The strong interaction limit of continuous-time weakly self-avoiding walk

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Dedicated to Erwin Bolthausen and Jürgen Gärtner on the occasion of their 65th and 60th birthday celebration

Abstract The strong interaction limit of the discrete-time weakly self-avoiding walk (or Domb–Joyce model) is trivially seen to be the usual strictly self-avoiding walk. For the continuous-time weakly self-avoiding walk, the situation is more delicate, and is clarified in this paper. The strong interaction limit in the continuous-time setting depends on how the fugacity is scaled, and in one extreme leads to the strictly self-avoiding walk, in another to simple random walk. These two extremes are interpolated by a new model of a self-repelling walk that we call the "quick step" model. We study the limit both for walks taking a fixed number of steps, and for the two-point function.

1 Domb–Joyce model: discrete time

The discrete-time weakly self-avoiding walk, or Domb–Joyce model [6], is a useful adaptation of the strictly self-avoiding walk that continues to be actively studied [1]. It is defined as follows. For simplicity, we restrict attention to the nearest-neighbour model on $\mathbb{Z}^d$, although a more general formulation is easy to obtain.

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Let $d \geq 1$ and $n \geq 0$ be integers, and let $\mathscr{W}_n$ denote the set of nearest-neighbour walks in $\mathbb{Z}^d$, of length n, which start from the origin. In other words, $\mathscr{W}_n$ consists of sequences $Y = (Y_0, Y_1, \dots, Y_n)$ with $Y_i \in \mathbb{Z}^d$, $Y_0 = 0$, $|Y_{i+1} - Y_i| = 1$ (Euclidean distance). Let $\mathscr{S}_n$ denote the set of nearest-neighbour self-avoiding walks in $\mathscr{W}_n$; these are the walks with $Y_i \neq Y_j$ for all $i \neq j$. Let c_n denote the cardinality of $\mathscr{S}_n$. For $Y \in \mathscr{W}_n$ and $x \in \mathbb{Z}^d$, let $n_x = n_x(Y) = \sum_{i=0}^n \mathbb{1}_{Y_i = x}$ denote the number of visits to x by Y. The Domb–Joyce model is the measure on $\mathscr{W}_n$ which assigns to a walk $Y \in \mathscr{W}_n$ the probability

$$P_{g,n}^{\mathrm{DJ}}(Y) = \frac{1}{c_n^{\mathrm{DJ}}(g)} e^{-g \sum_{x \in \mathbb{Z}^d} n_x(Y)(n_x(Y)-1)}, \tag{1}$$

where g is a positive parameter and

$$c_n^{\mathrm{DJ}}(g) = \sum_{Y \in \mathscr{W}_n} e^{-g \sum_{x \in \mathbb{Z}^d} n_x(Y)(n_x(Y)-1)}. \tag{2}$$

The Domb–Joyce model interpolates between simple random walk and self-avoiding walk. Indeed, the case $g = 0$ corresponds to simple random walk by definition, and also

$$\lim_{g \to \infty} e^{-g \sum_{x \in \mathbb{Z}^d} n_x(Y)(n_x(Y)-1)} = \mathbb{1}_{Y \in \mathscr{S}_n} \tag{3}$$

and hence

$$\lim_{g \to \infty} P_{g,n}^{\mathrm{DJ}}(Y) = \frac{1}{c_n} \mathbb{1}_{Y \in \mathscr{S}_n}. \tag{4}$$

This shows that the strong interaction limit of the Domb–Joyce model is the uniform measure on $\mathscr{S}_n$. (For an analogous result for weakly self-avoiding lattice trees, which is more subtle than for self-avoiding walks, see [2].)

A standard subadditivity argument (see, e.g., [10, Lemma 1.2.2]) implies that the limits

$$\mu(g) = \lim_{n \to \infty} c_n^{\mathrm{DJ}}(g)^{1/n}, \qquad \mu = \lim_{n \to \infty} c_n^{1/n} \tag{5}$$

exist and obey $c_n^{\mathrm{DJ}}(g) \geq \mu(g)^n$ and $c_n \geq \mu^n$ for all n. The number of walks that take steps only in the positive coordinate directions is d^n, and such walks are self-avoiding, so $c_n \geq d^n$, Also, it follows from (2) that if $0 \leq g < g_0$ then $(2d)^n \geq c_n^{\mathrm{DJ}}(g) \geq c_n^{\mathrm{DJ}}(g_0) \geq c_n \geq d^n$, and hence $2d \geq \mu(g) \geq \mu(g_0) \geq \mu \geq d$. In particular, by monotonicity, $\lim_{g \to \infty} \mu(g)$ exists in $[\mu, 2d]$. If we take the limit $g \to \infty$ in the inequality $c_n^{\mathrm{DJ}}(g) \geq \mu(g)^n \geq \mu^n$, we obtain $c_n \geq (\lim_{g \to \infty} \mu(g))^n \geq \mu^n$. Taking n^{th} roots and then the limit $n \to \infty$ then gives

$$\lim_{g \to \infty} \mu(g) = \mu. \tag{6}$$

Let $\mathscr{W}_n(x)$ denote the subset of $\mathscr{W}_n$ consisting of walks that end at $x \in \mathbb{Z}^d$. Let $\mathscr{S}_n(x) = \mathscr{S}_n \cap \mathscr{W}_n(x)$, and let $c_n(x)$ denote the cardinality of $\mathscr{S}_n(x)$. Let

$$c_{n,g}^{\mathrm{DJ}}(x) = \sum_{Y \in \mathscr{W}_n(x)} e^{-g \sum_{\tilde{x} \in \mathbb{Z}^d} n_{\tilde{x}}(Y)(n_{\tilde{x}}(Y)-1)}. \tag{7}$$

Let $z \geq 0$. The two-point functions of the Domb–Joyce and self-avoiding walk models are defined as follows:

$$G^{\mathrm{DJ}}_{g,z}(x) = \sum_{n=0}^{\infty} c^{\mathrm{DJ}}_{n,g}(x) z^n, \qquad G_z(x) = \sum_{n=0}^{\infty} c_n(x) z^n. \tag{8}$$

These series converge for $z < \mu(g)^{-1}$ and $z < \mu^{-1}$ respectively. Presumably they converge also for $z = \mu(g)^{-1}$ and $z = \mu^{-1}$ but this is a delicate question that is unproven except in high dimensions (in fact, the decay of the two-point function with $z = \mu^{-1}$ is known in some cases [4, 8, 9]). The following proposition shows that the strong interaction limit of $G^{\mathrm{DJ}}_{g,z}(x)$ is $G_z(x)$.

Proposition 1. *For $z \in [0, \mu^{-1})$ and $x \in \mathbb{Z}^d$,*

$$\lim_{g\to\infty} G^{\mathrm{DJ}}_{g,z}(x) = G_z(x). \tag{9}$$

Proof. Fix $z \in [0, \mu^{-1})$. By (6), if g_0 is sufficiently large then $z < \mu(g_0)^{-1}$. Thus, since $c^{\mathrm{DJ}}_n(g)$ is nonincreasing in g, there are $r < 1$ and $C > 0$ such that $c^{\mathrm{DJ}}_n(g) z^n \leq c^{\mathrm{DJ}}_n(g_0) z^n \leq Cr^n$ for all n, uniformly in $g \geq g_0$. Thus, for all $g \geq g_0$,

$$G^{\mathrm{DJ}}_{g,z}(x) \leq \sum_{x\in\mathbb{Z}^d} G^{\mathrm{DJ}}_{g,z}(x) = \sum_{n=0}^{\infty} c^{\mathrm{DJ}}_n(g) z^n \leq \frac{C}{1-r} < \infty. \tag{10}$$

By (3), $\lim_{g\to\infty} c^{\mathrm{DJ}}_{n,g}(x) = c_n(x)$, and the desired result then follows by dominated convergence. □

2 The continuous-time weakly self-avoiding walk

Our goal is to study the analogues of (4) and Proposition 1 for the continuous-time weakly self-avoiding walk. The continuous-time model is a lattice version of the Edwards model [7]. It has been useful in particular due to its representation in terms of functional integrals [5] that have been employed in renormalisation group analyses.

2.1 Fixed-length walks

We first consider the case of fixed-length walks, in which a fixed number n of steps is taken by the walk. We will find that the strong interaction limit depends on how an auxiliary parameter ρ is scaled, where e^{ρ} plays the role of a fugacity. The scaling is parametrized by $a \in [-\infty, \infty]$. The case $a = \infty$ leads to the strictly self-avoiding walk, the case $a = -\infty$ leads to simple random walk, and the interpolating cases,

$a \in (-\infty, \infty)$, define a new model of a self-repelling walk that we call the "quick step" model.

Let X denote the continuous-time Markov process with state space $\mathbb{Z}^d$, in which uniformly random nearest-neighbour steps are taken after independent $\mathrm{Exp}(1)$ holding times. Let $\mathbb{E}$ denote expectation for this process started at 0. We distinguish between the continuous-time walk X and the sequence of sites visited during its first n steps, which we typically denote by $Y \in \mathscr{W}_n$. Conditioning on the first n steps of X to be Y is denoted by $\mathbb{E}(\cdot \mid Y)$.

For fixed-length walks, the continuous-time weakly self-avoiding walk is the measure $Q_{g,\rho,n}$ on $\mathscr{W}_n$ defined as follows. Here ρ is a real parameter at our disposal, which we allow to depend on $g > 0$. Let T_n denote the time of the $(n+1)^{\mathrm{st}}$ jump of X, and let $L_{x,n}(X) = \int_0^{T_n} \mathbb{1}_{X(s)=x} ds$ denote the local time at x up to time T_n. By definition, $\sum_{x \in \mathbb{Z}^d} L_{x,n} = T_n$. For $Y \in \mathscr{W}_n$, let

$$Q_{g,\rho,n}(Y) = \frac{1}{Z_n(g,\rho)} \mathbb{E}\left(e^{-g\sum_x L_{x,n}^2 + \rho \sum_x L_{x,n}} \mid Y\right), \tag{11}$$

where

$$Z_n(g,\rho) = \sum_{Y \in \mathscr{W}_n} \mathbb{E}\left(e^{-g\sum_x L_{x,n}^2 + \rho \sum_x L_{x,n}} \mid Y\right). \tag{12}$$

For $a \in \mathbb{R}$ and $m \in \mathbb{N}$, let

$$I_m(a) = \int_{-a}^{\infty} \frac{(a+u)^{m-1}}{(m-1)!} e^{-u^2} du. \tag{13}$$

Proposition 2. *Let $\alpha = \alpha(g,\rho) = \frac{1}{2} g^{-1/2}(\rho - 1)$, and let $\rho = \rho(g)$ be chosen in such a way that $a = \lim_{g\to\infty} \alpha(g,\rho(g))$ exists in $[-\infty, \infty]$. Let $n \geq 1$ and $Y \in \mathscr{W}_n$. Then*

$$\lim_{g\to\infty} Q_{g,\rho(g),n}(Y) = \begin{cases} \frac{1}{Z_a} \prod_{x\in Y} e^{a^2} I_{n_x(Y)}(a) & \text{if } a \in (-\infty,\infty), \\ \frac{1}{c_n} \mathbb{1}_{Y \in \mathscr{S}_n} & \text{if } a = \infty, \\ \frac{1}{(2d)^n} & \text{if } a = -\infty, \end{cases} \tag{14}$$

where Z_a is a normalisation constant, and the product over x is over the distinct vertices visited by Y.

Proof. As before, we write $n_x = n_x(Y)$ for the number of times that x is visited by Y. Thus $\sum_x n_x = n+1$ is the number of vertices visited by Y (with multiplicity). Since the sum of m independent $\mathrm{Exp}(1)$ random variables has a $\mathrm{Gamma}(m,1)$ distribution, we have

$$\mathbb{E}\left(e^{-g\sum_x L_{x,n}^2 + \rho \sum_x L_{x,n}} \mid Y\right) = \prod_{x\in Y} \int_0^{\infty} \frac{s_x^{n_x - 1}}{(n_x - 1)!} e^{-s_x} e^{-g s_x^2 + \rho s_x} ds_x, \tag{15}$$

where the product is over the *distinct* vertices visited by Y. We make the changes of variables $t_x = g^{1/2} s_x$ and then $u_x = t_x - \alpha$. After completing the square, this leads to

$$\mathbb{E}\left(e^{-g\sum_x L_{x,n}^2+\rho\sum_x L_{x,n}} \mid Y\right) = g^{-(n+1)/2}\prod_{x\in Y} e^{\alpha^2} I_{n_x}(\alpha). \tag{16}$$

Case $a \in (-\infty,\infty)$: the quick step model. Suppose that $\alpha \to a \in (-\infty,\infty)$ as $g \to \infty$. In this case, by the continuity of $I_m(a)$ in a,

$$\mathbb{E}\left(e^{-g\sum_x L_{x,n}^2+\rho(g)\sum_x L_{x,n}} \mid Y\right) \sim g^{-(n+1)/2}\prod_{x\in Y} e^{a^2} I_{n_x}(a), \tag{17}$$

and thus

$$\lim_{g\to\infty} Q_{g,\rho(g),n}(Y) = \frac{1}{Z_a}\prod_{x\in Y} e^{a^2} I_{n_x(Y)}(a) \qquad (\alpha \to a \in (-\infty,\infty)). \tag{18}$$

Case $a = \infty$: limit is uniform on $\mathscr{S}_n$. Suppose that $\alpha \to \infty$ as $g \to \infty$. In this case, since α is nonzero we can use (16) to write

$$\begin{aligned}&\mathbb{E}\left(e^{-g\sum_x L_{x,n}^2+\rho\sum_x L_{x,n}} \mid Y\right)\\ &= (g^{-1/2}e^{\alpha^2})^{n+1}(\alpha e^{-\alpha^2})^{n+1-|Y|}\prod_{x\in Y}\int_{-\alpha}^{\infty}\frac{(1+u_x/\alpha)^{n_x-1}}{(n_x-1)!}e^{-u_x^2}\,du_x,\end{aligned} \tag{19}$$

where $|Y|$ denotes the number of distinct vertices visited by Y. Since the factor $(\alpha e^{-\alpha^2})^{n+1-|Y|}$ goes to zero unless Y is self-avoiding, in which case the factor is equal to 1 and $n_x = 1$ for the vertices visited by Y, and since also

$$\lim_{\alpha\to\infty}\int_{-\alpha}^{\infty} e^{-u_x^2}\,du_x = \sqrt{\pi}, \tag{20}$$

this gives

$$\mathbb{E}\left(e^{-g\sum_x L_{x,n}^2+\rho(g)\sum_x L_{x,n}} \mid Y\right) \sim (g^{-1/2}e^{\alpha^2}\sqrt{\pi})^{n+1}\mathbb{1}_{Y\in\mathscr{S}_n}. \tag{21}$$

When we take the normalisation into account we find that

$$\lim_{g\to\infty} Q_{g,\rho(g),n}(Y) = \frac{1}{c_n}\mathbb{1}_{Y\in\mathscr{S}_n} \qquad (\alpha\to\infty). \tag{22}$$

Case $a = -\infty$: limit is uniform on $\mathscr{W}_n$. Suppose that $\alpha \to -\infty$ as $g \to \infty$. We will show that, for $m \geq 1$,

$$e^{\alpha^2} I_m(\alpha) \sim (-2\alpha)^{-m} \quad \text{as } \alpha \to -\infty. \tag{23}$$

With (16), this claim implies that

$$\mathbb{E}\left(e^{-g\sum_x L_{x,n}^2+\rho(g)\sum_x L_{x,n}} \mid Y\right) \sim g^{-(n+1)/2}\prod_{x\in Y}(-2\alpha)^{-n_x} = (-2\alpha g^{-1/2})^{n+1}. \tag{24}$$

Since the right-hand side is independent of Y, this proves that the limiting measure is uniform on $\mathscr{W}_n$, as required. Finally, to prove (23), we set $b = -\alpha$ and obtain

$$(2b)^m e^{b^2} I_m(-b) = (2b)^m e^{b^2} \int_b^\infty \frac{(-b+u)^{m-1}}{(m-1)!} e^{-u^2}\, du = \int_0^\infty \frac{u^{m-1}}{(m-1)!} e^{-(u/(2b))^2 - u}\, du. \tag{25}$$

By dominated convergence, as $b \to \infty$, the integral on the right-hand side approaches 1 because it becomes the integral over the $\Gamma(m,1)$ probability density function. □

Proposition 2 shows that the case $\alpha \to \infty$ leads to the uniform measure on self-avoiding walks, whereas $\alpha \to -\infty$ leads to simple random walk. These two extremes are interpolated by the quick step walk, for $\alpha \to a \in (-\infty, \infty)$ (e.g., $a = 0$ if $|\rho| = o(g^{1/2})$ or $a = c$ if $\rho \sim 2cg^{1/2}$). The name "quick step walk" is intended to reflect that idea that the large g limit of the continuous-time walk should be dominated by quickly moving continuous-time walks. In fact, when $\rho = 2ag^{1/2}$, by completing the square the weight $e^{-\sum_x (gL_{x,n}^2 - \rho L_{x,n})}$ can be rewritten as $e^{\sum_x [-(g^{1/2} L_{x,n} - a)^2 + a^2]}$. Thus walks with smaller $L_{x,n}$ receive larger weight, and this effect grows in importance as $g \to \infty$.

The particular case of Proposition 2 for the choice

$$\rho(g) = (2g \log(g/\pi))^{1/2}, \tag{26}$$

which corresponds to $a = \infty$, was proved previously in [3].

For the case $a = 0$, evaluation of $I_{n_x(Y)}(0)$ in (18) gives

$$\lim_{g\to\infty} Q_{g,\rho(g),n}(Y) = \frac{1}{Z_0} \prod_{x\in Y} \frac{\Gamma(n_x(Y)/2)}{2\Gamma(n_x(Y))} \qquad (\alpha \to 0). \tag{27}$$

Large values of n_x are penalised under this limiting probability, so this is a model of a self-repelling walk. It is an interesting question whether the quick step walk is in the same universality class as the self-avoiding walk, for $a \in (-\infty, \infty)$. We do not have an answer to this question.

2.2 Two-point function

Now we show that when ρ is chosen carefully, depending on g, the two-point function for the continuous-time weakly self-avoiding walk converges, as $g \to \infty$, to the two-point function of the strictly self-avoiding walk. The two-point function of the continuous-time weakly self-avoiding walk can be written in two equivalent ways. This is discussed in a self-contained manner in [5], and we summarise the situation as follows.

The version of the two-point function that we will work with is written in terms of a modified Markov process $X = X(t)$, whose definition depends on a choice of

$\delta \in (0,1)$. The state space is $\mathbb{Z}^d \cup \{\partial\}$, where ∂ is an absorbing state called the cemetery. When X arrives at state x it waits for an $\mathrm{Exp}(1)$ holding time and then jumps to a neighbour of x with probability $(2d)^{-1}(1-\delta)$ and jumps to the cemetery with probability δ. The holding times are independent of each other and of the jumps. The two-point function is defined, for $x \in \mathbb{Z}^d$, to be

$$G^{\mathrm{CT}}_{g,\rho}(x) = \frac{1}{\delta}\mathbb{E}^{(\delta)}\left(e^{-g\sum_{v\in\mathbb{Z}^d} L_v^2 + \rho\zeta}\mathbb{1}_{X(\zeta^-)=x}\right), \tag{28}$$

where we leave implicit the dependence of G^{CT} on δ, where $\mathbb{E}^{(\delta)}$ denotes expectation with respect to the modified process, and where ρ is any real number for which the expectation is finite. The random number of steps taken by X before jumping to the cemetery is denoted η, and the independent sequence of holding times will be denoted $\sigma_0, \sigma_1, \ldots, \sigma_\eta$.

A special case of the conclusions of [5, Section 3.2] (there with $d_x = 1$ and $\pi_{x,\partial} = \delta$ for all x, and restricted to finite state space) is the equivalent formula

$$G^{\mathrm{CT}}_{g,\rho}(x) = \int_0^\infty \mathbb{E}\left(e^{-g\sum_{v\in\mathbb{Z}^d} L_{v,T}^2}\mathbb{1}_{X(T)=x}\right)e^{(\rho-\delta)T}\,dT, \tag{29}$$

where now X is the original continuous-time Markov process X without cemetery state, and $\mathbb{E}$ denotes its expectation when started from the origin of $\mathbb{Z}^d$. Here $L_{v,T} = \int_0^T \mathbb{1}_{X(s)=v}ds$ is the local time of X at $v \in \mathbb{Z}^d$ up to time T. We will work with (28) rather than (29).

As in Proposition 2, we write $\alpha = \alpha(g,\rho) = \frac{1}{2}g^{-1/2}(\rho-1)$. Throughout this section, we mainly choose $\rho = \rho(g)$ in such a way that

$$\lim_{g\to\infty} g^{-1/2}e^{\alpha^2(g,\rho(g))} = p \in [0,\infty) \tag{30}$$

For example, (30) holds for $p > 0$ when $\rho(g) = 2[g\log(p\sqrt{g})]^{1/2}$, which is a choice closely related to that in (26). Note that $\lim_{g\to\infty}\rho(g) = \infty$ when $p > 0$. It is natural to consider $\rho \to \infty$, because if ρ is fixed to a value such that $G^{\mathrm{CT}}_{g_0,\rho}(x) < \infty$ for some $g_0 > 0$, then by dominated convergence $\lim_{g\to\infty} G^{\mathrm{CT}}_{g,\rho}(x) = 0$. The conclusion of Proposition 3 shows that this trivial behaviour persists even when $\rho(g) \to \infty$ in such a way that $p = 0$.

Given $p \in [0,\infty)$, let

$$z = (2d)^{-1}(1-\delta)p\sqrt{\pi}. \tag{31}$$

The following proposition shows that, under the scaling (30), the strong interaction limit of the continuous-time weakly self-avoiding walk two-point function is the two-point function of the strictly self-avoiding walk defined in (8).

Proposition 3. *Let $\delta \in (0,1)$, $z \in [0,\mu^{-1})$, and $x \in \mathbb{Z}^d$. Suppose that* (30) *holds with the value of $p \in [0,\infty)$ specified by z via* (31)*. Then*

$$\lim_{g\to\infty} G^{\mathrm{CT}}_{g,\rho(g)}(x) = p\sqrt{\pi}G_z(x). \tag{32}$$

The proof of Proposition 3 uses three lemmas, and we discuss these next. For $m \in \mathbb{N}$ and $\alpha > 0$, let

$$J_m(\alpha) = \int_{-\alpha}^{\infty} \frac{(1+u/\alpha)^{m-1}}{(m-1)!} e^{-u^2}\, du. \tag{33}$$

Lemma 1. *Given any $\varepsilon > 0$ there exists $A_0 > 0$ such that for all $\alpha \geq A \geq A_0$ and $m \geq 1$,*

$$J_m(\alpha) \leq (1+\varepsilon) J_m(A). \tag{34}$$

Proof. For $m \geq 2$, $J_m(\alpha)$ is a non-increasing function of $\alpha \in (0,\infty)$ because

$$\begin{aligned}
\frac{dJ_m(\alpha)}{d\alpha} &= -\frac{1}{(m-2)!}\int_{-\alpha}^{\infty} \frac{u}{\alpha^2}(1+u/\alpha)^{m-2} e^{-u^2} du \\
&= -\frac{1}{(m-2)!}\bigg[\int_{\alpha}^{\infty} \frac{u}{\alpha^2}(1+u/\alpha)^{m-2} e^{-u^2} du \\
&\quad + \int_0^{\alpha} \frac{u}{\alpha^2}[(1+u/\alpha)^{m-2} - (1-u/\alpha)^{m-2}] e^{-u^2} du\bigg] \\
&\leq 0
\end{aligned} \tag{35}$$

(note that in the first line the contribution from differentiating the limit of integration vanishes), and thus (34) holds even with $\varepsilon = 0$. For the remaining case $m = 1$, since J_1 is increasing and $\lim_{\alpha\to\infty} J_1(\alpha) = \sqrt{\pi}$ (see (20)), given any $\varepsilon > 0$ there exists $A_0 > 0$ such that if $\alpha \geq A \geq A_0$ then $1 \leq J_1(\alpha)/J_1(A) \leq 1+\varepsilon$. □

Recall that η is the random number of steps taken by X before jumping to the cemetery state. For $x \in \mathbb{Z}^d$, let

$$w_n(g,\rho;x) = \frac{1}{\delta}\mathbb{E}^{(\delta)}[e^{-g\sum_v L_v^2 + \rho\zeta} \mathbb{1}_{X(\zeta^-)=x} \mathbb{1}_{\eta=n}], \tag{36}$$

$$w_n(g,\rho) = \frac{1}{\delta}\mathbb{E}^{(\delta)}[e^{-g\sum_v L_v^2 + \rho\zeta} \mathbb{1}_{\eta=n}]. \tag{37}$$

Let $w_n(g;x) = w_n(g,\rho(g);x)$ and $w_n(g) = w_n(g,\rho(g))$ with $\rho(g)$ chosen according to (30).

Lemma 2. *Suppose that* (30) *holds with $p > 0$, and let z be given by* (31)*. Then for $n \geq 0$ and $x \in \mathbb{Z}^d$,*

$$\lim_{g\to\infty} w_n(g;x) = p\sqrt{\pi} c_n(x) z^n. \tag{38}$$

Proof. Given that $\eta = n$, let $Y \in \mathscr{W}_n(x)$ denote the sequence of jumps made by X before landing in the cemetery, and let $|Y|$ denote the cardinality of the range of Y. By conditioning on Y and using (19), we see that, as $g \to \infty$,

$$
\begin{aligned}
w_n(g;x) &= [(2d)^{-1}(1-\delta)]^n (g^{-1/2}e^{\alpha^2})^{n+1} \sum_{Y\in\mathscr{W}_n(x)} (\alpha e^{-\alpha^2})^{n+1-|Y|} \prod_{v\in Y} J_{n_v}(\alpha) \\
&\sim [(2d)^{-1}(1-\delta)]^n p^{n+1} \sum_{Y\in\mathscr{W}_n(x)} (\alpha e^{-\alpha^2})^{n+1-|Y|} \prod_{v\in Y} J_{n_v}(\alpha),
\end{aligned} \tag{39}
$$

where the product is over the distinct vertices visited by Y and $|Y|$ denotes the number of such vertices. It suffices to show that, for any $Y \in \mathscr{W}_n(x)$,

$$
\lim_{g\to\infty} (\alpha e^{-\alpha^2})^{n+1-|Y|} \prod_{v\in Y} J_{n_v}(\alpha) = \mathbb{1}_{Y\in\mathscr{S}_n} \pi^{(n+1)/2}. \tag{40}
$$

Since $p > 0$, we have $\alpha \to \infty$, and so $\alpha e^{-\alpha^2} \to 0$. Therefore, the above limit is zero unless $n+1 = |Y|$, which corresponds to $Y \in \mathscr{S}_n$; the product over v remains bounded as $\alpha \to \infty$ and poses no difficulty. Since $J_1(\alpha) \to \sqrt{\pi}$ as in (20), the result follows. $\square$

Lemma 3. *Suppose that* (30) *holds with* $p \in (0,\infty)$, *and let* z *be specified by* (31). *Let*

$$
\mu(g,\rho) = \limsup_{n\to\infty} w_n(g,\rho)^{1/n}. \tag{41}
$$

Then

$$
\limsup_{g\to\infty} \mu(g,\rho(g)) \le z\mu. \tag{42}
$$

Proof. Let $L_{x,[i,j]} = \sum_{k=i}^{j} \sigma_k \mathbb{1}_{Y_k=x}$, where the σ_k are the exponential holding times. Let $\mathbb{E}_y^{(\delta)}$ denote the expectation for the process started in state y instead of state 0. For integers $n \ge 1$ and $m \ge 1$, an elementary argument using the strong Markov property leads to

$$
\begin{aligned}
w_{n+m}(g,\rho) &\le \frac{1}{\delta}\mathbb{E}^{(\delta)}[e^{-g\sum_x L_{x,[0,n]}^2 + \rho\sum_x L_{x,[0,n]}} e^{-g\sum_x L_{x,[n+1,n+m]}^2 + \rho\sum_x L_{x,[n+1,n+m]}} \mathbb{1}_{\eta=n+m}] \\
&= \sum_y \mathbb{E}^{(\delta)}[e^{-g\sum_x L_{x,[0,n]}^2 + \rho\sum_x L_{x,[0,n]}} \mathbb{1}_{Y_{n+1}=y}] \frac{1}{\delta}\mathbb{E}_y^{(\delta)}[e^{-g\sum_x L_x^2 + \rho\sum_x L_x} \mathbb{1}_{\eta=m-1}] \\
&= \frac{1-\delta}{\delta}\mathbb{E}^{(\delta)}[e^{-g\sum_x L_{x,[0,n]}^2 + \rho\sum_x L_{x,[0,n]}} \mathbb{1}_{\eta=n}] w_{m-1}(g,\rho) \\
&\le w_n(g,\rho) w_{m-1}(g,\rho).
\end{aligned} \tag{43}
$$

It is straightforward to adapt the proof of [10, Lemma 1.2.2] to obtain from this approximate subadditivity the equality

$$
\mu(g,\rho) = \inf_{n\ge 1} w_n(g,\rho)^{1/(n+1)}. \tag{44}
$$

Then we have

$$
w_n(g,\rho)^{1/(n+1)} \ge \mu(g,\rho). \tag{45}
$$

We let $g \to \infty$ in the above inequality, with $\rho(g)$ chosen as in (30); note that $\alpha \to \infty$ since $p > 0$. By Lemma 2, for $n \geq 0$,

$$\lim_{g\to\infty} w_n(g) = p\sqrt{\pi} c_n z^n. \tag{46}$$

By (45), this gives

$$(p\sqrt{\pi}c_n)^{1/(n+1)} z^{n/(n+1)} \geq \limsup_{g\to\infty} \mu(g,\rho(g)). \tag{47}$$

Now we take $n \to \infty$ to get

$$\mu z \geq \limsup_{g\to\infty} \mu(g,\rho(g)), \tag{48}$$

as required. □

Proof of Proposition 3. We consider separately the cases $p > 0$ and $p = 0$.
Case $p > 0$. We write $\rho = \rho(g)$. By (28), and by (36) with $\rho = \rho(g)$,

$$G^{\mathrm{CT}}_{g,\rho}(x) = \sum_{n=0}^{\infty} w_n(g;x). \tag{49}$$

By Lemma 2, the result of taking the limit $g \to \infty$ under the summation gives the desired result

$$p\sqrt{\pi} \sum_{n=0}^{\infty} c_n(x) z^n, \tag{50}$$

and it suffices to justify the interchange of limit and summation. For this, we will use dominated convergence. Since $w_n(g;x) \leq w_n(g)$, it suffices to find a $g_0 > 0$ and a summable sequence B_n such that, for $g \geq g_0$ and $n \in \mathbb{N}_0$,

$$w_n(g;x) \leq B_n. \tag{51}$$

This will follow if we show the stronger statement that for large g

$$w_n(g) \leq B_n. \tag{52}$$

Since $z\mu < 1$, there exists $\varepsilon > 0$ such that $(1+\varepsilon)^2(\mu z+\varepsilon) < 1$. Since $g^{-1/2}e^{\alpha^2} \to p > 0$, there is a (large) g_0 such that if $g \geq g_0$ then $g^{-1/2}e^{\alpha^2} \leq g_0^{-1/2}e^{\alpha_0^2}(1+\varepsilon)$, where α_0 is the value of α corresponding to $g = g_0$; also $\alpha e^{-\alpha^2} \leq \alpha_0 e^{-\alpha_0^2}$. Therefore, by (39), and by Lemma 1 (increasing g_0 if necessary),

$$\begin{aligned} w_n(g) &= [(2d)^{-1}(1-\delta)]^n (g^{-1/2}e^{\alpha^2})^{n+1} \sum_{Y\in\mathscr{W}_n} (\alpha e^{-\alpha^2})^{n+1-|Y|} \prod_{v\in Y} J_{n_v}(\alpha) \\ &\leq [(2d)^{-1}(1-\delta)]^n (g_0^{-1/2}e^{\alpha_0^2}(1+\varepsilon)^2)^{n+1} \sum_{Y\in\mathscr{W}_n} (\alpha_0 e^{-\alpha_0^2})^{n+1-|Y|} \prod_{v\in Y} J_{n_v}(\alpha_0) \\ &= (1+\varepsilon)^{2(n+1)} w_n(g_0). \end{aligned} \tag{53}$$

We set $B_n = (1+\varepsilon)^{2(n+1)} w_n(g_0)$. Then

$$\limsup_{n\to\infty} B_n^{1/n} = (1+\varepsilon)^2 \mu(g_0, \rho(g_0)) \leq (1+\varepsilon)^2 (z\mu + \varepsilon) < 1, \tag{54}$$

by taking g_0 larger if necessary and applying Lemma 3. Therefore $\sum_n B_n$ converges, and the proof is complete for the case $p > 0$.

Case $p = 0$. We will prove that

$$\lim_{g\to\infty} \sum_{n=0}^{\infty} w_n(g) = 0. \tag{55}$$

By (49), this is more than sufficient. We again write $\rho = \rho(g)$. By conditioning on Y and using (16), for $n \geq 0$ we have

$$w_n(g) = [(2d)^{-1}(1-\delta)]^n \sum_{Y\in\mathscr{W}_n} \prod_{x\in Y} g^{-n_x/2} e^{\alpha^2} I_{n_x}(\alpha). \tag{56}$$

The change of variables $s = a + u$ in (13) gives, for $m \geq 1$,

$$\begin{aligned} e^{\alpha^2} I_m(\alpha) &= e^{\alpha^2} \int_0^\infty \frac{s^{m-1}}{(m-1)!} e^{-(s-\alpha)^2}\, ds \\ &\leq e^{\alpha^2} \int_0^\infty \frac{s^{m-1}}{(m-1)!} e^{-s} \left(\sup_{s\in\mathbb{R}} e^{s-(s-\alpha)^2} \right) ds = e^{\alpha^2+\alpha+1/4}. \end{aligned} \tag{57}$$

Let $\varepsilon > 0$. Since $g^{-1/2}e^{\alpha^2} \to p = 0$, we can find $g(\varepsilon)$ such that for $g \geq g(\varepsilon)$ and $m \geq 2$,

$$g^{-1/2} e^{\alpha^2} \sqrt{\pi} \leq \varepsilon, \qquad g^{-m/2} e^{\alpha^2+\alpha+1/4} \leq \varepsilon^m. \tag{58}$$

Henceforth we assume that $g \geq g(\varepsilon)$. By (57),

$$g^{-m/2} e^{\alpha^2} I_m(\alpha) \leq \varepsilon^m \quad \text{for } m \geq 2. \tag{59}$$

For $m = 1$, we obtain an upper bound by extending the range of the integral in the first line of (57) to the entire real line, whereupon it evaluates to $\sqrt{\pi}$. Thus, by (58), $g^{-1/2}e^{\alpha^2} I_1(\alpha) \leq \varepsilon$. By (56) and the fact that the number of walks in $\mathscr{W}_n$ is $(2d)^n$, for $n \geq 0$ we then have

$$w_n(g) \leq [(2d)^{-1}(1-\delta)]^n \sum_{Y\in\mathscr{W}_n} \prod_{v\in Y} \varepsilon^{n_v} = (1-\delta)^n \varepsilon^{n+1}. \tag{60}$$

(The case $n = 0$ corresponds to $m = 1$ because the number of visits to state 0 is $n_0 = 1$.) Therefore $\limsup_{g\to\infty} \sum_{n=0}^{\infty} w_n(g) = O(\varepsilon)$. Since ε is arbitrary, this proves (55), and the proof is complete. □

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