lambda_{j_{ab}-1}$ for all $j \geq j_{ab}$, so we assume that $j < j_{ab} - 1$. To estimate $\check{v}_{\lambda,j}$, we use the fact that $|\lambda_j| \leq C_{\mathcal{D}}$ by assumption, and apply [7, Lemma 6.2] and [22, (1.78)] to see that $w_j^{(1)} \prec L^{2j}$ and $|\nu_{j+1} - \nu_j^+| \prec L^{-2j} \chi_j \tilde{g}_j^2$, and also $|\nu_j| w_j^{(1)} \prec L^{-2j} g_j w_j^{(1)} \prec \chi_j \tilde{g}_j$. The factor $v_{\lambda,j}$ is bounded via (4.17), and the last term on the right-hand side of (4.27) is similarly bounded (without any need for cancellation in the δ term). We conclude that $|\check{v}_{\lambda,j}| \prec \chi_j \tilde{g}_j^2$, as required. With (3.20), this leads to $|\lambda_{j+1}| = 1 + O(g_0) \leq C_{\mathcal{D}}$ (since we have assumed above (3.25) that $C_{\mathcal{D}} \geq 2$). This establishes that λ_{j+1} obeys the condition required in the definition of the domain $\mathbb{D}_{j+1}$, and all necessary properties for λ_{j+1} have been established.

By (4.29) and (4.18) with IH_j , (4.22) holds with $v_{q,j}$ obeying (4.24). This advances the induction and completes the proof of part (i).

(ii) The N -independence of $\lambda_j, \delta q_j$ follows exactly as in the proof of [6, Proposition 8.1], so we only sketch the argument. By (4.4)–(4.5), λ_{pt} and δq_{pt} are independent of N . Moreover, by [22, Proposition 1.18(i)], $R_+(V, K)$ is independent of N provided that V is independent of N and that the family K has Property $\mathbb{Z}^d$ defined in [22]. That the renormalisation group map preserves Property $\mathbb{Z}^d$ for K is shown in [22, Proposition 1.17].

To show that $\lambda_j, \delta q_j$ (and thus also q_j) are continuous as functions of $(m^2, g_0) \in [0, \delta]^2$, assuming that $V_0 = V_0^c(m^2, g_0)$, we can proceed exactly as in [6, Section 8.2]. The definition of *continuous functions of the renormalisation group coordinates at scale- j* , provided by [6, Definition 8.2] for the bulk coordinates, applies literally also to the renormalisation group coordinates with observables. By Theorem 4.2 for R_+ and [7, Lemma 6.2] for V_{pt} , $\lambda_+, \delta q_+$ are both continuous functions of the renormalisation group coordinates at scale- j . By [6, Proposition 8.3], which also applies literally with observables, we conclude continuity of $\lambda_j, \delta q_j$ for all j . This completes the proof. ■

The next two lemmas apply Proposition 4.3 to study limits of the sequences λ_j, q_j . By Proposition 4.3(ii), λ_j, q_j are independent of N (larger than j_{ab}), and we can therefore assume that they are defined for all $j \in \mathbb{N}_0$. We make their dependence on (m^2, g_0) explicit by writing $\lambda_j = \lambda_j(m^2, g_0)$ and $q_j = q_j(m^2, g_0)$. They also depend on a, b but we leave this implicit in the notation.

Lemma 4.4. *For $(m^2, g_0) \in [0, \delta]^2$, the limit*

$$\lambda_{\infty}(m^2, g_0) = \lim_{|a-b| \rightarrow \infty} \lambda_{j_{ab}-1}(m^2, g_0) \quad (4.30)$$

exists, is continuous, and

$$|\lambda_{\infty}(m^2, g_0) - \lambda_{j_{ab}-1}(m^2, g_0)| \prec \chi_{j_{ab}} \bar{g}_{j_{ab}}. \quad (4.31)$$

Proof. Since $j_{ab} \rightarrow \infty$ as $|a - b| \rightarrow \infty$, it follows from (4.21), (4.24), $|\nu_j|w_j^{(1)} \prec \chi_j \tilde{g}_j$, and (3.20) that

$$\lambda_\infty = \lim_{j_{ab} \rightarrow \infty} \lambda_{j_{ab}-1} = 1 + \sum_{k=0}^{\infty} \check{v}_{\lambda,k} = 1 + O(g_0). \quad (4.32)$$

In particular, the limit exists. Also, for $j < j_{ab}$, by (3.20),

$$\lambda_j - \lambda_\infty = -\nu_j w_j^{(1)} \lambda_j - \sum_{k=j}^{\infty} \check{v}_{\lambda,k} = O(\chi_j \bar{g}_j). \quad (4.33)$$

By Proposition 4.3, all bounds are uniform in (m^2, g_0) and for each k the term under the sum over k is continuous in (m^2, g_0) . This implies that $\lambda_\infty(m^2, g_0)$ is continuous, and the proof is complete. $\blacksquare$

Lemma 4.5. *For $(m^2, g_0) \in [0, \delta)^2$, the limit*

$$q_\infty(m^2, g_0) = \lim_{j \rightarrow \infty} q_j(m^2, g_0), \quad (4.34)$$

exists, is continuous, and

$$q_\infty(0, g_0) = \lambda_\infty^2(0, g_0)(-\Delta_{\mathbb{Z}^4})_{ab}^{-1} \left(1 + O\left(\frac{1}{\log|a-b|}\right) \right) \quad \text{as } |a-b| \rightarrow \infty. \quad (4.35)$$

Proof. By (4.22),

$$q_j = \sum_{i=j_{ab}-1}^{j-1} (\lambda_{j_{ab}-1}^2 C_{i+1;a,b} + v_{q,i}). \quad (4.36)$$

Since $C_{i+1;a,b} = 0$ for $i < j_{ab}$, we can restore the scales $i < j_{ab}$ to the sum in the first term on the right-hand side, to obtain the complete finite-range decomposition for the inverse Laplacian on $\mathbb{Z}^4$ as in (3.1). As $j \rightarrow \infty$, this sum converges to

$$\lambda_{j_{ab}-1}^2 \sum_{i=j_{ab}-1}^{\infty} C_{i+1;a,b} = \lambda_{j_{ab}-1}^2 \sum_{i=0}^{\infty} C_{i+1;a,b} = \lambda_{j_{ab}-1}^2 (-\Delta_{\mathbb{Z}^4} + m^2)_{ab}^{-1}. \quad (4.37)$$

The dependence of $\lambda_{j_{ab}-1}$ is continuous in $[0, \delta)^2$ by Proposition 4.3, and $(-\Delta_{\mathbb{Z}^4} + m^2)_{ab}^{-1}$ is continuous in $m^2 \in [0, \delta)$. By Proposition 4.3, in the limit $j \rightarrow \infty$ the sum of $v_{q,i}$ on the right-hand side of (4.36) is a uniformly convergent sum of terms that are continuous. Therefore the sum $q_\infty(m^2, g_0)$ is also continuous, and

$$q_\infty = \lambda_{j_{ab}-1}^2 (-\Delta_{\mathbb{Z}^4} + m^2)_{ab}^{-1} + \sum_{i=j_{ab}}^{\infty} v_{q,i}. \quad (4.38)$$

By (4.18),

$$\sum_{i=j_{ab}}^{\infty} |v_{q,i}| \prec |a-b|^{-2} \sum_{i=j_{ab}}^{\infty} 4^{-(i-j_{ab})} \chi_i \tilde{g}_i \prec |a-b|^{-2} \chi_{j_{ab}} \bar{g}_{j_{ab}}. \quad (4.39)$$

Therefore,

$$|q_\infty - \lambda_{j_{ab}-1}^2(-\Delta_{\mathbb{Z}^4} + m^2)_{ab}^{-1}| \prec \chi_{j_{ab}} \bar{g}_{j_{ab}} |a - b|^{-2}. \quad (4.40)$$

By Lemma 4.4,

$$|\lambda_\infty^2 - \lambda_{j_{ab}-1}^2| \prec \chi_{j_{ab}} \bar{g}_{j_{ab}}. \quad (4.41)$$

With (4.40) and $|(-\Delta_{\mathbb{Z}^4} + m^2)_{ab}^{-1}| \prec |a - b|^{-2}$, this gives

$$|q_\infty - \lambda_\infty^2(-\Delta_{\mathbb{Z}^4} + m^2)_{ab}^{-1}| \prec \chi_{j_{ab}} \bar{g}_{j_{ab}} |a - b|^{-2}. \quad (4.42)$$

uniformly in (m^2, g_0) . By (3.19), $\chi_{j_{ab}} \bar{g}_{j_{ab}} \prec j_{ab}^{-1} \prec (\log |a - b|)^{-1}$. In particular, the limit $q_\infty(0, g_0)$ obeys (4.35). $\blacksquare$

4.4 Proof of main result

We now prove Theorem 1.1. In addition to the study of q_j , which provides the leading contribution, this requires the extraction of information from the norm estimate (4.23) for K_N . Recall from [22, (1.31)] that in the $\mathcal{W}_j$ norm each occurrence of σ or $\bar{\sigma}$ gives rise to a weight

$$\ell_{\sigma,j} = 2^{(j-j_{ab})+} L^{(j \wedge j_{ab})} \tilde{g}_j. \quad (4.43)$$

There is therefore a factor $\ell_{\sigma,j}^2$ inside the norm of $\pi_{ab} K_j$. Using this, it is shown in [22, (1.64)] that (4.23) implies that

$$|K_{N;\sigma\bar{\sigma}}(\Lambda)| \prec \frac{1}{4^{N-j_{ab}}} \frac{1}{|a - b|^2} \chi_N \bar{g}_N, \quad (4.44)$$

uniformly in $m^2 \in [\delta L^{-2(N-1)}, \delta]$.

Proof of Theorem 1.1. For small $g, \varepsilon > 0$, set $\nu = \nu_c(g) + \varepsilon$, and let $(m^2, g_0, \nu_0, z_0) = (\tilde{m}^2, \tilde{g}_0, \tilde{\nu}_0, \tilde{z}_0)$ be the functions of (g, ε) given by Proposition 3.2. Since $z_0 = \tilde{z}_0(g, \varepsilon) \rightarrow \tilde{z}_0(g, 0)$ as $\varepsilon \downarrow 0$, (4.1) gives

$$G_{g,\nu_c}(a, b) = (1 + \tilde{z}_0(g, 0)) \lim_{\varepsilon \downarrow 0} \lim_{N \rightarrow \infty} Z_{N;\sigma\bar{\sigma}}. \quad (4.45)$$

As discussed below (3.4), the final renormalisation group map implements the action of $\mathbb{E}_{C_{N,N}}$, not $\mathbb{E}_{C_{N,N}}\theta$. This means that the fields ϕ, ψ are to be set to zero in I_N, K_N . Thus $I_N(\Lambda) = 1$, and for $K_N(\Lambda)$ only dependence on $\sigma, \bar{\sigma}$ remains. From (3.13) we obtain

$$Z_N = e^{q_N \sigma \bar{\sigma}} (I_N(\Lambda) + K_N(\Lambda)) = e^{q_N \sigma \bar{\sigma}} (1 + K_N(\Lambda)). \quad (4.46)$$

Equating the coefficients of $\sigma \bar{\sigma}$ on both sides gives

$$Z_{N;\sigma\bar{\sigma}} = q_N (1 + \pi_\emptyset K_N(\Lambda)) + K_{N;\sigma\bar{\sigma}}. \quad (4.47)$$

Since $\varepsilon > 0$ by assumption, it follows that $m^2 > 0$, by Proposition 3.2. Therefore, for N sufficiently large, the bounds (3.22) and (4.44) hold. In particular, by (3.19),

$$\lim_{N \rightarrow \infty} \pi_\emptyset K_N(\Lambda) = 0, \quad \lim_{N \rightarrow \infty} K_{N;\sigma\bar{\sigma}} = 0, \quad (4.48)$$

and therefore

$$\lim_{N \rightarrow \infty} Z_{N; \sigma \bar{\sigma}} = \lim_{N \rightarrow \infty} q_N = q_\infty. \quad (4.49)$$

With (4.45) and Lemma 4.5, this gives

$$\begin{aligned} G_{g, \nu_c}(a, b) &= (1 + \tilde{z}_0(g, 0)) \lim_{\varepsilon \downarrow 0} q_\infty \\ &= (1 + \tilde{z}_0(g, 0)) \lambda_\infty^2 (-\Delta_{\mathbb{Z}^4})_{ab}^{-1} \left(1 + O \left(\frac{1}{\log |a - b|} \right) \right). \end{aligned} \quad (4.50)$$

Since $(-\Delta_{\mathbb{Z}^4})_{ab}^{-1} = \text{const} |a - b|^{-2} (1 + O(|a - b|^{-1}))$ [46], the proof is complete. $\blacksquare$

Note that

$$G_{g, \nu_c}(a, b) = (1 + \tilde{z}_0(g, 0)) q_\infty \quad (4.51)$$

is an *equality*, and not merely an asymptotic formula. As such, it contains all information about the two-point function, including not just the leading asymptotic behaviour but also all higher-order corrections.

Remark 4.6. Equations (4.43)–(4.44) provide corrections to [15, (109)–(111)], which contain erroneous powers of $\bar{g}_N$ in the upper bounds. In [15, (109), (111)], the $\bar{g}_N^3$ in the upper bound should be $\bar{g}_N$, and in [15, (110)] a factor $\bar{g}_N$ is missing on the right-hand side (it is present in (4.43)). The above proof shows that the correct powers here remain sufficient to prove (4.50).

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