# DIEUDONNÉ THEORY VIA COHOMOLOGY OF CLASSIFYING STACKS II

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Abstract. In this paper, we prove two results: first, we use classifying stacks to reconstruct the classical Dieudonné module without relying on the work of Fontaine and Berthelot–Breen–Messing. As a corollary, we reprove the isomorphism $\sigma^* M(G) \simeq \mathrm{Ext}^1(G, \mathcal{O}^{\mathrm{crys}})$ due to Berthelot–Breen–Messing using stacky methods combined with the theory of de Rham–Witt complexes. Additionally, we show that finite locally free group schemes of $p$-power rank over a fairly general base embed fully faithfully into the category of prismatic $F$-gauges, which extends the work of Anschütz and Le Bras on prismatic Dieudonné theory for $p$-divisible groups.

## 1. INTRODUCTION

Let $k$ be a perfect field of characteristic $p > 0$. Let $G$ be a finite group scheme over $k$ of $p$-power rank. Then $G$ admits a canonical decomposition $G = G^{\mathrm{uni}} \bigoplus G^{\mathrm{mul}}$, where $G^{\mathrm{mul}}$ is a local group scheme whose Cartier dual is étale. Classically, one defines the (contravariant) Dieudonné module of $G$ in the following manner. Let us first define $M(G^{\mathrm{uni}}) := \mathrm{Hom}(G, \varinjlim_{n,V} W_n)$. One can now define $M(G) := M(G^{\mathrm{uni}}) \bigoplus M((G^{\mathrm{mul}})^*)^*$.

A uniform construction without appealing to duality was first given by Fontaine [Fon77] using a more complicated formal group $CW$, which maybe realized as a completion of $\varinjlim_{n,V} W_n$ in a certain sense. Another uniform construction is due to the work of Berthelot–Breen–Messing [BBM82] in terms of crystalline Dieudonné theory; they proved that $\sigma^* M(G) \simeq \mathrm{Ext}^1(G, \mathcal{O}^{\mathrm{crys}})$, where the last Ext group is computed in the large crystalline site. Their proof crucially relies on Fontaine's work and in particular certain explicit computations done in the crystalline site to understand the somewhat complicated object $CW$. In [Mon21], it was shown that $\sigma^* M(G) \simeq H^2_{\mathrm{crys}}(BG)$, where the proof relied on the work of Berthelot–Breen–Messing.

In this paper, we directly prove that $\sigma^* M(G) \simeq H^2_{\mathrm{crys}}(BG)$ without using the work of Fontaine or Berthelot–Breen–Messing. Instead, our techniques use cohomology of algebraic stacks and the de Rham–Witt complex. Broadly speaking, the main new ingredient is the usage of geometric techniques such as differential forms, deformation theory in the study of Dieudonné modules by means of the classifying stack $BG$.

**Theorem 1.1.** *Let $G$ be a finite commutative $p$-power rank group scheme over a perfect field $k$ of characteristic $p > 0$. We have a canonical isomorphism $\sigma^* M(G) \simeq H^2_{\mathrm{crys}}(BG)$.*

As a corollary, we obtain a new proof of

**Corollary 1.2** ([BBM82, Thm. 4.2.14])**.** *Let $G$ be a finite group scheme over $k$ of $p$-power rank. Then $\sigma^* M(G) \simeq \mathrm{Ext}^1_{\mathrm{CRYS}}(G, \mathcal{O}^{\mathrm{crys}})$.*

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Motivated by the above two results, in this paper, we treat $H^2_{\mathrm{crys}}(BG)$ as the main invariant of interest, and regard the classical Dieudonné module as its de Rham–Witt realization and the formula $\mathrm{Ext}^1_{\mathrm{CRYS}}(G, \mathcal{O}^{\mathrm{crys}})$ as its sheaf theoretic incarnation.

Pursing the above perspective further, we show that in the mixed characteristic context, for a quasisyntomic algebra $S$, the functor $\mathcal{M}^\vee$ (see Definition 3.8 for the precise definition) induced by sending a $p$-divisible group $G$ to $H^2_{\Delta}(BG^\vee)$ gives a fully faithful embedding of the category $\mathrm{BT}(S)$ of $p$-divisible groups over $S$ into the category of prismatic $F$-gauges over $S$; the latter is defined as the category of quasicoherent sheaves on the stack $\mathrm{Spf}(S)^{\mathrm{syn}}$.

**Theorem 1.3** (Proposition 3.32)**.** *There exists a functor* $\mathcal{M}^\vee : \mathrm{BT}(S) \to \mathrm{Vect}(S^{\mathrm{syn}})$ *induced by* $G \mapsto$ "$H^2_{\Delta}(BG^\vee)$" *which is fully faithful. Here* $\mathrm{BT}(S)$ *is the category of* $p$*-divisible groups over the quasisyntomic algebra* $S$.

The above result gives a different proof of fully faithfulness result of [ALB23], where the authors work with "admissible prismatic Dieudonné modules", which can be shown to embed fully faithfully into the category of prismatic $F$-gauges. Our approach uses the formalism of quasi-coherent sheaves on $S^{\mathrm{syn}}$, and crucially uses the dualizability of $\mathcal{M}^\vee(G) = \mathcal{M}(G^\vee)$ along with the compatibility with Cartier duality (Proposition 3.31). We take a different approach to establishing the dualizability based on a direct computation of the cotangent complex $\mathbb{L}_{BG}$ (Proposition 3.22).

The main advantage for working with the category of prismatic $F$-gauges, and a description of the Dieudonné module in terms of classifying stacks, is that it can be used to classify finite locally free commutative group schemes of $p$-power rank as well, which was not addressed by the approach taken in [ALB23]. In some sense, the category of "admissible prismatic Dieudonné module" considered in loc. cit. is not flexible enough for classifying finite flat group schemes. However, in mixed charactertistic, under the presence of torsion, it turns out that even $H^2_{\Delta}(BG)$ is not the right invariant. This is a "pathology" that does not occur when $S$ has characteristic $p$ (see Remark 3.2). To resolve this, we use the 2-stack $B^2G$.

**Theorem 1.4** (Proposition 3.21)**.** *There exists a canonical functor* $\mathcal{M}^\vee : \mathrm{FFG}(S) \to D_{\mathrm{perf}}(S^{\mathrm{syn}})$ *from the category of finite locally free commutative group schemes over a quasisyntomic algebra* $S$ *of* $p$*-power rank to perfect complexes of prismatic* $F$*-gauges induced by*

$$G \mapsto \text{“}\tau_{[-2,-3]} R\Gamma_{\Delta}(B^2G^\vee)[3]\text{”}$$

*which is fully faithful.*

**Remark 1.5.** For a precise definition of $\mathcal{M}(G) = \mathcal{M}^\vee(G^\vee)$, see Definition 3.9. As noted in Proposition 3.12, the Dieudonné crystal $\mathcal{M}(G)$ that we define can also be described alternatively as

$$\mathcal{M}(G) \simeq \tau_{\geq 0} \mathcal{R}Hom_{D(S^{\mathrm{syn}},\mathbb{Z})}(G^{\mathrm{syn}}, \mathbb{G}_a[1]).$$

The above formula is somewhat similar to the notion of Dieudonné crystal introduced in [BBM82].

A key data in our approach, that appears somewhat implicitly, is the divided Frobenius (which has previously appeared in the work of Lau [Lau18]); this is not necessary in the case of $p$-divisible groups and was not considered in [ALB23], but it plays an essential role in the presence of torsion. See Remark 3.7 for an elaboration on this point, and the results appearing before Remark 3.7, which explains what data is exactly necessary to functorially reconstruct a finite flat group scheme $G$ from the prismatic cohomology of $BG$ or $B^2G$. From our perspective, the Dieudonné module of a finite locally free group scheme or a $p$-divisible group $G$ should be regarded as a "$p$-adic motive", which is realized by the cohomology of $BG$ or $B^2G$. The work of Drinfeld [Dri21] and Bhatt–Lurie ([BL22a], [BL22b], [Bha23]) on prismatic $F$-gauges allow one to precisely formulate the desired category as quasi-coherent sheaves on the stack $S^{\mathrm{syn}}$. We will now give a very brief explanation of how to work with prismatic $F$-gauges, relevant in our set up.

By quasisyntomic descent (see [BMS19]), in order to understand $S^{\mathrm{syn}}$ one can restrict attention to the case when $S = \mathrm{Spf}(R)$ (see [Bha23]) for a quasiregular semiperfectoid algebra $R$. In this case, we concretely spell out the stack $(\mathrm{Spf}(R))^{\mathrm{syn}}$ to give a sense of what kind of objects we are working with, and how it keeps track of the divided Frobenius.

**Construction 1.6.** Let $R$ be a quasiregular semiperfectoid algebra. Define $\mathrm{Spf}(R)^{\Delta} := \mathrm{Spf}(\Delta_R)$, where $\Delta_R$ is the prism associated to $R$. Define

$$\mathrm{Spf}(R)^{\mathrm{Nyg}} := \mathrm{Spf}\left(\bigoplus_{i\in\mathbb{Z}} \mathrm{Fil}^i_{\mathrm{Nyg}}\Delta_R\{i\}\right)/\mathbb{G}_m.$$

The Nygaard filtration provides a map of graded rings $\bigoplus_{i\in\mathbb{Z}} \mathrm{Fil}^i_{\mathrm{Nyg}}\Delta_R\{i\} \to \bigoplus_{i\in\mathbb{Z}} \Delta_R\{i\}$, which induces a map

$$\mathrm{can} : \mathrm{Spf}(R)^{\Delta} \to \mathrm{Spf}(R)^{\mathrm{Nyg}}.$$

Also, the divided Frobenius defines a map of graded rings $\oplus_{i\in\mathbb{Z}}(\varphi_i) : \bigoplus_{i\in\mathbb{Z}} \mathrm{Fil}^i_{\mathrm{Nyg}}\Delta_R\{i\} \to \bigoplus_{i\in\mathbb{Z}} \Delta_R\{i\}$, which induces a map

$$\varphi : \mathrm{Spf}(R)^{\Delta} \to \mathrm{Spf}(R)^{\mathrm{Nyg}}.$$

One defines

$$\mathrm{Spf}(R)^{\mathrm{syn}} := \mathrm{coeq}\left( \mathrm{Spf}(R)^{\Delta} \underset{\varphi}{\overset{\mathrm{can}}{\rightrightarrows}} \mathrm{Spf}(R)^{\mathrm{Nyg}} \right).$$

In this set up, the graded module $\bigoplus_{i\in\mathbb{Z}} \mathrm{Fil}^{i-1}_{\mathrm{Nyg}}\Delta_R\{i-1\}$ defines a vector bundle on $\mathrm{Spf}(R)^{\mathrm{Nyg}}$, which descends to a vector bundle on $\mathrm{Spf}(R)^{\mathrm{syn}}$– this will be called the Breuil–Kisin twist and will be denoted by $\mathcal{O}\{-1\}$. For any $M \in \mathrm{Spf}(S)^{\mathrm{syn}}$, we use $M\{-n\}$ to denote $M \otimes_{\mathcal{O}} \mathcal{O}\{-1\}^{\otimes n}$. See Remark 3.18 for a discussion on what kind of structures are encoded on quasicoherent sheaves on $\mathrm{Spf}(R)^{\mathrm{syn}}$.

Our main technique for proving Theorem 1.4 above is using the formalism of quasi-cohrent sheaves on the stack $S^{\mathrm{syn}}$ and certain other computations of cohomology of $BG$ involving the Breuil–Kisin twist $\mathcal{O}\{1\}$. A key role is played by the dualizability of $\mathcal{M}^\vee(G)$ as a prismatic $F$-gauge. In order to prove the latter statement, we work with the concrete description appearing in Construction 1.6, and directly work with the full Nygaard filtration on prismatic cohomology of $BG$.

**Notations and conventions.** We will use the language of $\infty$-categories as in [Lur09], more specifically, the language of stable $\infty$-categories [Lur17]. For an ordinary commutative ring $R$, we will let $D(R)$ denote the derived $\infty$-category of $R$-modules, so that it is naturally equipped with a $t$-structure and $D_{\geq 0}(R)$ (resp. $D_{\leq 0}(R)$) denotes the connective (resp. coconnective) objects, following the homological convention. We work with a fixed prime $p$. We will let $R\Gamma_{\mathrm{crys}}(\cdot)$ denote derived crystalline cohomology, which reduces to derived de

Rham cohomology $\mathrm{dR}(\cdot)$ modulo $p$ (see [Bha12]). We freely use the quasisyntomic descent techniques using quasiregular semiperfectoid algebras introduced in [BMS19]. We also freely use the formalism of prismatic cohomology developed by Bhatt–Scholze [BS19], as well as the stacky approach to prismatic cohomology developed by Drinfeld [Dri21] and Bhatt–Lurie [BL22a], [BL22b], [BL]. At the moment, our main reference for working with prismatic $F$-gauges is based on Bhatt's lecture notes [Bha23]. For a group scheme $G$, we let $BG$ denote the classifying stack and $B^2G$ denote the 2-stack $K(G,2)$.

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## 2. Crystalline Dieudonné theory

One of the key ingredients in our proof of Theorem 1.1 is the following construction based on the de Rham–Witt complex. Construction 2.1 below extends a construction from [Ill79, § 6] to stacks, which was originally introduced by Illusie to produce torsion in crystalline cohomology by using the de Rham–Witt complex.

**Construction 2.1.** We discuss the construction of a map

$$T : R\Gamma_{\mathrm{qsyn}}(\mathcal{Y}, \varinjlim_{n,V} W_n) \to \sigma_* R\Gamma_{\mathrm{crys}}(\mathcal{Y})[1]$$

for a stack $\mathcal{Y}$ over a perfect field $k$ of characteristic $p>0$, where the left hand side denotes cohomology computed in the quaisyntomic site [BMS19]. By considering sheafification in the quasisyntomic topology (see [BMS19, Example 5.12]) one may reduce this to the case of affine schemes. Further, by considering simplicial resolutions using polynomial algebras (see [SD23, Example 2.13]), this amounts to constructing a natural map

$$R\Gamma_{\mathrm{qsyn}}(X, \varinjlim_{n,V} W_n) \to \sigma_* R\Gamma_{\mathrm{crys}}(X)[1]$$

for an affine scheme $X = \operatorname{Spec} R$, where $R$ is a polynomial algebra over $k$. Note that we have an exact sequence

$$0 \to W \xrightarrow{V^n} W \to W_n \to 0$$

of group schemes over $k$. Let $W\Omega^*_X$ denote the de Rham Witt complex of $X$. Define another complex $W\Omega^*_X(n) := W\mathcal{O}_X \xrightarrow{F^n d} W\Omega^1_X \xrightarrow{d} W\Omega^2_X \xrightarrow{d} \ldots$. There is a natural map $W\Omega^*_X \to W\Omega^*_X(n)$ induced by $V^n$ (since $FdV = d$). We obtain an exact sequence of complexes

$$0 \to W\Omega^*_X \to W\Omega^*_X(n) \to W_n\mathcal{O}_X \to 0. \tag{2.0.1}$$

Taking direct limits produce an exact sequence of complexes

$$0 \to W\Omega^*_X \to \varinjlim_{n,V} W\Omega^*_X(n) \to \varinjlim_{n,V} W_n\mathcal{O}_X \to 0. \tag{2.0.2}$$

Passing to the derived category, using the comparison of de Rham–Witt and crystalline cohomology and keeping track of the $W(k)$-module structure, we obtain the map $R\Gamma_{\mathrm{qsyn}}(X, \varinjlim_{n,V} W_n) \to \sigma_* R\Gamma_{\mathrm{crys}}(X)[1]$ as desired.

**Remark 2.2.** Let us give further explanation about the previous construction, which is based on the construction of the natural map $W_n\mathcal{O}_X \to R\Gamma_{\mathrm{crys}}(X)[1]$. Note that such a map arises from a map $W_n\mathcal{O}_X \to R\Gamma_{\mathrm{crys}}(X)/p^n$ via composition with $R\Gamma_{\mathrm{crys}}(X)/p^n \to R\Gamma_{\mathrm{crys}}(X)[1]$. The desired map $W_n\mathcal{O}_X \to R\Gamma_{\mathrm{crys}}(X)/p^n$ can be thought of as simply arising from the more general conjugate filtration $\mathrm{Fil}^*_{\mathrm{conj}} R\Gamma_{\mathrm{crys}}(X)/p^n$ whose graded pieces can be understood by the $\varphi^n$-linear higher Cartier isomorphism

$$\mathbb{L}W_n\Omega^i_X[-i] \simeq \mathrm{gr}^i_{\mathrm{conj}}\left(R\Gamma_{\mathrm{crys}}(X)/p^n\right).$$

We will apply the map $T$ from Construction 2.1 in the case $\mathcal{Y} = BG$. To this end, let us note a few general remarks about cohomology of $BG$. Let CRYS denote the big crystalline topos, and $\mathrm{CRYS}_n$ denote the $n$-truncated variant. Let $\mathcal{F}$ be any object in the derived category of quasisyntomic sheaves. By Cech descent along the effective epimorphism $* \to BG$, we obtain

$$R\Gamma(BG, \mathcal{F}) \simeq \varprojlim_{m\in\Delta} R\Gamma(G^{[m]}, \mathcal{F}),$$

where $G^{[m]}$ is the Cech nerve of $* \to BG$. Applying this to $\mathcal{F} = R\Gamma_{\mathrm{crys}}(\cdot)$ (resp. $R\Gamma_{\mathrm{crys}}(\cdot)/p^n$), we obtain

$$R\Gamma_{\mathrm{crys}}(BG, \mathcal{F}) \simeq \varprojlim_{m\in\Delta} R\Gamma_{\mathrm{crys}}(G^{[m]}, \mathcal{F}) \simeq R\mathrm{Hom}_{\mathrm{CRYS}}(\varinjlim_{n\in\Delta^{op}} \mathbb{Z}[G^{[m]}], \mathcal{O}^{\mathrm{crys}}).$$

Let us define $\mathbb{Z}[BG] := \varinjlim_{n\in\Delta^{op}} \mathbb{Z}[G^{[m]}]$. This way, we get a spectral sequence with $E_2$-page

$$E_2^{i,j} = \mathrm{Ext}^i_{\mathrm{CRYS}}(H^{-j}(\mathbb{Z}[BG]), \mathcal{O}^{\mathrm{crys}}) \implies H^{i+j}_{\mathrm{crys}}(BG) \tag{2.0.3}$$

(resp. a similar spectral sequence in $\mathrm{CRYS}_n$ converging to $H^*_{\mathrm{crys}}(BG/W_n)$).

**Lemma 2.3.** *Let $G$ be a group scheme of order $p^m$. Then for any $i > 0$, the group $H^i_{\mathrm{crys}}(BG)$ is killed by a power of $p$.*

*Proof.* This is a consequence of the above $E_2$-spectral sequence and the fact that an $n$-torsion ordinary abelian group $T$, the group homology $H_i(T, \mathbb{Z}) = H_i(\mathbb{Z}[BT])$ is $n$-torsion. $\square$

Note that, by definition, for any stack $\mathcal{Y}$, we have an exact sequence

$$0 \to H^i_{\mathrm{crys}}(\mathcal{Y})/p^n \to H^i_{\mathrm{crys}}(\mathcal{Y}/W_n) \to H^{i+1}_{\mathrm{crys}}(\mathcal{Y})[p^n] \to 0. \tag{2.0.4}$$

**Lemma 2.4.** *Let $G$ be a group scheme of order $p^m$. Then $H^1_{\mathrm{crys}}(BG) = 0$*

*Proof.* We choose a large enough $n$ such that $H^1_{\mathrm{crys}}(BG)[p^n] = H^1_{\mathrm{crys}}(BG)$ and apply (2.0.4) for $i = 0$. $\square$

**Lemma 2.5.** *Let $G$ be a finite group scheme of $p$-power order. Then for all $n \gg 0$, the map $H^1_{\mathrm{crys}}(BG/W_n) \to H^2_{\mathrm{crys}}(BG)$ is an isomorphism.*

*Proof.* We just need to choose $n$ large enough such that $H^2_{\mathrm{crys}}(BG)$ is killed by $p^n$ and apply (2.0.4) for $i = 1$ along with the previous lemma. $\square$

**Lemma 2.6.** *Let $k'/k$ be an extension of perfect fields. Then*

$$R\Gamma_{\mathrm{crys}}(BG_{k'}) \simeq R\Gamma_{\mathrm{crys}}(BG) \otimes_{W(k)} W(k').$$

*Proof.* By derived $p$-completeness, one can reduce modulo $p$. By the de Rham–crystalline comparison, it would be enough to prove that $R\Gamma_{\mathrm{dR}}(BG_{k'}) \simeq R\Gamma_{\mathrm{dR}}(BG) \otimes_k k'$. Since $R\Gamma_{\mathrm{dR}}(G_{k'}) \simeq R\Gamma_{\mathrm{dR}}(G) \otimes_k k'$ the result follows from descent along $* \to BG$ and using the fact that totalization of bounded below cochain complexes commute with filtered colimits. □

**Lemma 2.7.** *Let $0 \to G' \to G \to G'' \to 0$ be an exact sequence of finite group schemes over $k$. Then we have an exact sequence $0 \to H^2_{\mathrm{crys}}(BG'') \to H^2_{\mathrm{crys}}(BG) \to H^2_{\mathrm{crys}}(BG')$ of $W(k)$-modules.*

*Proof.* We pick an $n$ large enough so that $H^2_{\mathrm{crys}}(BG), H^2_{\mathrm{crys}}(BG'), H^2_{\mathrm{crys}}(BG'')$ are all killed by $p^n$. Then the desired exactness of the maps follow from Lemma 2.5 and the fact that $H^1_{\mathrm{crys}}(BH/W_n) = \mathrm{Ext}^0_{\mathrm{QSyn}}(H, R\Gamma(\cdot)/p^n)$. □

**Lemma 2.8.** *Let $G$ be a finite group scheme of $p$-power rank over a perfect field $k$. Then $H^2_{\mathrm{crys}}(BG)$ is a finite length $W(k)$-module.*

*Proof.* By Lemma 2.6, one may assume that $k$ is algebraically closed. In that case, one can argue by induction using Lemma 2.7, which reduces us to the statement for the simple group schemes $\mathbb{Z}/p, \alpha_p$ and $\mu_p$. This follows from Remark 3.19 below. □

**Proposition 2.9** (cf. [ABM21])**.** *$H^2_{\mathrm{crys}}(BH) = k$ when $H$ is either $\mathbb{Z}/p, \mu_p$ or $\alpha_p$.*

*Proof.* By base change, it is enough to argue when $k = \mathbb{F}_p$. First note that for $n \gg 0$, we have isomorphisms $\mathrm{Ext}^0_{\mathrm{QSyn}}(H, R\Gamma(\cdot)/p^n) \simeq H^1_{\mathrm{crys}}(BH/W_n) \simeq H^2_{\mathrm{crys}}(BH)$. Since $H$ is $p$-torsion, it follows by looking at the left term above that $H^2_{\mathrm{crys}}(BH)$ is $p$-torsion. Therefore, it suffices to show that $H^1_{\mathrm{dR}}(BH) = k$ as a $k$-vector space.

(1) $H = \mathbb{Z}/p$ : Using descent along $* \to B\mathbb{Z}/p$, one sees that $H^*_{\mathrm{dR}}(B\mathbb{Z}/p) \simeq H^*(\mathbb{Z}/p, k)$, where the latter denotes group cohomology. Thus, the claim follows.
(2) $H = \mu_p$ : Since $\mu_p$ lifts to $\mathbb{Z}/p^2$ as a group scheme (along with lift of the Frobenius, which is simply the zero map), the conjugate filtration on $R\Gamma_{\mathrm{dR}}(B\mu_p)$ splits. Since $\mathbb{L}_{B\mu_p} = \mathcal{O} \oplus \mathcal{O}[-1]$, our claim follows by noting that $H^{>0}(B\mu_p, \mathcal{O}) = 0$.
(3) $H = \alpha_p$ : We use the conjugate spectral sequence. Note that $H^1(B\alpha_p, \mathcal{O}) = \mathrm{Hom}(\alpha_p, \mathbb{G}_a) = k$. Note once again that $\mathbb{L}_{B\alpha_p} = \mathcal{O} \oplus \mathcal{O}[-1]$. It would be enough to prove that the map $d$ given by the differential $k = H^0(B\alpha_p, \mathbb{L}_{B\alpha_p}) \to H^2(B\alpha_p, \mathcal{O})$ is injective. Note that there is an obstruction class $c \in \mathrm{Ext}^2(\mathbb{L}_{B\alpha_p}, \mathcal{O}) = H^2(B\alpha_p, \mathcal{O}) \oplus H^3(B\alpha_p, \mathcal{O})$, which is nonzero since $B\alpha_p$ does not lift to $\mathbb{Z}/p^2$. To see the latter statement, note that if $B\alpha_p$ lifted to $\mathbb{Z}/p^2$, due to smoothness of $B\alpha_p$ as a stack, we would be able to lift the map $* \to B\alpha_p$ too, thus ultimately, producing a lifting of the group scheme $\alpha_p$ to $\mathbb{Z}/p^2$, which is impossible. Now, the map $d$ is obtained by applying $H^0$ to the map $\mathbb{L}_{B\alpha_p} \to \mathcal{O}[2]$ parametrizing the obstruction class $c$. Using the map $B\alpha_p \to B\mathbb{G}_a$, by functoriality of obstruction class (see [FGI+05, 8.5.10]), we know that the obstruction classes to lifting for $B\alpha_p$ and $B\mathbb{G}_a$ have the same image in $H^2(B\alpha_p, \mathcal{O}[1]) = H^3(B\alpha_p, \mathcal{O})$; but that must be zero, since $B\mathbb{G}_a$ is liftable. This implies that projection of $c$ on $H^3(B\alpha_p, \mathcal{O})$ is zero. Since $c$ is nonzero itself, we see that the map $d$ must be nonzero. This gives the claim that $H^1_{\mathrm{dR}}(B\alpha_p) = k$.

This ends the proof. □

**Proposition 2.10.** *Let $k$ be a perfect field. Then as $W(k)$-modules, we have a canonical isomorphism $H^2_{\text{crys}}(B\mu_{p^m}) \simeq \sigma^*(W(k)/p^m)$.*

*Proof.* Chern class of the line bundle corresponding to the map $B\mu_{p^m} \to B\mathbb{G}_m$ defines a canonical $p^m$-torsion class on $H^2_{\text{crys}}(B\mu_{p^m})$. Since $H^1_{\text{crys}}(B\mu_{p^m}) = 0$, we obtain a canonical map

$$\sigma^* W(k) \oplus \sigma^* W(k)/p^m[-2] \to R\Gamma_{\text{crys}}(B\mu_{p^m})$$

in the derived category of $W(k)$-modules. Let $C$ denote the cofiber. It will be enough to prove that $C \in D^{\geq 3}(W(k))$. Since $C$ is derived $p$-complete it is enough to prove the same for $C/p$. It is therefore enough to prove that the induced map

$$k^{(1)} \oplus k^{(1)}[-1] \oplus k^{(1)}[-2] \to R\Gamma_{\text{dR}}(B\mu_{p^m})$$

has cofiber in $D^{\geq 3}(k)$. To this end, we will use the conjugate spectral sequence. Since $\mu_{p^m}$ lifts to $W_2(k)$ as a group scheme along with a lift of the Frobenius (which is just multiplication by $p$), the conjugate filtration splits. The claim now follows from the fact that $\mathbb{L}_{B\mu_{p^m}} = \mathcal{O} \oplus \mathcal{O}[-1]$ and $H^{>0}(B\mu_{p^m}, \mathcal{O}) = 0$, as the map constructed above is seen to induce isomorphism on $i$-th cohomology for $i \leq 2$. □

**Construction 2.11.** Now we can use the map $T$ from Construction 2.1 to obtain a map

$$C : H^1(BG, \varinjlim_n W_n) \to H^2_{\text{crys}}(BG).$$

Assume $G$ is unipotent. Since $H^1(BG, \varinjlim_n W_n) = \text{Hom}(G, \varinjlim_n W_n)$, we get a natural map

$$C^{\text{uni}} : \sigma^* M(G) \to H^2_{\text{crys}}(BG). \tag{2.0.5}$$

Suppose that $G$ is a local group scheme over $k$ of order $p^k$ whose Cartier dual $G^*$ is étale. We will produce a natural map

$$C^{\text{mult}} : H^2_{\text{crys}}(BG) \to \sigma^* M(G). \tag{2.0.6}$$

To this end, in the remarks below, we recall certain constructions.

**Remark 2.12** (Duality)**.** Let $\text{Mod}^{\text{fl}}_{W(k)}$ denote the category of finite length $W(k)$-modules. The functor that sends $M \mapsto \text{Hom}_{W(k)}(M, W(k)[\frac{1}{p}]/W(k))$ induces an antiequivalence of $\text{Mod}^{\text{fl}}_{W(k)}$. This duality also extends to the set up of Dieudonné modules whose underlying $W(k)$-module is finite length.

**Remark 2.13** (Galois descent)**.** Let $\overline{k}$ be an algebraic closure of $k$. Let $(\text{Mod}^{\text{fl}}_{W(\overline{k})})^{\text{Gal}(\overline{k}/k)}$ denote the category of finite length $W(\overline{k})$-modules equipped with a (semilinear) action of $\text{Gal}(\overline{k}/k)$. By Galois descent, we obtain an equivalence of categories $(\text{Mod}^{\text{fl}}_{W(\overline{k})})^{\text{Gal}(\overline{k}/k)} \simeq \text{Mod}^{\text{fl}}_{W(k)}$ induced by the functors that send $M \mapsto M \otimes_{W(k)} W(\overline{k}) \in (\text{Mod}^{\text{fl}}_{W(\overline{k})})^{\text{Gal}(\overline{k}/k)}$, and $N \to N^{\text{Gal}(\overline{k}/k)}$. To check this, one needs to prove that the natural map $N^{\text{Gal}(\overline{k}/k)} \otimes_{W(k)} W(\overline{k}) \to N$ is an isomorphism. Suppose first that $N$ is $p$-torsion. Then $N$ corresponds to a vector bundle in the étale site of $\text{Spec}\, k$, which must be trivial by descent; i.e., $N \simeq M \otimes_k \overline{k}$ for some finite dimensional $k$-vector space $N$, which implies that the desired map is an isomorphism. The case of general $N$ follows by considering the (finite) $p$-adic filtration on $N$ and using that $H^{>0}(\text{Gal}(\overline{k}/k), \overline{k}) = 0$.

Now, let $G$ be such that $G^*$ is étale. Note that for $n \gg 0$, we have

$$M(G^*_{\overline{k}}) \simeq \mathrm{Hom}(G^*_{\overline{k}}, \mathbb{Z}/p^n) \otimes_{\mathbb{Z}_p} W(\overline{k}) \simeq \mathrm{Hom}(\mu_{p^n}, G_{\overline{k}}) \otimes_{\mathbb{Z}_p} W(\overline{k}),$$

where the last step follows from Cartier duality. By functoriality of crystalline cohomology and Proposition 2.10, we obtain a map

$$\mathrm{Hom}(\mu_{p^n}, G_{\overline{k}}) \otimes_{\mathbb{Z}_p} W(\overline{k}) \to \mathrm{Hom}^{\mathrm{Gal}(\overline{k}/k)}_{W(\overline{k})}(\sigma_* H^2_{\mathrm{crys}}(BG_{\overline{k}}), W(\overline{k})/p^n);$$

the latter denotes maps taken in $(\mathrm{Mod}^{\mathrm{fl}}_{W(\overline{k})})^{\mathrm{Gal}(\overline{k}/k)}$. Taking Galois fixed points, we obtain a map $M(G^*) \to (\sigma_* H^2_{\mathrm{crys}}(BG))^*$. Applying duality now produces a map

$$\sigma_* H^2_{\mathrm{crys}}(BG) \to M(G^*)^* = M(G).$$

This constructs $C^{\mathrm{mult}}$ as desired in (2.0.6).

**Lemma 2.14.** *Let $G$ be unipotent. The map $C^{\mathrm{uni}}$ is injective.*

*Proof.* By Construction 2.1, we have the following commutative diagram where the bottom row is exact:

$$\begin{array}{ccccc} & & H^1(BG, \varinjlim W_n) & \longrightarrow & H^2_{\mathrm{crys}}(BG) \\ & & \downarrow & & \downarrow \\ H^1(BG, \varinjlim_V W) & \longrightarrow & H^1(BG, \varinjlim W_n) & \longrightarrow & H^2(BG, W) \end{array}$$

Therefore, to prove the injectivity of $C^{\mathrm{uni}}$, it suffices to show that $H^1(BG, \varinjlim_V W) = \mathrm{Hom}(G, \varinjlim_V W) = 0$. But this follows because $W$ is $V$-torsion free, and $G$, being finite and unipotent, is killed by a power of $V$. □

**Proposition 2.15.** *Let $G$ be a finite commutative $p$-power rank group scheme over $k$. We have a canonical isomorphism $\sigma^* M(G) \simeq H^2_{\mathrm{crys}}(BG)$.*

*Proof.* We use the natural maps $C^{\mathrm{uni}}$ and $C^{\mathrm{mult}}$ constructed before and argue separately. To check that the natural maps are isomorphisms, we may assume that $k$ is algebraically closed (Lemma 2.6). We have an exact sequence $0 \to G' \to G \to G'' \to 0$, where one may assume that $G'$ is simple. Since $k$ is algebraically closed, $G'$ must be either $\mathbb{Z}/p, \mu_p$ or $\alpha_p$. By Lemma 2.7, we have the following diagram where the rows are exact:

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^2_{\mathrm{crys}}(BG'') & \longrightarrow & H^2_{\mathrm{crys}}(BG) & \longrightarrow & H^2_{\mathrm{crys}}(BG') \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \sigma^* M(G'') & \longrightarrow & \sigma^* M(G) & \longrightarrow & \sigma^* M(G') \end{array}$$

One sees directly that in this case, the map $\sigma^* M(G) \to \sigma^* M(G')$ is surjective; thus the claim follows by Remark 3.19 and induction on the length of $G$. □

**Proposition 2.16.** *Let $G$ be a finite commutative $p$-power rank group scheme over $k$. We have a canonical isomorphism $H^2_{\mathrm{crys}}(BG) \simeq H^3_{\mathrm{crys}}(B^2 G)$.*

*Proof.* Using descent along $* \to B^2G$, we obtain an $E_1$-spectral sequence

$$E_1^{i,j} = H^j_{\mathrm{crys}}((BG)^i) \implies H^{i+j}_{\mathrm{crys}}(B^2G).$$

The claim now follows by using this spectral sequence and Lemma 2.4. □

**Lemma 2.17.** *For any group scheme $G$ over $k$, one has $H^3_{\mathrm{crys}}(B^2G) \simeq \mathrm{Ext}^1_{\mathrm{CRYS}}(G, \mathcal{O}^{\mathrm{crys}})$.*

*Proof.* By applying descent along $* \to B^2G$, similar to (2.0.3), we obtain an $E_2$-spectral sequence

$$E_2^{i,j} = \mathrm{Ext}^i_{\mathrm{CRYS}}(H^{-j}(\mathbb{Z}[B^2G]), \mathcal{O}^{\mathrm{crys}}) \implies H^{i+j}_{\mathrm{crys}}(B^2G).$$

The claim now follows from the fact that $H^{-1}(\mathbb{Z}[B^2G]) = 0$, $H^{-2}(\mathbb{Z}[B^2G]) = G$ (by e.g., Hurewicz theorem) and $H^{-3}(\mathbb{Z}[B^2G]) = 0$; the latter vanishing can be seen by applying the Serre fibration spectral sequence for the fibration $K(G,n) \to * \to K(G,n+1)$. □

Combining the three propositions before, we obtain

**Corollary 2.18** (Berthelot–Breen–Messing)**.** *Let $G$ be a finite group scheme over $k$ of $p$-power rank. Then $\sigma^* M(G) \simeq \mathrm{Ext}^1_{\mathrm{CRYS}}(G, \mathcal{O}^{\mathrm{crys}})$.*

## 3. Classification of finite locally free group schemes and $p$-divisible groups

In this section, we prove that the Dieudonné module functor induced by $G \mapsto H^2_{\Delta}(BG)$ give a fully faithful functor. However, one cannot expect this functor to be fully fathful without carefully analyzing what other extra data on $H^2_{\Delta}(BG)$ one needs to remember. Let us first take a step back to explain our perspective on Dieudonné theory taken in this paper.

By Pontryagin duality, for a finite discrete abelian grouop $G$, the functor $G \mapsto \mathrm{Hom}(G, \mathbb{S}^1)$ gives an equivalence of categories. Note that in this case, since $\mathbb{S}^1 = K(\mathbb{Z}, 1)$ and $G$ is finite, one has a natural isomorphism $\mathrm{Hom}(G, \mathbb{S}^1) = H^1(BG, \mathbb{Z}[1]) = H^2(BG, \mathbb{Z})$, where the latter denotes Betti cohomology. Our main point here is that it is possible to reconstruct a group (or rather its Pontryagin dual) from the cohomology of $BG$.

Now, let $G$ be a finite locally free commutative group scheme over a base scheme $S$. By Cartier duality, the functor $G \to G^\vee := \mathcal{H}om(G, \mathbb{G}_m)$ gives an antiequivalence of categories. Note that $\mathcal{H}om(G, \mathbb{G}_m) = \mathcal{H}^1(BG, \mathbb{G}_m)$. Further, one may write $\mathbb{G}_m = \mathbb{Z}(1)[1]$, where $\mathbb{Z}(1)$ is the Tate twist. Thus, $\mathcal{H}^1(BG, \mathbb{G}_m) = \mathcal{H}^2(BG, \mathbb{Z}(1))$; the latter recovers the group scheme $G$ (or rather, its Cartier dual). The notion of Tate twists and other related twists would play a very important role in our approach.

Since the goal of Dieudonné theory is to classify finite locally free or $p$-divisible group schemes by linear algebraic data, it is natural to look for a more linearized way to recover the Tate twist $\mathbb{Z}(1)$. To this end, we now entirely specialize to the $p$-adic set up. We take $S$ to be a $p$-complete quasisyntomic formal scheme. The $p$-adic Tate twist $\mathbb{Z}_p(1)$ maybe obtained from prismatic cohomology by means of the following formula:

$$\mathrm{R}\Gamma(\mathcal{Y}, \mathbb{Z}_p(1)) \simeq \mathrm{Fib}\left(\mathrm{Fil}^1_{\mathrm{Nyg}} R\Gamma_{\Delta}(\mathcal{Y})\{1\} \xrightarrow{\varphi_1 - \mathrm{can}} R\Gamma_{\Delta}(\mathcal{Y})\{1\}\right), \tag{3.0.1}$$

where the curly brackets denote the Breuil–Kisin twist. Let us specialize to $\mathcal{Y} = BG$. Note the following vanishings, that would allow us to simply the above formula.

**Lemma 3.1.** *Let $G$ be a $p$-divisible group. We have $\mathcal{H}^1_{\Delta}(BG) = 0$.*

*Proof.* Note that any class in prismatic cohomology is killed quasisyntomic locally, thus it is enough to prove that $\mathcal{H}om(G, \mathbb{\Delta}_{(\cdot)}) = 0$ in the quasisyntomic topos. When $G$ is $p$-divisible, this follows because $\mathbb{\Delta}_{(\cdot)}$ is derived $p$-complete. □

**Remark 3.2.** Even if we are working over a quasisyntomic base ring $S$, for a finite locally group $G$ scheme over $S$, the above vanishing need not hold. One can take $S$ to be a quasiregular semiperfectoid algebra such that $\mathbb{\Delta}_S$ has $p$-torsion. Then $G = \mathbb{Z}/p\mathbb{Z}_S$ gives such an example. Let us point out that if $S$ has characteristic $p > 0$, this "pathology" does not occur since for any quasiregular semiperfect ring $S$, the ring $\mathbb{A}_{\text{crys}}(S)$ is $p$-torsion free. It is precisely to circumvent this pathology that we will be working with $B^2G$ for finite locally free commutative group schemes, whereas for $p$-divisible groups $G$, the stack $BG$ suffices. This makes sense from a conceptual view point: formation of the stack $BG$ does not require $G$ to be commutative, while $B^2G$ does.

**Lemma 3.3.** *Let $G$ be finite locally free group scheme of $p$-power rank. Then*

$$\mathcal{H}^2(BG, \mathbb{Z}_p(1)) = \mathcal{E}xt^1(G, \mathbb{Z}_p(1)) = G^\vee.$$

*Proof.* Since $G$ is killed by a power of $p$, it follows that the natural map $\mathbb{G}_m \to \mathbb{Z}_p(1)[1]$ induces an isomorphism $\mathcal{E}xt^1(G, \mathbb{Z}_p(1)) = \mathcal{H}om(G, \mathbb{G}_m) = G^\vee$. To see that $\mathcal{H}^2(BG, \mathbb{Z}_p(1)) = \mathcal{E}xt^1(G, \mathbb{Z}_p(1))$, it suffices to show that $\mathcal{H}om(G \wedge G, \mathbb{Z}_p(1)) = 0$; the latter follows from the fact that $\mathcal{H}om(G, \mathbb{Z}_p(1)) = \mathcal{E}xt^{-1}(G, \mathbb{G}_m) = 0$. □

**Proposition 3.4.** *Let $G$ be a finite locally free group scheme of $p$-power rank over a quasisyntomic ring $S$. Then $\mathcal{H}^2(B^2G, \mathbb{Z}_p(1)) = 0$ and $\mathcal{H}^3(B^2G, \mathbb{Z}_p(1)) = G^\vee$.*

*Proof.* For any quasisyntomic sheaf $\mathcal{F}$ such that $H^i(*, \mathcal{F}) = 0$ for $i > 0$, we have $H^2(B^2G, \mathcal{F}) = \text{Hom}(G, \mathcal{F})$ and $H^3(B^2G, \mathcal{F})) = \text{Ext}^1(G, \mathcal{F}) = 0$. We prove the latter statement. Indeed, $H^3(B^2G, \mathcal{F})) = \pi_0\text{Maps}_*(B^2G, B^3\mathcal{F})$, where the latter denotes pointed maps. By delooping twice, that corresponds to homotopy classes $\mathbb{E}_2$-group objects $G \to B\mathcal{F}$. However, since the objects are 1-truncated, that corresponds to $\text{Ext}^1(G, \mathcal{F})$. Applying this to $\mathcal{F} = \mathbb{Z}_p(1)$, the conclusion follows from Lemma 3.3. □

**Lemma 3.5.** *Let $G$ be a $p$-divisible group. Then*

$$H^2(BG, \mathbb{Z}_p(1)) = \mathcal{E}xt^1(G, \mathbb{Z}_p(1)) = T_p(G^\vee).$$

*Proof.* Follows from the above lemma by taking inverse limits. □

**Lemma 3.6.** *Let $G$ be a finite locally free group scheme or a $p$-divisible group. Then $\mathcal{E}xt^2(G, \mathbb{Z}_p(1)) = 0$.*

*Proof.* By considering inverse limits, one reduces to the finite locally free case. The fiber of the natural map $\mathbb{G}_m \to \mathbb{Z}_p(1)[1]$ is uniquely $p$-divisible. But since $G$ is killed by a power of $p$, we must have $\mathcal{E}xt^2(G, \mathbb{Z}_p(1)) = \mathcal{E}xt^1(G, \mathbb{G}_m)$. To show vanishing of the latter, let $n$ be an integer that kills $G$. Using the exact sequence $0 \to \mu_n \to \mathbb{G}_m \xrightarrow{n} \mathbb{G}_m \to 0$, one sees that any class $u \in \text{Ext}^1(G, \mathbb{G}_m)$ arises from a class $v \in \text{Ext}^1(G, \mu_n)$. Let us represent $v$ by an exact sequence $0 \to \mu_n \to H \to G \to 0$. The class $u$ can be described via pushout of $\mu_n \to H$ along $\mu_n \to \mathbb{G}_m$. The class $u$ can be killed if there exists a map $H \to \mathbb{G}_m$ such that the composition $\mu_n \to H \to \mathbb{G}_m$ is the natural inclusion. However, by Cartier duality, we have a surjection of group schemes $H^\vee \to \mu_n^\vee$; therefore, $u$ can be killed syntomic locally. □

Let us now specialize (3.0.1) to $\mathcal{Y} = BG$ and apply $H^2(\cdot)$. For a finite locally free group scheme $G$ for which $\mathcal{H}^1_{\Delta}(BG) = 0$, we get (by using the lemmas above)

$$0 \to G^\vee \to \mathcal{H}^2_{\Delta}(\mathrm{Fil}^1_{\mathrm{Nyg}} BG)\{1\} \xrightarrow{\varphi_1 - \mathrm{can}} \mathcal{H}^2_{\Delta}(BG)\{1\} \to 0. \tag{3.0.2}$$

For a $p$-divisible group $G$, we obtain

$$0 \to T_p(G^\vee) \to \mathcal{H}^2_{\Delta}(\mathrm{Fil}^1_{\mathrm{Nyg}} BG)\{1\} \xrightarrow{\varphi_1 - \mathrm{can}} \mathcal{H}^2_{\Delta}(BG)\{1\} \to 0. \tag{3.0.3}$$

If $G$ is a finite flat group scheme, by taking $\mathcal{Y} = B^2G$, we have the following fiber sequence of $D(\mathbb{Z})$-valued quasisyntomic sheaves on $S$

(3.0.4)

$$\tau_{[-3,-2]} R\Gamma(B^2G_{(\cdot)}, \mathbb{Z}_p(1)) \to \tau_{[-3,-2]} \mathrm{Fil}^1_{\mathrm{Nyg}} R\Gamma_{\Delta}(B^2G_{(\cdot)})\{1\} \xrightarrow{\varphi_1 - \mathrm{can}} \tau_{[-3,-2]} R\Gamma_{\Delta}(B^2G_{(\cdot)})\{1\}.$$

The above follows because $\mathcal{H}^1_{\Delta}(B^2G) = 0$ and the vanishing from Lemma 3.6. By using Proposition 3.4, one may further simply it to the following fiber sequence of $D(\mathbb{Z})$-valued quasisyntomic sheaves

(3.0.5)

$$R\Gamma_{\mathrm{qsyn}}((\cdot), G^\vee)[-3] \to \tau_{[-3,-2]} \mathrm{Fil}^1_{\mathrm{Nyg}} R\Gamma_{\Delta}(B^2G_{(\cdot)})\{1\} \xrightarrow{\varphi_1 - \mathrm{can}} \tau_{[-3,-2]} R\Gamma_{\Delta}(B^2G_{(\cdot)})\{1\}.$$

**Remark 3.7.** One of the tricky feature of Dieudonné theory over general base rings is to keep track of all the necessary data. A puzzling aspect of this theory is that certain extra data, under specific situations, become conditions, or properties. More concretely, certain data such as a "divided Frobenius" can often be ignored in torsion free cases, but they become essential to record in torsion cases. Further, certain filtration data can be ignored in the locally free cases (corresponding to Dieudonné theory for $p$-divisible groups), but become essential in the general cases (e.g., finite flat group schemes).

The above exact sequences make it clear that it is possible to reconstruct $G$ from prismatic cohomology, and what other *extra data* is essential for the purpose of reconstruction. Namely, we use $\mathcal{H}^2_{\Delta}(BG)$ or $\tau_{[-2,-3]} R\Gamma_{\Delta}(B^2G)$, viewed as a prismatic crystal, as well as the Nygaard filtration, and the *divided* Frobenius. If one wanted to fully faithfully embed $p$-divisible groups and finite locally free group schemes, one naturally looks for a certain category where all of these data make sense. This is given by the derived category of the stack $S^{\mathrm{syn}}$, introduced by Drinfeld and Bhatt–Lurie, which gives the category of coefficients for prismatic cohomology along with the other natural data such as the Nygaard filtration, divided Frobenius, etc. We will see that the formalism of quasi-coherent sheaves on these stacks (which automatically keeps track of the relevant data) gives a very convenient framework for Dieudonné theory.

The advantage of the stacky approach to Dieudonné theory taken in this paper is that it makes to following construction of the Dieudonné module functor easy, thanks to the stacky approach of Drinfeld and Bhatt–Lurie.

**Definition 3.8** (Dieudonné crystal of a $p$-divisible group)**.** Let $G$ be a $p$-divisible group over $S$. By functoriality of the stacky approach, we obtain a natural map

$$v : BG^{\mathrm{syn}} \to S^{\mathrm{syn}}.$$

The Dieudonné crystal of $G$, denoted as $\mathcal{M}(G)$, is defined to be $R^2 v_* \mathcal{O}_{BG^{\mathrm{syn}}} \in D_{\mathrm{qc}}(S^{\mathrm{syn}})$.

**Definition 3.9** (Dieudonné crystal)**.** Let $G$ be a $p$-divisible group or a finite locally free commutative group scheme of $p$-power rank over $S$. We have a natural map $u : B^2G^{\mathrm{syn}} \to S^{\mathrm{syn}}$. The Dieudonné crystal of $G$, denoted as $\mathcal{M}(G)$, is defined to be

$$\mathcal{M}(G) := \tau_{[-3,-2]}Rv_*\mathcal{O}_{B^2G^{\mathrm{syn}}}[3].$$

**Remark 3.10.** Let us explain why the above two definitions are consistent in the case of a $p$-divisible group. Since $\mathrm{Fil}^i\mathbb{\Delta}_{(\cdot)}$ is derived $p$-complete as a quasisyntomic sheaf for all $i$, similar to the proof of Lemma 3.1, it follows that $\mathcal{H}^1(\mathrm{Fil}^i_{\mathrm{Nyg}}R\Gamma_{\mathbb{\Delta}}BG) = 0$. In the notation of the above two definitions, this implies by quasisyntomic descent that $R^1v_*\mathcal{O} = 0 = R^2u_*\mathcal{O}$. Similar to Proposition 2.16, one sees that $R^3u_*\mathcal{O} \simeq R^2v_*\mathcal{O}$.

It would be useful to have an alternative description of $\mathcal{M}(G)$. Below, we work in the category of sheaves for the $p$-completely faithfully flat topology on $S^{\mathrm{syn}}$ with values in $D(\mathbb{Z})$; this will be denoted by $D(S^{\mathrm{syn}}, \mathbb{Z})$. Note that $G^{\mathrm{syn}}$ is a commutative group stack, and can be viewed naturally as an object of $D(S^{\mathrm{syn}}, \mathbb{Z})$.

**Proposition 3.11.** *Let $G$ be a $p$-divisible group over $S$. Then $\mathcal{M}(G) \simeq \mathcal{E}xt^1_{D(S^{\mathrm{syn}},\mathbb{Z})}(G^{\mathrm{syn}}, \mathbb{G}_a)$.*

*Proof.* Note that $\mathcal{M}(G) \simeq \pi_0\mathcal{M}aps_*(B^2G^{\mathrm{syn}}, K(\mathbb{G}_a, 3))$, where the latter denotes internal mapping space as pointed objects. By delooping twice, it is equivalent to $\pi_0\mathcal{M}aps_{\mathbb{E}_2}(G^{\mathrm{syn}}, K(\mathbb{G}_a, 1))$, where the latter denote maps as $\mathbb{E}_2$-group objects. Since the group objects themselves are 1-truncated, we see that it is isomorphic to $\mathcal{E}xt^1(G^{\mathrm{syn}}, \mathbb{G}_a)$, which gives the claim. □

**Proposition 3.12.** *Let $G$ be a $p$-divisible group or a finite locally free commutative group scheme over $S$. Then*

$$\mathcal{M}(G) \simeq \tau_{\geq 0}\mathcal{R}Hom_{D(S^{\mathrm{syn}},\mathbb{Z})}(G^{\mathrm{syn}}, \mathbb{G}_a[1]).$$

*Proof.* Let $P$ be an ordinary abelian group. We have a natural map $\mathbb{Z} \to \mathbb{Z}[B^2P]$ whose cofiber will be denoted as $\mathbb{Z}^{\mathrm{red}}[B^2P]$. There is a natural map $\mathbb{Z}^{\mathrm{red}}[B^2P] \to P[2]$. By animation, one obtains a natural map $\mathbb{Z}[B^2P] \to P[2]$ for any $P \in D(\mathbb{Z})_{\geq 0}$. Using this, we obtain a natural map

$$\mathcal{R}Hom(G^{\mathrm{syn}}[2], \mathbb{G}_a) \to \mathcal{R}Hom(\mathbb{Z}[B^2G^{\mathrm{syn}}], \mathbb{G}_a).$$

By the proof of Proposition 3.11, we see that $\mathcal{E}xt^i(\mathbb{Z}[B^2G^{\mathrm{syn}}], \mathbb{G}_a) \simeq \mathcal{E}xt^{i-2}(G^{\mathrm{syn}}, \mathbb{G}_a)$ for $i \in \{2, 3\}$ . This yields the desired statement. □

Before we proceed to prove that the functor induced by $\mathcal{M}$ is fully faithful, we will prove certain basic properties of the functor $\mathcal{M}$. To this end, we note an important computation and a duality compatibility.

**Proposition 3.13.** *Let $\psi : B\mathbb{G}_m^{\mathrm{syn}} \to S^{\mathrm{syn}}$ be the structure map. Then $R^2\psi_*\mathcal{O} \simeq \mathcal{O}\{-1\} \in D_{\mathrm{qc}}(S^{\mathrm{syn}})$.*

*Proof.* Chern class of the tautological line bundle on $B\mathbb{G}_m$ gives a map $\mathcal{O} \to R^2\psi_*\mathcal{O}_{B\mathbb{G}_m^{\mathrm{syn}}}\{1\}$, which, as one can check, is an isomorphism. We give sketch of an argument. by quasisyntomic descent, we may reduce to the case when $S$ is a quasiregular semiperfectoid algebra. Let us also choose a map $R \to S$, where $R$ is a perfectoid ring mapping onto $S$. It would suffice to prove that the natural map

$$\bigoplus_{i\in\mathbb{Z}} \mathrm{Fil}^{i-1}_{\mathrm{Nyg}}\mathbb{\Delta}_S\{i-1\} \to \bigoplus_{i\in\mathbb{Z}} H^2(\mathrm{Fil}^i_{\mathrm{Nyg}}R\Gamma_{\mathbb{\Delta}}(B\mathbb{G}_m)\{i\})$$

of graded of modules over the graded ring $\oplus_{i\in\mathbb{Z}}\mathrm{Fil}^i_{\mathrm{Nyg}}\Delta_S\{i\}$ induced by the tautological Chern class $c_1 \in H^2(\mathrm{Fil}^1_{\mathrm{Nyg}}R\Gamma_{\Delta}(B\mathbb{G}_m)\{1\})$ is an isomorphism. It suffices to prove that the map on each component

$$\mathrm{Fil}^{i-1}_{\mathrm{Nyg}}\Delta_S\{i-1\} \to H^2(\mathrm{Fil}^i_{\mathrm{Nyg}}R\Gamma_{\Delta}(B\mathbb{G}_m)\{i\})$$

is an isomorphism. By reducing to Hodge–Tate cohomology and using the fact that $\mathbb{L}_{B\mathbb{G}_m/S} = \mathcal{O}[-1]$, we can deduce that the above map is an isomorphism for $i = 0$. From now on in this proof, we omit the Breuil–Kisin twists. By using the fact $\mathrm{gr}^i_{\mathrm{Nyg}}R\Gamma_{\Delta}(B\mathbb{G}_m) \simeq \mathrm{Fil}^i_{\mathrm{conj}}R\Gamma_{\overline{\Delta}}(B\mathbb{G}_m)$, where the latter denotes $i$-th conjugate conjugate filtration on absolute Hodge–Tate cohomology. Using the Nygaard filtration and induction on $i$, we would be done if we prove that the induced map

$$\mathrm{Fil}^{i-1}_{\mathrm{conj}}\overline{\Delta}_S \to H^2(\mathrm{Fil}^i_{\mathrm{conj}}R\Gamma_{\overline{\Delta}}(B\mathbb{G}_m))$$

is an isomorphism. To check this, we use the fact that $\mathrm{gr}^i_{\mathrm{conj}}R\Gamma_{\overline{\Delta}}(B\mathbb{G}_m) \simeq \wedge^i\mathbb{L}_{B\mathbb{G}_m/R}[-i]$, and the isomorphism $\mathbb{L}_{B\mathbb{G}_m/R} \simeq \mathbb{L}_{S/R} \oplus \mathcal{O}[-1]$. The latter implies that

$$\wedge^i\mathbb{L}_{B\mathbb{G}_m/R}[-i] \simeq \bigoplus_{m\geq 0}\wedge^{i-m}\mathbb{L}_{S/R}[m-i][-2m] \simeq \bigoplus_{m\geq 0}\mathrm{gr}^{i-m}_{\mathrm{conj}}\overline{\Delta}_S[-2m],$$

which yields the desired claim. □

**Proposition 3.14.** *Let $\pi : A \to S$ be an abelian scheme. Then $R^1\pi^{\mathrm{syn}}_*\mathcal{O}_{A^{\mathrm{syn}}}$ is a vector bundle of rank $2\dim A$ as an object of $D_{\mathrm{qc}}(S^{\mathrm{syn}})$. Further, let $\pi^\vee : A^\vee \to S$ be the dual abelian scheme. Then we have $R^1\pi^{\vee\,\mathrm{syn}}_*\mathcal{O} \simeq (R^1\pi^{\mathrm{syn}}_*\mathcal{O})^*\{-1\}$.*

*Proof.* The fact that $R^1\pi^{\mathrm{syn}}_*\mathcal{O}$ is a vector bundle of rank $2\dim A$ can be checked in a way similar to Proposition 3.29, by computing the Nygaard filtration by reducing to conjugate filtration, and computing the latter by further using the fact that $\mathbb{L}_{A/S} \simeq \Omega_A$ is a locally free $S$-module of rank $\dim A$. We explain a different proof using the classifying stack $BA$, which in fact gives a proof identical to Proposition 3.29. Let $\pi\times\pi : A\times A \to S$. Then the natural map $R^1\pi^{\mathrm{syn}}_*\mathcal{O} \oplus R^1\pi^{\mathrm{syn}}_*\mathcal{O} \to R^1(\pi\times\pi)^{\mathrm{syn}}_*\mathcal{O}$ is an isomorphism. To check this, by quasisyntomic descent, we may assume that $S$ is a quasiregular semiperfectoid algebra. It would suffice to show that $H^1(\mathrm{Fil}^i_{\mathrm{Nyg}}R\Gamma_{\Delta}(A)) \oplus H^1(\mathrm{Fil}^i_{\mathrm{Nyg}}R\Gamma_{\Delta}(A)) \simeq H^1(\mathrm{Fil}^i_{\mathrm{Nyg}}R\Gamma_{\Delta}(A\times A))$. The latter follows by induction on $i$, starting from the case $i = 0$, using the description of graded pieces of the Nygaard filtration in terms of conjugate filtered Hodge–Tate cohomology, where we use the fact that for any perfectoid ring $R$ mapping onto $S$, we have $\mathbb{L}_{A/R} = \mathbb{L}_{S/R} \oplus \Omega_A$. Now using the $E_1$-spectral sequence associated to applying descent along $* \to BA$, we see that $R^1\pi^{\mathrm{syn}}_*\mathcal{O} \simeq R^2v^{\mathrm{syn}}_*\mathcal{O}$, where $v : BA \to S$ is the structure map. It is therefore enough to prove that $R^2v_*\mathcal{O}$ is a vector bundle on $S^{\mathrm{syn}}$ of rank $2\dim A$. This proof is identical to Proposition 3.29, where we deal with the more general case of $p$-divisible groups.

For the formula for duality, we use the tautological map $A \times A^\vee \to B\mathbb{G}_m$ to obtain a map

$$\mathcal{O}\{-1\} \to R^1\pi^{\mathrm{syn}}_*\mathcal{O} \otimes R^1\pi^{\vee\,\mathrm{syn}}_*\mathcal{O}$$

which gives the claim. □

**Remark 3.15.** Let $R \to S$ be a perfectoid ring $R$ mapping to a quasiregular semiperfectoid ring $S$ and let $A$ be an abelian variety over $S$. As carried out later in the more general case of $p$-divisible groups, by using the vanishing $H^3(BA, \mathcal{O}) = 0$ for any abelian variety $A$, and the fact $\mathbb{L}_{BA/R} \simeq \mathbb{L}_{S/R} \oplus \Omega_A[-1]$, one can show that the Hodge–Tate cohomology $H^3_{\overline{\Delta}}(BA) = 0$. This implies that $H^3_{\Delta}(BA) = 0$. Similar to the case of $p$-divisible groups, we have $H^3(\mathrm{Fil}^i_{\mathrm{Nyg}} R\Gamma_{\Delta}(BA)) = 0$ (resp. $H^1(\mathrm{Fil}^i_{\mathrm{Nyg}} R\Gamma_{\Delta}(BA)) = 0$) for all $i \geq 0$. In the quasisyntomic topos of $S$, this implies $\mathrm{Ext}^2(A, \mathrm{Fil}^i_{\mathrm{Nyg}} \Delta_{(\cdot)}) = 0$ (resp. $\mathrm{Hom}(A, \mathrm{Fil}^i_{\mathrm{Nyg}} \Delta_{(\cdot)}) = 0$) for all $i \geq 0$. Further, by the above proof, we have

$$H^1(\mathrm{Fil}^i_{\mathrm{Nyg}} R\Gamma_{\Delta}(A)) = H^2(\mathrm{Fil}^i_{\mathrm{Nyg}} R\Gamma_{\Delta}(BA)) = \mathrm{Ext}^1(A, \mathrm{Fil}^i_{\mathrm{Nyg}} \Delta_{(\cdot)}).$$

This has the following consequence

**Proposition 3.16** (Duality)**.** *Let $G$ be a finite locally free group scheme of $p$-power rank over $S$. Then $\mathcal{M}(G)$ is a dualizable object of dimension* 0 *of Tor amplitude in homological degrees* $[0, 1]$. *Further, if $G^\vee$ is the Cartier dual of $G$, then we have a natural isomorphism*

$$\mathcal{M}(G^\vee) \simeq \mathcal{M}(G)^* \{-1\} [1].$$

*Proof.* We can assume that $S$ is a quasiregular semiperfectoid algebra. For the first statement, we may work Zariski locally on $S$ and assume that there is an exact sequence $0 \to G \to A' \to A \to 0$, where $\pi : A \to S$ and $\pi' : A' \to S$ are abelian schemes. By the Ext-vanishings from Remark 3.15, it follows that we have a fiber sequence

$$R^1\pi^{\mathrm{syn}}_* \mathcal{O} \to R^1\pi'^{\mathrm{syn}}_* \mathcal{O} \to \mathcal{M}(G).$$

Applying Proposition 3.14 now gives the claim.

The duality compatibility follows once again by locally embedding $G$ in abelian varieties, using Proposition 3.14, and varying over all such local embeddings, to get a natural map. ☐

**Definition 3.17.** Let $D_{\mathrm{perf}}(S^{\mathrm{syn}})$ denote the $\infty$-category of dualizable $\mathcal{O}_{S^{\mathrm{syn}}}$-modules. We will construct a functor

$$\mathrm{T}(\cdot)(n) : D_{\mathrm{perf}}(S^{\mathrm{syn}}) \to D_{\mathrm{qsyn}}(S, \mathbb{Z}),$$

that we think of as a certain Tate module functor of weight $n$. Let $M \in D_{\mathrm{perf}}(S^{\mathrm{syn}})$. For a scheme $f : R \to S$, we may consider $f^*M \in D_{\mathrm{perf}}(R^{\mathrm{syn}})$. The association $R \mapsto R\Gamma(R^{\mathrm{syn}}, f^*M\{n\})$ defines a $D(\mathbb{Z})$-valued quasisyntomic sheaf on $S_{\mathrm{qsyn}}$, which determines the functor $T(\cdot)(n)$

For our purposes, we will only need to use $\mathcal{H}^0(T(\cdot)(1))$, which we denote as $T^0(\cdot)(1)$.

**Remark 3.18.** Suppose that $S$ is a quasiregular semiperfectoid algebra. Using the concrete descriptions from Construction 1.6, let us explain how to describe $T(M)(n)$ for $n \geq 0$. Any such $M \in D_{\mathrm{qc}}(S^{\mathrm{syn}})$ arises from a (derived) graded module $M' := \bigoplus_{i \in \mathbb{Z}} M_i$ over the graded ring $\bigoplus_{i \in \mathbb{Z}} \mathrm{Fil}^i_{\mathrm{Nyg}} \Delta_S \{i\}$. We assume that the associated maps $M_i \to M_{i-1}\{1\}$ are isomorphisms for $i \leq 0$. Realizing $M'$ as an object of $D_{\mathrm{qc}}(\mathrm{Spf}(S)^{\mathrm{Nyg}})$, the pullback of $M'\{n\}$ along the map "can" can be identified with $M_0\{n\}$ considered as a $\Delta_S$-module. By applying the (derived) global section functor, the latter identifications recover the map $i : M_n \to M_0\{n\}$. Since $M'\{n\}$ descends to $S^{\mathrm{syn}}$, we have an isomorphism $\varphi^*M'\{n\} \simeq \mathrm{can}^*M'\{n\}$. By using the previous identification, applying the global section functor, we obtain a map $\nu_n : M_n \to M_0\{n\}$, which maybe regarded as the "$n$-th divided Frobenius". It follows that

$$R\Gamma(S^{\mathrm{syn}}, M\{n\}) \simeq \mathrm{fib}\left(M_n \xrightarrow{\nu_n - i} M_0\{n\}\right).$$

**Remark 3.19.** Let $S$ be a quasiregular semiperfectoid algebra and let $G$ be a finite flat locally free commutative group scheme of $p$-power rank over $S$. As a consequence of (3.0.5) and Remark 3.18, we obtain an isomorphism

$$R\Gamma_{\mathrm{qsyn}}(S, G^\vee) \simeq R\Gamma(S^{\mathrm{syn}}, \mathcal{M}(G)\{1\}).$$

By quasisyntomic descent, the same result remains true if we only assume $S$ to be quasisyntomic; this explains how to compute cohomology with coefficients in $G$ in terms of coherent cohomology of the Dieudonné crystal of $G$.

Similarly, for a $p$-divisible group $G$ over $S$, we have

$$R\Gamma_{\mathrm{qsyn}}(S, T_p(G^\vee)) \simeq R\Gamma(S^{\mathrm{syn}}, \mathcal{M}(G)\{1\}).$$

Let $D_{\mathrm{perf}}(S^{\mathrm{syn}})^{\mathrm{c}}_{\leq 1}$ denote the full subcategory of dualizable $\mathcal{O}_{S^{\mathrm{syn}}}$-modules $M$ with Tor amplitude in homological degrees $\leq 1$ for which $T(M)(1)$ is coconnective and $T^0M(1)$ is representable by a finite locally free group scheme of $p$-power rank. Let $\mathrm{FFG}(S)$ denote the category of finite locally free group schemes of $p$-power rank over $S$.

**Proposition 3.20.** *The functor $\mathcal{M}^\vee : \mathrm{FFG}(S) \to D_{\mathrm{perf}}(S^{\mathrm{syn}})^{\mathrm{c}}_{\leq 1}$ that is determined by $G \mapsto \mathcal{M}(G^\vee)$ is left adjoint to $T^0(M)(1)$.*

*Proof.* In the proof below, we simply write Hom to mean connective cover of "RHom" in the relevant stable $\infty$-categories. Let $G \in \mathrm{FFG}(S)$ and $M \in D_{\mathrm{perf}}(S^{\mathrm{syn}})^{\mathrm{c}}_{\leq 1}$. We note that

$$\begin{aligned}
\mathrm{Hom}_{D(S^{\mathrm{syn}},\mathcal{O})}(\mathcal{M}(G^\vee), M) &\xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}}(M^*[1], \mathcal{M}(G^\vee)^*[1]) \\
&\xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}}(M^*[1], \mathcal{M}(G)\{1\}) \\
&\xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}}(M^*[1], \mathrm{R}\mathcal{H}om_{D(S^{\mathrm{syn}},\mathbb{Z})}(G^{\mathrm{syn}}, \mathcal{O}[1])\{1\}) \\
&\xrightarrow{\sim} \mathrm{Hom}_{\mathbb{Z}}(G^{\mathrm{syn}}, \mathrm{R}\mathcal{H}om_{\mathcal{O}}(\mathcal{O}, M\{1\})) \\
&\xrightarrow{\sim} \mathrm{Hom}_{D_{\mathrm{qsyn}}(S,\mathbb{Z})}(G, T^0(M)(1)).
\end{aligned}$$

In the above, the first isomorphism is a consequence of dualizability as objects of $D(S^{\mathrm{syn}}, \mathcal{O})$, the second one follows from the Cartier duality compatibility from Proposition 3.16, the third one comes from Proposition 3.12, the fourth one is a consequence of adjunction, and the final one follows from Yoneda lemma and Definition 3.17.

This finishes the proof. □

**Proposition 3.21.** *The functor $\mathcal{M}^\vee : \mathrm{FFG}(S) \to D_{\mathrm{perf}}(S^{\mathrm{syn}})$ is fully faithful.*

*Proof.* In view of Proposition 3.20, it would be enough to show that for any $G \in \mathrm{FFG}(S)$, we have $T^0(\mathcal{M}(G^\vee))(1) \simeq G$. This follows from Remark 3.18 and (3.0.5). □

Now, we turn to the case of a $p$-divisible group over $S$. In the spirit of our stacky approach, we will begin by an understanding of the cotangent complex of the classifying stack of a $p$-divisible group (cf. [Ill85]). At first, we assume that $p^N S = 0$, and we consider an $n$-truncated Barsotti–Tate group $G$ for $n \geq N$ over $S$. Let $\ell_G$ denote the co-Lie complex of $G$, which is a perfect complex of $S$-modules with Tor amplitude in $[0, 1]$. Its dual $\ell_G^\vee$ is called the lie complex. We let $\omega_G := H^0(\ell_G)$, which is a finite locally free $S$-module, whose rank is called the dimension of $G$.

By a result of Grothendieck, one has $\ell_G^\vee = \tau_{\geq -1} R\mathcal{H}om_{\mathbb{Z}}(G^*, \mathbb{G}_a)$. Note that there is a natural map $\phi_n : \mathcal{E}xt^1(G^*, \mathbb{G}_a) \to \mathcal{H}om(G^*, \mathbb{G}_a)$ obtained as follows: an element of $\mathcal{E}xt^1(G^*, \mathbb{G}_a)$ determines a map $u : G^* \to \mathbb{G}_a[1]$. Applying $(\cdot) \otimes^L_{\mathbb{Z}} \mathbb{Z}/p^n$ and noting that $G^*$

is killed by $p^n$, we obtain a map $G^*[1] \otimes G^* \to \mathbb{G}_a[2] \oplus \mathbb{G}_a[1]$; applying $\pi_1$ gives the desired map $\phi_n$. If $G$ is an $n$-truncated Barsotti–Tate group, then $\phi_n$ is an isomorphism. Suppose now that $G = \{G_n\}$ is a $p$-divisible group over $S$. Let $f : G_{n+1} \to G_n$ be any map of group schemes. By construction, we have a commutative diagram

$$\begin{array}{ccc} \mathcal{E}xt^1(G_n, \mathbb{G}_a) & \xrightarrow{\phi_n} & \mathcal{H}om(G_n, \mathbb{G}_a) \\ \big\downarrow f & & \big\downarrow pf \\ \mathcal{E}xt^1(G_{n+1}, \mathbb{G}_a) & \xrightarrow{\phi_{n+1}} & \mathcal{H}om(G_{n+1}, \mathbb{G}_a). \end{array}$$

**Proposition 3.22.** *Let $S$ be a bounded $p^\infty$-torsion, $p$-complete ring and $G = \{G_n\}$ be a $p$-divisible group over $S$. Then, we have a natural isomorphism*

$$\mathbb{L}_{BG/S} := \varprojlim \mathbb{L}^{\wedge p}_{BG_n/S} \simeq \omega_G[-1],$$

*where $\omega_G$ is a locally free $S$-module of finite rank.*

*Proof.* First, we argue over $S/p^N S$. The above commutative diagram shows that the map "$p$" : $G^*_{n+1} \to G^*_n$ induces an ind-object $\mathcal{E}xt^1(G^*_n, \mathbb{G}_a)$ that is equivalent to zero. By duality, this implies that the pro-object $H^{-1}(\ell_G)$ is zero. Since $\mathbb{L}_{BG_n} \simeq \ell_{G_n}[-1]$, we see that $\mathbb{L}_{BG_n}$ is naturally pro-isomorphic to $\omega_{G_n}[-1]$; since $\omega_G := \varprojlim \omega_{G_n}$ is locally free, this gives the claim in that case. Since $S$ has bounded $p$-power torsion, the pro-objects (in the category of animated rings) $S \otimes^L_{\mathbb{Z}} \mathbb{Z}/p^k$ and $S/p^k S$ are pro-isomorphic, and our claim in the $p$-complete case follows from base change properties of cotangent complex and taking limits. □

**Remark 3.23.** Note that for any map $g : G_n \to G_{n+1}$, one has a similar diagram

$$\begin{array}{ccc} \mathcal{E}xt^1(G_{n+1}, \mathbb{G}_a) & \xrightarrow{\phi_{n+1}} & \mathcal{H}om(G_{n+1}, \mathbb{G}_a) \\ \big\downarrow pg & & \big\downarrow g \\ \mathcal{E}xt^1(G_n, \mathbb{G}_a) & \xrightarrow{\phi_n} & \mathcal{H}om(G_n, \mathbb{G}_a). \end{array}$$

Let $S$ be such that $p^N S = 0$. For a $p$-divisible group $G = \{G_n\}$ over $S$, the above diagram implies that the structure maps $i : G_n \to G_{n+1}$ induces a pro-system $\mathcal{H}om(G_n, \mathbb{G}_a)$ that is pro-zero. We have already seen in the above proof that the maps "$p$" : $G_{n+1} \to G_n$ induces a direct system $\mathcal{E}xt^1(G_n, \mathbb{G}_a)$ that is equivalent to zero. However, the pro-system $\mathcal{E}xt^1(G_n, \mathbb{G}_a)$ induced by the maps $i : G_n \to G_{n+1}$ is nonzero; in fact, we have $\mathcal{E}xt^1(G, \mathbb{G}_a) \simeq \omega^\vee_{G^*}$, the latter will be denoted by $t_{G^*}$.

**Proposition 3.24.** *Let $R \to S$ be a perfectoid ring surjecting onto a quasiregular semiperfectoid algebra $S$. Let $G$ be a $p$-divisible group over $S$. Then we have*

$$\mathbb{L}_{BG/R} \simeq \mathbb{L}_{S/R} \oplus \omega_G[-1].$$

*Proof.* Follows from the transitivity fiber sequence of cotangent complex associated to the maps $R \to S \to BG$ and the fact that $\omega_G$ is locally free. □

Using the above proposition, one can fully describe the conjugate filtration on absolute Hodge–Tate cohomology $H^2_{\overline{\Delta}}(BG)$. Below, let $t_{G^*} := H^2(BG, \mathcal{O})$, which is a locally free $S$-module.

**Proposition 3.25** (Hodge–Tate sequence)**.** *Let $G$ be a $p$-divisible group over a quasiregular semiperfectoid ring $S$. Then there is a canonical exact sequence*

$$0 \to \overline{\Delta}_S \otimes_S t_{G^*} \to H^2_{\overline{\Delta}}(BG) \to (\overline{\Delta}_S \otimes_S \omega_G)\{-1\} \to 0.$$

*Proof.* Note that $H^2_{\Delta}(BG)$ can be computed as relative prismatic cohomology of $BG_{\overline{\Delta}_S}$ with respect to the canonical prism $(\Delta_S, I)$ associated to $S$. The claim now follows from the relative Hodge–Tate filtration and Proposition 3.22. □

**Remark 3.26.** The above sequence can be split non-canonically. It also implies that $H^2_{\overline{\Delta}}(BG)$ is a locally free $\Delta_S$-module of rank height$(G)$. Using the fact that $H^3(BG, \mathcal{O}) = 0$ (resp. $H^1(BG, \mathcal{O}) = 0$) and Proposition 3.22, one can obtain that $H^3_{\overline{\Delta}}(BG) = 0$ (resp. $H^1_{\overline{\Delta}}(BG) = 0$).

**Proposition 3.27.** *Let $G$ be a $p$-divisible group over a quasiregular semiperfectoid ring $S$. Then $H^2_{\Delta}(BG)$ is a locally free $\Delta_S$-module of rank* height$(G)$.

*Proof.* Let $(\Delta_S, I)$ be the prism associated to $I$. In this situation, $I = (d)$ for some element $d$. Using the fact that $R\Gamma_{\Delta}(BG)$ is derived $d$-complete, by a limit argument, one deduces that $H^3_{\Delta}(BG) = 0$. Using the fact that $H^1_{\overline{\Delta}}(BG) = 0$, and the universal coefficient theorem, one see that $H^2_{\Delta}(BG) \otimes^L_{\Delta_S} (\Delta_S/d) = H^2_{\overline{\Delta}}(BG)$. Invoking Proposition 3.25 now gives the claim. □

Let $G$ be a $p$-divisible group over a quasiregular semiperfectoid ring $S$. Let $R\Gamma_{\Delta}(BG)$ denote the absolute prismatic cohomology. Below, we use the following notations: $F^k_{\mathrm{N}} := H^2(\mathrm{Fil}^k_{\mathrm{Nyg}} R\Gamma_{\Delta}(BG))$ and $F^k_{\mathrm{conj}} := H^2(\mathrm{Fil}^k_{\mathrm{conj}} R\Gamma_{\overline{\Delta}}(BG))$.

**Lemma 3.28.** *Let $R \to S$ be a perfectoid ring mapping surjectively onto the quasiregular semiperfectoid algebra $S$. In the above notations, the natural maps $F^{k-1}_{\mathrm{conj}} \to F^k_{\mathrm{conj}}$ are injective and*

$$F^k_{\mathrm{conj}}/F^{k-1}_{\mathrm{conj}} \simeq (\wedge^k \mathbb{L}_{S/R}[-k] \otimes_S t_{G^*}) \oplus (\wedge^{k-1}\mathbb{L}_{S/R}[-k+1] \otimes_S \omega_G).$$

*Proof.* We have a fiber sequence

$$\mathrm{Fil}^{k-1}_{\mathrm{conj}} R\Gamma_{\overline{\Delta}}(BG) \to \mathrm{Fil}^k_{\mathrm{conj}} R\Gamma_{\overline{\Delta}}(BG) \to \mathrm{gr}^k_{\mathrm{conj}} R\Gamma_{\overline{\Delta}}(BG). \tag{3.0.6}$$

Note that

$$\mathrm{gr}^k_{\mathrm{conj}} R\Gamma_{\overline{\Delta}}(BG) \simeq \wedge^k \mathbb{L}_{S/R}[-k] \simeq \bigoplus_{u+v=k} \wedge^u \mathbb{L}_{S/R}[-u] \otimes_S \mathrm{Sym}^v \omega_G[-2v].$$

It follows that $H^2(\mathrm{gr}^k_{\mathrm{conj}} R\Gamma_{\overline{\Delta}}(BG)) \simeq (\wedge^k \mathbb{L}_{S/R}[-k] \otimes_S t_{G^*}) \oplus (\wedge^{k-1}\mathbb{L}_{S/R}[-k+1] \otimes_S \omega_G)$. We also note that $H^1(\mathrm{gr}^k_{\mathrm{conj}} R\Gamma_{\overline{\Delta}}(BG)) = 0$. Moreover, using the vanishing $H^3(BG, \mathcal{O}) = 0$, it follows that $H^3(\mathrm{gr}^k_{\mathrm{conj}} R\Gamma_{\overline{\Delta}}(BG)) = 0$. Inductively, we obtain that $H^3(\mathrm{Fil}^k_{\mathrm{conj}} R\Gamma_{\overline{\Delta}}(BG)) = 0$ for all $k$. Applying $H^2(\cdot)$ to (3.0.6) now gives the desired claim. □

**Proposition 3.29** (Dualizability)**.** *Let $G$ be a $p$-divisible group over $S$ of height $h$. Then $\mathcal{M}(G)$ is a vector bundle on* $\mathrm{Spf}(S)^{\mathrm{syn}}$ *of rank $h$.*

*Proof.* In order to prove this, by quasisyntomic descent, we may reduce to the case where $S$ is a quasiregular semiperfectoid algebra. In this case, it suffices to prove that the filtered object $F^k_{\mathrm{N}} := H^2_{\Delta}(\mathrm{Fil}^k_{\mathrm{Nyg}} R\Gamma_{\Delta}(BG))$ determines a vector bundle of rank $h$ over the stack $\mathrm{Spf}(S)^{\mathrm{Nyg}} = \mathrm{Spf} \bigoplus_{i\in\mathbb{Z}} (\mathrm{Fil}^i_{\mathrm{Nyg}} \Delta_S)/\mathbb{G}_m$.

Note that we have a short exact sequence $0 \to F^1_{\mathrm{N}} \to H^2_{\Delta}(BG) \to t_{G^*} \to 0$. This gives a natural surjection $H^2_{\Delta}(BG) \otimes_{\Delta_S} S \to t_{G^*}$. Further, one sees that the kernel admits a surjection to $H^2(BG, \mathbb{L}_{BG/S}[-1]) = \omega_G$. By Proposition 3.27, this gives a natural short exact sequence $0 \to \omega_G \to H^2_{\Delta}(BG) \otimes_{\Delta_S} S \to t_{G^*} \to 0$. We recall that the modules $w_G$ and $t_{G^*}$ are locally free. Let us choose a splitting $H^2_{\Delta}(BG) \otimes_{\Delta_S} S \simeq \omega_G \oplus t_{G^*}$. Since the surjection $\Delta_S \to S$ is henselian (see [ALB23, Lem. 4.1.28]), it is possible to choose an isomorphism $H^2_{\Delta}(BG) \simeq W \oplus T$, such that $W \otimes_{\Delta_S} S \simeq \omega_G$, $T \otimes_{\Delta_S} S \simeq t_{G^*}$, and lifting the isomorphism $H^2_{\Delta}(BG) \otimes_{\Delta_S} S \simeq \omega_G \oplus t_{G^*}$. It follows that under these identifications, $F^1_N = (\mathrm{Fil}^1_{\mathrm{Nyg}} \Delta_S \otimes_{\Delta_S} T) \oplus W$. Let us define

$$G^k_{\mathrm{N}} := (\mathrm{Fil}^k_{\mathrm{Nyg}} \Delta_S \otimes_{\Delta_S} T) \oplus (\mathrm{Fil}^{k-1}_{\mathrm{Nyg}} \Delta_S \otimes_{\Delta_S} W).$$

We have a natural map $G^k_{\mathrm{N}} \to F^k_{\mathrm{N}}$ of (decreasing) filtered objects. We will show that it is an isomorphism. Since the underlying objects are isomorphic, we need to check that it induces isomorphism on graded pieces. The graded pieces for the left hand side are given by

$$T^k := \mathrm{gr}^k(G^\bullet_{\mathrm{N}}) \simeq (\mathrm{Fil}^k_{\mathrm{conj}} \overline{\Delta}_S \{k\} \otimes_S t_{G^*}) \oplus (\mathrm{Fil}^{k-1}_{\mathrm{conj}} \overline{\Delta}_S \{k-1\} \otimes_S \omega_G).$$

Note that we have a fiber sequence

$$\mathrm{Fil}^{k+1}_{\mathrm{Nyg}} R\Gamma_{\Delta}(BG) \to \mathrm{Fil}^k_{\mathrm{Nyg}} R\Gamma_{\Delta}(BG) \to \mathrm{Fil}^k_{\mathrm{conj}} R\Gamma_{\overline{\Delta}}(BG)\{k\}. \tag{3.0.7}$$

Since $H^1(\mathrm{Fil}^k_{\mathrm{conj}} R\Gamma_{\overline{\Delta}}(BG)) = 0$, it follows that $F^{k+1}_{\mathrm{N}} \to F^k_{\mathrm{N}}$ is injective. This gives a map $T^k\{-k\} \to \mathrm{F}^k_{\mathrm{conj}}$ of (increasing) filtered objects. By Proposition 3.25, the underlying objects are isomorphic. Thus, to prove that $T^k\{-k\} \to \mathrm{F}^k_{\mathrm{conj}}$ is an isomorphism, we are reduced to checking isomorphism on graded pieces. However, this follows from Lemma 3.28. Now, we have maps

$$T^k \to F^k_{\mathrm{N}}/F^{k+1}_{\mathrm{N}} \to F^k_{\mathrm{conj}}\{k\}$$

such that the composition is an isomorphism. From (3.0.7), we see that the map in the right is injective. Therefore, $T^k = \mathrm{gr}^k(G^\bullet_{\mathrm{N}}) \simeq \mathrm{gr}^k(F^\bullet_{\mathrm{N}})$, which implies that there is an isomorphism

$$F^k_{\mathrm{N}} \simeq (\mathrm{Fil}^k_{\mathrm{Nyg}} \Delta_S \otimes_{\Delta_S} T) \oplus (\mathrm{Fil}^{k-1}_{\mathrm{Nyg}} \Delta_S \otimes_{\Delta_S} W).$$

Note that $\bigoplus_k F^k_N$ corresponds to the pullback of the associated vector bundle on $\mathrm{Spf}(S)^{\mathrm{Nyg}}$ along the faithfully flat map $\mathrm{Spf} \bigoplus_{i\in S} (\mathrm{Fil}^i_{\mathrm{Nyg}} \Delta_S) \to \mathrm{Spf}(S)^{\mathrm{Nyg}}$, and is locally free (since $T$ and $W$ are locally free by choice). By faithfully flat descent, we conclude that $\mathcal{M}(G)$ is a vector bundle on of rank height$(G)$. □

**Remark 3.30.** As a consequence of the above proof, when $S$ is a quasiregular semiperfectoid algbra, using (3.0.7), we see that the map $F^k_{\mathrm{N}} \to F^k_{\mathrm{conj}}$ is surjective. Using the fact that $H^3_{\Delta}(BG) = 0$, we inductively obtain that $H^3(\mathrm{Fil}^k_{\mathrm{Nyg}} R\Gamma_{\Delta}(BG)) = 0$ for all $k$. Using the the $E_2$-spectral sequence (where the Ext-groups are computed in the quasisyntomic topos)

$$E_2^{i,j} = \mathrm{Ext}^i(H^{-j}(\mathbb{Z}[BG]), \mathrm{Fil}^k_{\mathrm{Nyg}} \Delta_{(\cdot)}) \implies H^{i+j}(\mathrm{Fil}^k_{\mathrm{Nyg}}(R\Gamma_{\Delta}(BG)),$$

the vanishing $H^3(\mathrm{Fil}^k_{\mathrm{Nyg}} R\Gamma_{\Delta}(BG)) = 0$ implies that $\mathrm{Ext}^2(G, \mathrm{Fil}^k_{\mathrm{Nyg}} \Delta_{(\cdot)}) = 0$. Similarly, using $H^1(\mathrm{Fil}^k_{\mathrm{Nyg}} R\Gamma_{\Delta}(BG)) = 0$, we also obtain $\mathrm{Hom}(G, \mathrm{Fil}^k_{\mathrm{Nyg}} \Delta_{(\cdot)}) = 0$. Further, one also has

$$H^2(\mathrm{Fil}^i_{\mathrm{Nyg}} R\Gamma_{\Delta}(BG)) = \mathrm{Ext}^1(G, \mathrm{Fil}^i_{\mathrm{Nyg}} \Delta_{(\cdot)}).$$

**Proposition 3.31** (Duality)**.** *Let $G$ be a $p$-divisible group over $S$ of height $h$. If $G^\vee$ is the Cartier dual of $G$, then we have a natural isomorphism*

$$\mathcal{M}(G^\vee) \simeq \mathcal{M}(G)^* \{-1\}.$$

*Proof.* Let us write $G = \{G_n\}$. We have a fiber sequence $G_n \to G \xrightarrow{p^n} G$. Applying the vanishing from Remark 3.30 gives us a fiber sequence

$$\mathcal{M}(G) \xrightarrow{p^n} \mathcal{M}(G) \to \mathcal{M}(G_n).$$

Dualizing, we obtain that $\mathcal{M}(G_n)^* \simeq (\mathcal{M}(G)^*/p^n)[-1]$. Now, we have

$$\mathcal{M}(G)^\vee \simeq \varprojlim \mathcal{M}(G_n^\vee) \simeq \varprojlim \mathcal{M}(G_n)^* \{-1\} [1],$$

where the last step follows from Proposition 3.16. Further,

$$\varprojlim \mathcal{M}(G_n)^* \{-1\} [1] \simeq \varprojlim (\mathcal{M}(G)^* \{-1\})/p^n \simeq \mathcal{M}(G)^* \{-1\}.$$

This finishes the proof. □

**Proposition 3.32.** *Let $S$ be a quasisyntomic algebra and let $\mathrm{BT}(S)$ denote the category of $p$-divisible groups over $S$. The functor $\mathcal{M}^\vee : \mathrm{BT}(S) \to \mathrm{Vect}(S^{\mathrm{syn}})$ is fully faithful.*

*Proof.* Let $G, H \in \mathrm{BT}(S)$. We have

$$\begin{aligned}
\mathrm{Hom}_{D(S^{\mathrm{syn}},\mathcal{O})}(\mathcal{M}(G^\vee), M(H^\vee)) &\xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}}(M(H^\vee)^*, \mathcal{M}(G^\vee)^*) \\
&\xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}}(M(H^\vee)^*, \mathcal{M}(G)\{1\}) \\
&\xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}}(M(H^\vee)^*, \mathrm{R}\mathcal{H}om_{D(S^{\mathrm{syn}},\mathbb{Z})}(G^{\mathrm{syn}}, \mathcal{O}[1])\{1\}) \\
&\xrightarrow{\sim} \mathrm{Hom}_{\mathbb{Z}}(G^{\mathrm{syn}}, \mathrm{R}\mathcal{H}om_{\mathcal{O}}(\mathcal{O}, M(H^\vee)[1]\{1\})) \\
&\xrightarrow{\sim} \mathrm{Hom}_{D_{\mathrm{qsyn}}(S,\mathbb{Z})}(G[-1], \tau_{\geq -1} T(M(H^\vee))(1)).
\end{aligned}$$

Note that, by Lemma 3.5, it follows that $T^0(M(H^\vee))(1) \simeq T_p(H)$. Using the fiber sequence $T_p(G) \to \varprojlim_p G \to G$ and the fact that $T(M(H^\vee))(1)$ is derived $p$-complete, it follows further that

$$\begin{aligned}
\mathrm{Hom}_{D_{\mathrm{qsyn}}(S,\mathbb{Z})}(G[-1], \tau_{\geq -1} T(M(H^\vee))(1)) &\xrightarrow{\sim} \mathrm{Hom}_{D_{\mathrm{qsyn}}(S,\mathbb{Z})}(T_p(G), \tau_{\geq -1} T(M(H^\vee))(1)) \\
&\xrightarrow{\sim} \mathrm{Hom}(T_p(G), T_p(H)) \\
&\xrightarrow{\sim} \mathrm{Hom}_{\mathrm{BT}(S)}(G, H).
\end{aligned}$$

This finishes the proof. □

**Definition 3.33** (Hodge–Tate weights)**.** Let us suppose that $S$ is a quasiregular semiperfectoid algebra. We have a natural map of graded rings

$$\bigoplus_{i \in \mathbb{Z}} \mathrm{Fil}^i_{\mathrm{Nyg}} \Delta_S \{i\} \to S,$$

which is obtained by quotenting the left hand side by the graded ideal

$$I := \left( \bigoplus_{i \neq 0} \mathrm{Fil}^i_{\mathrm{Nyg}} \Delta_S \{i\} \right) \oplus \mathrm{Fil}^1_{\mathrm{Nyg}} \Delta_S.$$

This defines a map $\mathrm{Spf}(S) \times B\mathbb{G}_m \to \mathrm{Spf}(S)^{\mathrm{Nyg}}$. By quasisyntomic descent, this defines a map

$$j : \mathrm{Spf}(S) \times B\mathbb{G}_m \to \mathrm{Spf}(S)^{\mathrm{Nyg}}$$

for any quasisyntomic $S$ (cf. [Bha23, Remark 5.3.14]). For any $M \in D_{\mathrm{qc}}(\mathrm{Spf}(S)^{\mathrm{Nyg}})$, the pullback $j^*M$ can be identified with $\oplus_{i \in \mathbb{Z}} M_i \in D_{\mathrm{qc}}(\mathrm{Spf}(S)^{\mathrm{Nyg}})$. The set of integers $i$ such that $M_i \neq 0$ is called the set of Hodge–Tate weights of $M$. For $M \in D_{\mathrm{qc}}(\mathrm{Spf}(S)^{\mathrm{syn}})$, the set of Hodge–Tate weights of $M$ is defined as the set of Hodge–Tate weights of the pullback of $M$ along $\mathrm{Spf}(S)^{\mathrm{Nyg}} \to \mathrm{Spf}(S)^{\mathrm{syn}}$.

**Remark 3.34** (Essential image of $p$-divisible groups, cf. [Kis06])**.** Let $S$ be a quasisyntomic algebra and let $G$ be a $p$-divisible group over $S$. We know that $\mathcal{M}(G)$ is a vector bundle on $S^{\mathrm{syn}}$. Pullback of $\mathcal{M}(G)$ to $\mathrm{Spf}(S) \times B\mathbb{G}_m$ is given by $\bigoplus_{i \in \mathbb{Z}} H^2(BG, \wedge^i \mathbb{L}_{BG/S}[-i])$ (see [Bha23, Remark 5.3.14]). By the calculation of cotangent complex from Proposition 3.22, it follows that $\mathcal{M}(G)$ has Hodge–Tate weights in $\{0, 1\}$. This implies that the functor $\mathcal{M}^\vee$ from Proposition 3.32 lands in the subcategory $\mathrm{Vect}_{\{0,1\}}(S^{\mathrm{syn}})$ of vector bundles on $S^{\mathrm{syn}}$ with Hodge–Tate weights in $\{0, 1\}$. In fact, as shown in [ALB23] (by reducing to the case when $S$ is perfectoid [SW20, Theorem 17.5.2]), the essential image can be exactly identified with $\mathrm{Vect}_{\{0,1\}}(S^{\mathrm{syn}})$ under the identification of admissible Prismatic Dieudonné modules over $S$ with $\mathrm{Vect}_{\{0,1\}}(S^{\mathrm{syn}})$.

**Remark 3.35** (Essential image of finite locally free $p$-power rank group schemes, cf. [Kis06])**.** Let $S$ be a quasisyntomic algebra. We will determine the full subcategory of $D_{\mathrm{perf}}(S^{\mathrm{syn}})$ given by the essential image of finite locally free group schemes of $p$-power rank over $S$. Let $D_{\mathrm{perf}}(S^{\mathrm{syn}})_{\mathrm{iso}}$ be the full subcategory of $D_{\mathrm{perf}}(S^{\mathrm{syn}})$ spanned by objects $M$ satisfying the following properties:

(1) There exists a Zariski open cover $(S_i)_{i \in I}$ of $S$ such that $M_i := M|_{S_i{}^{\mathrm{syn}}}$ is isomorphic to $\mathrm{cofib}(V_i \xrightarrow{f_i} V_i')$, for some $V_i, V_i' \in \mathrm{Vect}(S^{\mathrm{syn}})$ and $f_i : V_i \to V_i'$.
(2) The vector bundles $V_i, V_i'$ appearing above have Hodge–Tate weights in $\{0, 1\}$ for all $i \in I$.
(3) The map $f_i$ appearing above has the property that it is an isomorphism when viewed in the category $\mathrm{Vect}(S^{\mathrm{syn}}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.

Let $G \in \mathrm{FFG}(S)$. By a result of Raynaud [BBM82, Thm. 3.1.1], one can Zariski locally realize $G$ as a kernel of an isogeny $A' \to A$ of abelian varieties. As in the proof of Proposition 3.16, it follows that (locally) we have a fiber sequence

$$\mathcal{M}(A) \to \mathcal{M}(A') \to \mathcal{M}(G).$$

Since $\mathcal{M}(A)$ and $\mathcal{M}(A')$ are vector bundles of Hodge–Tate weights in $\{0, 1\}$, it follows that $\mathcal{M}(G)$ indeed lies in $D_{\mathrm{perf}}(S^{\mathrm{syn}})_{\mathrm{iso}}$. Now we can formulate the following.

**Proposition 3.36.** *The functor*

$$\mathcal{M}^\vee : \mathrm{FFG}(S) \to D_{\mathrm{perf}}(S^{\mathrm{syn}})_{\mathrm{iso}} \tag{3.0.8}$$

*induces an equivalence of categories.*

*Proof.* Since we have already shown that $T^0(\mathcal{M}(G^\vee))(1) \simeq G$ (see proof of Proposition 3.21), it would be enough to prove that if $M \in D_{\mathrm{perf}}(S^{\mathrm{syn}})_{\mathrm{iso}}$, then the quasi-syntomic sheaf $T^0(M)(1)$ lies in $\mathrm{FFG}(S)$, and $\mathcal{M}^\vee(T^0(M)(1)) \simeq M$. To this end, we may work locally and assume without loss of generality that there exists $V, V' \in \mathrm{Vect}_{\{0,1\}}(S^{\mathrm{syn}})$ and a map $f : V \to V'$ such that we have a fiber sequence $V \to V' \to M$ and $f[\frac{1}{p}]$ is an isomorphism. By construction (see Definition 3.17), we have a fiber sequence

$$T(V)(1) \to T(V')(1) \to T(M)(1) \tag{3.0.9}$$

By Remark 3.34, the map $f$ corresponds to an isogeny $\underline{f} : G' \to G$ of $p$-divisible groups such that $\mathcal{M}(G'), \mathcal{M}(G)$ identify with $V', V$ respectively. We have a fiber sequence $H^\vee \to G^\vee \to G'^\vee$ where $H$ is a finite locally free group scheme of $p$-power rank. It follows that $H^\vee$ is killed by a power of $p$. Passing to derived $p$-completion, we obtain a fiber sequence

$$H^\vee \to T_p(G^\vee)[1] \to T_p(G'^\vee)[1],$$

which maybe rewritten as a fiber sequence

$$T_p(G^\vee) \to T_p(G'^\vee) \to H^\vee \tag{3.0.10}$$

of quasi-syntomic sheaves. Using (3.0.9), (3.0.10) and Remark 3.19, it follows that we have a natural identification $R\Gamma((\cdot), H^\vee) \simeq T(M)(1)$ as $D(\mathbb{Z})$-valued quasisyntomic sheaves. This shows that $T^0(M)(1) \simeq H^\vee$ as quasisyntomic sheaf of abelian groups. Now, $\mathcal{M}^\vee(T^0(M)(1)) \simeq \mathcal{M}^\vee(H^\vee) \simeq \mathcal{M}(H) \simeq \mathrm{cofib}(\mathcal{M}(G) \to \mathcal{M}(G'))$, where the last isomorphism follows from the fiber sequence $H \to G' \to G$. Since $\mathcal{M}(G'), \mathcal{M}(G)$ naturally identify with $V', V$ respectively, and we have a fiber sequence $V \to V' \to M$, we see that $\mathcal{M}(H) \simeq M$. Thus, we obtain $\mathcal{M}^\vee(T^0(M)(1)) \simeq M$, which finishes the proof. □

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