Solutions to exercises in "The lace expansion and its applications" by Gordon Slade

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with solutions by
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Foreword

This document contains solutions to all the exercises in my lecture notes on the lace expansion. All references in the solutions are to Version 3 (Final) of "The Lace Expansion and its Applications" dated August 9, 2005.

The solutions were prepared by students in my course at the PIMS/UBC Summer School on Probability which took place June 6–30, 2005 in Vancouver, with the exception of Exercise 3.8 for which an essential ingredient was provided in Saint–Flour in July 2004. I thank Jeremy Flowers, Jesse Goodman, Jeffrey Hood, Sandra Kliem, Richard Liang, Terry Soo and Amanda Turner for their contributions and for their permission to include their work in this document. I am particularly grateful to Sandra Kliem and Richard Liang, who compiled, edited and proof-read the solutions, and prepared this document.

Gordon Slade

Solution to Exercise 1.1 by Terry Soo.

(a) This is an application of Fubini's theorem and is a general fact about Fourier transforms. We introduce some notation: for $f \in L^1(\mu)$, define $\hat{f}(s) = \int f(x)e^{is\cdot x}d\mu$. In our case μ is a counting measure. Now let $f, g \in L^1$. We first need to show that $f * g$ is indeed in L^1. Observe that by Fubini's theorem and a change of variables:

$$\int \left(\int |f(y)||g(x-y)|d\mu(y) \right) d\mu(x) = \int |f(x)|d\mu(x) \int |g(x)|d\mu(x).$$

So we may proceed to compute $\widehat{f * g}$. Again we use Fubini's theorem, do a change of variables and also use the fact that $e^{x+y} = e^x e^y$:

$$\widehat{f * g}(s) = \int \left(\int f(y)g(x-y)d\mu(y) \right) e^{is\cdot x}d\mu(x)$$

and

$$\int \int f(y)g(x-y)e^{is\cdot x}d\mu(x)d\mu(y) = \int f(y)e^{is\cdot y}d\mu(y) \int g(z)e^{is\cdot z}d\mu(z).$$

(b) Without loss of generality assume that the radius of convergence of F, G is R. To make things similar to the previous notation, let f, g denote the sequences f_n, g_n. We also extend f_n, g_n, so that they are zero when n is negative. We see that (with counting measure dn) $h = f * g$. We can think of $F(z)$ as a transform of the function f in the same way as we think of $\hat{f}(z)$, except that $f \in l^1$ alone is not enough to define the transform, since we can't just bound the z^n term above as we can an e^{inz} term. To this end, we define the measure $\mu_r(A) = \int_A I[n \geq 0]r^n dn$; in this notation, $f, g \in L^1(\mu_r)$ for all $0 < r < R$. Now we will use the important property that $x^{m+n} = x^m x^n$. Observe that by Fubini's theorem and a change of variables:

$$\begin{aligned} \int \left[\int |f(k)||g(n-k)|dk \right] d\mu_r(n) &= \int \int |f(k)||g(n-k)|r^n dn\, dk \\ &= \|f\|_{L^1(\mu_r)} \|g\|_{L^1(\mu_r)}. \end{aligned}$$

So indeed $h \in L^1(\mu_r)$ and H is well defined. And again the following calculation is justified by Fubini's theorem:

$$\int \left[\int f(k)g(n-k)dk \right] z^n dn = \int \int f(k)g(n-k)z^n dn\, dk = F(z)G(z).$$

So indeed $H(z) = F(z)G(z)$ for all $z < R$ and we are done.

Solution to Exercise 1.2 by Terry Soo.

Applying the definition of the Fourier Transform we get that:

$$\hat{D}(k) = \sum_{x \in \mathbb{Z}^d} \frac{1}{|\Omega|} I[x \in \Omega] e^{ik \cdot x} = \frac{1}{|\Omega|} \sum_{x \in \Omega} e^{ik \cdot x}.$$

(a) For the nearest-neighbour model we use the fact that Ω is simply the unit vectors in $\mathbb{Z}^d$. Let e_i be the standard unit basis vectors in $\mathbb{Z}^d$. Then $\Omega = \{e_i\}_{i=1}^d \cup \{-e_i\}_{i=1}^d$ and $|\Omega| = 2d$. Observe that $e^{ik \cdot e_j} + e^{-ik \cdot e_j} = 2\cos(k \cdot e_j) = 2\cos(k_j)$. So rewriting the sum with respect to this grouping of the reflections we get the required form:

$$\hat{D}(k) = \frac{1}{d} \sum_{j=1}^{d} \cos(k_j).$$

For the spread out model, we need to recall from our previous experiences with Fourier analysis that the Dirichlet kernel is defined to be:

$$M_N(t) = \sum_{n=-N}^{N} e^{int}.$$

Just for future reference, we easily see by the orthogonality conditions that:

$$\|M_N\|_2^2 = 2N + 1.$$

By summing a geometric series and with trigonometric identities and some pain we can obtain a useful form:

$$M_N(t) = \frac{\sin\Big(\big(N + \frac{1}{2}\big)\, t\Big)}{\sin\Big(\frac{t}{2}\Big)}, \quad t \neq 0 \bmod 2\pi.$$

We will rewrite the sum to make use of this identity. Recall that $x \in \Omega$ iff $x \neq 0$ and for all $1 \leq i \leq d$, $\|x_i\| \leq L$. To make things more visually appealing we'll take $d = 2$ and $\Omega_0 = \Omega \cup \{0\}$, then:

$$\sum_{x \in \Omega_0} e^{ik \cdot x} = \sum_{m=-L}^{L} \sum_{n=-L}^{L} e^{ik \cdot (m,n)} = \sum_{m=-L}^{L} e^{ik_1 m} \sum_{n=-L}^{L} e^{ik_2 n} = M(k_1) M(k_2).$$

We easily see that this generalizes to any d. To obtain the correct expression for $\hat{D}$ we recall the factor of $\frac{1}{|\Omega|}$ and that we should actually sum over Ω and not Ω_0, hence correct bookkeeping leads to:

$$\hat{D}(k) = \frac{1}{|\Omega|}\left[\left(\prod_{j=1}^{d} M(k_j)\right) - 1\right].$$

(b) For the nearest-neighbour, recall that the elements of Ω are unit vectors, therefore:

$$\sigma^2 = \sum_{x\in\mathbb{Z}^d} |x|^2 D(x) = \frac{1}{|\Omega|}\sum_{x\in\Omega} |x|^2 = 1.$$

In the case of the spread-out model, we recall the Parseval relation equating the L^2 norms:

$$\left\|\hat{f}\right\|_2^2 = \|f\|_2^2.$$

An easy corollary of this, noting that $-i\nabla\hat{f}$ is the Fourier transform of $xf(x)$, is

$$\left\|\nabla\hat{f}\right\|_2^2 = \sum_{x\in\mathbb{Z}^d} |x|^2|f(x)|^2.$$

Hence:

$$\frac{1}{|\Omega|}\sigma^2 = \sum_{x\in\mathbb{Z}^d} |x|^2||D(x)|^2 = \left\|\nabla\hat{D}\right\|_2^2.$$

For simplicity we'll do the case $d = 1$ first. Observe that:

$$\hat{D}'(k) = M'(k) = \frac{1}{|\Omega|}\sum_{j=-L}^{L} ije^{ijk}.$$

The orthogonality conditions give us that:

$$\|\hat{D}'(k)\|_2^2 = \frac{2}{|\Omega|^2}\sum_{j=0}^{L} j^2.$$

We recall that $|\Omega| = 2L$ and that the sum of squares is asymptotic to a multiple of L^3, hence we obtain that σ^2 is asymptotic to L^2, so we're done.

For future use define:

$$S_L := 2\sum_{j=0}^{L} j^2.$$

For the case of d dimensions:

$$\partial_j \hat{D}(k) = \frac{1}{|\Omega|}\left(\prod_{\substack{r=1\\ r\neq j}}^{d} M(k_r)\right) M'(k_j).$$

Now we need to compute $\left\|\partial_j \hat{D}(k)\right\|_2^2$. To do this we need to apply Fubini's theorem, which tells us that:

$$\left\|\partial_j \hat{D}(k)\right\|_2^2 = \frac{1}{|\Omega|^2}\left\|M'(k_j)\right\|_2^2 \prod_{\substack{r=1\\ r\neq j}}^{d} \left\|M(k_r)\right\|_2^2.$$

Now substituting our previous computations involving the Dirichlet Kernel, we get that:

$$\left\|\partial_j \hat{D}(k)\right\|_2^2 = \frac{1}{|\Omega|^2} S_L (2L+1)^{d-1}.$$

Hence we sum over j to obtain that:

$$\left\|\nabla \hat{D}\right\|_2^2 = \frac{d}{|\Omega|^2} S_L (2L+1)^{d-1}.$$

So by counting all the L's we obtain the result $\|\nabla \hat{D}\|_2^2 |\Omega| = \frac{d}{|\Omega|} S_L (2L+1)^{d-1}$. We recall that $|\Omega| = (2L+1)^d - 1$ and S_L is asymptotic to a multiple of L^3, so that indeed everything works out and we obtain that σ^2 is asymptotic to L^2 and we're done.

Solution to Exercise 1.3 by Terry Soo.

Proof of (1.15). Note that: two things should make this a bit easier than a standard proof of the central limit. First, the symmetry of Ω makes the characteristic functions real valued, the mean (already) zero, and it also makes the covariance matrix a constant times the identity. Second, the finiteness of Ω makes it so that $\hat{D}$ is C^∞. The only thing that might be cumbersome is the fact that we work in d dimensions. Let us first be content with a discussion of the one dimension case, then we will try to move on from there. After some algebra we see that what we need to prove is:

$$\left[\hat{D}\left(\frac{k}{\sigma\sqrt{n}}\right)\right]^n \to e^{-\frac{k^2}{2}}.$$

The standard proof requires us to take a Taylor expansion:

$$\hat{D}(t) = 1 - \frac{\sigma^2 t^2}{2} + o\left(t^2\right).$$

(Recall that the mean is zero and that the variance is $\sigma^2 = -\hat{D}''(0)$.) Hence we have that:

$$\left[\hat{D}\left(\frac{k}{\sigma\sqrt{n}}\right)\right]^n = \left(1 - \frac{k^2}{2n} + o\left(n^{-1}\right)\right)^n \to e^{-\frac{k^2}{2}}.$$

In our case, since it is real-valued we could have also taken a log on both sides and shown instead that:

$$n \log\left[\hat{D}\left(\frac{k}{\sigma\sqrt{n}}\right)\right] \to -\frac{k^2}{2}.$$

We can get this using two iterations of l'Hôpital's rule. The generality of the previous argument will prove useful later.

Now we need to do some preparatory work for the d dimensional case. We recall that the covariance matrix Γ of a random vector X with zero mean is given by:

$$\Gamma_{ij} = \mathbb{E}X^{(i)}X^{(j)}.$$

For our case, let X be a random variable with probability distribution D. Then note that $\Gamma = \frac{\sigma^2}{d}I$, where I is the identity matrix in d dimensions. To see this note that by symmetry, $\mathbb{E}X^{(i)}X^{(i)} = \mathbb{E}X^{(j)}X^{(j)}$ and hence $\mathbb{E}|X|^2 = \sigma^2 = d\,\mathbb{E}X^{(i)}X^{(i)}$. Also by symmetry we must have that for $i \neq j$, $\mathbb{E}X^{(i)}X^{(j)} = 0$. Now to bootstrap from the one dimensional case observe that $k \cdot X = Y_k$ is a random variable in one dimension with mean zero and variance exactly $\sum_i \sum_j k_i k_j \Gamma_{ij} = |k|^2 \frac{\sigma^2}{d}$. But also observe that $\mathbb{E}(e^{iY_k}) = \hat{Y_k}(1) = \hat{X}(k) = \hat{D}(k)$. So that similarly we can see that:

$$\left[\hat{D}\left(\frac{k}{\sigma\sqrt{n}}\right)\right]^n = \left[\hat{Y_k}\left(\frac{1}{\sigma\sqrt{n}}\right)\right]^n.$$

After rewriting and some algebra so that we can use the one dimensional central limit we have that:

$$\left[\hat{Y_k}\left(\frac{1}{\sigma\sqrt{n}}\right)\right]^n = \left[\hat{Y_k}\left(\frac{\frac{|k|}{\sqrt{d}}}{|k|\frac{\sigma}{\sqrt{d}}\sqrt{n}}\right)\right]^n \to e^{-\frac{|k|^2}{2d}}.$$

Note that we could have done this directly by using a Taylor expansion in d dimensions:

$$\hat{D}(k) = 1 - \frac{1}{2}(k \cdot \Gamma k) + o\Big(|k|^2\Big) = 1 - \frac{\sigma^2 |k|^2}{2d} + o\Big(|k^2|\Big),$$

so the proof is exactly as in the one dimensional case. We'll find this Taylor expansion useful in the future.

Proof of (1.16). This is a matter of the product rule. Recall that in d dimensions:

$$\nabla^2(fg) = g\nabla^2 f + 2\nabla f \cdot \nabla g + f\nabla^2 g.$$

We recognize that

$$\sum_x |x_j|^2 c_n^{(0)}(x) = -\partial_j^2 \sum_x c_n^{(0)}(x) e^{ik\cdot x}\Big|_{k=0} = -\partial_j^2 \left[\hat{c}_1^{(0)}(k)\right]^n\Big|_{k=0}.$$

Also, since $\sum_x c_n^{(0)}(x) = |\Omega|^n$:

$$\frac{\sum_x |x_j|^2 c_n^{(0)}(x)}{\sum_x c_n^{(0)}(x)} = -\partial_j^2 \hat{D}^n\Big|_{k=0}.$$

Hence:

$$\frac{\sum_x |x|^2 c_n^{(0)}(x)}{\sum_x c_n^{(0)}(x)} = -\nabla^2 \hat{D}^n|_{k=0}.$$

To compute the second derivative we use the product rule and note that by symmetry $\nabla\hat{D}(0) = i\mu = i\mathbb{E}X = 0$ and recall that we already know that $\sigma^2 = -\nabla^2\hat{D}(0)$. Hence we have by the product rule that:

$$-\nabla^2 \hat{D}^n|_{k=0} = -\nabla^2[\hat{D}^{n-1}\hat{D}]|_{k=0} = \sigma^2 - \nabla^2 \hat{D}^{n-1}|_{k=0}.$$

So clearly we may deduce that:

$$-\nabla^2 \hat{D}^n|_{k=0} = n\sigma^2$$

as desired.

Solution to Exercise 1.4 by Terry Soo.

Some notation for this question: write $\int_{[-\pi,\pi]^d} f(k)\frac{d^dk}{(2\pi)^d} = \int_{\mathbb{T}^d} f(k)\mu(dk)$.

We have a slightly easier job than usual since again our characteristic functions are real valued. The first thing we should observe is that by monotone convergence we have that:

$$\lim_{z\uparrow z_c} C_z(x) = \lim_{z\uparrow z_c}\int_{\mathbb{T}^d}\frac{e^{-ik\cdot x}}{1-z|\Omega|\hat{D}(k)}\mu(dk) = \lim_{z\uparrow z_c}\sum c_n^{(0)}(x)z^n = \sum c_n^{(0)}(x)z_c^n.$$

So what we need to show is that we can indeed bring the limit inside. Usually this requires some sort of dominated convergence theorem. Note that $\left|\hat{D}(k)\right| \leq 1$ and $|e^{is}| \leq 1$. So, all we really need to show is that:

$$\lim_{t\uparrow 1}\int_{\mathbb{T}^d}\frac{1}{1-t\hat{D}(k)}\mu(dk).$$

First, observe that for $t \in \left[\frac{1}{2}, 1\right]$:

$$\frac{1}{1-t\hat{D}(k)} = \frac{1}{t\left(\frac{1}{t}-\hat{D}(k)\right)} \leq \frac{t^{-1}}{1-\hat{D}(k)} \leq \frac{2}{1-\hat{D}(k)}.$$

So all we need to do is show that:

$$\frac{1}{1-\hat{D}(k)} \in L^1\left(\mathbb{T}^d\right).$$

An area of concern is at (a neighborhood of) $k=0$. It is true that if the walk is aperiodic and irreducible then $\hat{D}(k) = 1$ iff $k = 0$, so that this is the only area of concern. To see why this is true recall that for aperiodic and irreducible walks we have that for all x there exists an n_x sufficiently large so that $D_n(x) = \frac{c_n^{(0)}(x)}{|\Omega|^n} > 0$, for all $n \geq n_x$. If $k \neq 0$, clearly there must exist an y so that $y \cdot k \not\equiv 2\pi \bmod 0$. But also if $\hat{D}(k) = 1$, then $\left[\hat{D}(k)\right]^{n_y} = 1$, which would require that $x \cdot k \equiv 2\pi \bmod 0$ for all x where $D_n(x) > 0$. However the point y is a counterexample, hence we must have that $\hat{D}(k) = 1$ iff $k = 0$.

(Note that of course the simple random walk is not aperiodic, but it is easy to see from Exercise 1.2 that $\hat{D}(k) = 1$ iff $k = 0$.)

For aperiodic and irreducible distributions it is possible to obtain bounds of the form:

$$1 - \hat{D}(k) \geq \lambda|k|^2.$$

A bound of this sort is obtained directly for the case of the simple symmetric random walk in Exercise 1.6 part (c). Once we have something like this, we just need to establish that $\frac{1}{k^2}$ is integrable in $\mathbb{T}^d$ to bring the limit inside. Observe that this is exactly true when $d > 2$. For our purpose the job is easy since we have as many derivatives as we want for $\hat{D}$. As in the proof of the central limit theorem,

$$\hat{D}(k) = 1 - \frac{\sigma^2|k|^2}{2} + o\Big(|k|^2\Big)\,,$$

so we have that:

$$1 - \hat{D}(k) = \frac{\sigma^2|k|^2}{2} + o\Big(|k|^2\Big) = |k|^2\left[\frac{\sigma^2}{2} + o(1)\right].$$

Hence we break up $1 - \hat{D}(k)$ into two parts: one in a sufficiently small neighborhood about $k = 0$ and the other outside of the critical neighborhood. We get the standard bound and moreover we find that the required integrand is indeed integrable if and only if $d > 2$. So that we don't leave any loose ends also observe that by Fatou we always have that:

$$\lim_{t\uparrow 1} \int_{\mathbb{T}^d} \frac{1}{1 - t\hat{D}(k)}\mu(dk) \geq \int_{\mathbb{T}^d} \frac{1}{1 - \hat{D}(k)}\mu(dk).$$

So clearly $C_{z_c}(0) = \infty$ for $d \leq 2$. Note that in fact it must blow up for all x. We can see this by taking the probabilistic interpretation via Exercise 1.6. We require that the random walk be irreducible, then by the SMP we must have that if 0 is visited infinitely, so must any other point by translation invariance. Analytically we note that since C_z is real-valued for z real and since our characteristic functions are real valued we have that:

$$\begin{aligned} C_{z_c}(x) &= \lim_{t\uparrow 1} \int_{\mathbb{T}^d} \frac{e^{-ik\cdot x}}{1 - t\hat{D}(k)}\mu(dk) = \mathrm{Re}\lim_{t\uparrow 1} \int_{\mathbb{T}^d} \frac{e^{-ik\cdot x}}{1 - t\hat{D}(k)}\mu(dk) \\ &= \lim_{t\uparrow 1} \int_{\mathbb{T}^d} \frac{\cos(k\cdot x)}{1 - t\hat{D}(k)}\mu(dk). \end{aligned}$$

Now considering a sufficiently small neighbourhood about $k = 0$ we have that $\cos(k\cdot x) \geq \frac{1}{2}$ so hence again by Fatou we get the required result.

Solution to Exercise 1.5 by Terry Soo.

This is a calculation from the definition that requires us to use the symmetric properties of Ω.

$$\begin{aligned}\triangle f(x) &= \frac{1}{2}\frac{1}{|\Omega|}\sum_{y\in\Omega}\partial_y^-\partial_y^+ f(x) = \frac{1}{2}\frac{1}{|\Omega|}\sum_{y\in\Omega}\partial_y^-(f(x+y)-f(x)) \\ &= \frac{1}{2}\frac{1}{|\Omega|}\sum_{y\in\Omega}[(f(x+y)-f(x))+(f(x-y)-f(x))] \\ &= \frac{1}{|\Omega|}\sum_{y\in\Omega} f(x+y)-f(x).\end{aligned}$$

We used symmetry to get the last equality. Now we have to verify that indeed $-\triangle C_{z_c} = \delta_{0,x}$. In accordance with the previous exercise we should put ourselves in the case where $d > 2$.

$$A_{C_{z_c}}(x) := \frac{1}{|\Omega|}\sum_{y\in\Omega} C_{z_c}(x+y) = \frac{1}{|\Omega|}\sum_{y\in\Omega}\sum_n c_n^{(0)}(x+y)z_c^n.$$

Recall that $z_c = \frac{1}{|\Omega|}$. We interchange summations and use the symmetry of Ω to obtain that:

$$\frac{1}{|\Omega|}\sum_n z_c^n \sum_{y\in\Omega} c_n^{(0)}(x+y) = \sum_n z_c^{n+1}c_{n+1}^{(0)}(x) = \sum_{n=1} z_c^n c_n^{(0)}(x).$$

Therefore:

$$A_{C_{z_c}}(x) = C_{z_c}(x) - c_0^{(0)}(0,x) = C_{z_c}(x) - \delta_{0,x}.$$

So indeed we have that $-\triangle C_{z_c} = \delta_{0,x}$.

Solution to Exercise 1.6 by Terry Soo.

(a) This is an exercise in the strong Markov property. We define τ_n to be the time it takes to return to 0 for the n^{th} time. Recall by the SMP that $\mathbb{P}(\tau_n < \infty) = \mathbb{P}(\tau_1 < \infty)^n$. To use this we observe that:

$$N = \sum_{n=0}^{\infty} I[\omega(n) = 0] = \sum_{m=0}^{\infty} I[\tau_m < \infty].$$

Note that in this notation $\mathbb{P}(\tau_1 < \infty) = u$. Hence we have that:

$$\mathbb{E}N = \sum_n u^n = \frac{1}{1-u}$$

which is what we need to show the sufficience and necessity of the condition $m = \infty$ for the walk to be recurrent.

(b) We recognize that $\mathbb{P}(\omega(n) = 0) = \frac{c_n^{(0)}(0)}{|\Omega|^n}$, therefore we have that $\mathbb{E}N = C_{z_c}(0)$, hence by the Fourier inversion formula and our justification of it at $z = z_c$ we're done.

(c) This is an application of Exercise 1.3. However, in the case of the nearest neighbour model (that is, simple symmetric random walk), we can obtain the required bounds on $1 - \hat{D}(k)$ more directly, though one may argue that the following is based on Taylor series. Observe that $1 - \cos(x) \geq \frac{x^2}{\pi^2}$ on $\mathbb{T}$, so we have that:

$$1 - \hat{D}(k) = \frac{1}{d}\sum_{j=1}^{d}(1 - \cos(k_j)) \geq \frac{1}{d\pi^2}\sum_{j=1}^{d} k_j^2 = \frac{1}{d\pi^2}|k|^2.$$

Hence we easily obtain that for $d > 2$ we have transience. As for $d \leq 2$, similarly observe that $1 - \cos(x) \leq Kx^2$ for some large $K > 0$. So, we obtain that the integral is infinite when $d \leq 2$ and the walk is recurrent.

Solution to Exercise 1.7 by Terry Soo.

We rewrite:

$$\begin{aligned} X &= \sum_m \sum_n I\Big[\omega^{(1)}(m)\omega^{(2)}(n)\Big] \\ &= \sum_m \sum_n \sum_a I\Big[\Big(\omega^{(1)}(n) = a\Big) \cap \Big(\omega^{(2)}(m) = a\Big)\Big]. \end{aligned}$$

Hence using independence:

$$\mathbb{E}X = \sum_n \sum_m \sum_a \mathbb{P}\Big(\omega^{(1)}(n) = a\Big)\,\mathbb{P}\Big(\omega^{(2)}(m) = a\Big).$$

Changing the order of summation and recognizing that

$$\sum_m \mathbb{P}\Big(\omega^{(2)}(m) = a\Big) = C_{z_c}(a)$$

we get

$$\mathbb{E}X = \sum_a (C_{z_c}(a))^2 = \|C_{z_c}\|_2^2 = \left\|\hat{C_{z_c}}\right\|_2^2.$$

So we've obtained the proper expression. We know from Exercise 1.3 that basically $\frac{1}{1-\hat{D}(k)}$ behaves like $\frac{1}{|k|^2}$ so that $\left[\frac{1}{\hat{D}(k)}\right]^2$ behaves like $\frac{1}{|k|^4}$, which is integrable iff $d > 4$.

Solution to Exercise 2.1 by Terry Soo.

Observe that from the definition it suffices to find/draw a m step self avoiding walk that has no continuation; that is, it gets trapped by its own tail. Once we draw such a m step random walk ω, with the uniform distribution we must have that $\mathbb{P}_m(\omega) > 0$, but since this walk has no continuation, we have that for any $n > m$ and for all $\rho > \omega$ (that is, ρ whose first m steps agree with ω), $\mathbb{P}_n(\rho) = 0$.

Solution to Exercise 2.2 by Terry Soo.

Observe that in 1 dimension there is only one self avoiding walk to any point on $\mathbb{Z}$. Hence we have that $G_z(x) = z^{|x|}$. Taking the Fourier transform we find that:

$$\hat{G}_z(k) = \sum_{x\in\mathbb{Z}} z^{|x|} e^{ikx}.$$

This is exactly the Poisson kernel of Fourier analysis and PDE's and we recognize it as being exactly the required expression. We can compute this by rewriting it as the sum of two geometric series starting at zero and subtracting 1. We then easily derive the identity by taking a common denominator and using Euler's trigonometric identity.

Solution to Exercise 2.4 by Terry Soo.

Define $T(z)$ via:

$$T(z) := \sum_{x,y} \sum_{\omega^{(1)}\in\mathcal{S}(0,x)} \sum_{\omega^{(2)}\in\mathcal{S}(x,y)} z^{|\omega^{(1)}|+|\omega^{(2)}|} I\Big[\omega^{(1)} \cap \omega^{(2)} \neq \{x\}\Big].$$

We need to show that:

$$T(z) \leq Q(z)[B(z) - 1].$$

We follow the suggested relaxations on T to get this. First some notation: we let the last site of intersection $w = \omega^{(2)}(l) = \omega_l$ be represented by a. The first thing to do is to rewrite T in terms of this last point. Note that $a \neq x$ and observe that:

$$T(z) = \sum_{x,y} \sum_{\omega^{(1)}\in\mathcal{S}(0,x)} \sum_{\omega^{(2)}\in\mathcal{S}(x,y)} \sum_{a\neq x} z^{|\omega^{(1)}|+|\omega^{(2)}|} I[\omega_l = a].$$

Now we split up the walks using the point a. The notation will be such that $\omega^{(1)}$ will be split into $\omega^{(10)}$ and $\omega^{(11)}$:

$$\begin{aligned} T(z) &= \sum_{\substack{x,y \\ a\neq x}} \sum_{\substack{\omega^{(10)}\in\mathcal{S}(0,a) \\ \omega^{(11)}\in\mathcal{S}(a,x)}} \sum_{\substack{\omega^{(20)}\in\mathcal{S}(x,a) \\ \omega^{(21)}\in\mathcal{S}(a,y)}} z^{(|\omega^{(10)}|+|\omega^{(11)}|+|\omega^{(20)}|+|\omega^{(21)}|)} \\ &\quad \times I\Big[\omega^{(10)}\cap\omega^{(11)}=\{a\}\Big]\, I\Big[\omega^{(20)}\cap\omega^{(21)}=\{a\}\Big] \\ &\quad \times I\Big[\omega^{(21)}\cap\omega^{(10)}=\{a\}\Big]\, I\Big[\omega^{(21)}\cap\omega^{(11)}=\{a\}\Big]. \end{aligned}$$

We have the usual restrictions to make sure that when the pieces are put together the whole is a random walk. The last two restrictions on $\omega^{(21)}$ are there to ensure that the last piece never intersects with the $\omega^{(1)}$ walk.

Now we begin to apply the relaxations. Relaxing the mutual avoidance of the two parts means that we can replace the first two indicators with 1. That the last part need not avoid the part of $\omega^{(1)}$ linking a to x means that we can replace the last indicator with 1. Finally this leaves us with one indicator and gives us that:

$$T(z) \leq \sum_{\substack{x,y \\ a\neq x}} \sum_{\substack{\omega^{(10)}\in\mathcal{S}(0,a) \\ \omega^{(21)}\in\mathcal{S}(a,y)}} \sum_{\substack{\omega^{(11)}\in\mathcal{S}(a,x) \\ \omega^{(20)}\in\mathcal{S}(x,a)}} z^{|\omega^{(10)}|+|\omega^{(21)}|} z^{|\omega^{(11)}|} z^{|\omega^{(20)}|} I\Big[\omega^{(21)}\cap\omega^{(10)}=\{a\}\Big].$$

In different notation this is just:

$$T(z) \leq \sum_{\substack{x\neq a \\ y,a}} \sum_{\substack{\omega^{(10)}\in\mathcal{S}(0,a) \\ \omega^{(21)}\in\mathcal{S}(a,y)}} G_z(a-x)G_z(x-a) I\Big[\omega^{(21)}\cap\omega^{(10)}=\{a\}\Big].$$

Note that actually, $G(a-x)=G(x-a)$ by symmetry and also $\sum_x G(x)^2 = \sum_x G(x-a)^2$, hence we should do the sum over x. This gives us that:

$$T(z) \leq [B(z)-1] \sum_{y,a} \sum_{\substack{\omega^{(10)}\in\mathcal{S}(0,a) \\ \omega^{(21)}\in\mathcal{S}(a,y)}} I\Big[\omega^{(21)}\cap\omega^{(10)}=\{a\}\Big].$$

Recalling the expression that was earlier derived for Q we obtain that:

$$T(z) \leq [B(z)-1]Q(z)$$

as desired.

Solution to Exercise 3.1 by Richard Liang.

We iterate the procedure outlined in equations (3.3) and (3.4). When we repeat the inclusion-exclusion procedure on the second term of (3.4), we are counting the number of diagrams that consist of a self-avoiding loop of length s followed by a self-avoiding walk from 0 to x, where this loop and walk are not mutually avoiding.

To count this using inclusion-exclusion, we let t be the first time $\omega^{(2)}(t) = v \in \omega^{(3)}$ and correct for its removal, leading to the expression given for $\pi_m^{(2)}(v)$, and pictured diagrammatically in the notes.

The next iteration of inclusion-exclusion is considering counting walks like those counted by $\pi_m^{(2)}(v)$ (call this walk $\omega^{(4)}$) with a self-avoiding tail (call this $\omega^{(5)}$) coming out of the point v that also avoids the last "loop" of $\omega^{(4)}$. To count these, again we let t be the time of the first contact between $\omega^{(5)}$ and the last part of $\omega^{(4)}$. (The reason we are only avoiding this part is because in the previous step we considered walks which consist of a loop from 0 to 0 and then added on a self-avoiding walk, considering only the *first* time it touches the loop. So after this the remainder of the walk does not have to remember the loop part of the walk. In the next iteration, we only have to avoid the last part, for the same reason.) So, summing over the possible locations for $\omega^{(5)}(t)$ and encoding all of the self and mutual avoidances, we arrive at:

$$\begin{aligned}\pi_m^{(3)}(v) \;=\; & \sum_{(m_1,\ldots,m_5)\in M(m)} \sum_{x\in\mathbb{Z}^d} \sum_{\left(\omega^{(1)},\ldots,\omega^{(5)}\right)\in W((m_1,\ldots,m_5),x,v)} \\ & I\Big(\omega^{(1)},\omega^{(2)},\omega^{(3)},\omega^{(4)}\Big)\, I\Big(\omega^{(3)},\omega^{(4)},\omega^{(5)}\Big),\end{aligned}$$

where

$$\begin{aligned}M(m) \;=\; & \{(m_1,\ldots,m_5) : m_1+\cdots+m_5 = m; \\ & m_1, m_2, m_4, m_5 > 0; m_3 \geq 0\}\end{aligned}$$

where

$$\begin{aligned}W((m_1,\ldots,m_5),x,v) \;=\; & \Big\{\Big(\omega^{(1)},\omega^{(2)},\omega^{(3)},\omega^{(4)},\omega^{(5)}\Big) : \\ & \omega^{(1)} \in \mathcal{S}_{m_1(0,x)}, \omega^{(2)} \in \mathcal{S}_{m_2(0,x)}, \\ & \omega^{(3)} \in \mathcal{S}_{m_3(0,v)}, \omega^{(4)} \in \mathcal{S}_{m_3(v,x)}, \\ & \omega^{(5)} \in \mathcal{S}_{m_3(x,v)}\Big\}\end{aligned}$$

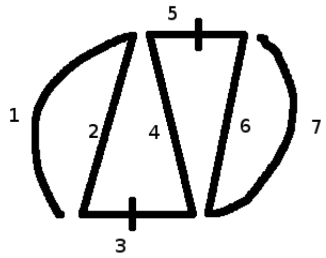


Figure 1: The diagram for $\pi_m^{(4)}(v)$, with mutual avoidances [1234][3456][567], where the square brackets indicate that all of the subwalks inside them are mutually avoiding save for their mutual endpoints. As in the notes, the slashed lines can have length 0.

and where the function I is 1 if all of its parameters $\omega^{(i)}$ are pairwise mutually avoiding apart from their endpoints, and 0 otherwise.

The picture for $\pi_m^{(4)}(v)$ is shown in Figure 1.

Solution to Exercise 3.3 by Richard Liang.

Expanding the product in (3.7) and collecting up all of the terms that are products of j U_{st} terms, we get

$$K[a,b] = 1 + \sum_{j=1}^{\infty} \sum_{s_1t_1,\ldots,s_jt_j \text{ distinct}} \mathcal{U}_{s_1t_1} \cdots \mathcal{U}_{s_jt_j}.$$

Notice that we can represent the product as being over all the edges in a graph of j edges and the second sum as summing over all possible such graphs. So we rewrite:

$$K[a,b] = 1 + \sum_{j=1}^{\infty} \sum_{\Gamma \in \mathcal{B}^{(j)}[a,b]} \prod_{st \in \Gamma} \mathcal{U}_{st}.$$

The leading term 1 can also be absorbed into this sum by noting that there is exactly one graph Γ with no edges, and the product of $\mathcal{U}_{st}$ over all of its edges is empty, yielding a factor of 1:

$$K[a,b] = \sum_{j=0}^{\infty} \sum_{\Gamma \in \mathcal{B}^{(j)}[a,b]} \prod_{st \in \Gamma} \mathcal{U}_{st}.$$

Finally, collecting all of the graphs together into one sum, we get

$$K[a,b] = \sum_{\Gamma \in \mathcal{B}[a,b]} \prod_{st \in \Gamma} \mathcal{U}_{st}$$

as desired.

Solution to Exercise 3.6 by Richard Liang.

We need to show

$$L \in \mathcal{L}^{(N)}[a,b] \Leftrightarrow (3.15).$$

First, we prove the forward direction. Let $L \in \mathcal{L}^{(N)}[a,b]$ ($N \geq 2$) and write $L = \{s_1t_1, \ldots, s_Nt_N\}$ where $s_l < t_l \, \forall \, l$. We call an edge $st \in \Gamma$ *removable* if removing it does not disconnect Γ. Implicitly it is also assumed that $s_i < s_{i+1} \, \forall \, i = 1, \ldots, N-1$; that is, that the edges are ordered by their start point, and there can be no ambiguity as if two edges have the same start point, the shorter of the two is removable, which contradicts the minimality of the lace. Given this, we can then see that $t_i < t_{i+1} \, \forall \, i = 1, \ldots, N-1$, for if $t_i \geq t_{i+1}$ for some i, then the $(i+1)^{\text{th}}$ edge is completely subsumed by the i^{th}, and is thus removable.

Clearly, then, $a = s_1$ or else the graph cannot be connected as it will not have an edge with a as an endpoint. Also by the above, $s_1 < s_2$, so we have shown the first statement in (3.15). Similarly, $t_N = b$ and $t_{N-1} < t_N$, so we have shown the last part of (3.15).

It remains to show that $s_{j+1} < t_j$ for all $j = 1, \ldots, N-1$ and that $t_j \leq s_{j+2}$ for all $j = 1, \ldots, N-2$. To show the first of these statements, suppose that there is some l such that $s_{l+1} \geq t_l$. If this were the case and the graph were connected, since the edges are sorted by ascending order of their start points there would have to be some other $k < l$ such that $t_k > s_{l+1}$. But then

$$t_l \leq s_{l+1} < t_k,$$

which contradicts $t_k < t_l$, proving the first statement.

The second statement is again shown by contradiction. Assume that there is some other $l \in \{1, \ldots, N-2\}$ such that $t_l > s_{l+2}$. But if this is the case, then the $(l+1)^{\text{th}}$ edge is removable, since $s_l < s_{l+1}$. This contradicts minimality of L, proving the second statement. Thus we have proved the forward direction.

To prove the reverse direction, assume that $L = \{s_1t_1, \ldots, s_Nt_N\}$ satisfies (3.15), and as above we can assume the implicit ordering of the edges by their start points, so $a = s_1$ and $t_N = b$.

L is connected: to see this, suppose that $y \in (a, b)$ is not contained in any interval (s, t) where $st \in L$. Pick l maximal so that $s_l < y \leq s_{l+1}$. Since y is not covered by any edge, $t_l \leq y$ also. But then this contradicts $s_{j+1} < t_j \;\forall\, j = 1, \ldots, N-1$.

L is minimal: removing s_1t_1 will make it so that no edge has a as an endpoint, and removing s_Nt_N will do the same for b, so neither edge is removable. Recall that an equivalent definition of a graph Γ in $\mathcal{G}[a, b]$ being connected is that $\cup_{st\in\Gamma}(s, t) = (a, b)$. Suppose some other edge s_lt_l is removable (for some l in $\{2, \ldots, N-1\}$); then since we have ordered the edges by their start points and by (3.15) the end points are also strictly increasing, this means that for every y covered by s_lt_l (i.e. for every $y \in (s_l, t_l)$), $s_{l+1} < y < t_{l-1}$, which contradicts (3.15).

Thus we have proved $L \in \mathcal{L}^{(N)}[a, b]$.

Solution to Exercise 3.7 by Richard Liang.

First we show the forward direction. Suppose that L is built out of Γ by the prescribed algorithm. Clearly $L \subset \Gamma$ because the only edges that are included in L are edges that originally occurred in Γ.

L is a lace: if L has one edge, this is easy because by the algorithm, since Γ was connected, this edge has to be ab. If L has $N > 1$ edges, we apply the previous exercise. $a = s_1$ by definition. $s_1 < s_2$ because s_1t_1 is by definition the longest edge coming out of a, so if

$$s_2 = \min\{s : st_2 \in \Gamma\} = s_1 = a$$

then because

$$t_2 = \max\{t : \exists\, s < t_1 \text{ such that } st \in \Gamma\}$$

then $t_2 = t_1$, which means $s_1t_1 = s_2t_2$. Since Γ is connected, this means that t_2 must be the end of the region we build the graph on, i.e. that $t_2 = b$ and therefore $N = 1$. This is a contradiction. This same argument ensures that we can order the edges by their start points, and no two edges should share the same start point.

For $j = 1, \ldots, N-1$, $s_{j+1} < t_j$ by construction since by definition of t_{j+1} we know that s_{j+1} is the smallest number with $st_{j+1} \in \Gamma$, and there is some such $s < t_j$.

For $j = 1, \dots, N-2$, we must now show $t_j \le s_{j+2}$. By construction, $t_{j+2} > t_{j+1}$ for such j, since otherwise our algorithm would have terminated sooner. So if $t_l > s_{l+2}$ for some l in this range, then $t_{l+1} \ge t_{l+2}$ since $s_{l+2} < t_l$ and we would have chosen something at least as large as t_{l+2} for t_{l+1}, and again we arrive at a contradiction.

Finally, the last edge must satisfy $t_N = b$, and as above we have to have $t_{N-1} < t_N$ or else the algorithm would have terminated sooner. So (3.15) holds and therefore L is a lace.

It remains to show that $\Gamma \setminus L \subset \mathcal{C}(L)$. This comes by noting that the algorithm, at each step, picks from the eligible edges (those that start from before the furthest point the algorithm has reached with the edges already chosen) precisely the edge that reaches the furthest from the point to which it has so far reached while still remaining connected, breaking ties by picking the longest of these edges. Thus $\Gamma \setminus L$ consists of exactly the edges of Γ that, when eligible to be chosen in our algorithm, either do not reach as far as some edge in L or are not as long, and so applying the algorithm to $L \cup \{st\}$ for $st \in \Gamma \setminus L$ will omit this edge again, leaving us only with L. This completes the forward direction.

The reverse direction is easier. Suppose $L \subset \Gamma$ is a lace with $\Gamma \setminus L \subset \mathcal{C}(L)$. Writing $\mathsf{L}_{\Gamma'}$ for the lace generated by our algorithm from a connected graph Γ', we write $\mathsf{L}_{\Gamma} = \mathsf{L}_{L \cup (\Gamma \setminus L)}$, and we need to show that $\mathsf{L}_{L \cup (\Gamma \setminus L)} = \mathsf{L}_L$. As explained above in the proof of the forward direction, our algorithm chooses from the eligible edges, at each step, the one that reaches the furthest while still remaining connected, breaking ties by choosing the longest of these edges. By definition, every edge in $\mathcal{C}(L)$ will not change the output of the algorithm; therefore every edge in $\mathcal{C}(L)$, when it is eligible to be chosen, either does not reach as far as some edge in L or is shorter, so it will be passed over in the algorithm. Since $\Gamma \setminus L \subset \mathcal{C}(L)$, the result of applying the algorithm to $L \cup (\Gamma \setminus L)$ will just be L again. This completes the reverse direction.

Solution to Exercise 3.8 by Richard Liang and Amanda Turner.

(a) The first step is chosen from $|\Omega|$ possibilities, and since the walk only remembers its previous step, every subsequent step is chosen from $|\Omega| - 1$ possibilities. Therefore

$$\hat{c}_n(0) = |\Omega|(|\Omega| - 1)^{n-1}$$

for $n \ge 1$, and $\hat{c}_0(0) = 1$ obviously.

(b) We'll use formulas (3.23) and (3.25) to calculate first $\pi_m^{(N)}(x)$ which then gives $\pi_m(x)$. We start by considering $J^{(N)}[0,m]$: we rewrite formula (3.21)

$$J^{(N)}[0,m] = \sum_{L\in\mathcal{L}^{(N)}[0,m]} \prod_{st\in L} \mathcal{U}_{st} \prod_{s't'\in\mathcal{C}(L)} (1+\mathcal{U}_{s't'}).$$

Note that $\mathcal{U}_{st} = 0$ if $|s-t| \neq 2$, so we can immediately forget about any laces with edges of any other length. But then the only lace remaining is $\{02, 13, 24, ..., (m-3)(m-1), (m-2)m\}$, which we denote by L_m. So, by equation (3.22), we see that

$$J[0,m] = J^{(m-1)}[0,m] = \prod_{i=0}^{m-2} \mathcal{U}_{i(i+2)} \prod_{s't'\in\mathcal{C}(L_m)} (1+\mathcal{U}_{s't'}).$$

The edges that are compatible with L_m are exactly the edges of length 1, and for these edges $\mathcal{U}$ is equal to 0 by definition, so the rightmost product just gives a factor of 1. So:

$$J[0,m] \;=\; (-1)^{m-1}\, I((\omega(i) = \omega(i+2),\ i = 0,\dots,m-2)).$$

Using (3.22), (3.23), and (3.25) now, we see that

$$\begin{aligned}
\pi_m(x) &= \sum_{N=1}^{\infty} \sum_{\omega\in\mathcal{W}_m(0,x)} J^{(N)}[0,m] \\
&= \sum_{\omega\in\mathcal{W}_m(0,x)} J[0,m] \\
&= \sum_{\omega\in\mathcal{W}_m(0,x)} (-1)^{m-1}\, I((\omega(i) = \omega(i+2),\ i = 0,\dots,m-2))
\end{aligned}$$

and this is exactly equal to $-|\Omega|\delta_{x,0}$ when m is even and equal to $I((x\in\Omega))$ when m is odd.

(c) Denote the mean-square displacement by ρ_n, and note that

$$\rho_n = -\frac{\nabla^2 \hat{c}_n(0)}{\hat{c}_n(0)}.$$

By applying the Laplacian to the lace expansion's Fourier transform, we see

$$\begin{aligned}\nabla^2\hat{c}_n(0) &= |\Omega|\nabla^2\hat{D}(0)\hat{c}_{n-1}(0) + 2|\Omega|\nabla\hat{D}(0)\cdot\nabla\hat{c}_{n-1}(0)\\ &\quad + |\Omega|\hat{D}(0)\nabla^2\hat{c}_{n-1}(0) + \sum_{m=2}^{n}\Big(\nabla^2\hat{\pi}_m(0)\hat{c}_{n-m}(0)\\ &\quad + 2\,\nabla\hat{\pi}_m(0)\cdot\nabla\hat{c}_{n-m}(0) + \hat{\pi}_m(0)\nabla^2\hat{c}_{n-m}(0)\Big)\,.\end{aligned}$$

Note that the sum starts at $m = 2$, because $\pi_1^{(N)}(x) = 0$ for all N as the only lace on $[0, 1]$ is the single-edge graph 01 and by definition $\mathcal{U}_{01} = 0$, and therefore $\pi_1(x) = 0$. It should also be noted that $\pi_0^{(N)}(x) = 0$ for all N because it is defined as a sum over all laces over the interval $[0, 0]$ (see equation (3.23)); since there are no such laces the sum is empty. Hence $\pi_0(x) = 0$.

The cross terms cancel because the gradient of $\hat{D}$ is 0 at $k = 0$, as is that of $\hat{\pi}_m$ for all m by part (b). We substitute $\hat{D}(0) = 1$, $\sigma^2 = -\nabla^2\hat{D}(0)$, $\hat{c}_0(0) = 1$ and $|\Omega|(|\Omega| - 1)^{n-1} = \hat{c}_n(0)$ and separate off the last term in the sum; simplifying we get

$$\begin{aligned}\nabla^2\hat{c}_n(0) &= -|\Omega|\sigma^2|\Omega|(|\Omega|-1)^{n-2} + |\Omega|\nabla^2\hat{c}_{n-1}(0)\\ &\quad + \sum_{m=2}^{n-1}\Big(\nabla^2\hat{\pi}_m(0)|\Omega|(|\Omega|-1)^{n-m-1} + \hat{\pi}_m(0)\nabla^2\hat{c}_{n-m}(0)\Big)\\ &\quad + \nabla^2\hat{\pi}_n(0). \end{aligned}\tag{1}$$

Now,

$$\begin{aligned}\hat{\pi}_m(k) = \sum_{x\in\mathbb{Z}^d}\pi_m(x)e^{ik\cdot x} &= \begin{cases}\sum_{x\in\mathbb{Z}^d}(-|\Omega|\delta_{x,0})e^{ik\cdot x} & m \text{ even}\\ \sum_{x\in\mathbb{Z}^d} I(x\in\Omega)e^{ik\cdot x} & m \text{ odd}\end{cases}\\ &= \begin{cases}-|\Omega| & m \text{ even}\\ |\Omega|\hat{D}(k) & m \text{ odd,}\end{cases}\end{aligned}\tag{2}$$

which gives

$$\begin{aligned}\hat{\pi}_m(0) &= (-1)^{m-1}|\Omega|,\\ \nabla^2\hat{\pi}_m(0) &= \begin{cases}0 & m \text{ even}\\ -|\Omega|\sigma^2 & m \text{ odd.}\end{cases}\end{aligned}\tag{3}$$

So, substituting (3) into (1), we get

$$\begin{aligned}
\nabla^2 \hat{c}_n(0) &= -|\Omega|^2(|\Omega|-1)^{n-2}\sigma^2 + |\Omega|\nabla^2\hat{c}_{n-1}(0) \\
&\quad + \sum_{m=2}^{n-1}\Big(-|\Omega|^2(|\Omega-1|)^{n-m-1}\sigma^2 I(m \text{ odd}) \\
&\quad + (-1)^{m-1}|\Omega|\nabla^2\hat{c}_{n-m}(0)\Big) - |\Omega|\sigma^2 I(n \text{ odd}) \\
&= \frac{|\Omega|^2(|\Omega|-1)^{n-2}\sigma^2}{1-\delta^2}(\delta^2-1) \\
&\quad - \Big(|\Omega|^2(|\Omega|-1)^{n-2}\sigma^2\Big)\sum_{m=2}^{n-1}(|\Omega|-1)^{-m+1} I(m \text{ odd}) \\
&\quad + \sum_{m=1}^{n-1}(-1)^{m-1}|\Omega|\nabla^2\hat{c}_{n-m}(0) - |\Omega|\sigma^2 I(n \text{ odd}), \qquad (4)
\end{aligned}$$

where in the second equation we have incorporated the $|\Omega|\nabla^2\hat{c}_{n-1}(0)$ term into the second sum.

The first sum is a geometric sum (recall $\delta = (|\Omega|-1)^{-1}$):

$$\begin{aligned}
\sum_{m=2}^{n-1}(|\Omega|-1)^{-m+1} I(m \text{ odd}) &= \sum_{m=2}^{n-1}\delta^{m-1} I(m \text{ odd}). \\
\text{Replace } m \text{ by } m+1: \quad &= \sum_{m=1}^{n-2}\delta^{m} I(m \text{ even}) \\
&= \sum_{m=1}^{\lfloor \frac{n}{2}\rfloor -1}\delta^{2m} = \sum_{m=1}^{\lfloor \frac{n}{2}\rfloor -1}(\delta^2)^m \\
&= \frac{1-(\delta^2)^{\lfloor \frac{n}{2}\rfloor -1+1}}{1-\delta^2} - 1 = \frac{\delta^2 - \delta^{2\lfloor \frac{n}{2}\rfloor}}{1-\delta^2}.
\end{aligned}$$

Plugging this into (4) and replacing $(|\Omega|-1) = \delta^{-1}$,

$$\begin{aligned}
(4) &= \frac{|\Omega|^2\delta^{-(n-2)}\sigma^2}{1-\delta^2}\left(-\delta^2 + \delta^{2\lfloor \frac{n}{2}\rfloor} - 1 + \delta^2\right) \\
&\quad + \sum_{m=1}^{n-1}(-1)^{m-1}|\Omega|\nabla^2\hat{c}_{n-m}(0) - |\Omega|\sigma^2 I(n \text{ odd}).
\end{aligned}$$

Dividing through by $\hat{c}_n(0) = |\Omega|\delta^{-(n-1)}$,

$$\begin{aligned}\frac{\nabla^2 \hat{c}_n(0)}{\hat{c}_n(0)} &= \frac{\delta|\Omega|\sigma^2}{1-\delta^2}\left(\delta^{2\lfloor\frac{n}{2}\rfloor} - 1\right) \\ &\quad + \sum_{m=1}^{n-1}(-1)^{m-1}\delta^m|\Omega|\frac{\nabla^2\hat{c}_{n-m}(0)}{\hat{c}_{n-m}(0)} - \delta^{n-1}\sigma^2 I(n \text{ odd}),\end{aligned}$$

where in the second line you can check that $\hat{c}_{n-m}(0) = \hat{c}_n(0)\delta^m$.

Note that $\delta|\Omega| = 1+\delta$, so factoring $1-\delta^2 = (1+\delta)(1-\delta)$ and multiplying the last term by $\frac{1-\delta}{1-\delta}$, the first and last terms combine:

$$\begin{aligned}&\frac{\delta|\Omega|\sigma^2}{1-\delta^2}\left(\delta^{2\lfloor\frac{n}{2}\rfloor} - 1\right) - \delta^{n-1}\sigma^2 I(n \text{ odd}) \\ &\quad = \frac{\sigma^2}{1-\delta}\left(\delta^{2\lfloor\frac{n}{2}\rfloor} - 1 - \delta^{n-1}I(n \text{ odd}) + \delta^n I(n \text{ odd})\right) \\ &\quad = \frac{\sigma^2}{1-\delta}\left(\delta^n - 1\right).\end{aligned}$$

Multiplying through by -1, replacing $-\frac{\nabla^2\hat{c}_j(0)}{\hat{c}_j(0)}$ with ρ_j, replacing $\delta|\Omega| = 1+\delta$ and simplifying, we get

$$\begin{aligned}\rho_n &= \frac{\sigma^2}{1-\delta}(1-\delta^n) + (1+\delta)\sum_{m=1}^{n-1}(-\delta)^{m-1}\rho_{n-m} \\ &= \frac{\sigma^2}{1-\delta}(1-\delta^n) - (1+\delta)(-1)^n\delta^{n-1}\sum_{m=1}^{n-1}\left(-\delta^{-1}\right)^{n-m}\rho_{n-m} \\ &= \frac{\sigma^2}{1-\delta}(1-\delta^n) - (1+\delta)(-1)^n\delta^{n-1}\sum_{m=1}^{n-1}\left(-\delta^{-1}\right)^{m}\rho_{m}, \qquad (5)\end{aligned}$$

where we have re-indexed the sum in the last line.

Splitting off the last term in the sum,

$$\begin{aligned}\rho_n &= \frac{\sigma^2}{1-\delta}(1-\delta^n) - (-1)^n(1+\delta)\delta^{n-1}(-\delta)^{-(n-1)}\rho_{n-1} \\ &\quad + \delta(1+\delta)(-1)^{n-1}\delta^{n-2}\sum_{m=1}^{n-2}\left(-\delta^{-1}\right)^m\rho_m\end{aligned}$$

and we recognize the last term as resembling the last term in (5).

Using this, we can write

$$\begin{aligned}
\rho_n &= \frac{\sigma^2}{1-\delta}(1-\delta^n) + (1+\delta)\rho_{n-1} + \delta\left(\frac{\sigma^2}{1-\delta}\left(1-\delta^{n-1}\right) - \rho_{n-1}\right) \\
&= \frac{\sigma^2}{1-\delta}(1-\delta^n+\delta-\delta^n) + \rho_{n-1},
\end{aligned}$$

which yields

$$\rho_n - \rho_{n-1} = \frac{1+\delta}{1-\delta}\sigma^2 - \frac{2\sigma^2\delta^n}{1-\delta}.$$

Using $\rho_0 = 0$, we solve this difference equation to get

$$\rho_n = \sum_{m=1}^{n}(\rho_m - \rho_{m-1}) = \sigma^2\left[\frac{1+\delta}{1-\delta}n - \frac{2\delta}{(1-\delta)^2}(1-\delta^n)\right]$$

as desired.

Noting that as $n \longrightarrow \infty$, $\delta^n = o(n)$ since $\delta = \frac{1}{|\Omega|-1}$ and $|\Omega| > 2$, and writing δ in terms of $|\Omega|$, we get

$$\rho_n \sim \left(\frac{\sigma^2|\Omega|}{|\Omega|-2}\right)n.$$

(d) This follows from equation (2) and the Fourier transform of the generating function of $\pi_m(x)$,

$$\hat{\Pi}_z(k) = \sum_{m=0}^{\infty}\hat{\pi}_m(k)z^m.$$

As mentioned in the solution of (c), $\pi_0(x) = \pi_1(x) = 0$, so we can forget about the $m = 0, 1$ terms in the sum and we thus have

$$\begin{aligned}
\hat{\Pi}_z(k) &= \sum_{m=2}^{\infty}\hat{\pi}_m(k)z^m \\
&= \sum_{m=2}^{\infty}(-|\Omega|)I(m \text{ even})z^m + \sum_{m=2}^{\infty}(|\Omega|\hat{D}(k))I(m \text{ odd})z^m \\
&= (-|\Omega|)\sum_{m=1}^{\infty}z^{2m} + z|\Omega|\hat{D}(k)\sum_{m=1}^{\infty}z^{2m} \\
&= \left(-|\Omega| + z|\Omega|\hat{D}(k)\right)\left(\frac{1}{1-z^2} - 1\right) = \left(-|\Omega| + z|\Omega|\hat{D}(k)\right)\frac{z^2}{1-z^2}.
\end{aligned}$$

Plugging this into equation (3.30) yields the desired result.

Solution to Exercise 4.3 by Richard Liang.

First, note that $J_x^{(N)}[0,m]$ is meant to count diagrams that resemble those of Figure 4.2, except that the walks that compose the closed loops of the diagram are mutually avoiding in the same way that the diagrams for $\pi_m^{(N)}(x)$ are. Also note that the very last part of the walk this is representing is only self-avoiding and has no knowledge of the part before it. The intuition behind the exercise is to show that there are fewer such diagrams than ones where you relax the mutual avoidances for the last loop and instead ask for a diagram with one less loop, so you essentially replace the last loop (which must visit x) by two self-avoiding walks: one to x and one back to some point on the previous loop.

The algebra behind it is involved: we start with (4.24).

$$\begin{aligned} J_x^{(N)}[0,m] &= \sum_{j=0}^{m} K[j,m] \sum_{L\in\mathcal{L}^{(N-1)}[0,j]} \sum_{i=t_{N-2}}^{j-1} \delta_{x,\omega(i)} \\ &\qquad \times \prod_{st\in L}(-U_{st}) \prod_{s't'\in\mathcal{C}(L)} (1+U_{s't'}), \end{aligned}$$

where $L=\{s_1t_1, s_2t_2, \ldots, s_{N-1}t_{N-1}\}$.

We explicitly split off the last edge in the sum and replace the sum with three sums, replacing the last factor $-U_{s_{N-1}t_{N-1}}$ using (4.30):

$$\begin{aligned} J_x^{(N)}[0,m] &= \sum_{j=0}^{m} K[j,m] \sum_{t_{N-2}=0}^{j-1} \sum_{L\in\mathcal{L}^{(N-2)}[0,t_{N-2}]} \sum_{s_{N-1}=t_{N-3}}^{t_{N-2}-1} \sum_{i=t_{N-2}}^{j-1} \delta_{x,\omega(i)} \\ &\qquad \times \prod_{st\in L}(-U_{st}) \sum_{u\in\mathbb{Z}^d} \delta_{\omega(s_{N-1}),u}\delta_{u,\omega(j)} \prod_{s't'\in\mathcal{C}(L\cup\{s_{N-1}t_{N-1}\})} (1+U_{s't'}), \end{aligned}$$

where we are using $t_{N-1}=j$.

We recognize that $\mathcal{C}(L\cup\{s_{N-1}t_{N-1}\})$ contains both the set of compatible edges for L as well as any edge in the interval $[t_{N-2}, t_{N-1}]$, and omitting terms in this product will only make the result larger, so we have

$$\begin{aligned} J_x^{(N)}[0,m] &\le \sum_{j=0}^{m} K[j,m] \sum_{t_{N-2}=0}^{j-1} \sum_{L\in\mathcal{L}^{(N-2)}[0,t_{N-2}]} \sum_{s_{N-1}=t_{N-3}}^{t_{N-2}-1} \sum_{i=t_{N-2}}^{j-1} \delta_{x,\omega(i)} \\ &\quad \times \prod_{st\in L}(-U_{st}) \sum_{u\in\mathbb{Z}^d} \delta_{\omega(s_{N-1}),u}\delta_{u,\omega(j)} \\ &\quad \times \prod_{s't'\in\mathcal{C}(L)}(1+U_{s't'}) \prod_{t_{N-2}\le s'<t'\le t_{N-1}}(1+U_{s't'}). \end{aligned}$$

Immediately we see that the last product is $K[t_{N-2},j] \le K[t_{N-2},i]K[i,j]$. We rearrange the order of the sums and bring terms as far forward as we can to make things clearer:

$$\begin{aligned} J_x^{(N)}[0,m] &\le \sum_{u\in\mathbb{Z}^d}\sum_{j=0}^{m} K[j,m]\delta_{u,\omega(j)} \sum_{i=0}^{j-1} K[i,j]\delta_{\omega(i),x} \\ &\quad \times \left[\sum_{t_{N-2}=0}^{i} K[t_{N-2},i] \sum_{L\in\mathcal{L}^{(N-2)}[0,t_{N-2}]} \sum_{s_{N-1}=t_{N-3}}^{t_{N-2}-1} \delta_{\omega(s_{N-1}),u} \right. \\ &\quad \left. \times \prod_{st\in L}(-U_{st}) \prod_{s't'\in\mathcal{C}(L)}(1+U_{s't'}) \right]. \end{aligned}$$

We recognize the bracketed term as $J_u^{(N-1)}[0,i]$, arriving at

$$J_x^{(N)}[0,m] \le \sum_u \sum_{0\le i<j\le m} J_u^{(N-1)}[0,i]K[i,j]\delta_{\omega(i),x}K[j,m]\delta_{u,\omega(j)}$$

as desired.

Solution to Exercise 4.5 by Richard Liang.

We have

$$\sum_{u\in\mathbb{Z}^d} P_z^{(N)}(u,u+w) = \sum_{u\in\mathbb{Z}^d} P_z^{(N)}(-u,-u+w) \tag{6}$$

by replacing u by $-u$.

I claim that $P_z^{(N)}(x,y) = P_z^{(N)}(-x,-y)$; this combines with equation (6) to finish the exercise. We prove that $p_m^{(N)}(x,y) = p_m^{(N)}(-x,-y)$ for all $N \ge 2$, and to do this we proceed by induction. By symmetry of $c_j(x)$ this holds for the base case $N = 2$ (see equation (4.16)). For the induction step, we assume that our hypothesis holds for some $N \geq 2$.

$$\begin{aligned} p^{(N+1)}(-x,-y) &= \sum_{u,v\in\mathbb{Z}^d}\sum_{i=0}^{m} p_i^{(N)}(u,v)a_{m-i}(u,v,-x,-y) \\ &= \sum_{u,v\in\mathbb{Z}^d}\sum_{i=0}^{m} p_i^{(N)}(-u,-v)a_{m-i}(-u,-v,-x,-y), \end{aligned} \tag{7}$$

where in the last line we have again replaced u, v by $-u, -v$. By the induction hypothesis, $p_i^{(N)}(-u,-v) = p_i^{(N)}(u,v)$ for all u, v, and for any $k \geq 1$,

$$\begin{aligned} a_k(-u,-v,-x,-y) &= \delta_{-v,-x}\sum_{l=1}^{k} c_l(v-u)c_{k-l}(u-y) \\ &= \delta_{v,x}\sum_{l=1}^{k} c_l(u-v)c_{k-l}(y-u) \\ &= a_k(u,v,x,y). \end{aligned} \tag{8}$$

Putting (7) and (8) together, we get $p^{(N+1)}(-x,-y) = p^{(N+1)}(x,y)$. This completes our proof.

Solution to Exercise 4.7 by Richard Liang.

This is very similar to the case outlined in the notes where you associate the $\|\cdot\|_\infty$ norm to f_{2M-1}. Applying the $M = 1$ case to $\mathcal{H}_M\mathcal{M}_M F_{M-1}$ (in particular, the third line of (4.42)) allows you to write down as an upper bound

$$\| f_{2M} \|_\infty \, \| f_{2M-1} * F_{M-1} \|_\infty \, .$$

Let $\widetilde{\mathcal{H}}'_{M-1}$ denote a convolution by the non-negative even function $f_{2M-1} * f_{2M-2}$, which results in

$$f_{2M-1} * F_{M-1} = \widetilde{\mathcal{H}}'_{M-1}\mathcal{M}_{M-1}F_{M-2} \tag{9}$$

and now as in the notes we apply the induction hypothesis to get a bound on the ∞-norm of the right hand side of (9), associating the ∞-norm to $\widetilde{\mathcal{H}}'_{M-1}$, and this gives the desired estimate.

Solution to Exercise 5.4 by Richard Liang.

(This proof is largely borrowed from Madras and Slade.) We proceed as in the proof for the spread-out model. First, we find a bound on $1-\hat{D}(k)$ in the region $[-\pi,\pi]^d$. For the nearest neighbour model, first recall $1-\cos(x) \geq \frac{2}{\pi^2}x^2$ for $x \in [-\pi,\pi]$, so that we can write this as

$$1-\hat{D}(k) = \frac{1}{d}\sum_{j=1}^{d}(1-\cos(k_j)) \geq \frac{1}{d}\sum_{j=1}^{d}\frac{2}{\pi^2}k_j^2 = \frac{2}{\pi^2 d}|k|^2$$
$$\therefore\ \hat{D}(k) \leq 1-\frac{2}{\pi^2 d}|k|^2.$$

We next aim to show that $\|\hat{D}^n\|_1 \leq \left(\frac{\pi d}{4n}\right)^{d/2}$. By symmetry,

$$\begin{aligned}\|\hat{D}^n\|_1 &= \frac{1}{\pi^d}\int_{[0,\pi]^d}|\hat{D}(k)^n|d^dk \\ &= \frac{1}{\pi^d}\sum_{m=0}^{d}\binom{d}{m}\int_{[0,\pi/2]^m}dk_1\ldots dk_m \\ &\quad\times\int_{[\pi/2,\pi]^{d-m}}dk_{m+1}\ldots dk_d\,|\hat{D}(k)^n|. \end{aligned}\tag{10}$$

In the domain of integration on the right, noting that $|\cos(x)| \leq 1-\frac{4}{\pi^2}x^2$ for $x \in [0,\pi/2]$ we get

$$\begin{aligned}|\hat{D}(k)| &\leq \frac{1}{d}\sum_{l=1}^{d}|\cos k_l| \\ &\leq 1-\frac{4}{\pi^2 d}\left[\sum_{l=1}^{m}k_l^2+\sum_{l=m+1}^{d}(\pi-k_l)^2\right] \\ &\leq \exp\left(-\frac{4}{\pi^2 d}\left[\sum_{l=1}^{m}k_l^2+\sum_{l=m+1}^{d}(\pi-k_l)^2\right]\right), \end{aligned}\tag{11}$$

where we used $1-x \leq e^{-x}$ to obtain the last inequality.

Combining equations (10) and (11) and changing variables,

$$\begin{aligned}\|\hat{D}^n\|_1 &\leq \left(\frac{2}{\pi}\right)^d\int_{[0,\pi/2]^d}\exp[-4n|k|^2\pi^{-2}d^{-1}]\,d^dk \\ &\leq \left(\frac{2}{\pi}\right)^d\int_{\mathbb{R}_+^d}\exp[-4n|k|^2\pi^{-2}d^{-1}]\,d^dk \\ &= \left(\frac{\pi d}{4n}\right)^{d/2}. \end{aligned}\tag{12}$$

Now, we write as in the proof of Proposition 5.3

$$\begin{aligned}\int_{[-\pi,\pi]^d} \frac{\hat{D}(k)^2}{[1-\hat{D}(k)]^2}\frac{d^dk}{(2\pi)^d} &= \sum_{j=1}^{\infty}\int_{[-\pi,\pi]^d} j\hat{D}(k)^{j+1}\frac{d^dk}{(2\pi)^d}\\ &\le \sum_{n=1}^{d-2} n\int \hat{D}(k)^{n+1}\frac{d^dk}{(2\pi)^d}\\ &\quad + \sum_{n=d-1}^{\infty} n\left(\frac{\pi d}{4(n+1)}\right)^{d/2}\end{aligned}$$

using our bound (12).

For the first sum, all of the integrands with odd powers integrate to zero, and we can write, using the fact $|\hat{D}| \le 1$,

$$\begin{aligned}\sum_{n=1}^{d-2} n\int \hat{D}(k)^{n+1}\frac{d^dk}{(2\pi)^d} &\le \int \hat{D}(k)^2\frac{d^dk}{(2\pi)^d} + 3\int \hat{D}(k)^4\frac{d^dk}{(2\pi)^d}\\ &\quad + d^2\int \hat{D}(k)^6\frac{d^dk}{(2\pi)^d},\end{aligned}$$

where in the last term we have collected all of the integrals with integrands of higher power. These three integrals are the probabilities that simple random walk returns to the origin after 2, 4, and 6 steps, yielding probabilities on the order of $1/d$, $1/d^2$, and $1/d^3$ respectively, so this term is $O(d^{-1})$, with a universal constant.

The second sum is bounded by using the following inequality: for $a > 1$,

$$\sum_{n=N}^{\infty}\frac{1}{n^a} \le \int_{N-1}^{\infty}\frac{1}{t^a}dt = \frac{(N-1)^{-a+1}}{a-1}. \tag{13}$$

This gives

$$\begin{aligned}\sum_{n=d-1}^{\infty} n\left(\frac{\pi d}{4(n+1)}\right)^{d/2} &= \left(\frac{\pi d}{4}\right)^{d/2}\sum_{n=d-1}^{\infty}\frac{n}{(n+1)^{d/2}}\\ &\le \left(\frac{\pi d}{4}\right)^{d/2}\sum_{n=d-1}^{\infty}\frac{1}{(n+1)^{d/2-1}}.\\ \text{Re-index } n \text{ by } n-1: &= \left(\frac{\pi d}{4}\right)^{d/2}\sum_{n=d}^{\infty}\frac{1}{n^{d/2-1}}.\\ \text{Apply equation (13):} &\le \left(\frac{\pi}{4}\right)^{d/2}\left(\frac{d}{d-1}\right)^{d/2}(d-1)^2\frac{2}{d-4}.\end{aligned}$$

The $(\pi/4)^{d/2}$ exponential decay term dominates the $(d-1)^2$ term, so we can bound their product by a constant. And

$$\left(\frac{d}{d-1}\right)^{d/2} = \left(1+\frac{1}{d-1}\right)^{d/2} \to e^{\frac{1}{2}}$$

and so we can bound this term by a constant also, not depending on d. We are left with $O((d-4)^{-1})$, whose constant does not depend on d.

Solution to Exercise 5.6 by Richard Liang.

We write $[1-\hat{D}(k)]^{-1}$ and $[1-\hat{D}(k)]^{-2}$ as power series, and multiply by $\hat{D}(k)$ and $\hat{D}(k)^2$ respectively. We get

$$\begin{aligned}\frac{\hat{D}(k)}{1-\hat{D}(k)} &= \sum_{n=1}^{\infty} \hat{D}(k)^n, \\ \frac{\hat{D}(k)^2}{[1-\hat{D}(k)]^2} &= \sum_{n=2}^{\infty} (n-1)\hat{D}(k)^n.\end{aligned}$$

All of the terms with integrands having odd powers integrate to zero. The only remaining terms have integrands with even powers, and these are all therefore strictly positive real. These terms all have larger coefficients in the second series than in the first (except for the $n=2$ coefficients, which are equal), showing the inequality.

Solution to Exercise 5.15 by Richard Liang.

Fix $r < z_c$; as in the proof for f_2 we find a uniform bound on the derivative in z for $z < r$.

$$f_3(z) = \sup_{k,l\in[-\pi,\pi]^d} \frac{\frac{1}{2}|\triangle_k \hat{G}_z(l)|}{U_{p(z)}(k,l)},$$

so it suffices to show that

$$g_3(z) = \frac{-\frac{1}{2}\triangle_k \hat{G}_z(l)}{U_{p(z)}(k,l)}$$

is equicontinuous instead.

We take the derivative:

$$\begin{aligned} g_3'(z) &= \frac{1}{U_{p(z)}(k,l)^2}\left[\frac{d}{dz}\left(-\frac{1}{2}\triangle_k \hat{G}_z(l)\right) U_{p(z)}(k,l)\right. \\ &\quad \left. - \left(-\frac{1}{2}\triangle_k \hat{G}_z(l)\, \frac{d}{dp} U_p(k,l)\Big|_{p=p(z)} \frac{d}{dz} p(z)\right)\right]. \end{aligned} \tag{14}$$

To bound this, we reuse the bounds from the proof of Lemma 5.14, and from these we also get (after a lot of algebra):

$$\begin{aligned}
\frac{d}{dz}\left(-\frac{1}{2}\triangle_k \hat{G}_z(l)\right) &\leq 2\chi'(r), \\
-\frac{1}{2}\triangle_k \hat{G}_z(l) &\leq 2\chi(r), \\
\frac{12}{\chi(r)} \leq U_{p(z)}(k,l) &\leq 96\chi(r)^2, \\
\frac{d}{dp}U_p(k,l) &\leq 192|\Omega|\chi(r)^4.
\end{aligned}$$

Plugging all of these bounds into (14) we get

$$g_3'(z) \leq \frac{4}{3}\chi(r)^4\chi'(r) + \frac{8}{3}\chi(r)^7\chi'(r),$$

which suffices.

Solution to Exercise 5.17 by Richard Liang.

(a) This comes directly from (5.47). Since $\pi_m^{(N)}(x) > 0$ always, $\Pi_z^{(N)}(x)$ must be increasing in z. So, if $N \geq 2$,

$$\begin{aligned}
\sum_{x\in\mathbb{Z}^d}[1-\cos(k\cdot x)]\Pi_z^{(N)}(x) &\leq \lim_{z\to z_c^-}\sum_{x\in\mathbb{Z}^d}[1-\cos(k\cdot x)]\Pi_z^{(N)}(x) \\
&\leq \lim_{z\to z_c^-}\hat{C}_{p(z)}(k)^{-1}N^2\left(c_K'\beta\right)^{N-1}
\end{aligned}$$

and in this limit, $\hat{C}_{p(z)}(k)^{-1} \to (1-\hat{D}(k))$.

Summing this up as in the proof of Lemma 5.11 shows that we can replace the $\hat{C}_{p(z)}(k)^{-1}$ by $1-\hat{D}(k)$, yielding

$$\sum_{x\in\mathbb{Z}^d}[1-\cos(k\cdot x)]\,|\Pi_z(x)| \leq \overline{c}_K\beta\left(1-\hat{D}(k)\right). \tag{15}$$

(b) If we expand the left hand side of equation (15) and apply de Moivre's formula, we obtain:

$$\sum_{x\in\mathbb{Z}^d}|\Pi_z(x)| - \frac{1}{2}\left(\sum_{x\in\mathbb{Z}^d}e^{ik\cdot x}\,|\Pi_z(x)| + \sum_{x\in\mathbb{Z}^d}e^{-ik\cdot x}\,|\Pi_z(x)|\right) = \widehat{|\Pi_z|}(0) - \widehat{|\Pi_z|}(k),$$

where in the last step we have used symmetry of $\Pi_z(x)$.

Therefore, for all k,

$$\widehat{|\Pi_z|}(0) - \widehat{|\Pi_z|}(k) \leq \overline{c}_K \beta \left(1 - \hat{D}(k)\right) \tag{16}$$

and therefore, for $j = 1, 2, \ldots, d$,

$$\frac{2\left(\widehat{|\Pi_z|}(0) - \widehat{|\Pi_z|}(t\mathbf{e}_j)\right)}{t^2} \leq \frac{2\overline{c}_K \beta \left(1 - \hat{D}(t\mathbf{e}_j)\right)}{t^2},$$

which, since $\widehat{|\Pi_z|}(k)$ is an even function and noticing that $\hat{D}(0) = 1$, leads us to take the limit as $t \to 0$ and we get

$$\begin{aligned} -\frac{\partial^2}{\partial k_j^2}\widehat{|\Pi_z|}(0) &\leq -\overline{c}_K \beta \frac{\partial^2 \hat{D}}{\partial k_j^2}(0) \\ \therefore\ -\nabla^2 \widehat{|\Pi_z|}(0) &\leq \overline{c}_K \beta \left(-\nabla^2 \hat{D}(0)\right). \end{aligned}$$

Recognizing that $\sum_x |x|^2 |\Pi_z(x)| = -\nabla^2 \widehat{|\Pi_z|}(0)$, furthermore substituting $\sigma^2 = \sum_{x \in \mathbb{Z}^d} |x|^2 D(x) = -\nabla^2 \hat{D}(0)$ and noting that the constant $\overline{c}_K$ is independent of z, we are done.

(c) We begin by recognizing that

$$\xi_2(z)^2 = -\frac{\nabla^2 \hat{G}_z(0)}{\hat{G}_z(0)}$$

and so a natural starting point is to take the Laplacian of the lace expansion (3.29).

As usual the cross terms cancel because the gradient of $\hat{D}(k)$ and of $\hat{\Pi}_z(k)$ are both zero. Evaluating at $k = 0$, substituting in σ^2 for $-\nabla^2 \hat{D}(0)$ and 1 for $\hat{D}(0)$, we are left with

$$\begin{aligned} \nabla^2 \hat{G}_z(0) &= -z|\Omega|\sigma^2 \hat{G}_z(0) + z|\Omega|\nabla^2 \hat{G}_z(0) \\ &\quad + \nabla^2 \hat{\Pi}_z(0) \hat{G}_z(0) + \hat{\Pi}_z(0) \nabla^2 \hat{G}_z(0). \end{aligned}$$

Collecting like terms,

$$\nabla^2 \hat{G}_z(0) \left(1 - z|\Omega| - \hat{\Pi}_z(0)\right) = \hat{G}_z(0) \left(-z|\Omega|\sigma^2 + \nabla^2 \hat{\Pi}_z(0)\right).$$

The term in parentheses on the left hand side is exactly $\hat{G}_z(0)^{-1}$, so multiplying through by -1,

$$\xi_2(z)^2 = -\frac{\nabla^2 \hat{G}_z(0)}{\hat{G}_z(0)} = \chi(z)\left(z|\Omega|\sigma^2 - \nabla^2\hat{\Pi}_z(0)\right).$$

We can bound the absolute value of the $\nabla^2\hat{\Pi}_z(0)$ term by $\sum_x |x|^2|\Pi_z(x)|$, which we showed above to be $O(\beta\sigma^2)$, so for β sufficiently small, we can forget about this term. As $z \to z_c^-$, the term $z|\Omega|\sigma^2$ just goes to a constant, and so we can conclude that $\xi_2(z)^2 \simeq (1-z/z_c)^{-1}$ since $\chi(z)$ is. Take square roots to finish.

Solution to Exercise 5.18 by Richard Liang.

(a) Using the given hint and Fubini's theorem (this is why we need $d > 2m$; otherwise the integral does not exist),

$$\int_{[-\pi,\pi]^d} \frac{1}{\left(1-\hat{D}(k)\right)^m} \frac{d^dk}{(2\pi)^d} = \frac{1}{\Gamma(m)} \int_0^\infty du\, u^{m-1} \int_{[-\pi,\pi]^d} \frac{d^dk}{(2\pi)^d} e^{-u\left[1-\hat{D}(k)\right]}.$$

Writing $1-\hat{D}(k) = \frac{1}{d}\sum_{j=1}^d (1-\cos k_j)$ and splitting up the d-dimensional integral into a product of d identical ones, we get

$$\text{r.h.s.} = \frac{1}{\Gamma(m)} \int_0^\infty du\, u^{m-1} \left(\int_{-\pi}^{\pi} \frac{d\theta}{2\pi} e^{-\frac{u}{d}(1-\cos\theta)}\right)^d$$

and we recognize the term in parentheses as the $L_{1/d}$ norm of the function $e^{-(1-\cos\theta)}$ with respect to the probability measure given by $\frac{d\theta}{2\pi}$ over the region $[-\pi,\pi]$, which is nonincreasing in $d > 2m$.

(b) $H_z^{(j)}(x)$ is the generating function for a self-avoiding walk from 0 to x that takes at least j steps. Relaxing some of the self-avoidances in the walk will only increase the number of them, so we can replace the first j steps by simple random walk. This gives

$$H_z^{(j)}(x) \le (z|\Omega|D)^{*j} * G_z(x), \tag{17}$$

where $(z|\Omega|D)^{*j}$ represents j-fold convolution with itself.

Using the fact that $\| f \|_\infty \leq \| \hat{f} \|_1$, we take the Fourier transform of equation (17):

$$\begin{aligned}
\| H_z^{(j)} \|_\infty &\leq (z|\Omega|)^j \| \hat{D}^j \hat{G}_z \|_1 \\
&\leq (z|\Omega|)^j \| \hat{D}^{2j} \|_1^{1/2} \| \hat{G}_z^2 \|_1^{1/2} \text{ by Cauchy-Schwarz.} \qquad (18)
\end{aligned}$$

The $\| \hat{D}^{2j} \|_1$ is exactly the probability that a $2j$-step simple random walk starting at 0 ends at 0. This probability is given by

$$\begin{aligned}
\mathbb{P}(2j\text{-step SRW } 0 \to 0) &= \frac{|\{\omega : \omega(2j) = 0\}|}{(2d)^{2j}} \\
&\leq \frac{1}{(2d)^{2j}} \sum_{l=1}^{j} \binom{d}{l} (2l)^{2j},
\end{aligned}$$

where we have essentially conditioned on the number of different dimensions such a walk can occupy in the second line.

Bounding the binomial coefficient above by $\frac{d^l}{l!} \leq \frac{d^j}{l!}$,

$$\text{r.h.s.} \leq \frac{1}{(2d)^{2j}} d^j \underbrace{(2j)^{2j} \sum_{l=1}^{j} \frac{1}{l!}}_{=\text{const}(j)} = \frac{\text{const}(j)}{d^j},$$

so the $\| \hat{D}^{2j} \|_1^{1/2}$ contributes an $O(d^{-j/2})$ to equation (18).

The $\| \hat{G}_z^2 \|_1$ only contributes a constant:

$$\begin{aligned}
\int \hat{G}_z^2(k) \frac{d^d k}{(2\pi)^d} &\leq K^2 \int \hat{C}_{p(z)}^2(k) \frac{d^d k}{(2\pi)^d} \text{ by the infrared bound in (5.36)} \\
&= K^2 \sum_x C_{p(z)}(x)^2 \text{ by Parseval} \\
&\leq K^2 \sum_x C_{\frac{1}{2d}}(x)^2 \\
&= K^2 \int_{[-\pi,\pi]^d} \frac{1}{\left(1 - \hat{D}(k)\right)^2} \frac{d^d k}{(2\pi)^d}, \qquad (19)
\end{aligned}$$

which by part (a) we know to be bounded above by a constant. Plugging in $z = z_c$ in equation (18), since $z_c|\Omega| \leq 2$, we get $\| H_{z_c}^{(j)} \|_\infty \leq O\left(d^{-j/2}\right)$ as desired.

(c) Our goal is to show that we can truncate the series

$$\hat{\Pi}^{(1)}_{z_c}(0) = \sum_{m=1}^{\infty} \hat{\pi}^{(1)}_m(0) z^m$$

at some j and bound the error.

The sum $\sum_{m=j}^{\infty} \hat{\pi}^{(1)}_m(0) z^m$ is the generating function for all self-avoiding polygons that take at least j steps, and

$$\begin{aligned} \sum_{m=j}^{\infty} \hat{\pi}^{(1)}_m(0) z^m &= \left(H^{(j-1)}_z * z|\Omega|D\right)(0) \\ &= \sum_y z|\Omega|D(y) H^{(j-1)}_z(y) \text{ by symmetry.} \end{aligned} \tag{20}$$

Breaking up $H^{(j-1)}_z(y)$ by bounding it above as in equation (17) and using symmetry again, we identify

$$\text{r.h.s.} \le \left((z|\Omega|D)^{*j} * G_z\right)(0),$$

which we showed above to be bounded by $O\left(d^{j/2}\right)$ when you put $z = z_c$.

So, truncating the series at $j = 6$ and noting that $\hat{\pi}^{(1)}_2(0)$ is $2d$ and that $\hat{\pi}^{(1)}_4(0)$ is $2d(2d-2)$, we get

$$\hat{\Pi}^{(1)}_{z_c}(0) = 2dz_c^2 + 2d(2d-2)z_c^4 + O\left(\frac{1}{(2d)^3}\right) \tag{21}$$

and plugging in $z_c = \frac{1}{|\Omega|} + O\left(\frac{1}{|\Omega|^2}\right) = \frac{1}{2d} + O\left(\frac{1}{(2d)^2}\right)$ we get first that $\hat{\Pi}^{(1)}_{z_c}(0) = 1/(2d) + O((2d)^2)$. Using the bound (5.46), we have

$$\sum_{N=M}^{\infty} \sum_x \Pi^{(N)}_z(x) \le O\left((2d)^{-M}\right) \tag{22}$$

and so

$$\hat{\Pi}_{z_c}(0) = -\hat{\Pi}^{(1)}_{z_c}(0) + O\left(\frac{1}{(2d)^2}\right) = -\frac{1}{2d} + O\left(\frac{1}{(2d)^2}\right),$$

and plugging this into equation (5.67) of the notes,

$$z_c = \frac{1}{2d} + \frac{1}{(2d)^2} + O\left(\frac{1}{(2d)^3}\right). \tag{23}$$

Plugging this back into equation (21) gets us

$$\hat{\Pi}_{z_c}^{(1)}(0) = \frac{1}{2d} + \frac{3}{(2d)^2} + O\left(\frac{1}{(2d)^3}\right).$$

(d) For this part of the exercise, we are dealing with the 2-loop diagrams (or *theta* diagrams) depicted in Section 3.1 of the notes. We aim to show that we can again truncate the series $\hat{\Pi}_{z_c}^{(2)}(0) = \sum_{m=1}^{\infty} \hat{\pi}_m^{(2)}(0) z^m$ at $j = 6$ and be left with an error of $O(d^{-3})$.

Since there are no theta diagrams of length 6, we are left to only consider those of length 7 and greater. There are three possible cases for such diagrams of length m: a walk that forms a polygon with its first $m-1$ steps and then reverses its last step; a walk that forms a polygon with its first $m-1$ steps and then follows its *first* step; and a walk that makes a theta diagram with two loops having length greater than 3, since any other theta diagram of length at least 7 must have two such loops.

In the first two cases, we denote $\mathcal{P}_m$ as the set of walks that form self-avoiding polygons of length at least m. The generating function then for the theta diagrams that consist of these polygons plus an extra step to either their first step or back to their last step is

$$\sum_{\omega \in \mathcal{P}_m} z_c^{|\omega|} z_c = O\left(d^{-m/2}\right) \frac{1}{d} = O\left(\frac{1}{d^{m/2+1}}\right),$$

where the $O\left(d^{-m/2}\right)$ term comes from the fact that $\sum_{\omega \in \mathcal{P}_m} z_c^{|\omega|}$ is exactly the generating function for self-avoiding polygons of length at least m; i.e. $\sum_{j=m}^{\infty} \hat{\pi}_m(0) z_c^m$, which we dealt with in part (c). The extra z_c factor in the sum is from the last step, and by equation (23) this contributes a factor of $1/d$.

In the remaining case, we have a diagram that looks like the one depicted in Figure 2, where two of the walks (say, the first and third) have length at least 3 (in the picture, the leftmost and the rightmost ones). Relaxing the mutual avoidances between the walks to get an upper bound, we are left with two cases.

First, there is the case where x is a neighbour of 0. In this case, call $x = e$. The generating function of such a diagram (with mutual avoidances ignored) is

$$\sum_e H_{z_c}^{(3)}(e) H_{z_c}^{(1)}(e) H_{z_c}^{(3)}(e).$$

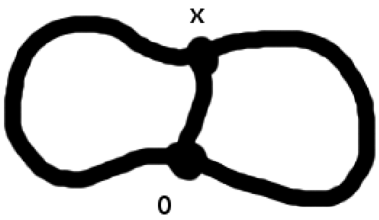


Figure 2: The third type of theta diagram described in Exercise 5.18d. If we number the walks 1,2, and 3 from left to right, we intend for walks 1 and 3 to have length at least 3.

The $H^{(3)}_{z_c}(e)$ terms each contribute a $O\left(d^{-3/2}\right)$. The $H^{(1)}_{z_c}(e)$ gives a $1/d$, because $H^{(1)}_{z_c}(e) = z_c + H^{(3)}_{z_c}(e)$ (there is no 2-step walk to a neighbour of 0), and $z_c = 1/d + O(d^{-2})$ and $H^{(3)}_{z_c}(e) = O\left(d^{-3/2}\right)$. Summing over all neighbours of 0 gives a factor of d, leaving us with a bound $O(d^{-3})$.

If x is *not* a neighbour of 0, the second walk must have length at least 2, so the analogous generating function is now

$$\begin{aligned}\sum_x H^{(3)}_{z_c}(x) H^{(2)}_{z_c}(x) H^{(3)}_{z_c}(x) &\leq O\left(d^{-1}\right) \sum_x H^{(3)}_{z_c}(x) H^{(3)}_{z_c}(x) \\ &= \frac{1}{d}\left(H^{(3)}_{z_c}(x)\right)^{*2}(0).\end{aligned}$$

Again doing the same trick of relaxing the self-avoidances in the first 3 steps of each of the first and third subwalks, we get that $\left(H^{(3)}_{z_c}(x)\right)^{*2}(0) \leq \left((z_c|\Omega|D)^{*6} * G^{*2}_{z_c}\right)(0)$.

To bound this, recall $\| f \|_\infty \leq \| \hat{f} \|_1$, so factoring out the $(z_c|\Omega|)^6$, the $\| D^{*6} * G^{*2}_{z_c} \|_\infty$ term is bounded above by $\int |\hat{D}|^6 \hat{G}^2_{z_c}(k) \frac{d^d k}{(2\pi)^d}$, which is

$$\begin{aligned}&\left((z_c|\Omega|D)^{*6} * G^{*2}_{z_c}\right)(0) \\ &\leq (z_c|\Omega|)^6 \left(\int |\hat{D}(k)|^{12} \frac{d^d k}{(2\pi)^d}\right)^{1/2} \left(\int \hat{G}^4_{z_c}(k) \frac{d^d k}{(2\pi)^d}\right)^{1/2}\end{aligned}$$

by the Cauchy-Schwarz inequality. The $(z_c|\Omega|)^6$ is just a constant, so we don't worry about it. The first integral term is $O\left(d^{-6/2}\right)$ and the second is bounded by a constant by part (a). (To see this, mimic the argument in equation (19) using 4 as the exponent.) So we get that this whole term is $O(d^{-3})$.

So, putting the three cases together, we have shown that

$$\sum_{m=6}^{\infty} \hat{\pi}_m^{(2)}(0) z_c^m = O\Big(d^{-3}\Big). \tag{24}$$

Proceeding as in part (c), noting that there are exactly $2d$ theta diagrams with 3 steps, and $3(2d(2d-2))$ theta diagrams with 5 steps,

$$\hat{\Pi}_{z_c}^{(2)}(0) \quad = \quad 2dz_c^3 + 3(2d(2d-2))z_c^5 + O\Big(d^{-3}\Big).$$

Plugging in equation (23) gives

$$\hat{\Pi}_{z_c}^{(2)}(0) = \frac{1}{(2d)^2} + O\bigg(\frac{1}{(2d)^3}\bigg).$$

(e) Using the bound (22) we have

$$\begin{aligned}\hat{\Pi}_{z_c}(0) &= -\hat{\Pi}_{z_c}^{(1)}(0) + \hat{\Pi}_{z_c}^{(2)}(0) + O\Big(d^{-3}\Big) \\ &= -\frac{1}{2d} - \frac{2}{(2d)^2} + O\bigg(\frac{1}{(2d)^3}\bigg).\end{aligned}$$

Again plugging this back into equation (5.67) in the notes, we get

$$z_c = \frac{1}{2d} + \frac{1}{(2d)^2} + \frac{2}{(2d)^3} + O\bigg(\frac{1}{(2d)^4}\bigg).$$

Expand $\mu = \frac{1}{z_c}$ (using a computer will help) to get

$$\mu = 2d - 1 - \frac{1}{2d} + O\bigg(\frac{1}{(2d)^2}\bigg)$$

as desired.

Solution to Exercise 5.19 by Richard Liang.

Consider the function $(z\chi(z))^{-1}$: differentiating yields the differential equation

$$\frac{d}{dz}\left[\frac{1}{z\chi(z)}\right] = -\frac{1}{z^2\chi(z)^2}\frac{d(z\chi(z))}{dz} \sim -\frac{V(z_c)}{z^2} \sim -\frac{V(z_c)}{z_c^2}$$

by (5.76).

We can integrate asymptotic relationships, so we integrate over (z, z_c). In the limit as z goes to z_c, $\frac{1}{z\chi(z)} \to 0$; so we are left with

$$-\frac{1}{z\chi(z)} \sim -\frac{V(z_c)}{z_c^2}(z_c - z).$$

Rearranging and noting that $z_c/z \sim 1$, we get

$$\chi(z) \sim \frac{1}{V(z_c)}\frac{1}{1 - z/z_c}.$$

Finally,

$$V(z_c) \quad = \quad z_c \frac{d\hat{\Pi}_{z_c}(0)}{dz} + \left(1 - \hat{\Pi}_{z_c}(0)\right)$$

and by (5.66) in the notes, the term in parentheses is $z_c|\Omega|$.

So, we put it all together to see that

$$\chi(z) \sim \frac{1}{z_c\left[|\Omega| + \frac{d}{dz}\hat{\Pi}_{z_c}(0)\right]}\frac{1}{1 - z/z_c}$$

as desired.

Solution to Exercise 6.4 by Jeffrey Hood.

(a) $\sum_{n=0}^{\infty} 2p_{vn}(x)$ can be bounded above and below by multiples of

$$2\int_{n=0}^{\infty} \frac{d^{d/2}}{(2\pi vn)^{d/2}} e^{-d|x|^2/2vn} dn \simeq |x|^{2-d}$$

for $d > 2$ by substituting $u = d|x|^2/2vn$ to obtain a du gamma integral. Summing the error term, and letting the value of K change from line to line, we have for $d > 2$

$$\begin{aligned}
&\sum_{n=1}^{\infty} K\left[\frac{1}{\sqrt{n}}p_{vn}(x) + \frac{1}{n^{d/2}}\sum_{j=1}^{n/2} jp_{vj}(x)\right] \\
&\leq K\sum_{n=1}^{\infty} n^{-d/2}n^{-1/2}e^{-d|x|^2/2vn} + K\sum_{j=1}^{\infty} j^{-d/2+1}e^{-d|x|^2/2vj}\sum_{n=2j}^{\infty} n^{-d/2} \\
&\leq K\sum_{j=1}^{\infty} j^{-(d+1)/2}e^{-d|x|^2/2vj} + K\sum_{j=1}^{\infty} j^{-d/2+1}e^{-d|x|^2/2vj}(2j)^{-d/2+1} \\
&\leq K\sum_{j=1}^{\infty}\left(j^{-(d+1)/2} + j^{2-d}\right)e^{-d|x|^2/2vj},
\end{aligned}$$

which for $d > 3$ can be bounded by a multiple of

$$\begin{aligned}\int_0^\infty \left(j^{-(d+1)/2} + j^{2-d}\right) e^{-d|x|^2/2vj} dj &= \left(\frac{2v}{d}\right)^{(d-1)/2} \Gamma((d-1)/2)|x|^{1-d} \\ &\quad + \left(\frac{2v}{d}\right)^{d-3} \Gamma(d-3)|x|^{6-2d}.\end{aligned}$$

Since $\mu = \lim_{n\to\infty} \left(c_n^{(\lambda)}\right)^{1/n}$,

$$G_{z_c}^{(\lambda)}(x) = \sum_{n=0}^\infty \frac{c_n^{(\lambda)}(x)}{\mu^n} \simeq \sum_{n=0}^\infty \frac{c_n^{(\lambda)}(x)}{c_n^{(\lambda)}} \simeq |x|^{2-d}$$

because the error term is dominated by $|x|^{2-d}$ if $d > 4$. Therefore, the critical exponent in (2.27) is $\eta = 0$.

(b) As $d > 4$ we obtain

$$\begin{aligned}\frac{c_n^{(\lambda)}(x)}{c_n^{(\lambda)}} &\le 2p_{vn}(x) + K\left[\frac{1}{\sqrt{n}} p_{vn}(x) + \frac{1}{n^{d/2}} \sum_{j=1}^{n/2} j p_{vj}(x)\right] \\ &\le 2\frac{d^{d/2}}{(2\pi v)^{d/2}} e^0 n^{-d/2} \\ &\quad + K\left[\frac{d^{d/2}}{(2\pi v)^{d/2}} e^0 n^{-(d+1)/2} + \frac{1}{n^{d/2}} \sum_{j=1}^{n/2} \frac{d^{d/2}}{(2\pi v)^{d/2}} e^0 j^{1-d/2}\right] \\ &\le K\left[n^{-d/2} + n^{-(d+1)/2} + n^{-d/2}\right] \\ &\le K n^{-d/2}.\end{aligned}$$

Solution to Exercise 7.1 by Jeffrey Hood and Sandra Kliem.

First, for any tree T, let T_0 be the tree obtained by subtracting $\bar{x}_T$ from each vertex in T. While most likely not a lattice tree, T_0 has center of mass 0 and we obtain $\sum_{y\in T_0} y = 0$. Then

$$\begin{aligned}R(T)^2 &= \frac{1}{|T|+1} \sum_{y\in T_0} |y|^2 \\ &= \frac{1}{2} \frac{1}{|T|+1} \left(\sum_{y\in T_0} |y|^2 + \sum_{z\in T_0} |z|^2\right)\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2}\frac{1}{|T|+1}\left(\sum_{y\in T_0}|y|^2 - 2\sum_{y\in T_0} y \sum_{z\in T_0} z + \sum_{z\in T_0}|z|^2\right) \\
&= \frac{1}{2}\frac{1}{(|T|+1)^2}\sum_{y,z\in T_0}|y-z|^2 \\
&= \frac{1}{2}\frac{1}{(|T|+1)^2}\sum_{y,z\in T}|y-z|^2,
\end{aligned}$$

and so

$$\begin{aligned}
R_n^2 &= \frac{1}{t_n^{(1)}}\sum_{T:|T|=n,T\ni 0}\frac{1}{2}\frac{1}{(|T|+1)^2}\sum_{y,z\in T}|y-z|^2 \\
&= \frac{1}{2\hat{t}_n^{(2)}(0)}\frac{1}{n+1}\sum_{T:|T|=n,T\ni 0}\sum_{y,z\in T}|y-z|^2,
\end{aligned}$$

where we used $\hat{t}_n^{(2)}(0) = (n+1)t_n^{(1)}(0)$. Therefore, it suffices to show that

$$\sum_{T:|T|=n,T\ni 0}\sum_{y,z\in T}|y-z|^2 = (n+1)\sum_{T:|T|=n,T\ni 0}\sum_{x\in T}|x|^2$$

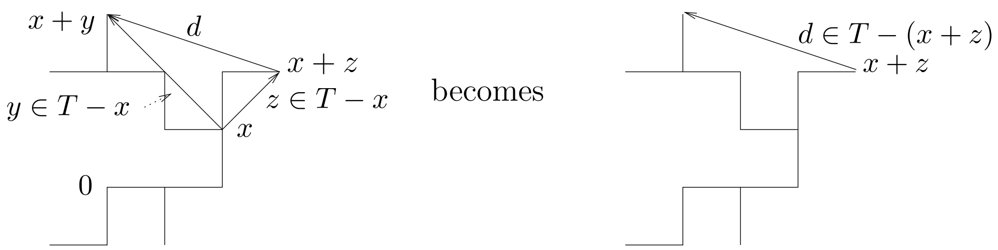
as

$$\sum_{T:|T|=n,T\ni 0}\sum_{x\in T}|x|^2 = \sum_{x}|x|^2 t_n^{(2)}(x).$$

First split the sum over all trees with n bonds (i.e. $n+1$ vertices) into two sums by grouping trees together which are translations of each other. This yields

$$\begin{aligned}
\sum_{T:|T|=n,T\ni 0}\sum_{y,z\in T}|y-z|^2 &= \sum_{\substack{T:|T|=n,T\ni 0,\\ \text{not equal via translations}}}\sum_{x\in T}\sum_{y,z\in T-x}|y-z|^2 \\
&= \sum_{\substack{T:|T|=n,T\ni 0,\\ \text{not equal via translations}}}\sum_{x\in T}\sum_{z\in T-x}\sum_{d\in T-x-z}|d|^2.
\end{aligned}$$

The following picture illustrates the next step of calculation:



Writing $\tilde{x} = x + z$ we get

$$\begin{aligned}
\sum_{T:|T|=n, T\ni 0} \sum_{y,z\in T} |y-z|^2 &= \sum_{\substack{T:|T|=n, T\ni 0,\\ \text{not equal via translations}}} \sum_{x\in T} \sum_{\tilde{x}\in T} \sum_{d\in T-\tilde{x}} |d|^2 \\
&= \sum_{\substack{T:|T|=n, T\ni 0,\\ \text{not equal via translations}}} \sum_{\tilde{x}\in T} \sum_{d\in T-\tilde{x}} |d|^2 \sum_{x\in T} 1 \\
&= \sum_{T:|T|=n, T\ni 0} \sum_{d\in T} |d|^2 (n+1),
\end{aligned}$$

which completes the proof.

Solution to Exercise 7.5 by Jeffrey Hood and Sandra Kliem.

The following is adapted from [95]. For the definition of "ribs" and "backbone" refer to the beginning of paragraph 8.1. In this context we'll speak of "T_x minimal rib at u" if we consider the backbone of $T_x \ni x, u$ and the resulting rib at u after removal of the bonds of the backbone is the "minimal rib" resp. tree $\{u\}$.

We look for a lower bound for $\frac{d[z\chi(z)]}{dz} = \sum_{x,y} G_z^{(3)}(x,y)$, where

$$\begin{aligned}
G_z^{(3)}(x,y) &= \sum_u \sum_{\substack{T_0\ni 0,u\\ T_x\ni x,u\\ T_y\ni y,u}} z^{|T_0|+|T_x|+|T_y|} I[T_x, T_y \,\text{minimal ribs at}\, u] \\
&\quad \times I[T_0\cap T_x \cap T_y = \{u\}]\,. \qquad (25)
\end{aligned}$$

By inclusion-exclusion,

$$\begin{aligned}
&I[T_0\cap T_x\cap T_y = \{u\}] \\
&\geq 1 - I[T_0\cap T_x \supsetneq \{u\}] - I[T_0\cap T_y \supsetneq \{u\}] - I[T_x\cap T_y \supsetneq \{u\}]\,. \qquad (26)
\end{aligned}$$

Plugging this into (25) we obtain for the first term

$$\begin{aligned}
&\sum_u \sum_{\substack{T_0\ni 0,u\\ T_x\ni x,u\\ T_y\ni y,u}} z^{|T_0|+|T_x|+|T_y|} I[T_x, T_y \,\text{minimal ribs at}\, u] \\
&\geq \sum_u G_z(u) \left(\sum_{T_x\ni x,u} z^{|T_x|} I[T_x \,\text{minimal rib at}\, u] \right) \\
&\qquad \times \left(\sum_{T_y\ni y,u} z^{|T_y|} I[T_y \,\text{minimal rib at}\, u] \right).
\end{aligned}$$

Now

$$\sum_{T_x \ni x,u} z^{|T_x|} I[T_x \,\text{minimal rib at}\, u] \geq \frac{1}{g_z} G_z(x-u)$$

yields

$$\begin{aligned}
&\sum_u \sum_{\substack{T_0 \ni 0,u \\ T_x \ni x,u \\ T_y \ni y,u}} z^{|T_0|+|T_x|+|T_y|} I[T_x, T_y \,\text{minimal ribs at}\, u] \\
&\geq \frac{1}{g_z^2} \sum_u G_z(u) G_z(x-u) G_z(y-u)
\end{aligned}$$

and thus summation over x and y gives

$$\begin{aligned}
&\sum_{x,y} \sum_u \sum_{\substack{T_0 \ni 0,u \\ T_x \ni x,u \\ T_y \ni y,u}} z^{|T_0|+|T_x|+|T_y|} I[T_x, T_y \,\text{minimal ribs at}\, u] \\
&\geq \frac{1}{g_z^2} \sum_{x,y} \sum_u G_z(u) G_z(x-u) G_z(y-u) \\
&= \frac{1}{g_z^2} \chi(z)^3.
\end{aligned}$$

Let's have a look at the remaining three subtracted terms resulting from inclusion-exclusion. Plugging (26) into (25) and using that

$$\sum_{\substack{T_0 \ni 0,u \\ T_x \ni x,u}} z^{|T_0|+|T_x|} I[T_0 \cap T_x \supsetneq \{u\}] \leq \sum_{w \neq u} G_z^{(3)}(u,w) G_z^{(3)}(x-u, w-u),$$

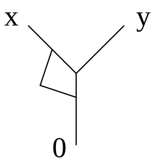
we obtain for the first of the three subtracted terms

$$\begin{aligned}
&\sum_u \sum_{\substack{T_0 \ni 0,u \\ T_x \ni x,u \\ T_y \ni y,u}} z^{|T_0|+|T_x|+|T_y|} I[T_x, T_y \,\text{minimal ribs at}\, u]\, I[T_0 \cap T_x \supsetneq \{u\}] \\
&\leq \sum_u \sum_{T_y \ni y,u} z^{|T_y|} \sum_{w \neq u} G_z^{(3)}(u,w) G_z^{(3)}(x-u, w-u).
\end{aligned}$$

Using that $\sum\limits_{T_y \ni y,u} z^{|T_y|} = G_z(u-y)$ this gives

$$\text{l.h.s.} \leq \sum_u G_z(u-y) \sum_{w \neq u} G_z^{(3)}(u,w) G_z^{(3)}(x-u, w-u),$$

which can be depicted as follows:



Using (7.24) we get

$$\begin{aligned}\text{l.h.s.} \quad \leq \quad & \sum_u G_z(u-y) \sum_{w \neq u} \sum_a G_z(a) G_z(u-a) G_z(w-a) \\ & \times \sum_b G_z(b) G_z(x-u-b) G_z(w-u-b).\end{aligned}$$

Let $\tilde{u} = u - a$, $\tilde{w} = w - a$ and $\tilde{b} = b - a$ to obtain

$$\begin{aligned}\text{l.h.s.} \quad \leq \quad & \sum_a \sum_{\tilde{u}} G_z(\tilde{u} + a - y) \sum_{\tilde{w} \neq \tilde{u}} G_z(a) G_z(\tilde{u}) G_z(\tilde{w}) \\ & \times \sum_{\tilde{b}} G_z(\tilde{b} + a) G_z(x - \tilde{u} - \tilde{b} - 2a) G_z(\tilde{w} - \tilde{u} - \tilde{b} - a).\end{aligned}$$

Furthermore replacing $\bar{b} = \tilde{b} + a + \tilde{u}$ yields

$$\begin{aligned}\text{l.h.s.} \quad \leq \quad & \sum_a \sum_{\tilde{u}} G_z(\tilde{u} + a - y) \sum_{\tilde{w} \neq \tilde{u}} G_z(a) G_z(\tilde{u}) G_z(\tilde{w}) \\ & \times \sum_{\bar{b}} G_z(\bar{b} - \tilde{u}) G_z(x - \bar{b} - a) G_z(\tilde{w} - \bar{b}).\end{aligned}$$

Next we sum this term over x and y to get

$$\begin{aligned}& \sum_{x,y} \sum_u \sum_{\substack{T_0 \ni 0,u \\ T_x \ni x,u \\ T_y \ni y,u}} z^{|T_0|+|T_x|+|T_y|} I[T_x, T_y \,\text{minimal ribs at}\, u]\, I[T_0 \cap T_x \supsetneq \{u\}] \\ & \leq \sum_{x,y} \sum_a \sum_{\tilde{u}} G_z(\tilde{u} + a - y) \sum_{\tilde{w} \neq \tilde{u}} G_z(a) G_z(\tilde{u}) G_z(\tilde{w}) \\ & \quad \times \sum_{\bar{b}} G_z(\bar{b} - \tilde{u}) G_z(x - \bar{b} - a) G_z(\tilde{w} - \bar{b}).\end{aligned}$$

Using that $\sum_x G_z(x) = \chi(z)$ this gives

$$\begin{aligned}
\text{l.h.s.} &\le \sum_a \sum_{\tilde u} \chi(z) \sum_{\tilde w \neq \tilde u} G_z(a) G_z(\tilde u) G_z(\tilde w) \\
&\quad \times \sum_{\bar b} G_z(\bar b - \tilde u) \chi(z) G_z(\tilde w - \bar b) \\
&= \chi(z)^3 \sum_{\tilde u} \sum_{\tilde w \neq \tilde u} G_z(\tilde u) G_z(\tilde w) \sum_{\bar b} G_z(\bar b - \tilde u) G_z(\tilde w - \bar b) \\
&= \chi(z)^3 \left(S(z) - \sum_{\tilde u} G_z(\tilde u)^2 \sum_{\bar b} G_z(\bar b - \tilde u) G_z(\tilde u - \bar b) \right)
\end{aligned}$$

and as $\sum_x G_z(x) \ge G_z(0) = g_z$ we finally get

$$\text{l.h.s.} \le \chi(z)^3 \left(S(z) - \left(G_z(0)^2 \right)^2 \right) = \chi(z)^3 \left(S(z) - g_z^4 \right).$$

The third and forth term of (26) plugged into (25) can be treated similarly. Taking all together the derivation of (7.28) is complete.

Solution to Exercise 7.7 by Jeffrey Hood.

(a) Following [62], note that $|z|^{-(n+1)} \le e$ on the contour $|z| = 1 - 1/n$, and thus

$$\begin{aligned}
|a_n| &= \left| \frac{1}{2\pi i} \oint_\Gamma f(z) \frac{dz}{z^{n+1}} \right| \\
&\le C \int_{-\pi}^{\pi} \left| f\left(\left(1 - n^{-1}\right) e^{i\theta} \right) \right| d\theta \;\le\; C \int_0^{\pi} \left| 1 - \left(1 - n^{-1}\right) e^{i\theta} \right|^{-b} d\theta \\
&= C \left(\int_0^{\pi/2} + \int_{\pi/2}^{\pi} \right) \left| 1 - \left(1 - n^{-1}\right) e^{i\theta} \right|^{-b} d\theta \\
&\le C \int_0^{\pi/2} \left| 1 - \left(1 - n^{-1}\right) e^{i\theta} \right|^{-b} d\theta + C \cdot 1 \cdot \frac{\pi}{2},
\end{aligned}$$

where $b \ge 1$. To bound the integrand on $\left[0, \frac{\pi}{2}\right]$, note that

$$\operatorname{Re}\left(1 - \left(1 - n^{-1}\right) e^{i\theta}\right) = 1 - \left(1 - n^{-1}\right) \cos\theta \ge n^{-1}$$

and, for $n \ge 2$,

$$\left| \operatorname{Im}\left(1 - \left(1 - n^{-1}\right) e^{i\theta}\right) \right| = \left| \left(1 - n^{-1}\right) \sin\theta \right| \ge \left(1 - n^{-1}\right) \frac{2\theta}{\pi} \ge \frac{\theta}{\pi}.$$

Therefore,

$$\left|1 - \left(1 - n^{-1}\right) e^{i\theta}\right| \ge \frac{1}{2}\left(\frac{1}{n} + \frac{\theta}{\pi}\right),$$

and so

$$\int_0^{\pi/2} \left|1 - \left(1 - n^{-1}\right) e^{i\theta}\right|^{-b} d\theta \le 2^b \int_0^{\pi/2} \left(\frac{1}{n} + \frac{\theta}{\pi}\right)^{-b} d\theta \le \begin{cases} K n^{b-1} & \text{if } b > 1, \\ K \log n & \text{if } b = 1. \end{cases}$$

(b) We have

$$\begin{aligned} z\chi(z) &= z \sum_x G_z(x) = z \sum_x \sum_{n=0}^{\infty} t_n^{(2)}(x) z^n \\ &= \sum_{n=0}^{\infty} \sum_x t_n^{(2)}(x) z^{n+1} = \sum_{n=0}^{\infty} \hat{t}_n^{(2)}(0) z^{n+1} \end{aligned}$$

and therefore

$$\frac{d\left[z\chi(z)\right]}{dz} = \frac{d}{dz} \sum_{n=0}^{\infty} \hat{t}_n^{(2)}(0) z^{n+1} = \sum_{n=0}^{\infty} (n+1) \hat{t}_n^{(2)}(0) z^n = \sum_{n=0}^{\infty} (n+1)^3 t_n z^n.$$

By using part (a) on the error term and expanding the first term of (7.41) in a binomial series, we have that

$$\begin{aligned} (n+1)^3 t_n &= \frac{A\sqrt{\pi}}{2} \lambda^n \binom{-\frac{3}{2}}{n} (-1)^n + \lambda^n O\Big(n^{-(-3/2+\epsilon)-1}\Big) \\ &= \frac{A\sqrt{\pi}}{2} \lambda^n \frac{(2n+1)!}{2^{2n}(n!)^2} + \lambda^n n^{1/2} O\Big(n^{-\epsilon}\Big) \\ &= \frac{A\sqrt{\pi}}{2} \lambda^n \frac{(2n+1)^{2n+1} e^{-2n-1} \sqrt{2\pi(2n+1)}}{2^{2n} n^{2n} e^{-2n} 2\pi n} + \lambda^n n^{1/2} O\Big(n^{-\epsilon}\Big) \\ &= \frac{A\sqrt{\pi}}{2} \lambda^n \frac{(2n)^{2n+1} \left(1 + \frac{1}{2n}\right)^{2n+1} e^{-1}}{(2n)^{2n} \sqrt{\pi n}} + \lambda^n n^{1/2} O\Big(n^{-\epsilon}\Big) \\ &= A \lambda^n n^{1/2} \left(1 + O\Big(n^{-\epsilon}\Big)\right) \end{aligned}$$

or

$$t_n = A \lambda^n n^{-5/2} \left(1 + O\Big(n^{-\epsilon}\Big)\right).$$

Solution to Exercise 8.1 by Jeffrey Hood.

As in (3.8) we get

$$K[0,b] = \sum_{\Gamma \in \mathcal{B}[0,b]} \prod_{st\in\Gamma} \mathcal{U}_{st}.$$

Split the sum's index set into three disjoint parts: (1) the set of graphs that do not contain 0 in an edge—$\mathcal{B}[1,b]$, (2) the set of graphs that are connected—$\mathcal{G}[0,b]$, and (3) the set A of remaining graphs. The contribution from (1) is $K[1,b]$, and the contribution from (2) is $J[0,b]$. For (3), associate with each Γ the number $j(\Gamma) = \max\{i : \Gamma|_{[0,i]} \in \mathcal{G}[0,i]\}$. If $\Gamma \in A$, then $1 \le j(\Gamma) \le b-1$, and so

$$\begin{aligned}
\sum_{\Gamma\in A} \prod_{st\in\Gamma} \mathcal{U}_{st} &= \sum_{i=1}^{b-1} \sum_{\substack{\Gamma\in\mathcal{B}[0,b]:\\ j(\Gamma)=i}} \prod_{st\in\Gamma} \mathcal{U}_{st} \\
&= \sum_{i=1}^{b-1} \sum_{\Gamma_1\in\mathcal{G}[0,i]} \sum_{\Gamma_2\in\mathcal{B}[i+1,b]} \prod_{st\in\Gamma_1} \mathcal{U}_{st} \prod_{qr\in\Gamma_2} \mathcal{U}_{qr} \\
&= \sum_{i=1}^{b-1} \left(\sum_{\Gamma\in\mathcal{G}[0,i]} \prod_{st\in\Gamma} \mathcal{U}_{st} \right) \left(\sum_{\Gamma\in\mathcal{B}[i+1,b]} \prod_{qr\in\Gamma} \mathcal{U}_{qr} \right) \\
&= \sum_{i=1}^{b-1} J[0,i] K[i+1,b].
\end{aligned}$$

Note that no over-counting is done in obtaining the right-hand side of the second equation because in each iteration of i, a gap is required in the union of Γ_1 and Γ_2 ensuring that such a graph will not be counted in later iterations. Taking this together we obtain the assertion.

Solution to Exercise 8.3 by Jeffrey Hood.

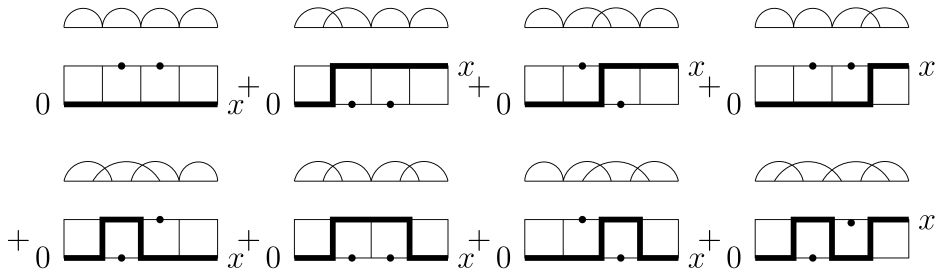
a) For a lace $L = (0t_1, s_2t_2, s_3|\omega|)$ consisting of exactly three edges, there are four generic configurations possible: $s_2 = t_1$, $s_3 = t_2$; $s_2 = t_1$, $s_3 < t_2$; $s_2 < t_1$, $s_3 = t_2$; and $s_2 < t_1$, $s_3 < t_2$. We illustrate the method for the first case. The contribution to $\Pi_z^{(2)}(x)$ due to these laces can be written

$$\sum_{\substack{\omega\in\mathcal{W}(0,x)\\ |\omega|\ge 3}} z^{|\omega|} \left[\prod_{i=0}^{|\omega|} \sum_{R_i \ni \omega(i)} z^{|R_i|} \right] \sum_{1\le s<t\le|\omega|-1} U_{0s} U_{st} U_{t|\omega|} \prod_{s't'\in\mathcal{C}(0s,st,t|\omega|)} (1+U_{s't'}).$$

For $U_{0s} \ne 0$, R_0 and R_s must intersect. Let y be a vertex where R_0 and R_s intersect, and let ω_{R_s} be the backbone of R_s joining $\omega(s)$ and y. For $U_{st} \ne 0$,

R_s and R_t must intersect, and hence there must be a rib emanating from a vertex on the backbone ω_{R_s} which intersects R_t. Finally, there must be a rib emanating from a vertex on the backbone ω_{R_t} which intersects $R_{|\omega|}$. By summing over the locations where these interections occur, neglecting the avoidances of the intermediate ribs with R_0, R_s, R_t, and $R_{|\omega|}$, and arguing in a similar fashion to the case $N = 1$, the contribution from this type of lace can be bounded by the Feynman diagram at far right in Figure 8.4. Note that an extra vertex occurs in the center loop because when summing over which ribs on backbones ω_{R_s} and ω_{R_t} result in intersections with later ribs in addition to the point where these ribs intersect, a pentagon is created in the center loop.

(b) The diagrams for $N = 4$ are shown below:



Solution to Exercise 9.1 by Jeremy Flowers.

(a) We approach this using two methods, each being instructive in a different way.

(Method 1) As the hint suggests (see [87, page 173]) we have $\eta = 1 - \theta(p)$ equal to the smallest root of

$$\mathbb{E}_p\left[s^X\right] = s,$$

where X is a binomial $(n = 2, p)$ random variable. It will be more useful to use $t = 1 - s$ and take the largest root of the equation. Calculating the expectation gives

$$\begin{aligned}
& (1-t)^2\mathbb{P}_p(X=2) + (1-t)\mathbb{P}_p(X=1) + \mathbb{P}_p(X=0) = 1-t \\
\Longleftrightarrow \quad & \left[(1-t)p + (1-p)\right]^2 = 1-t \\
\Longleftrightarrow \quad & \left[tp - 1\right]^2 = 1-t \\
\Longleftrightarrow \quad & t(p^2 t - 2p + 1) = 0.
\end{aligned}$$

Taking the largest root thus gives the result of the exercise,

$$\theta(p) = \max\left\{0, \frac{2p-1}{p^2}\right\}$$

which is 0 if $p \le 1/2$ and $(2p-1)/p^2$ if $p \ge 1/2$.

(Method 2) First we will need to condition on the two closest vertices to the origin. Let X_1 and X_2 be the occupation states, 0 or 1, for the bonds $\{0,1\}$ and $\{0,2\}$, respectively. There are 4 occupation states possible for the two bonds, their probabilities are listed below.

$$\begin{aligned}
\mathbb{P}(X_1 = 1, X_2 = 1) &= p^2 \\
\mathbb{P}(X_1 = 1, X_2 = 0) &= p(1-p) \\
\mathbb{P}(X_1 = 0, X_2 = 1) &= (1-p)p \\
\mathbb{P}(X_1 = 0, X_2 = 0) &= (1-p)^2.
\end{aligned}$$

We wish to calculate $\eta(p) = 1 - \theta(p) = \mathbb{P}(|C(0)| < \infty)$. Conditioning on the above events we see that:

$$\begin{aligned}
\eta &= \mathbb{P}(|C(0)| < \infty | X_1 = 1, X_2 = 1)\mathbb{P}(X_1 = 1, X_2 = 1) \\
&\quad + \mathbb{P}(|C(0)| < \infty | X_1 = 0, X_2 = 1)\mathbb{P}(X_1 = 0, X_2 = 1) \\
&\quad + \mathbb{P}(|C(0)| < \infty | X_1 = 1, X_2 = 0)\mathbb{P}(X_1 = 1, X_2 = 0) \\
&\quad + \mathbb{P}(|C(0)| < \infty | X_1 = 0, X_2 = 0)\mathbb{P}(X_1 = 0, X_2 = 0).
\end{aligned}$$

Let $C'(i) = \tilde{C}^{(0,i)}(i)$ be the set of vertices connected to i after the bond to 0 had been cut, for $i = 1, 2$. Understanding that the conditioning above is equivalent to restarting the system at the vertices closest to 0, we get:

$$\begin{aligned}
\eta &= \mathbb{P}(|C'(1)| < \infty, |C'(2)| < \infty | X_1 = 1, X_2 = 1)p^2 \\
&\quad + \mathbb{P}(|C'(1)| < \infty | X_1 = 1, X_2 = 0)p(1-p) \\
&\quad + \mathbb{P}(|C'(2)| < \infty | X_1 = 0, X_2 = 1)p(1-p) + (1-p)^2.
\end{aligned}$$

Noting that the events $\{|C'(1)| < \infty\}$, $\{|C'(2)| < \infty\}$ are independent and that $\mathbb{P}(|C'(1)| < \infty) = \mathbb{P}(|C'(2)| < \infty) = \mathbb{P}(|C(0)| < \infty) = \eta$ gives

$$\begin{aligned}
&\eta = (\eta p)^2 + 2(\eta p)(1-p) + (1-p)^2 \\
\iff\quad &\eta - 1 + 1 = ((\eta p) + (1-p))^2 = ((\eta - 1)p + 1)^2.
\end{aligned}$$

Replacing $\theta = 1 - \eta$ and expanding gives

$$0 = p^2\theta^2 + (1-2p)\theta,$$

so that $\theta = 0$ or$(1-2p)/p^2$.

The second method only assures us that $\theta = 0$ for $p \leq 1/2$ and that θ may or may not be $(1-2p)/p^2$ for $p > 1/2$. However, part (b) allows us to assert that $\pi_c = 1/2$ and Exercise 9.6 will allow us to assert $\pi_c = p_c$. From this $p_c = 1/2$ follows and the result of the exercise follows. The reader should be aware that the theorem which allows us to assert that $\pi_c = p_c$ is far more grandiose in nature than that of method one, which is why both methods have been taken into consideration here.

(b) Proceeding with a similar calculation to above we have:

$$\begin{aligned}
\chi(p) &= \mathbb{E}(|C(0)|)\mathbb{E}(|C(0)||X_1=1,X_2=1)\mathbb{P}(X_1=1,X_2=1) \\
&\quad + \mathbb{E}(|C(0)||X_1=0,X_2=1)\mathbb{P}(X_1=0,X_2=1) \\
&\quad + \mathbb{E}(|C(0)||X_1=1,X_2=0)\mathbb{P}(X_1=1,X_2=0) \\
&\quad + \mathbb{E}(|C(0)||X_1=0,X_2=0)\mathbb{P}(X_1=0,X_2=0) \\
&= \mathbb{E}(|C'(1)|+|C'(2)|+1|X_1=1,X_2=1)p^2 \\
&\quad + \mathbb{E}(|C'(2)|+1|X_1=0,X_2=1)p(1-p) \\
&\quad + \mathbb{E}(|C'(1)|+1|X_1=1,X_2=0)p(1-p) \\
&\quad + \mathbb{E}(1|X_1=0,X_2=0)(1-p)^2 \\
&= (2\mathbb{E}(|C(0)|)+1)p^2 + 2(\mathbb{E}(|C(0)|)+1)p(1-p) + 1(1-p)^2 \\
&= (2\chi+1)p^2 + 2(\chi+1)p(1-p) + 1(1-p)^2.
\end{aligned}$$

It follows readily that $\chi = \frac{1}{1-2p}$.

Solution to Exercise 9.6 by Jeremy Flowers.

(a) By Markov's inequality we have that

$$\mathbb{P}(|C(0)| = \infty) \leq \mathbb{P}(|C(0)| \geq N) \leq \frac{1}{N}\mathbb{E}(|C(0)|).$$

Letting $N \to \infty$ gives that $\theta(p) = 0$ whenever $\chi(p) < \infty$ and implies $\pi_c \leq p_c$.

(b) Assume $\pi_c < p < p_c$. Pick $\epsilon > 0$ small enough so that $\pi_c + \epsilon < p$. By definition of π_c, $\pi_c + \epsilon$ is such that $\chi^f(p) = \infty$ since for $p < p_c$ there is almost surely no infinite cluster. By Theorem 9.5 either $\theta(\pi_c+\epsilon)>0$, or $\theta(\pi_c+\epsilon)=0$ and $\theta(p)\geq\frac{1}{2p}(p-(\pi_c+\epsilon))>0$. Both of these cases would give a contradiction to the definition of p_c and the result, $\pi_c=p_c$, follows.

Solution to Exercise 9.14 by Jeremy Flowers.

We first show

$$\begin{aligned}&\{(u,v)\text{ pivotal for }0\leftrightarrow x\}\\&\subset\left\{0\leftrightarrow u\text{ in }\tilde{C}^{(u,v)}(0)\right\}\cap\left\{v\leftrightarrow x\text{ in }\mathbb{Z}^d\setminus\tilde{C}^{(u,v)}(0)\right\}.\end{aligned}$$

Let $\omega\in\{(u,v)\text{ pivotal for }0\leftrightarrow x\}$: then it follows from the definition of pivotal that $0\leftrightarrow u$ and $v\leftrightarrow x$ after the bond has been cut. It is immediate that $u\in\tilde{C}^{(u,v)}(0)$, so all that remains is to show $v\notin\tilde{C}^{(u,v)}(0)$. Suppose $v\in\tilde{C}^{(u,v)}(0)$; then $v\leftrightarrow y$ where $y\in\tilde{C}^{(u,v)}(0)$ which implies that $x\leftrightarrow v\leftrightarrow y\leftrightarrow 0$ after the bond has been cut. $x\leftrightarrow 0$ is a contradiction because (u,v) is pivotal and the inclusion above follows.

Secondly, we show

$$\begin{aligned}&\left\{0\leftrightarrow u\text{ in }\tilde{C}^{(u,v)}(0)\right\}\cap\left\{v\leftrightarrow x\text{ in }\mathbb{Z}^d\setminus\tilde{C}^{(u,v)}(0)\right\}\\&\subset\{(u,v)\text{ pivotal for }0\leftrightarrow x\}.\end{aligned}$$

Suppose $\omega\in\left\{0\leftrightarrow u\text{ in }\tilde{C}^{(u,v)}(0)\right\}\cap\left\{v\leftrightarrow x\text{ in }\mathbb{Z}^d\setminus\tilde{C}^{(u,v)}(0)\right\}$. Then, $0\leftrightarrow u$ and $v\leftrightarrow x$ after the bond has been cut is obvious. It remains to show that 0 is not connected to x after the cut. Assume that $0\leftrightarrow x$ after the bond has been cut. $0\leftrightarrow x$ and $x\leftrightarrow v$ imply that $0\leftrightarrow v$; that is, $v\in\tilde{C}^{(u,v)}(0)$. But then $v\in\mathbb{Z}^d\setminus\tilde{C}^{(u,v)}(0)$ for this configuration, which is a contradiction and so the inclusion above follows.

Solution to Exercise 9.16 by Jeremy Flowers.

First note that

$$\chi(p,\gamma)=\sum_{n=1}^{\infty}n(1-\gamma)^nP_n(p)$$

and

$$\tau_{p,\gamma}(0,x)=\mathbb{P}_{p,\gamma}(0\leftrightarrow x,0\not\leftrightarrow G).$$

Conditioning on possible lattice animals connected to 0 with n vertices and not connected to G gives

$$\begin{aligned}
&\sum_{x\in\mathbb{Z}^d}\tau_{p,\gamma}(0,x)\\
&=\sum_{x\in\mathbb{Z}^d}\sum_{n=1}^{\infty}\sum_{a\in\mathcal{A}_n}\mathbb{E}_{p,\gamma}(I(0\leftrightarrow x, 0\not\leftrightarrow G)|C(0)=a)\,\mathbb{P}_{p,\gamma}(C(0)=a)\\
&=\sum_{n=1}^{\infty}\sum_{a\in\mathcal{A}_n}\mathbb{E}_{p,\gamma}\left(\sum_{x\in\mathbb{Z}^d}I(0\leftrightarrow x, 0\not\leftrightarrow G)\middle| C(0)=a\right)\mathbb{P}_{p,\gamma}(C(0)=a).
\end{aligned}$$

There are n vertices and hence only n values of x for which the indicator above is not 0. Thus,

$$\sum_{x\in\mathbb{Z}^d}\tau_{p,\gamma}(0,x)=\sum_{n=1}^{\infty}n\sum_{a\in\mathcal{A}_n}\mathbb{E}_{p,\gamma}(I(0\not\leftrightarrow G)|C(0)=a)\,\mathbb{P}_{p,\gamma}(C(0)=a).$$

The n fixed vertices not connected to G carry a weight of $(1-\gamma)^n$, so we have

$$\sum_{x\in\mathbb{Z}^d}\tau_{p,\gamma}(0,x)=\sum_{n=1}^{\infty}n(1-\gamma)^n\sum_{a\in\mathcal{A}_n}\mathbb{P}_{p,\gamma}(C(0)=a).$$

Summing on all animals with n vertices gives

$$\sum_{x\in\mathbb{Z}^d}\tau_{p,\gamma}(0,x)=\sum_{n=1}^{\infty}n(1-\gamma)^n\mathbb{P}_{p,\gamma}(|C(0)|=n),$$

and the result follows by equation (9.29).

Solution to Exercise 9.18 by Jeremy Flowers.

Recall the definition of $F_{(u,v)}$:

$$F_{(u,v)}=\{(u,v)\text{ pivotal for }0\leftrightarrow G\}\cap\{(u,v)\text{ is occupied}\}\cap\{v\Leftrightarrow G\}.$$

First, we show:

$$\begin{aligned}
&\{(u,v)\text{ pivotal for }0\leftrightarrow G\}\cap\{(u,v)\text{ is occupied}\}\cap\{v\Leftrightarrow G\}\\
&\subset\left\{0\leftrightarrow u\text{ in }\mathbb{Z}^d\setminus\tilde{C}^{(u,v)}(v)\right\}\cap\left\{0\not\leftrightarrow G\text{ in }\mathbb{Z}^d\setminus\tilde{C}^{(u,v)}(v)\right\}\\
&\qquad\cap\{(u,v)\text{ is occupied}\}\cap\{v\Leftrightarrow G\text{ after }\{\text{u,v}\}\text{ is made vacant}\}.
\end{aligned}$$

(u,v) pivotal for $0\leftrightarrow G$ and $v\Leftrightarrow G$ implies that $v\Leftrightarrow G$ after (u,v) has been cut. This is because if one of the connections from v to G went through

(u,v) then $u \leftrightarrow G$ after the bond has been cut. But (u,v) is pivotal and we have a contradiction.

Also, if $0 \leftrightarrow y$, $y \in \tilde{C}^{(u,v)}(v)$, then $0 \leftrightarrow v$ after the pivotal bond has been cut, which is a contradiction. Thus we must have $0 \leftrightarrow u$ in $\mathbb{Z}^d \setminus \tilde{C}^{(u,v)}(v)$.

For $0 \not\leftrightarrow G$ in $\mathbb{Z}^d \setminus \tilde{C}^{(u,v)}(v)$ we assume the contrary again. This would imply that $0 \leftrightarrow y \leftrightarrow G$, $y \in \tilde{C}^{(u,v)}(v)$. But this would mean that we have a connection from 0 to G that does not contain (u,v), a contradiction since (u,v) is pivotal.

Second, we show:

$$
\begin{aligned}
&\left\{0 \leftrightarrow u \text{ in } \mathbb{Z}^d \setminus \tilde{C}^{(u,v)}(v)\right\} \cap \left\{0 \not\leftrightarrow G \text{ in } \mathbb{Z}^d \setminus \tilde{C}^{(u,v)}(v)\right\} \\
&\cap \{(u,v) \text{ is occupied}\} \cap \{v \Leftrightarrow G \text{ after } \{\text{u,v}\} \text{ is made vacant}\} \\
&\qquad \subset \{(u,v) \text{ pivotal for } 0 \leftrightarrow G\} \cap \{(u,v) \text{ is occupied}\} \cap \{v \Leftrightarrow G\}\,.
\end{aligned}
$$

$v \Leftrightarrow G$ is obvious as there will still be a double connection if we make (u,v) occupied.

For (u,v) pivotal for $0 \leftrightarrow G$ we need to show that after the bond is cut, $0 \not\leftrightarrow G$, $0 \leftrightarrow u$ and $v \leftrightarrow G$.

The connection of v to G has already been established. Note that $0 \leftrightarrow u$ in $\mathbb{Z}^d \setminus \tilde{C}^{(u,v)}(v)$ implies at least that 0 is connected to u (and v since the bond is occupied) before the bond is cut. If 0 is not connected to u after the bond is cut, we must have that 0 is connected to v after the cut. This is a contradiction because the connection from 0 to u cannot have any vertices that would be connected to v after the bond has been cut.

For $0 \not\leftrightarrow G$ after the cut, suppose the contrary. But $0 \not\leftrightarrow G$ in $\mathbb{Z}^d \setminus \tilde{C}^{(u,v)}(v)$ would imply that we must go through $\tilde{C}^{(u,v)}(v)$ to get a connection to G. However, we already have $0 \leftrightarrow y \leftrightarrow u$, $y \in \mathbb{Z}^d \setminus \tilde{C}^{(u,v)}(v)$. This would imply that y is connected to v, and we have a contradiction. The above inclusion then follows, asserting the claim of the exercise.

Solution to Exercise 11.5 by Jeremy Flowers.

(a) Consider

$$
\sum_{z \in \mathbb{Z}^d : |x-z| \le \frac{1}{2}|x|} \tau_{p_c}(z)\tau_{p_c}(y-z)\,\frac{\tau_{p_c}(x-z)}{\tau_{p_c}(x)}.
$$

From Theorem 11.2, we see that for large enough x we have the bound

$$\begin{aligned}
&\sum_{z\in\mathbb{Z}^d:|x-z|\le\frac{1}{2}|x|} \tau_{p_c}(z)\tau_{p_c}(y-z)\,\frac{\tau_{p_c}(x-z)}{\tau_{p_c}(x)} \\
&\le \text{const} \sum_{z\in\mathbb{Z}^d:|x-z|\le\frac{1}{2}|x|} \frac{|x|^{d-2}}{|z|^{d-2}|y-z|^{d-2}}\,\tau_{p_c}(x-z).
\end{aligned}$$

Notice that $||x|-|z|| \le |x-z| \le \frac{1}{2}|x|$ implies $\frac{1}{2}|x| \le |z| \le \frac{3}{2}|x|$. We may also take $\frac{1}{4}|x| \ge |y|$ so that $\frac{1}{4}|x| \le \frac{1}{2}|x| + |y| \le |y-z| \le |y| + \frac{3}{2}|x| \le 2|x|$. We thus have a further bound:

$$\text{l.h.s.} \le \text{const} \sum_{z\in\mathbb{Z}^d:|x-z|\le\frac{1}{2}|x|} \frac{|x|^{d-2}}{|x|^{d-2}|x|^{d-2}}\,\tau_{p_c}(x-z).$$

Reordering the summation and using Theorem 11.2 again,

$$\begin{aligned}
\text{l.h.s.} &= \text{const}\,\frac{1}{|x|^{d-2}} \sum_{u\in\mathbb{Z}^d:|u|\le\frac{1}{2}|x|} \tau_{p_c}(u) \\
&\le \text{const}\,\frac{1}{|x|^{d-2}} \sum_{u\in\mathbb{Z}^d:|u|\le\frac{1}{2}|x|} \frac{1}{|u|^{d-2}} \\
&\le \text{const}\,\frac{1}{|x|^{d-2}}\,\frac{|x|^d}{|x|^{d-2}} = \text{const}\,\frac{1}{|x|^{d-4}}.
\end{aligned}$$

The exercise follows by taking the limit as $|x| \to \infty$.

(b) Expanding the convolution and bounding above,

$$\begin{aligned}
|f * g(x)| &= \left|\sum_{y\in\mathbb{Z}^d} f(y)g(x-y)\right| \\
&\le \sum_{y\in\mathbb{Z}^d} (|y|+1)^{2-d}(|y-x|+1)^{2-d} \\
&= \sum_{y:|x-y|\le|y|} (|y|+1)^{2-d}(|y-x|+1)^{2-d} \\
&\quad + \sum_{y:|x-y|>|y|} (|y|+1)^{2-d}(|y-x|+1)^{2-d}.
\end{aligned}$$

Change variables in the second sum to $z = x - y$,

$$\begin{aligned}
\text{r.h.s.} &= \sum_{y:|x-y|\le|y|} (|y|+1)^{2-d}(|y-x|+1)^{2-d} \\
&\quad + \sum_{z:|z-x|<|z|} (|z-x|+1)^{2-d}(|z|+1)^{2-d} \\
&< 2\sum_{y:|x-y|\le|y|} (|y|+1)^{2-d}(|y-x|+1)^{2-d},
\end{aligned}$$

where we must add in $|y| = |z-x|$ above. Note that $|x-y| \le |y|$ implies $|y| \ge \frac{1}{2}|x|$. Thus we may split the sum into two intervals, $\frac{1}{2}|x| \le |y| \le \frac{3}{2}|x|$ and $|y| > \frac{3}{2}|x|$. This gives,

$$\begin{aligned}
\text{r.h.s.} &= 2\sum_{y:|x-y|\le|y|,\frac{1}{2}|x|\le|y|\le\frac{3}{2}|x|} (|y|+1)^{2-d}(|y-x|+1)^{2-d} \\
&\quad +2\sum_{y:|x-y|\le|y|,\frac{3}{2}|x|<|y|} (|y|+1)^{2-d}(|y-x|+1)^{2-d}.
\end{aligned}$$

Note that $|y| \ge \frac{3}{2}|x|$ implies $|y-x| \ge |y| - |x| \ge |y|/3$. This gives

$$\begin{aligned}
\text{r.h.s.} &\le 2\left[2^{d-2}\frac{1}{(|x|+1)^{d-2}}\sum_{y:|x-y|\le|y|\le\frac{3}{2}|x|} (|y-x|+1)^{2-d}\right. \\
&\quad \left. +3^{d-2}\sum_{y:|x-y|\le\frac{3}{2}|x|\le|y|} (|y|+1)^{2(2-d)}\right] \\
&\le 2\left[2^{d-2}\frac{1}{(|x|+1)^{d-2}}\sum_{u:|u|\le\frac{3}{2}|x|} (|u|+1)^{2-d}\right. \\
&\quad \left. +3^{d-2}\sum_{y:\frac{3}{2}|x|\le|y|} (|y|+1)^{2(2-d)}\right] \\
&\le \text{const}\frac{1}{(|x|+1)^{d-2}}\frac{(|x|+1)^d}{(|x|+1)^{d-2}} + \text{const}\frac{(|x|+1)^d}{(|x|+1)^{2(d-2)}} \\
&= \text{const}(|x|+1)^{4-d},
\end{aligned}$$

where in the second line we have summed over more terms, and the result follows. Note that we implicitly use $\left(\frac{|x|}{c}+1\right)^{2-d} = c^{d-2}(|x|+c)^{2-d} \leq c^{d-2}(|x|+1)^{2-d}$, for $c \geq 1, d > 1$.

(c) As stated in the hint, we have:

$$\begin{aligned}
&\lim_{x\to\infty} \sum_{z\in\mathbb{Z}^d:|x-z|>\frac{1}{2}|x|} \tau_{p_c}(z)\tau_{p_c}(y-z)\,\frac{\tau_{p_c}(x-z)}{\tau_{p_c}(x)} \\
&\leq \lim_{x\to\infty} \sup_{z\in\mathbb{Z}^d:|x-z|>\frac{1}{2}|x|} \frac{\tau_{p_c}(x-z)}{\tau_{p_c}(x)} \sum_{z\in\mathbb{Z}^d:|x-z|>\frac{1}{2}|x|} \tau_{p_c}(z)\tau_{p_c}(y-z) \\
&\leq \text{const} \sum_{z\in\mathbb{Z}^d} \tau_{p_c}(z)\tau_{p_c}(y-z).
\end{aligned}$$

Part (b) gives the desired contribution:

$$\sum_{z\in\mathbb{Z}^d:|x-z|>\frac{1}{2}|x|} \tau_{p_c}(z)\tau_{p_c}(y-z)\,\frac{\tau_{p_c}(x-z)}{\tau_{p_c}(x)} \leq \text{const}\frac{1}{(|y|+1)^{d-4}}.$$

Solution to Exercise 12.2 by Jeremy Flowers.

First note that (12.20) is an inverse Fourier transform and we need only show that the integrand is the Fourier transform of the triangle condition. Let $f(s,x) = \tau_{p,s}(x)$, $g(s,x) = f(-s,-x) = \tau_{p,s}(-x)$ and consider (12.18) as:

$$\nabla_p(n,x) = \sum_{(y,l)}\sum_{(w,m)} f(l,y)f(m-l,w-y)g(n-m,x-w) = (f*f*g)(n,x).$$

The Fourier transform of $\nabla_p(n,x)$ is

$$\widehat{\nabla_p}(k,\theta) = \hat{f}^2(k,\theta)\hat{g}(k,\theta).$$

Using translation invariance and reordering the summations we see that:

$$\hat{g}(k,\theta) = \sum_{x,n} f(-n,-x)e^{in\theta}e^{ik\cdot x} = \sum_{x,n} f(n,x)e^{in(-\theta)}e^{ik\cdot x} = \hat{f}(k,-\theta).$$

Setting $e^{i\theta} = s$ allows us to change from the the Fourier transform to the generating function. This gives the desired result,

$$\widehat{\nabla_p}(k,\theta) = \hat{t}_p(k,e^{i\theta})^2\,\hat{t}_p(k,e^{-i\theta}).$$

Now, the Riemann-Lebesgue Lemma states that

$$\lim_{|(x,n)|\to\infty} \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \int_{[-\pi,\pi]^d} \frac{d^d k}{(2\pi)^d} g(k,\theta) e^{-ik\cdot x} e^{-i\theta n} = 0,$$

provided g is integrable. Thus we need to consider

$$\int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \int_{[-\pi,\pi]^d} \frac{d^d k}{(2\pi)^d} \left|\hat{t}_p(k, e^{\pm i\theta})\right|^3.$$

The infrared bound implies

$$\begin{aligned}
&\int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \int_{[-\pi,\pi]^d} \frac{d^d k}{(2\pi)^d} \left|\hat{t}_p(k, e^{\pm i\theta})\right|^3 \\
&\leq \text{ const} \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \int_{[-\pi,\pi]^d} \frac{d^d k}{(2\pi)^d} \frac{1}{(|k|^2 + |\theta|)^3} \\
&= \text{ const} \int_{[-\pi,\pi]^d} \frac{d^d k}{(2\pi)^d} \left[\frac{1}{(|k|^4} - \frac{1}{(|k|^2+\pi)^2} \right].
\end{aligned}$$

The $\frac{1}{|k|^4}$ term implies that the integral is bounded for $d > 4$ and the triangle condition follows for these dimensions.

Solution to Exercise 12.4 by Jeremy Flowers.

(a) The hint implies that we must show that there is no, or at least a negligible, contribution to the sum of the triangle condition for $|(x,n)| \geq R$ and $m \leq cR$ for some value of c and large enough R. We will show this is true for $c = \frac{1}{4L}$ for the spread out model, $L = 1$ for the nearest neighbor model.

Notice that $\tau_k(z) = 0$ for $|z| > Lk$ because the furthest a random walk can go in k steps is Lk. For the problem at hand we see that the term

$$\tau_{p_c,l}(y)\tau_{p_c,m-l}(w-y)\tau_{p_c,m-n}(w-x)$$

will be zero off of the set defined by

$$\begin{aligned}
& |y| \leq Ll, \\
\iff\quad & |w-y| \leq L(m-l), \text{ and} \\
\iff\quad & |w-x| \leq L(m-n).
\end{aligned}$$

Also, there will only be a contribution for $m - n \geq 0$. Notice that $|x| \leq |x-w| + |w-y| + |y| \leq L(l + (m-l) + (m-n)) \leq Lm$. From this we see that there will be no contribution for $\left\{m \leq \frac{1}{L}|x|\right\}$.

First we consider $n^2 \leq \frac{3}{4}R$ and notice $|(x,n)| \geq R$ implies that $|x| \geq \frac{1}{2}R$. Thus there will be no contribution for $\left\{m \leq \frac{1}{2L}R\right\}$, so taking $c = \frac{1}{4L}$ will give no contribution.

Now considering $n^2 \geq \frac{3}{4}R$ together with the condition that $m \geq n$ and taking $c = \frac{1}{4L}$ obviously gives no contribution as well, for any L. Thus, $m \leq cR$ and $|(x,n)| \geq R$ will give no contribution in the triangle condition sum for $c = \frac{1}{4L}$.

(b) First consider $l \leq m/2$. Noting $\| \tau_{p_c,n} \|_\infty \leq \text{const}\, n^{-d/2}$ gives

$$\begin{aligned}
&\sum_{(w,m):\, m \geq cR} \sum_{(y,l):\, l \leq m/2} \tau_{p_c,l}(y) \tau_{p_c,m-l}(w-y) \tau_{p_c,m-n}(w-x) \\
&\leq \text{ const} \sum_{(w,m):\, m \geq cR} \sum_{(y,l):\, l \leq m/2} \tau_{p_c,l}(y)(m-l)^{-d/2} \tau_{p_c,m-n}(w-x) \\
&\leq \text{ const} \sum_{(w,m):\, m \geq cR} \sum_{(y,l):\, l \leq m/2} \tau_{p_c,l}(y)(m/2)^{-d/2} \tau_{p_c,m-n}(w-x).
\end{aligned}$$

$\| \tau_{p_c,n} \|_1 = O(1)$ gives

$$\begin{aligned}
&\sum_{(w,m):\, m \geq cR} \sum_{(y,l):\, l \leq m/2} \tau_{p_c,l}(y) \tau_{p_c,m-l}(w-y) \tau_{p_c,m-n}(w-x) \\
&\leq \text{ const} \sum_{(w,m): m \geq cR} m(m)^{-d/2} \tau_{p_c,m-n}(w-x) \\
&\leq \text{ const} \sum_{m \geq cR} (m)^{-(d-2)/2} \\
&\leq \text{ const} \frac{1}{(cR)^{(d-4)/2}}.
\end{aligned}$$

Letting $R \to \infty$, we see that this goes to 0 provided $d > 4$.

Now consider $l \geq m/2$:

$$\begin{aligned}
&\sum_{(w,m):m\geq cR}\sum_{(y,l):l\geq m/2}\tau_{p_c,l}(y)\tau_{p_c,m-l}(w-y)\tau_{p_c,m-n}(w-x)\\
&\leq \text{const}\sum_{(w,m):m\geq cR}\sum_{(y,l):l\geq m/2} l^{-d/2}\tau_{p_c,m-l}(w-y)\tau_{p_c,m-n}(w-x)\\
&\leq \text{const}\sum_{(w,m):m\geq cR}\sum_{l\geq m/2} l^{-d/2}\tau_{p_c,m-n}(w-x)\\
&\leq \text{const}\sum_{(w,m):m\geq cR} m^{-(d-2)/2}\tau_{p_c,m-n}(w-x)\\
&\leq \text{const}\sum_{m\geq cR} m^{-(d-2)/2} \leq \text{const}\frac{1}{(cR)^{(d-4)/2}}.
\end{aligned}$$

Taking the limit $R \to \infty$ gives the result for the second interval and the triangle condition follows.

Solution to Exercise 12.8 by Jeremy Flowers.

First we calculate the moment generating functions for exponential and size-biased exponential random variables.

For exponential random variables,

$$M_X(-t) = E\Big[e^{-tX}\Big] = \int_0^\infty e^{-tx}\lambda e^{-\lambda x}dx = \int_0^\infty \lambda e^{-(t+\lambda)x}dx = \frac{\lambda}{\lambda+t}.$$

By differentiating the moment generating function we see directly that

$$E\Big[X^l\Big] = (-1)^l\frac{d^l}{dt^l}M_X(-t)|_{t=0} = \lambda^{-l}l!.$$

This matches up with equation (12.43) and $\lambda = \frac{2}{A^2V}$.

For the size-biased random variables,

$$M_X(t) = E\Big[e^{-tX}\Big] = \int_0^\infty e^{-tx}\lambda^2 x e^{-\lambda x}dx = \lambda^2\int_0^\infty x e^{-(t+\lambda)x}dx.$$

Integrating by parts gives

$$M_X(t) = \lambda^2\int_0^\infty \frac{1}{\lambda+t}e^{-(t+\lambda)x}dx = \frac{\lambda^2}{(\lambda+t)^2}.$$

By differentiating the moment generating function we see directly that,

$$E\Big[X^l\Big] = (-1)^l\frac{d^l}{dt^l}M_X(t)|_{t=0} = \lambda^{-l}(l+1)!.$$

This matches up with equation (12.42) and $\lambda = \frac{2}{A^2 V}$.

Equations (12.42) and (12.43) imply that the moments converge to the moments of the specified random variables. See Theorems 30.1 and 30.2 in Billingsley's "Probability and Measure", 3rd edition, which state that the convergence of moments is a sufficient condition for weak convergence. The statements of Theorem 12.7 follow.

Solution to Exercise 13.1 by Jeremy Flowers.

Note that:

$$\begin{aligned}&\{b \text{ is occupied}\} \cap E'\Big(\bar{b}, (x,n); \tilde{C}^b(w,l)\Big)\\ &= \Big\{\bar{b} \overset{\tilde{C}^b(w,l)}{\longleftrightarrow} (x,n)\Big\} \cap \Big\{\nexists\, (u',v') \subset P_{(\bar{b},(x,n))} \text{ such that } \bar{b} \overset{\tilde{C}^b(w,l)}{\longleftrightarrow} u'\Big\}.\end{aligned}$$

First, we show

$$\begin{aligned}E\Big(b, (x,n); \tilde{C}^b(w,l)\Big) \quad \subset \quad & \{b \text{ is occupied}\} \cap \Big\{\bar{b} \overset{\tilde{C}^b(w,l)}{\longleftrightarrow} (x,n)\Big\}\\ & \cap \Big\{\nexists\, (u',v') \subset P_{(\bar{b},(x,n))} \text{ such that } \bar{b} \overset{\tilde{C}^b(w,l)}{\longleftrightarrow} u'\Big\}.\end{aligned}$$

For configurations in the left set, the fact that b is occupied is obvious. Now consider any connection from $\bar{b}$ to (x,n). We know (point 2) that $\bar{b} \to (y,m) \to (x,n)$ for some $(y,m) \in \tilde{C}^b(w,l)$. But this implies that $(x,n) \in \tilde{C}^b(w,l)$ and so $\bar{b} \overset{\tilde{C}^b(w,l)}{\longleftrightarrow} (x,n)$ follows.

For the last set notice that if there were a connection between $\bar{b}$ and u that went through $\tilde{C}^b(w,l)$ there would be a vertex on the backbone in $\tilde{C}^b(w,l)$ and the third point in the definition would be contradicted. The inclusion above then follows.

Next we show

$$\begin{aligned}E\Big(b, (x,n); \tilde{C}^b(w,l)\Big) \quad \supset \quad & \{b \text{ is occupied }\} \cap \Big\{\bar{b} \overset{\tilde{C}^b(w,l)}{\longleftrightarrow} (x,n)\Big\}\\ & \cap \Big\{\nexists\, (u',v') \subset P_{(\bar{b},(x,n))} \text{ such that } \bar{b} \overset{\tilde{C}^b(w,l)}{\longleftrightarrow} u'\Big\}.\end{aligned}$$

For a configuration on the right it is obvious that point 1 is satisfied. Point 2 is also obvious because $\bar{b} \overset{\tilde{C}^b(w,l)}{\longleftrightarrow} (x,n)$ implies any connection from $\bar{b}$ to (x,n) must go through $\tilde{C}^b(w,l)$ and we can see from Definition 9.13

that there must be at least one connection. So, there is at least one point $(y, m) \in \tilde{C}^b(w, l)$ such that $\bar{b} \to (y, m) \to (x, n)$.

If point 3 were not true then there would exist a pivotal bond, b', with a vertex in $\tilde{C}^b(w, l)$ on the back bone. But this would imply that $\underline{b}' \in \tilde{C}^b(w, l)$ which is a contradiction because all paths connecting $\bar{b}$ to $\underline{b}'$ must not contain a vertex in $\tilde{C}^b(w, l)$. The inclusion above and the result of the exercise then follow.

Solution to Exercise 15.3 by Jesse Goodman.

When $r = 2$, $2r - 3 = 1$ so $\vec{y}, \vec{m}, \vec{\zeta}, \bar{\imath}$ reduce to x, m, ζ, i. Also, there is only one possible σ so it can be omitted. The condition that (T, φ, i) is compatible with (σ, x, m) reduces as follows. Condition 1 is trivial, since β_B will either be the same as σ or else degenerate. Condition 2 requires that $|\, i\,| = m$ (including in the degenerate case where $i = 0$). Condition 3 requires that $\varphi(i) = x$. So (15.24) becomes

$$\begin{aligned} b^{(2)}_{z,\zeta}(x) &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} b^{(r)}_n(\sigma; x, m) z^n \zeta^m \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{(T,\varphi)\,:\,|\,T\,|=n} \mathbb{P}(T, \varphi) z^n \sum_{i \in T} I[|\, i\,| = m] I[\varphi(i) = x] \zeta^m \\ &= \sum_{n=0}^{\infty} \sum_{(T,\varphi)\,:\,|\,T\,|=n} \mathbb{P}(T, \varphi) z^n \sum_{i \in T} I[\varphi(i) = x] \zeta^{|\,i\,|} \\ &= \sum_{(T,\varphi)} \mathbb{P}(T, \varphi) z^{|\,T\,|} \sum_{i \in T} I[\varphi(i) = x] \zeta^{|\,i\,|} \end{aligned}$$

as in (15.16).

Solution to Exercise 15.6 by Sandra Kliem.

Figure 3 illustrates the given setting. The six distinct (T, φ) that correspond to the collection (T_a, φ_a) $(a = 1, 2, 3)$ as in the last paragraph of the proof of Theorem 15.5 are given in Figure 4.

Solution to Exercise 16.1 by Jesse Goodman.

$b^{(1)}_z$ satisfies $b^{(1)}_0 = 0$ and is a solution to

$$b^{(1)}_z e^{-b^{(1)}_z} = z e^{-1}. \tag{27}$$

Define $g(w) = w e^{-w}$ and note that $g'(w) = (1 - w) e^{-w} \neq 0$ except at $w = 1$. So g is locally invertible except at $w = 1$. By (27), $b^{(1)}_z$ can only equal 1 at

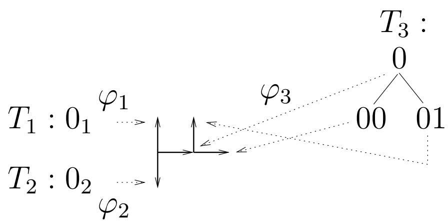


Figure 3: The setting of Exercise 15.6.

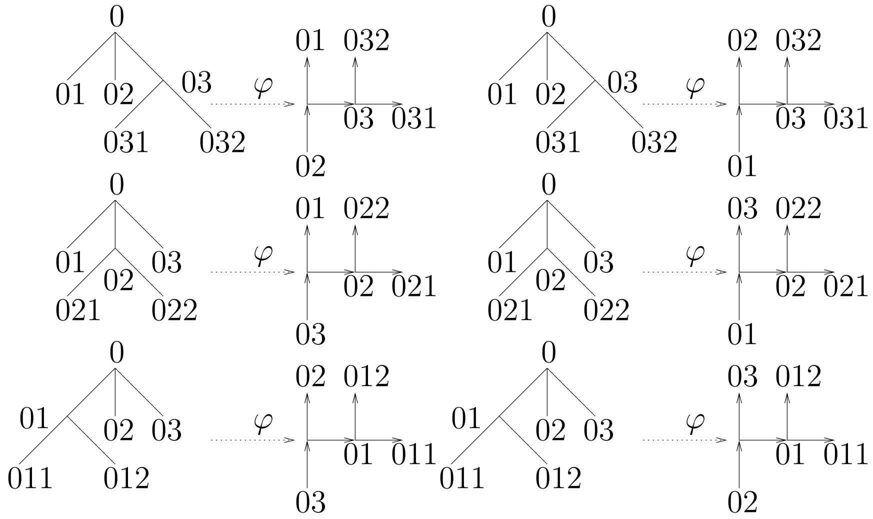


Figure 4: The six distinct (T, φ) corresponding to (T_a, φ_a) $(a = 1, 2, 3)$ as in the last paragraph of the proof of Theorem 15.5.

$z = 1$. So $b_z^{(1)}$ can be continued analytically to be defined in the cut plane $\mathbb{C}\backslash[1, \infty)$ (which is simply connected).

$g'(1) = 0$ but $g''(1) = -e^{-1}$, so near $w = 1$, $g(w) = g(1) - e^{-1}h(w)^2$ for some analytic h defined near $w = 1$ with $h(1) = 0$, $h'(1) \neq 0$ (the $-e^{-1}$ is a convenient scaling factor). Therefore $h^{-1}(\zeta)$ exists near $\zeta = h(1) = 0$. Differentiating twice gives

$$(-2 + w)e^{-w} = g''(w) = -e^{-1}[2h'(w)^2 + 2h(w)h''(w)],$$

and so $h'(1) = \pm 1/\sqrt{2}$, say $-1/\sqrt{2}$. So $(h^{-1})'(0) = -\sqrt{2}$ and $h^{-1}(\zeta) = 1 - \sqrt{2}\zeta + O(|\zeta|^2)$ near $\zeta = 0$.

So (27) becomes $e^{-1} - e^{-1}h(b_z^{(1)})^2 = ze^{-1}$ or $h(b_z^{(1)})^2 = 1 - z$. Therefore, in $\mathbb{C}\backslash[1,\infty)$, $h(b_z^{(1)}) = (1-z)^{1/2}$ and $b_z^{(1)} = h^{-1}((1-z)^{1/2}) = 1 - \sqrt{2}(1-z)^{1/2} + O(|\,1-z\,|)$ near $z = 1$, where $(1-z)^{1/2}$ is one of the branches of the square root. It remains to show that it is the correct branch. But by the power series expansion of Theorem 15.1, $b_z^{(1)}$ is an increasing function of z for $z \in [0, 1)$. As $z \to 1^-$, this can only happen if $(1-z)^{1/2}$ is the branch which is positive for $z < 1$.

Finally, it must be shown that the error term is $O(|\,1-z\,|)$ in the rest of $\mathbb{C}\backslash[1,\infty)$. The first two terms already are, so it is enough to show that $b_z^{(1)}$ is; and this will follow if $\frac{d}{dz}b_z^{(1)}$ is bounded as $z \to \infty$. Differentiating (27) gives

$$\begin{aligned} e^{-1} &= (1-b_z^{(1)})e^{-b_z^{(1)}}(b_z^{(1)})' = (b_z^{(1)})'(e^{-b_z^{(1)}} - ze^{-1}) \\ &= (b_z^{(1)})'\left(\frac{ze^{-1}}{b_z^{(1)}} - ze^{-1}\right) = (b_z^{(1)})'ze^{-1}\left(\frac{1}{b_z^{(1)}} - 1\right). \end{aligned}$$

As $z \to \infty$, $b_z^{(1)}$ cannot tend to 1 or to 0 since then (27) would imply that $z \to 1$ or $z \to 0$. So the last factor is bounded away from 0 and ∞. Since $z \to \infty$, it must in fact be that $(b_z^{(1)})' \to 0$; in particular, it is bounded as required.

Solution to Exercise 16.3 by Sandra Kliem.

We have already shown (equation (16.19)) that

$$h_n^{(r)}(\vec{k}n^{-1/4}) \sim \frac{1}{\sqrt{2\pi}}n^{r-5/2}\hat{A}^{(r)}(\vec{k})$$

holds in the non-degenerate case where all k_j are nonzero. Recall that by definition (16.15)

$$\hat{A}^{(r)}(\vec{k}) = \int_0^\infty dt_1 \cdots \int_0^\infty dt_{2r-3}\hat{a}^{(r)}(\vec{k},\vec{t}),$$

where

$$\hat{a}^{(r)}(\vec{k},\vec{t}) = \left(\sum_{i=1}^{2r-3} t_i\right) e^{-\left(\sum_{i=1}^{2r-3} t_i\right)^2/2} e^{-\sum_{i=1}^{2r-3}|k_i|^2 t_i/2d}.$$

$\hat{a}^{(r)}(\vec{k},\vec{t})$ is obviously continuous in $\vec{k}$ and $0 \le \hat{a}^{(r)}(\vec{k},\vec{t}) \le \hat{a}^{(r)}(\vec{0},\vec{t})$, where $\hat{a}^{(r)}(\vec{0},\vec{t})$ is integrable, so $\hat{A}^{(r)}(\vec{0}) < \infty$. Hence the dominated convergence theorem yields

$$\hat{A}^{(r)}(\vec{k}) = \lim_{m\to\infty}\hat{A}^{(r)}(\vec{k}_m), \tag{28}$$

where $\vec{k}$ is a vector of the degenerate case and $\vec{k}_m$ is a sequence of vectors of the non-degenerate case, approaching $\vec{k}$.

On the other hand (16.22) gave for arbitrary k_j's

$$h_n^{(r)}(\vec{k}n^{-1/4}) = \frac{1}{2\pi i} \oint\limits_{\partial D_{1/2}(0)} \frac{dz}{z^{n+1}} \prod_{j=1}^{2r-3} \frac{2}{|k_j|^2 n^{-1/2} + 2^{3/2}\sqrt{1-z}}.$$

Let $k_j^{(m)}$ be the j^{th} component of $\vec{k}_m$ and

$$f_m(z) = \frac{1}{z^{n+1}} \prod_{j=1}^{2r-3} \frac{2}{\left|k_j^{(m)}\right|^2 n^{-1/2} + 2^{3/2}\sqrt{1-z}},$$

then the f_m are continuous on the open annulus $D_{3/4}(0)\backslash\overline{D_{1/4}(0)}$, $\partial D_{1/2}(0)$ is a contour in the annulus and the f_m converge uniformly to

$$f(z) = \frac{1}{z^{n+1}} \prod_{j=1}^{2r-3} \frac{2}{|k_j|^2 n^{-1/2} + 2^{3/2}\sqrt{1-z}}$$

on the annulus as the $|f_m|$ are uniformly bounded above on the annulus by

$$\sup_{1/4\leq|z|\leq 3/4} \left(\frac{4}{3}\right)^{n+1} \prod_{j=1}^{2r-3} \left|\frac{2}{2^{3/2}\sqrt{1-z}}\right| < \infty$$

since $\sqrt{1-z}$ is analytic and nonzero on $\mathbb{C}\backslash[1,\infty)$. Hence the contour integrals also converge for $m \to \infty$ and we obtain

$$\begin{aligned} h_n^{(r)}(\vec{k}_m n^{-1/4}) &= \frac{1}{2\pi i} \oint\limits_{\partial D_{1/2}(0)} \frac{dz}{z^{n+1}} \prod_{j=1}^{2r-3} \frac{2}{\left|k_j^{(m)}\right|^2 n^{-1/2} + 2^{3/2}\sqrt{1-z}} \\ &\to \frac{1}{2\pi i} \oint\limits_{\partial D_{1/2}(0)} \frac{dz}{z^{n+1}} \prod_{j=1}^{2r-3} \frac{2}{|k_j|^2 n^{-1/2} + 2^{3/2}\sqrt{1-z}} \\ &= h_n^{(r)}(\vec{k}n^{-1/4}). \end{aligned}$$

Finally note that for $r \geq 3$ (i.e. $r - 5/2 > 0$)

$$\lim_{m\to\infty} \frac{h_n^{(r)}(\vec{k}_m n^{-1/4})}{\frac{1}{\sqrt{2\pi}} n^{r-5/2} \hat{A}^{(r)}(\vec{k}_m)}$$

$$= \lim_{m\to\infty} \frac{1}{\frac{1}{\sqrt{2\pi}} n^{r-5/2} \hat{A}^{(r)}(\vec{k}_m)} \oint_{\partial D_{1/2}(0)} \frac{dz}{z^{n+1} 2\pi i} \prod_{j=1}^{2r-3} \frac{2}{\left|k_j^{(m)}\right|^2 n^{-1/2} + 2^{3/2}\sqrt{1-z}}$$

$$= \frac{h_n^{(r)}(\vec{k} n^{-1/4})}{\frac{1}{\sqrt{2\pi}} n^{r-5/2} \hat{A}^{(r)}(\vec{k})}$$

uniformly in n for $r \geq 3$ as can be seen from the second equality. Hence we can exchange the limits with respect to n and m, which gives the assertion together with the asymptotic statement for the non-degenerate case. For $r = 2$ the vector $\vec{k}$ only consists of $2r - 3 = 1$ component, i.e. we are left to show that $h_n^{(2)}(0) \sim \frac{1}{\sqrt{2\pi}} n^{-1/2} \hat{A}^{(2)}(0)$. This case will be covered in Exercise 16.4.

Solution to Exercise 16.4 by Jesse Goodman and Sandra Kliem.
$h_n^{(r)}(\vec{0})$ is the coefficient of z^n in

$$H_{z,\vec{1}}^{(r)}(\vec{0}) = \prod_{j=1}^{2r-3} H_{z,1}^{(2)}(0) = \left(\frac{1}{2^{1/2}\sqrt{1-z}}\right)^{2r-3} = 2^{3/2-r}(1-z)^{-r+3/2}.$$

By the binomial theorem, this is

$$\begin{aligned}
h_n^{(r)}(\vec{0}) &= 2^{3/2-r}\binom{-r+3/2}{n}(-1)^n \\
&= \frac{1}{2^{r-3/2} n!}(-1)^n(-r+3/2)(-r+1/2)\cdots(-r+3/2-n+1) \\
&= \frac{1}{2^{r-3/2} n!}(r-3/2)(r-1/2)\cdots(r-5/2+n) \\
&= \frac{1}{2^{r-3/2} n! 2^n}(2r-3)(2r-1)\cdots(2r-5+2n) \\
&= \frac{1}{2^{r-3/2+n} n!}\frac{(2r-5+2n)!!}{(2r-5)!!}.
\end{aligned}$$

Applying the trick $(2m+1)!! = \frac{(2m+1)!}{2m(2m-2)\cdots 2} = \frac{(2m+1)!}{2^m m!}$ to $(2r-5+2n)!!$ we obtain

$$h_n^{(r)}(\vec{0}) = \frac{1}{2^{r-3/2+n} n!}\frac{(2r-5+2n)!}{2^{r-3+n}(r-3+n)!}\frac{1}{(2r-5)!!}.$$

Now we can apply Stirling's formula $m! \sim m^m e^{-m}\sqrt{2\pi m}$ for $m \to \infty$ to get

$$\begin{aligned} h_n^{(r)}(\vec{0}) &= \frac{1}{2^{2r-9/2+2n}} \frac{1}{n^n e^{-n}\sqrt{2\pi n}} \\ &\quad \times \frac{(2r-5+2n)^{2r-5+2n} e^{-(2r-5+2n)}\sqrt{2\pi(2r-5+2n)}}{(r-3+n)^{r-3+n} e^{-(r-3+n)}\sqrt{2\pi(r-3+n)}} \frac{1}{(2r-5)!!} \\ &\sim \frac{1}{2^{2r-9/2+2n}} \frac{e^{-(r-2)}}{n^n\sqrt{2\pi n}} \frac{2^{2r-5+2n}\sqrt{2}\,(r-5/2+n)^{2r-5+2n}}{(r-3+n)^{r-3+n}} \frac{1}{(2r-5)!!} \\ &\sim \frac{e^{-(r-2)}}{n^n\sqrt{2\pi n}} \frac{n^{2r-5+2n}}{n^{r-3+n}} \frac{(\frac{r-5/2+n}{n})^{2r-5+2n}}{(\frac{r-3+n}{n})^{r-3+n}} \frac{1}{(2r-5)!!} \\ &\sim \frac{e^{-(r-2)}}{\sqrt{2\pi}} n^{r-5/2} \frac{e^{2(r-5/2)}}{e^{r-3}} \frac{1}{(2r-5)!!} = \frac{1}{\sqrt{2\pi}} n^{r-5/2} \frac{1}{(2r-5)!!}. \end{aligned}$$

Comparing with (16.19) for $\vec{k} = 0$, (16.17) follows.

Solution to Exercise 17.1 by Jesse Goodman.

The $l = 1$ case follows from the discussion preceding this Exercise and the results of Proposition 16.2. We focus on the case $l \geq 2$.

Passing from (17.4) to (17.10) amounts to replacing

$$\hat{S}_{\bar{m}}^{(l+1)}(\bar{k}) = \sum_{(T,\varphi)} \mathbb{P}(T,\varphi) \sum_{\substack{i_1,\ldots,i_l \in T \\ |i_j| = m_j \ \forall j}} \sum_{\bar{x}} e^{i\bar{k}\cdot\bar{x}} I[\varphi(i_j) = x_j \ \forall j] \tag{29}$$

by

$$\sum_{\sigma\in\Sigma_{l+1}} \sum_{s\in J_{\bar{m}}(\sigma)} \sum_{N=0}^{\infty} \hat{b}_N^{(l+1)}(\sigma; \vec{k}(\sigma), \vec{s}). \tag{30}$$

Note that in (29), for each $(T, \varphi, \bar{\imath})$ there is a unique $\bar{x}$ which contributes, so

$$\hat{S}_{\bar{m}}^{(l+1)}(\bar{k}) = \sum_{(T,\varphi)} \mathbb{P}(T,\varphi) \sum_{\substack{i_1,\ldots,i_l \in T \\ |i_j| = m_j \ \forall j}} e^{i\bar{k}\cdot\varphi(\bar{\imath})}. \tag{31}$$

On the other hand, (30) becomes

$$\begin{aligned}
&\sum_{\sigma\in\Sigma_{l+1}}\sum_{s\in J_{\bar m}(\sigma)}\sum_{N=0}^{\infty}\sum_{(T,\varphi):|T|=N}\mathbb{P}(T,\varphi)\sum_{i_1,\ldots,i_l\in T}\\
&\sum_{\vec y} e^{i\vec k(\sigma)\cdot\vec y} I[(T,\varphi,\bar\imath)\text{ compatible with }(\sigma;\vec y,\vec s)]\\
&=\sum_{\sigma\in\Sigma_{l+1}}\sum_{s\in J_{\bar m}(\sigma)}\sum_{(T,\varphi)}\mathbb{P}(T,\varphi)\sum_{i_1,\ldots,i_l\in T} e^{i\bar k\cdot\varphi(\bar\imath)}\\
&\qquad\times I[(T,\varphi,\bar\imath)\text{ compatible with }(\sigma;\vec s)] \qquad (32)
\end{aligned}$$

(since as before there is only one choice of $\vec y$, and for that value of $\vec y$, $\vec k(\sigma)\cdot\vec y = \bar k\cdot\varphi(\bar\imath)$: in $\vec k(\sigma)\cdot\vec y$, there is a term $k_j\cdot y_i$ for each edge i on the path to vertex j, so the total k_j contribution is $k_j\cdot\sum_{i\in\omega_j} y_i = k_j\cdot\varphi(i_j)$ by compatibility)

$$\begin{aligned}
\text{r.h.s.} \;=\;& \sum_{(T,\varphi)}\mathbb{P}(T,\varphi)\sum_{\substack{i_1,\ldots,i_l\in T\\ |i_j|=m_j\ \forall j}} e^{i\bar k\cdot\varphi(\bar\imath)}\\
&\sum_{\sigma\in\Sigma_{l+1}}\sum_{s\in J_{\bar m}(\sigma)} I[(T,\varphi,\bar\imath)\text{ compatible with }(\sigma;\vec s)], \qquad (33)
\end{aligned}$$

since compatibility with $\vec s$ requires that the lengths $|\,i_j\,|$ should equal those coming from $\vec s$, which are the totals $\bar m_j(\sigma,\vec s) = \sum_{i\in\omega_j} s_i$, which are equal to $\bar m_j$ by definition of $J_{\bar m}(\sigma)$.

If the reduced skeleton of $(T,\varphi,\bar\imath)$ is non-degenerate, then there is only one choice of $(\sigma,\vec s)$ and consequently there is no overcounting in (33) compared to (31). Overcounting can only arise if the reduced skeleton is degenerate, and in that case there must be a non-trivial s_i which is zero. Therefore the error is at most

$$\begin{aligned}
|\,(33)-(31)\,| \;\le\;& \sum_{(T,\varphi)}\mathbb{P}(T,\varphi)\sum_{\substack{i_1,\ldots,i_l\in T\\ |i_j|=m_j\ \forall j}}(1)\sum_{\sigma\in\Sigma_{l+1}}\\
&\sum_{s\in J'_{\bar m}(\sigma)} I[(T,\varphi,\bar\imath)\text{ compatible with }(\sigma;\vec s)],
\end{aligned}$$

where $J'_{\bar m}(\sigma)$ is $J_{\bar m}(\sigma)$ with the extra constraint that some non-trivial s_i is 0. Consequently, J' is a union of at most $2l-1$ spaces of dimension one smaller than J; i.e., J' is $(l-2)$-dimensional. Now reverse the steps going from (33) to (30) to find:

$$
\begin{aligned}
|\,(33)-(31)\,| \;\le\;& \sum_{\sigma\in\Sigma_{l+1}} \sum_{s\in J'_{\bar m}(\sigma)} \sum_{(T,\varphi)} \mathbb{P}(T,\varphi) \\
& \sum_{\substack{i_1,\ldots,i_l\in T\\ |\,i_j\,|=m_j\ \forall j}} I[(T,\varphi,\bar\imath) \text{ compatible with } (\sigma;\vec s)] \\
=\;& \sum_{\sigma\in\Sigma_{l+1}} \sum_{s\in J'_{\bar m}(\sigma)} \sum_{N=0}^{\infty} \hat b_N^{(l+1)}(\sigma;0,\vec s).
\end{aligned}
$$

It will follow from (34) below that $\sum_{N=0}^{\infty} \hat b_N^{(l+1)}(\sigma;0,\vec s)$ is at most 1. So, writing $m_{\max}=\max m_j$,

$$
|\,(33)-(31)\,| \le \sum_{\sigma\in\Sigma_{l+1}} \left| J'_{\bar m}(\sigma)\cap\mathbb{Z}^{2l-1} \right| = O\Big(m_{\max}^{l-2}\Big),
$$

since J' is an $(l-2)$-dimensional set of diameter at most $m_{\max}$.

Now apply this to pass from (17.4) to (17.10): the error is at most

$$
\frac{1}{n^{l-1}} O\Big(\lfloor n\bar t\rfloor_{\max}^{l-2}\Big)\, \frac{1}{n^{l-1}} O\Big(n^{l-2}\Big)
$$

which tends to 0 as $n\to\infty$ as required. So (17.4) is the same in the limit as (17.10).

As mentioned, the term $\sum_{N=0}^{\infty} \hat b_N^{(l+1)}(\sigma;\vec k,\vec m)$ can be identified: it is the coefficient of $\prod_{i=1}^{2l-1}\zeta_i^{m_i}$ in the transformed $(l+1)$-point function of (15.24) with $z=1$, since

$$
\hat b_{z,\vec\zeta}^{(l+1)}(\sigma;\vec k) = \sum_{m_1,\ldots,m_{2l-1}=0}^{\infty} \prod_{i=1}^{2l-1} \zeta_i^{m_i} \sum_{N=0}^{\infty} z^N \hat b_N^{(l+1)}(\sigma;\vec k,\vec m).
$$

But

$$
\hat b_{1,\vec\zeta}^{(l+1)}(\sigma;\vec k) = (1)^{-2(l-1)} \prod_{i=1}^{2l-1} \frac{1}{1-\zeta_i(1)\hat D(k_i)} = \sum_{m_1,\ldots,m_{2l-1}=0}^{\infty} \prod_{i=1}^{2l-1} \zeta_i^{m_i} \hat D(k_i)^{m_i},
$$

so that

$$
\sum_{N=0}^{\infty} \hat b_N^{(l+1)}(\sigma;\vec k,\vec m) = \prod_{i=1}^{2l-1} \hat D(k_i)^{m_i} \tag{34}
$$

and in particular it is at most 1 as claimed before.

Some notation: for a degree 1 vertex j other than the root, let i_j be the edge adjacent to it. If $i = i_j$ for some j, define $\vec{c}_n$ by $c_{n,i} = nt_j - \lfloor nt_j \rfloor$ and $c_i = 0$ for all other i. (That is, c_{n,i_j} is the fractional part of nt_j.) To make (17.10) more closely resemble a Riemann sum over $J_{\bar{t}}(\sigma)$, make the substitution $\vec{u} = \frac{1}{n}\vec{s} + \frac{1}{n}\vec{c}_n$. Note that $\vec{u} \in J_{\bar{t}}(\sigma)$ for $\vec{s} \in J_{\lfloor n\bar{t} \rfloor}(\sigma)$ (since each c_{n,i_j} affects only the j^{th} component of $\bar{m}(\sigma, \vec{u})$). Write E_n for the image of $J_{\lfloor n\bar{t} \rfloor}(\sigma) \cap \mathbb{Z}^{2l-1}$ under this substitution, so $E_n \subset J_{\bar{t}}(\sigma)$. Then

$$\hat{M}_{\bar{t}}^{(l)}(\bar{k}) = \lim_{n\to\infty} \sum_{\sigma \in \Sigma_{l+1}} \frac{1}{n^{l-1}} \sum_{\vec{u} \in E_n} \prod_{i=1}^{2l-1} \hat{D}\Big(k_i(\sigma) n^{-1/2}\Big)^{nu_i - c_{n,i}}.$$

The factors on the right converge to Gaussians:

$$\begin{aligned} \hat{D}\Big(k_i(\sigma)n^{-1/2}\Big) &= \frac{1}{d}\sum_{r=1}^{d} \cos\Big(k_{i,r}(\sigma)n^{-1/2}\Big) \\ &= \frac{1}{d}\sum_{r=1}^{d}\Big(1 - \frac{1}{2}k_{i,r}(\sigma)^2 n^{-1} + O\Big(n^{-2}\Big)\Big) \\ &= 1 - \frac{|\,k_i(\sigma)\,|^2}{2dn} + O\Big(n^{-2}\Big), \end{aligned}$$

and therefore $\log\Big[\hat{D}\Big(k_i(\sigma)n^{-1/2}\Big)^{nu_i - cn,i}\Big] = (nu_i - c_{n,i})\left[-\frac{|\,k_i(\sigma)\,|^2}{2dn} + O(n^{-2})\right] = -\frac{|\,k_i(\sigma)\,|^2}{2d}u_i + O(n^{-1})$ and $\hat{D}\Big(k_i(\sigma)n^{-1/2}\Big) = e^{-|\,k_i(\sigma)\,|^2 u_i/2d} + O(n^{-1})$, all of this uniformly in $\vec{u}$ since $\vec{u}$ comes from a bounded region. After taking the product, the $O(n^{-1})$ terms contribute

$$\frac{1}{n^{l-1}} O\Big(n^{l-1}\Big)\, O\Big(n^{-1}\Big)$$

to the total sum, since there are $O(n^{l-1})$ terms in the sum over E_n (which is this time $(l-1)$-dimensional). Therefore these terms vanish and

$$\hat{M}_{\bar{t}}^{(l)}(\bar{k}) = \lim_{n\to\infty} \sum_{\sigma \in \Sigma_{l+1}} \frac{1}{n^{l-1}} \sum_{\vec{u} \in E_n} \prod_{i=1}^{2l-1} e^{-|\,k_i(\sigma)\,|^2 u_i/2d}. \tag{35}$$

This is almost a Riemann sum; there are only two obstacles left. First, at the "edge" of $J_{\bar{t}}$, the sum may overcount or undercount slightly if the partition points are improperly placed. This makes little difference, however.

More importantly, the integral is over an $(l-1)$-dimensional subset of $\mathbb{R}^{2l-1}$; it is troublesome to perform Riemann integration on such a region, because of the tilt of the region.

Instead, consider the integral as being an integral over a domain in $\mathbb{R}^{l-1}$, parametrized by $l-1$ of the $2l-1$ u-coordinates – say, the coordinates not corresponding to the edges i_j – with the remaining coordinates determined as functions of the first ones, determined by the constraint $\vec{u} \in J_{\bar{t}}$. (This way, there is no trouble with Jacobians involved in a surface integral. Indeed, the integral in (17.12) must also be interpreted in this way.) Construct a function as follows: for each point $\vec{u} \in E_n$, let $v_0 \in \mathbb{R}^{l-1}$ be the corresponding parametrization value. Define $f_n(v)$ to take the value $\prod_{i=1}^{2l-1} e^{-|\,k_i(\sigma)\,|^2 u_i/2d}$ for v in a cube of side length $1/n$ around v_0, and $f_n(v) = 0$ elsewhere. Then the sum inside (35) is precisely the integral of f_n. It is readily seen that f_n converges to $\prod_{i=1}^{2l-1} e^{-|\,k_i(\sigma)\,|^2 u_i/2d} I[\vec{u} \in J_{\bar{t}}]$ a.e. as $n \to \infty$ (since the product of Gaussians is uniformly continuous). Also, the f_n are bounded and supported on a bounded set. Hence

$$
\begin{aligned}
\hat{M}_{\bar{t}}^{(l)}(\bar{k}) &= \sum_{\sigma \in \Sigma_{l+1}} \lim_{n\to\infty} \int f_n(v)\,dv \\
&= \sum_{\sigma \in \Sigma_{l+1}} \int \prod_{i=1}^{2l-1} e^{-|\,k_i(\sigma)\,|^2 u_i/2d} I[\vec{u} \in J_{\bar{t}}]\,d\vec{u} \\
&= \sum_{\sigma \in \Sigma_{l+1}} \int_{J_{\bar{t}}} \prod_{i=1}^{2l-1} e^{-|\,k_i(\sigma)\,|^2 u_i/2d} d\vec{u}
\end{aligned}
$$

as required, where again this integral is interpreted as being parametrized by $l-1$ of the variables.

Solution to Exercise 17.2 by Jesse Goodman.

To begin, resum Σ_{l+1}. Given $\sigma \in \Sigma_{l+1}$, produce a subset $I \subset J$ and shapes $\sigma_I, \sigma_{J\setminus I}$ by removing the root and the edge from the root to the first branch point, and cutting the remainder at the original branch point. That is, splice the tree into 3 parts at the first branch point, and discard the portion (the 2-shape) containing the root. Make the old first branch point into the root of the new trees. Since σ is a shape, the results are also shapes.

Of these shapes, one of them contains the degree 1 vertex labelled 1. Call this shape $\sigma_{J\setminus I}$ and call the other one σ_I. Let I be the set of labels for the degree 1 vertices in σ_I. I must be non-empty, since σ is a shape, but $1 \notin I$.

Conversely, given σ_I and $\sigma_{J\backslash I}$ with labels drawn from I and $J\backslash I$, the original shape can be recovered by attaching the σ's at their roots and also reattaching a 2-shape to make a branch point and a root.

Write t for the component of $\vec{s}$ corresponding to the edge from the root. Then for $j \in I$, $\bar{m}_j(\sigma, \vec{s}) = t + \bar{m}_j(\sigma_I, \vec{s}_I)$ and $\bar{m}_j(\sigma_{J\backslash I}, \vec{s}_{J\backslash I}) = 0$. Conversely, for $j \notin I$, $\bar{m}_j(\sigma, \vec{s}) = t + \bar{m}_j(\sigma_{J\backslash I}, \vec{s}_{J\backslash I})$ and $\bar{m}_j(\sigma_I, \vec{s}_I) = 0$. So the linear system defining $J_{\bar{t}}(\sigma)$ decouples into two systems, $\bar{m}(\sigma_I, \vec{s}_I) = \bar{t}_I - t$ and $\bar{m}(\sigma_{J\backslash I}, \vec{s}_{J\backslash I}) = \bar{t}_{J\backslash I} - t$. Finally the non-negativity shows that we must take $t \leq \underline{t}$.

Similarly for $\vec{k}(\sigma)$: depending whether i is the edge from the root, an edge in σ_I, or an edge in $\sigma_{J\backslash I}$, $\vec{k}_i(\sigma)$ is either $k_1 + \cdots + k_l$, $\vec{k}_i(\sigma_I)$, or $\vec{k}_i(\sigma_{J\backslash I})$ respectively.

Finally,

$$
\begin{aligned}
\hat{M}^{(l)}_{\bar{t}}(\bar{k}) &= \sum_{\sigma \in \Sigma_{l+1}} \int \mathrm{d}\vec{s}\, I[\vec{s} \in J_{\bar{t}}(\sigma)] \prod_j e^{-\left|\vec{k}_j(\sigma)\right|^2 s_j/2d} \\
&= \sum_{I \subset J_1 : |I| \geq 1} \sum_{\sigma_I \in \Sigma_{|I|+1}} \sum_{\sigma_{J\backslash I} \in \Sigma_{l-|I|+1}} \int dt \int \mathrm{d}\vec{s}_I \int \mathrm{d}\vec{s}_{J\backslash I} \\
&\quad I[t \leq \underline{t}, \bar{m}(\sigma_I, \vec{s}_I) = \bar{t}_I - t, \bar{m}(\sigma_{J\backslash I}, \vec{s}_{J\backslash I}) = \bar{t}_{J\backslash I} - t] \\
&\quad \times e^{-|k_1 + \cdots + k_l|^2 s_j/2d} \cdot \prod_{j \sim \sigma_I} e^{-\left|\vec{k}_j(\sigma)\right|^2 s_j/2d} \cdot \prod_{j \sim \sigma_{J\backslash I}} e^{-\left|\vec{k}_j(\sigma)\right|^2 s_j/2d} \\
&= \int dt\, I[t \leq \underline{t}] e^{-|k_1 + \cdots + k_l|^2 s_j/2d} \sum_{I \subset J_1 : |I| \geq 1} \\
&\quad \left[\sum_{\sigma_I \in \Sigma_{|I|+1}} \int \mathrm{d}\vec{s}_I\, I[\bar{m}(\sigma_I, \vec{s}_I) = \bar{t}_I - t] \prod_{j \sim \sigma_I} e^{-\left|\vec{k}_j(\sigma)\right|^2 s_j/2d} \right] \\
&\quad \times \left[\sum_{\sigma_{J\backslash I} \in \Sigma_{l-|I|+1}} \int \mathrm{d}\vec{s}_{J\backslash I} I[\bar{m}(\sigma_{J\backslash I}, \vec{s}_{J\backslash I}) = \bar{t}_{J\backslash I} - t] \right. \\
&\quad \left. \times \prod_{j \sim \sigma_{J\backslash I}} e^{-\left|\vec{k}_j(\sigma)\right|^2 s_j/2d} \right]
\end{aligned}
$$

$$
\begin{aligned}
&= \int_0^{\underline{t}} dt\, \hat{M}_t^{(1)}(k_1 + \cdots + k_l) \sum_{I \subset J_1 : |\, I \,| \geq 1} \\
&\quad \sum_{\sigma_I \in \Sigma_{|\, I \,|+1}} \int_{J_{\bar{t}_I}(\sigma_I)} \mathrm{d}\vec{s}_I \prod_{j \sim \sigma_I} e^{-\left| \vec{k}_j(\sigma) \right|^2 s_j / 2d} \\
&\quad \times \sum_{\sigma_{J\setminus I} \in \Sigma_{l-|\, I \,|+1}} \int_{J_{\bar{t}_{J\setminus I}}(\sigma_{J\setminus I})} \mathrm{d}\vec{s}_{J\setminus I} \prod_{j \sim \sigma_{J\setminus I}} e^{-\left| \vec{k}_j(\sigma) \right|^2 s_j / 2d} \\
&= \int_0^{\underline{t}} dt\, \hat{M}_t^{(1)}(k_1 + \cdots + k_l) \sum_{I \subset J_1 : |\, I \,| \geq 1} \hat{M}^{(i)}_{\bar{t}_I - t}(\bar{k}_I) \hat{M}^{(l-i)}_{\bar{t}_{J\setminus I} - t}(\bar{k}_{J\setminus I})
\end{aligned}
$$

as required, where $j \sim \sigma_I$ means that j is an edge in σ_I, and i refers to the size of I in the last line.

Solution to Exercise 17.3 by Sandra Kliem.

We prove this by induction over l. For the base case $l = 1$: by (17.12) $\hat{M}_t^{(1)}(0) = 1 = \hat{M}_s^{(l)}(0)$.

For the inductive step $l \to l+1$: we'll use the recursion formula (17.15). We have $k_1 = 0$ and $\underline{t} = \min_i t_i$ as $t \geq s \geq \min_i t_i$ given. The formula therefore yields

$$
\begin{aligned}
\hat{M}^{(l)}_{(t,t_2,\ldots,t_l)}(0) &= \int_0^{\min_i t_i} du \hat{M}_u^{(1)}(0 + k_2 + \cdots + k_l) \\
&\quad \sum_{I \subset \{2,\ldots,l\}, |I| \geq 1} \hat{M}^{(|I|)}_{(t,t_2,\ldots,t_l)_I - u}((0, k_2, \ldots, k_l)_I) \\
&\quad \times \hat{M}^{(l-|I|)}_{(t,t_2,\ldots,t_l)_{J\setminus I} - u}\Big((0, k_2, \ldots, k_l)_{J\setminus I}\Big) .
\end{aligned}
$$

As $1 \notin I$, t has no influence on $\hat{M}^{(|I|)}_{(t,t_2,\ldots,t_l)_I - u}((0, k_2, \ldots, k_l)_I)$. The only term t might have an influence on is $\hat{M}^{(l-|I|)}_{(t,t_2,\ldots,t_l)_{J\setminus I} - u}\Big((0, k_2, \ldots, k_l)_{J\setminus I}\Big)$. As $l - |I| < l$ (since $|I| \geq 1$) and $t - u \geq \max_{i=2,\ldots,l} s_i - u \geq \max_{i \in J\setminus I} s_i - u$ induction gives that

$$
\hat{M}^{(l-|I|)}_{(t,t_2,\ldots,t_l)_{J\setminus I} - u}\Big((0, k_2, \ldots, k_l)_{J\setminus I}\Big) = \hat{M}^{(l-|I|)}_{(s,t_2,\ldots,t_l)_{J\setminus I} - u}\Big((0, k_2, \ldots, k_l)_{J\setminus I}\Big) .
$$

Following the calculations backwards, we therefore obtain the desired result.

Solution to Exercise 17.4 by Jesse Goodman.

Again, use strong induction: clearly $\hat{M}_t^{(1)}(0) = 1 = t^0 2^0 1!$. For $l > 1$, writing $i = |I|$,

$$
\begin{aligned}
\hat{M}^{(l)}_{t,\dots,t}(\bar{0}) &= \int_0^t d\tau\, \hat{M}^{(1)}_\tau(0) \sum_{I \subset J_1 : |\, I\,| \geq 1} \hat{M}^{(i)}_{t-\tau,\dots,t-\tau}(\bar{0}) \hat{M}^{(l-i)}_{t-\tau,\dots,t-\tau}(\bar{0}) \\
&= \int_0^t d\tau \sum_{I \subset J_1 : |\, I\,| \geq 1} (t-\tau)^{i-1} 2^{-i+1} i! (t-\tau)^{l-i-1} 2^{-l+i+1} (l-i)! \\
&= \sum_{i=1}^{l} \frac{(l-1)!}{i!(l-i-1)!} \int_0^t d\tau\, (t-\tau)^{l-2} 2^{-l+2} i! (l-i)! \\
&= 2^{-l+2}(l-1)! \frac{1}{l-2+1} t^{l-1} \sum_{i=1}^{l} (l-i) \\
&= 2^{-l+2}(l-1)! \frac{1}{l-1} t^{l-1} \frac{(l-1)(l-1+1)}{2} \\
&= 2^{-l+1}(l-1)! t^{l-1} l
\end{aligned}
$$

giving the induction step as required.

Solution to Exercise 17.6 by Sandra Kliem.

As Exercise 12.8. already proves the second part of Theorem 12.7(a), i.e. the weak convergence statement, we'll only prove that

$$\lim_{m\to\infty} \mathbb{E}_\infty\left[\left(\frac{N_m}{m}\right)^l\right] = \left(\frac{A^2 V}{2}\right)^l (l+1)!$$

under the given assumptions. Recall that by (12.40) $N_m = |\{y \in \mathbb{Z}^d : (0,0) \leftrightarrow (y,m)\}| = \sum_{y\in\mathbb{Z}^d} I[(0,0) \leftrightarrow (y,m)]$. Hence

$$
\begin{aligned}
\mathbb{E}_\infty\Big((N_m)^l\Big) &= \mathbb{E}_\infty\Bigg(\sum_{y_1,\dots,y_l} I[(0,0)\leftrightarrow(y_1,m)]\cdots I[(0,0)\leftrightarrow(y_l,m)]\Bigg) \\
&= \sum_{y_1,\dots,y_l} \mathbb{E}_\infty(I[(0,0)\leftrightarrow(y_i,m) \text{ for all } 1\leq i\leq l]).
\end{aligned}
$$

Using definition (12.29), which says $\mathbb{P}_\infty(E) = \lim_{n\to\infty} \frac{1}{\hat{\tau}_n(0)} \sum_{x\in\mathbb{Z}^d} \mathbb{P}(E \cap \{(0,0)\to(x,n)\})$, this gives

$$
\begin{aligned}
\mathbb{E}_\infty\Big((N_m)^l\Big) = &\sum_{y_1,\dots,y_l} \lim_{n\to\infty} \frac{1}{\hat{\tau}_n(0)} \\
&\times \sum_{x\in\mathbb{Z}^d} \mathbb{P}(\{(0,0)\leftrightarrow(y_i,m) \text{ for all } 1\leq i\leq l\} \cap \{(0,0)\to(x,n)\})\,.
\end{aligned}
$$

Note that $\lim_{n\to\infty} \hat{\tau}_n(0) = \lim_{n\to\infty} \hat{\tau}^{(2)}_{p_c,n}(0) = A\hat{M}^{(1)}_1(0) = A$ exists by Theorem 17.5. Exchanging the first sum and the limit (justified by what follows) and using definition (17.18) of the oriented percolation r-point functions we obtain

$$\begin{aligned}
\mathbb{E}_\infty\Big((N_m)^l\Big) &= \frac{1}{A}\lim_{n\to\infty}\sum_{y_1,\ldots,y_l}\sum_{x\in\mathbb{Z}^d} \\
&\qquad \mathbb{P}(\{(0,0)\leftrightarrow(y_i,m) \text{ for all } 1\le i\le l\}\cap\{(0,0)\to(x,n)\}) \\
&= \frac{1}{A}\lim_{n\to\infty}\sum_{y_1,\ldots,y_l}\sum_{x\in\mathbb{Z}^d}\tau^{(l+2)}_{p_c,(n,m,\ldots,m)}(x,y_1,\ldots,y_l) \\
&= \frac{1}{A}\lim_{n\to\infty}\hat{\tau}^{(l+2)}_{p_c,(n,m,\ldots,m)}(\bar{0}).
\end{aligned}$$

From a slight generalization of Theorem 17.5 (see [121, (2.52)] with $\sigma = 1$) we obtain

$$\hat{\tau}^{(r)}_{p_c,\lfloor m\bar{t}\rfloor}(\bar{k}/\sqrt{vm}) = \hat{\tau}^{(r)}_{\lfloor m\bar{t}\rfloor}(\bar{k}/\sqrt{vm}) = \Big(mVA^2\Big)^{r-2} A\left[\hat{M}^{(r-1)}_{\bar{t}}(\bar{k}) + O(m^{-\delta}c(\bar{t}))\right] \tag{36}$$

where $c(\bar{t})$ is the second largest component of $\bar{t}$. As the second largest component of $\frac{1}{m}(n,m,\ldots,m)$ is 1 for $n\to\infty$ we have

$$\mathbb{E}_\infty\Big((N_m)^l\Big) = \frac{1}{A}\lim_{n\to\infty}\Big(mVA^2\Big)^l A\left[\hat{M}^{(l+1)}_{(n,1,\ldots,1)}(\bar{0}) + O(m^{-\delta})\right].$$

Now we can apply Exercise 17.3 first to get $\hat{M}^{(l+1)}_{(n,1,\ldots,1)}(\bar{0}) = \hat{M}^{(l+1)}_{(1,1,\ldots,1)}(\bar{0})$ and then Exercise 17.4 to calculate $\hat{M}^{(l+1)}_{(1,1,\ldots,1)}(\bar{0}) = 1^l 2^{-l}(l+1)!$ and hence

$$\begin{aligned}
\lim_{m\to\infty}\mathbb{E}_\infty\left[\left(\frac{N_m}{m}\right)^l\right] &= \lim_{m\to\infty}\frac{1}{m^l}\Big(mVA^2\Big)^l\left[\hat{M}^{(l+1)}_{(n,1,\ldots,1)}(\bar{0}) + O(m^{-\delta})\right] \\
&= \Big(VA^2\Big)^l 2^{-l}(l+1)! = \left(\frac{A^2V}{2}\right)^l (l+1)!.
\end{aligned}$$

Solution to Exercise 17.7 by Jesse Goodman.

We have

$$\hat{\rho}^{(r)}_{\lfloor m\bar{t}\rfloor}\left(\frac{\bar{k}}{\sqrt{vm}}\right) = \lim_{n\to\infty}\frac{1}{\tau_n}\hat{\tau}^{(r+1)}_{(n,\lfloor m\bar{t}\rfloor)}\left(0,\frac{\bar{k}}{\sqrt{vm}}\right).$$

In [121, (2.52)] there is a result similar to Theorem 17.5 but which gives an estimate uniformly in the largest time component. It follows that

$$\hat{\tau}^{(r+1)}_{(n,\lfloor m\bar{t}\rfloor)}\left(0, \frac{\bar{k}}{\sqrt{vm}}\right) = A\left(A^2Vm\right)^{r-1}\left[\hat{M}^{(r)}_{(n/m,\lfloor m\bar{t}\rfloor/m)}(0,\bar{k}) + O((1+m)^{-\delta})\right]$$

uniformly in $n \geq m$. Since every $t_j \leq 1 \leq n/m$, by 17.3 the n/m can be replaced by 1 on the right-hand side. Substituting this into $\hat{\rho}$ gives

$$\begin{aligned}
\lim_{m\to\infty} \frac{\hat{\rho}^{(r)}_{\lfloor m\bar{t}\rfloor}\left(\frac{\bar{k}}{\sqrt{vm}}\right)}{(mA^2V)^{r-1}} &= \lim_{m\to\infty}\lim_{n\to\infty}\frac{A}{\tau_n}\left[\hat{M}^{(r)}_{(1,\lfloor m\bar{t}\rfloor/m)}(0,\bar{k}) + O((1+m)^{-\delta})\right] \\
&= \lim_{n\to\infty}\lim_{m\to\infty}\frac{A}{\tau_n}\left[\hat{M}^{(r)}_{(1,\lfloor m\bar{t}\rfloor/m)}(0,\bar{k}) + O((1+m)^{-\delta})\right] \\
&= \lim_{n\to\infty}\frac{A}{\tau_n}\hat{M}^{(r)}_{(1,\bar{t})}(0,\vec{k}) \\
&= \frac{A}{A}\hat{M}^{(r)}_{(1,\bar{t})}(0,\vec{k}) = \hat{M}^{(r)}_{(1,\bar{t})}(0,\vec{k})
\end{aligned}$$

as required, using the fact that the convergence is uniform in n to interchange the limits.