

# DIEUDONNÉ THEORY VIA COHOMOLOGY CLASSIFYING STACKS

## II

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ABSTRACT. In this paper, we prove two results: first, we use crystalline cohomology of classifying stacks to directly reconstruct the classical Dieudonné module of a finite,  $p$ -power rank, commutative group scheme  $G$  over a perfect field  $k$  of characteristic  $p > 0$ . As a consequence, we give a new proof of the isomorphism  $\sigma^*M(G) \simeq \text{Ext}^1(G, \mathcal{O}^{\text{crys}})$  due to Berthelot–Breen–Messing using stacky methods combined with the theory of de Rham–Witt complexes. Additionally, we show that finite locally free commutative group schemes of  $p$ -power rank over a quasisyntomic base ring embed fully faithfully into the category of prismatic  $F$ -gauges, which extends the work of Anschütz and Le Bras on prismatic Dieudonné theory.

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## 1. INTRODUCTION

Let  $p$  be a fixed prime number. Let  $k$  be a perfect field of characteristic  $p$ . Classically, Dieudonné theory offers a classification of arithmetic objects such as  $p$ -divisible groups and finite commutative group schemes of  $p$ -power rank in terms of certain linear algebraic data, called Dieudonné modules. Grothendieck suggested that the Dieudonné module should be viewed as a Dieudonné crystal, which has been realized by the work of Messing [Mes72], Mazur–Messing [MM74] and Berthelot–Breen–Messing [BBM82], [BM90]. This subject has received contributions from many authors and we refer to [Lau18] for a survey. Recently, over mixed characteristic base rings, Anschütz–Le Bras [ALB23] defined a notion of prismatic Dieudonné crystal associated to a  $p$ -divisible group and proved a classification result, using the theory of prismatic cohomology due to [BS19].

In this paper, we revisit aspects of crystalline Dieudonné theory [BBM82] from a modern perspective, using cohomology of classifying stacks, giving a short proof the main comparison result [BBM82, Thm. 4.2.14]. We also refine the notion of prismatic Dieudonné crystals due to [ALB23], and introduce a “prismatic Dieudonné  $F$ -gauge”  $\mathcal{M}(G)$  to any  $p$ -divisible group or finite locally free commutative group scheme  $G$  of  $p$ -power rank, which extends the classification results in [ALB23], covering the case of finite locally free group schemes as well, which was not addressed in loc. cit. Our work uses the notion of prismatic  $F$ -gauges, which gives a refined notion of coefficients for prismatic cohomology, recently introduced in the work of Drinfeld [Dri21a], and Bhatt–Lurie [Bha23], which may be regarded as a generalization of the notion of  $F$ -gauges due to Fontaine–Jannsen [FJ13].

Let us recall the classical construction (see [Dem72, §III]) of Dieudonné modules below.

**Definition 1.1** (Dieudonné). Let  $G$  be a finite commutative group scheme of  $p$ -power rank over  $k$ . Then  $G$  admits a canonical decomposition  $G = G^{\text{uni}} \oplus G^{\text{mul}}$ , where  $G^{\text{uni}}$  is unipotent and  $G^{\text{mul}}$  is a local group scheme whose Cartier dual  $(G^{\text{mul}})^{\vee}$  is étale. One defines the (contravariant) Dieudonné module of  $G$  in the following manner. Let us first define  $M(G^{\text{uni}}) := \varinjlim_{n,V} \text{Hom}(G, W_n)$ . One can now define  $M(G) := M(G^{\text{uni}}) \oplus M((G^{\text{mul}})^{\vee})^*$ .

A uniform construction without appealing to duality was first given by Fontaine [Fon77] using a more complicated formal group  $CW$ , which maybe realized as a completion of  $\varinjlim_{n,V} W_n$  in a certain sense. Another uniform construction is due to the seminal work of Berthelot–Breen–Messing [BBM82] in terms of crystalline Dieudonné theory; they proved that  $\sigma^*M(G) \simeq \text{Ext}^1(G, \mathcal{O}^{\text{crys}})$ , where the last Ext group is computed in the large crystalline site and  $\sigma$  is the Frobenius on  $W(k)$ . Their proof crucially relies on Fontaine’s work and in particular certain explicit computations done in the crystalline site to understand the somewhat complicated object  $CW$ . In [Mon21, Thm. 1.2], it was shown that  $\sigma^*M(G) \simeq H_{\text{crys}}^2(BG)$ , where the proof relied on the work of Berthelot–Breen–Messing.

In this paper, we directly prove that  $\sigma^*M(G) \simeq H_{\text{crys}}^2(BG)$  without relying on [Fon77] or [BBM82]. Instead, our techniques use cohomology of algebraic stacks and the de Rham–Witt complex [Ill79]. Broadly speaking, the main new ingredient is the usage of geometric techniques such as differential forms, deformation theory in the study of Dieudonné modules by means of the classifying stack  $BG$ .

**Theorem 1.2.** *Let  $G$  be a finite commutative  $p$ -power rank group scheme over a perfect field  $k$  of characteristic  $p > 0$ . We have a canonical isomorphism  $\sigma^*M(G) \simeq H_{\text{crys}}^2(BG)$ .*

As a corollary, we obtain a new proof of

**Corollary 1.3** ([BBM82, Thm. 4.2.14]). *Let  $G$  be a finite commutative group scheme over  $k$  of  $p$ -power rank. Then  $\sigma^*M(G) \simeq \mathrm{Ext}_{(k/W(k))_{\mathrm{Crys}}}^1(G, \mathcal{O}^{\mathrm{crys}})$ .*

Based on the above two results, one may regard the classical Dieudonné module as de Rham–Witt realization of  $H_{\mathrm{crys}}^2(BG)$  and the formula  $\mathrm{Ext}_{(k/W(k))_{\mathrm{Crys}}}^1(G, \mathcal{O}^{\mathrm{crys}})$  as its sheaf theoretic incarnation.

In Section 3, we study Dieudonné theory in the mixed characteristic set up, for  $p$ -divisible groups or finite locally free commutative group schemes of  $p$ -power rank over a quasisyntomic ring  $S$ . Let us recall the definition of such rings.

**Definition 1.4** (see [BMS19, § 4]). A ring  $S$  is called quasisyntomic if  $S$  is  $p$ -complete with bounded  $p^\infty$ -torsion and if the cotangent complex  $\mathbb{L}_{S/\mathbb{Z}_p}$  (see [Ill72]) has  $p$ -complete Tor-amplitude in homological degrees  $[0, 1]$ .

**Example 1.5.** Quasisyntomic rings form a fairly large class for rings. Any  $p$ -complete local complete intersection Noetherian ring is quasisyntomic. Any perfectoid ring, or  $p$ -completion of a smooth algebra over a perfectoid ring is also quasisyntomic. The class of rings in Definition 1.6 below form an important class, as they form a basis for the quasisyntomic topology on quasisyntomic rings (see [BMS19, § 4.4]).

**Definition 1.6** (see [BMS19, Rmk. 4.22]). A ring  $R$  is called quasiregular semiperfectoid if it is quasisyntomic and a quotient of some perfectoid ring.

In [ALB23], the authors introduced the notion of admissible prismatic Dieudonné modules over  $S$  in order to obtain a classification of  $p$ -divisible groups over  $S$  (see [ALB23, Thm. 1.4.4]). However, the category of admissible prismatic Dieudonné modules end up not being flexible enough to obtain a classification of finite locally free group schemes. In order to obtain a classification of the latter, one needs to work with a suitable category. Motivated by work of Lau [Lau18], in the desired category, one would like to have a notion of divided Frobenius in an appropriately filtered set up, which does not appear directly in a prismatic Dieudonné module. Further, in [Kis06, Thm. 0.3], Kisin obtained a classification of  $p$ -divisible groups over  $\mathcal{O}_K$  in terms of lattices in crystalline Galois representations with Hodge–Tate weights in  $[0, 1]$ . In their recent work [BS23, Thm. 1.2], Bhatt and Scholze explained how to recover lattices in crystalline Galois representations from prismatic  $F$ -crystals, or from the more refined notion of prismatic  $F$ -gauges (see [Bha23, Thm. 6.6.13]). Unwrapping the notion of a prismatic  $F$ -gauge algebraically, one naturally obtains certain filtered objects, and maps that play the role of divided Frobenii (see Remark 3.39). Moreover, as explained in Proposition 3.45, one can view admissible prismatic Dieudonné modules as prismatic  $F$ -gauges with Hodge–Tate weights in  $[0, 1]$ . As we will show, it turns out that prismatic  $F$ -gauges also provide the desired category for classifying locally free commutative group schemes of  $p$ -power rank.

The category of prismatic  $F$ -gauges over  $S$  is a rather elaborate piece of structure: they are defined to be the derived category of quasicoherent sheaves on certain  $p$ -adic formal stacks introduced in [Dri21a] and [Bha23], called the “syntomification of  $S$ ” and denoted by  $\mathrm{Spf}(S)^{\mathrm{syn}}$ . By quasisyntomic descent (see [BMS19, Lem. 4.28, Lem. 4.30]), in order to understand  $\mathrm{Spf}(S)^{\mathrm{syn}}$  one can restrict attention to the case when  $S = R$  for a quasiregular semiperfectoid algebra  $R$  (see Remark 3.17). In this case, we concretely spell out the stack  $\mathrm{Spf}(R)^{\mathrm{syn}}$  to give a sense of what kind of objects we are working with.

**Construction 1.7** (see [Bha23, Rmk. 5.5.16]). Let  $R$  be a quasiregular semiperfectoid algebra (qrsp). Define  $\mathrm{Spf}(R)^\Delta := \mathrm{Spf}(\Delta_R)$ , where  $\Delta_R$  is the prism associated to  $R$ . Define

$$\mathrm{Spf}(R)^{\mathrm{Nyg}} := \mathrm{Spf} \left( \bigoplus_{i \in \mathbb{Z}} \mathrm{Fil}_{\mathrm{Nyg}}^i \Delta_R \{i\} \right) / \mathbb{G}_m,$$

where  $\Delta_R \{i\}$  denotes the Breuil–Kisin twist. The Nygaard filtration provides a map of graded rings  $\bigoplus_{i \in \mathbb{Z}} \mathrm{Fil}_{\mathrm{Nyg}}^i \Delta_R \{i\} \rightarrow \bigoplus_{i \in \mathbb{Z}} \Delta_R \{i\}$ , which induces a map

$$\mathrm{can} : \mathrm{Spf}(R)^\Delta \rightarrow \mathrm{Spf}(R)^{\mathrm{Nyg}}.$$

Also, the divided Frobenius defines a map of graded rings  $\bigoplus_{i \in \mathbb{Z}} (\varphi_i) : \bigoplus_{i \in \mathbb{Z}} \mathrm{Fil}_{\mathrm{Nyg}}^i \Delta_R \{i\} \rightarrow \bigoplus_{i \in \mathbb{Z}} \Delta_R \{i\}$ , which induces a map

$$\varphi : \mathrm{Spf}(R)^\Delta \rightarrow \mathrm{Spf}(R)^{\mathrm{Nyg}}.$$

One defines

$$\mathrm{Spf}(R)^{\mathrm{syn}} := \mathrm{coeq} \left( \begin{array}{ccc} \mathrm{Spf}(R)^\Delta & \xrightarrow{\mathrm{can}} & \mathrm{Spf}(R)^{\mathrm{Nyg}} \\ & \xrightarrow{\varphi} & \end{array} \right).$$

In this set up, the graded module  $\bigoplus_{i \in \mathbb{Z}} \mathrm{Fil}_{\mathrm{Nyg}}^{i-1} \Delta_R \{i-1\}$  defines a vector bundle on  $\mathrm{Spf}(R)^{\mathrm{Nyg}}$ , which descends to a vector bundle on  $\mathrm{Spf}(R)^{\mathrm{syn}}$ —this will be called the Breuil–Kisin twist and will be denoted by  $\mathcal{O}\{-1\}$ . For any  $M \in \mathrm{Spf}(S)^{\mathrm{syn}}$ , we use  $M\{-n\}$  to denote  $M \otimes_{\mathcal{O}} \mathcal{O}\{-1\}^{\otimes n}$ .

**Definition 1.8** (Prismatic  $F$ -gauges, see [Bha23]). The category of prismatic  $F$ -gauges over  $S$ , denoted as  $F\text{-Gauge}_\Delta(S)$ , is defined to be the derived category of quasicoherent sheaves on  $\mathrm{Spf}(S)^{\mathrm{syn}}$ .

In Section 3.2, we will explain how to think of prismatic  $F$ -gauges as a certain filtered module over a filtered ring equipped with some extra structure in the quasisyntomic site of  $S$ ; this approach is closer in spirit to the notion of  $F$ -gauges in [FJ13] and has certain technical advantages. This approach will also crucially be used in our paper. However, for the introduction, it would be easier to explain our constructions from the stacky point of view.

In Section 3.3, we begin by defining the prismatic Dieudonné  $F$ -gauge  $\mathcal{M}(G)$  associated to a  $p$ -divisible group  $G$  over  $S$ . Our definition is directly motivated by Theorem 1.2, which we roughly explain below.

**Definition 1.9** (Prismatic Dieudonné  $F$ -gauge of a  $p$ -divisible group). Let  $G$  be a  $p$ -divisible group over  $S$ . By functoriality of the syntomification construction, we obtain a natural map

$$v : BG^{\mathrm{syn}} \rightarrow S^{\mathrm{syn}}.$$

The prismatic Dieudonné  $F$ -gauge of  $G$ , denoted as  $\mathcal{M}(G)$ , is defined to be  $R^2 v_* \mathcal{O}_{BG^{\mathrm{syn}}} \in F\text{-Gauge}_\Delta(S)$ .

**Remark 1.10.** Since the theory of  $t$ -structures on  $p$ -complete derived category of (derived)  $p$ -adic formal stacks is not well-behaved in general, the above definition of  $\mathcal{M}(G)$  should be regarded as heuristic and should be interpreted in the sense of Definition 3.47 (also see Remark 3.18). In fact, some work is necessary to prove that  $\mathcal{M}(G)$  is well-defined as a

prismatic  $F$ -gauge; this is proven in Proposition 3.56. If  $G$  is of height  $h$ , then  $\mathcal{M}(G)$  is a vector bundle of rank  $h$  when viewed as a quasicoherent sheaf on  $\mathrm{Spf}(S)^{\mathrm{syn}}$ .

We show the following:

**Theorem 1.11** (Proposition 3.92). *Let  $\mathrm{BT}(S)$  denote the category of  $p$ -divisible groups over a quasisyntomic ring  $S$  and  $F\text{-Gauge}_{\Delta}^{\mathrm{vect}}(S)$  denote the category of vector bundles on  $\mathrm{Spf}(S)^{\mathrm{syn}}$ . The prismatic Dieudonné  $F$ -gauge functor*

$$\mathcal{M} : \mathrm{BT}(S)^{\mathrm{op}} \rightarrow F\text{-Gauge}_{\Delta}^{\mathrm{vect}}(S),$$

*determined by  $G \mapsto \mathcal{M}(G)$  is fully faithful. The essential image of  $\mathcal{M}$  is the full subcategory of vector bundles with Hodge–Tate weights in  $[0, 1]$ .*

The above result gives a different proof of the fully faithfulness result of [ALB23, Thm. 1.4.4]. Our approach uses the formalism of quasi-coherent sheaves on  $S^{\mathrm{syn}}$ , and the dualizability of  $\mathcal{M}(G)$  along with the compatibility with Cartier duality (Proposition 3.88). In order to prove that  $\mathcal{M}(G)$  is a vector bundle, we use a direct computation of the cotangent complex  $\mathbb{L}_{BG}$  (Proposition 3.48). Under the equivalence (see Proposition 3.45) of admissible prismatic Dieudonné modules [ALB23, Thm. 1.4.4] and vector bundles on  $\mathrm{Spf}(S)^{\mathrm{syn}}$  with Hodge–Tate weights in  $[0, 1]$ , Theorem 1.11 is equivalent to [ALB23, Thm. 1.4.4].

We extend our approach using prismatic  $F$ -gauges to a classification of finite locally free commutative group schemes  $G$  of  $p$ -power rank as well. However, in mixed characteristic, under the presence of  $p$ -torsion, it turns out that  $R^2v_*\mathcal{O}_{BG^{\mathrm{syn}}}$  as in Definition 1.9 is not the right invariant. This is a “pathology” that does not occur when  $S$  has characteristic  $p$  (see Remark 3.4). To resolve this, we use the stack  $B^2G^{\mathrm{syn}}$ .

Below, we let  $\mathrm{FFG}(S)$  denote the category of finite locally free commutative group scheme of  $p$ -power rank over  $S$ . Note that in order to describe a prismatic  $F$ -gauge over  $S$ , one can restrict to attention to quasiregular semiperfectoid algebras; more precisely, we have

$$(1.0.1) \quad F\text{-Gauge}_{\Delta}(S) \simeq \lim_{\substack{S \rightarrow R; \\ R \text{ is qrsp}}} F\text{-Gauge}_{\Delta}(R).$$

**Definition 1.12** (Prismatic Dieudonné  $F$ -gauge of locally free group schemes). Let  $S$  be a quasisyntomic ring and let  $G \in \mathrm{FFG}(S)$ . Let  $S \rightarrow R$  be a map such that  $R$  is qrsp. By functoriality of the syntomification construction, we have a natural map

$$v : B^2G_R^{\mathrm{syn}} \rightarrow R^{\mathrm{syn}}.$$

The association  $R \mapsto (\tau_{[-2, -3]}Rv_*\mathcal{O})[3] \in D_{\mathrm{qc}}(R^{\mathrm{syn}})$  determines (via (1.0.1)) an object of  $F\text{-Gauge}_{\Delta}(S)$  that we call prismatic Dieudonné  $F$ -gauge of  $G$  and denote it by  $\mathcal{M}(G)$ .

**Remark 1.13.** A description similar to Definition 1.12 using  $B^2G^{\mathrm{syn}}$  in the case when  $G$  is a  $p$ -divisible group is still compatible with Definition 1.9. See Remark 3.72 and Remark 3.82.

**Remark 1.14.** Similar to Remark 1.10, the above construction in Definition 1.12 should be regarded as heuristic, and the precise construction is given in Construction 3.70 (cf. Remark 3.80). In fact, some work is necessary to prove that  $\mathcal{M}(G)$  is well-defined as a prismatic  $F$ -gauge; this is proven in Proposition 3.76. In Proposition 3.87, we show

that  $\mathcal{M}(G)$  is dualizable as a prismatic  $F$ -gauge and satisfies a compatibility formula with Cartier duality:

$$\mathcal{M}(G)^* \{-1\}[1] \simeq \mathcal{M}(G^\vee).$$

We show the following:

**Theorem 1.15** (Proposition 3.89). *Let  $S$  be a quasisyntomic ring. The prismatic Dieudonné  $F$ -gauge functor*

$$\mathcal{M} : \text{FFG}(S)^{\text{op}} \rightarrow F\text{-Gauge}_{\Delta}(S),$$

*determined by  $G \mapsto \mathcal{M}(G)$  is fully faithful.*

The proof of Theorem 1.15, as well as the compatibility with Cartier duality (Proposition 3.87) uses Proposition 3.84, which expresses cohomology with coefficient in a group scheme  $G$  as cohomology of  $\mathcal{M}(G^\vee)\{1\}$  viewed as a quasicoherent sheaf on  $\text{Spf}(S)^{\text{syn}}$ , i.e.,

$$R\Gamma_{\text{qsyn}}(S, G) \simeq R\Gamma(\text{Spf}(S)^{\text{syn}}, \mathcal{M}(G^\vee)\{1\}).$$

A formula as above is of independent interest, e.g., see [CS24, 1.1.6] for applications in purity. See Proposition 3.85 for a similar result for  $p$ -divisible groups. In Proposition 3.96, we give a description of the essential image of the embedding of Theorem 1.15.

In Proposition 3.99, we show that the étale realization functor for prismatic  $F$ -gauges ([Bha23, Cons. 6.3.1]) can be used to recover Galois representations associated to  $p$ -divisible groups or finite locally free commutative group schemes of  $p$ -power rank  $G$  from their prismatic Dieudonné  $F$ -gauge  $\mathcal{M}(G)$ . Results in this spirit (but in different framework) appeared in [Fal99], [Kim12], [Lau19] (see also [Bre00], [Zin01], [Zin02], [Kis09a], [Kis09b], [ALB23]).

**Remark 1.16.** Since the stack  $\text{BT}_n^h$  of  $n$ -truncated Barsotti–Tate groups of height  $h$  is a smooth stack ([Ill85], [Lau08, Thm. 2.1]), it seems plausible to use Theorem 1.15 to obtain a classification when the base is only assumed to be  $p$ -complete. In forthcoming work of Gardner–Madapusi–Mathew [GMM24], the authors prove that the assignment that sends  $X$  to vector bundles on  $X^{\text{syn}} \otimes \mathbb{Z}/p^n$  with Hodge–Tate weights in  $[0, 1]$  is representable by a smooth  $p$ -adic formal stack over  $\mathbb{Z}_p$ —this is used in their paper to obtain a classification of  $n$ -truncated Barsotti–Tate groups in a more general situation. In a joint work of Madapusi and the author [MM24], we hope to extend these to a more general classification of finite locally free commutative group schemes of  $p$ -power rank as well.

### Notations and conventions.

- (1) We will use the language of  $\infty$ -categories as in [Lur09a], more specifically, the language of stable  $\infty$ -categories [Lur17]. For an ordinary commutative ring  $R$ , we will let  $D(R)$  denote the derived  $\infty$ -category of  $R$ -modules, so that it is naturally equipped with a  $t$ -structure and  $D_{\geq 0}(R)$  (resp.  $D_{\leq 0}(R)$ ) denotes the connective (resp. coconnective) objects, following the homological convention. We will let  $\mathcal{S}$  denote the  $\infty$ -category of spaces, or anima. All tensor products are assumed to be derived.
- (2) For any presentable, stable, symmetric monoidal  $\infty$ -category  $\mathcal{C}$ , we will denote its associated  $\infty$ -category of  $\mathbb{Z}$ -indexed filtered objects by  $\text{Fil}(\mathcal{C})$ . Note that  $\text{Fil}(\mathcal{C})$  is again a presentable stable  $\infty$ -category, which can be regarded as a symmetric monoidal  $\infty$ -category under the Day convolution (see e.g., [GP18]). Using [Lur17, § 3], one can define the category  $\text{CAlg}(\text{Fil}(\mathcal{C}))$ , which may be called the  $\infty$ -category

of filtered commutative algebra objects of  $\mathcal{C}$ . Given any  $A \in \text{CAlg}(\text{Fil}(\mathcal{C}))$ , one can define the category  $\text{Mod}_A(\text{Fil}(\mathcal{C}))$ , which maybe called the  $\infty$ -category of filtered modules over the filtered commutative ring  $A$ .

- (3) Let  $\mathcal{C}$  be any Grothendieck site and let  $\mathcal{D}$  be any presentable  $\infty$ -category. Then one can define the category of “sheaves on  $\mathcal{C}$  with values in  $\mathcal{D}$ ” denoted by  $\text{Shv}_{\mathcal{D}}(\mathcal{C})$  as in [Lur18, Def. 1.3.1.4]. This agrees with the usual notion of sheaves when  $\mathcal{D}$  is a 1-category. Let  $\text{PShv}_{\mathcal{D}}(\mathcal{C}) := \text{Fun}(\mathcal{C}^{\text{op}}, \mathcal{D})$ . There is a natural inclusion functor  $\text{Shv}_{\mathcal{D}}(\mathcal{C}) \rightarrow \text{PShv}_{\mathcal{D}}(\mathcal{C})$ , that admits a left adjoint, which we will call sheafification and denote by  $\text{PShv}_{\mathcal{D}}(\mathcal{C}) \ni \mathcal{F} \mapsto \mathcal{F}^{\sharp}$ . For an ordinary ring  $R$ , the classical (triangulated) derived category obtained from complexes of sheaves of  $R$ -modules on  $\mathcal{C}$  is the homotopy category of the full subcategory of hypercomplete objects of  $\text{Shv}_{D(R)}(\mathcal{C})$ .
- (4) Let  $\mathcal{C}$  be any Grothendieck site and  $\mathcal{D}$  be any presentable, stable, symmetric monoidal  $\infty$ -category. Then  $\text{Shv}_{\mathcal{D}}(\mathcal{C})$  is naturally a presentable, stable, symmetric monoidal  $\infty$ -category. One can define the category  $\text{CAlg}(\text{Fil}(\text{Shv}_{\mathcal{D}}(\mathcal{C})))$ , which may be called the  $\infty$ -category of sheaves on  $\mathcal{C}$  with values in  $\text{CAlg}(\text{Fil}(\mathcal{D}))$ . For  $A \in \text{CAlg}(\text{Fil}(\text{Shv}_{\mathcal{D}}(\mathcal{C})))$ , one can define the category  $\text{Mod}_A(\text{Fil}(\text{Shv}_{\mathcal{D}}(\mathcal{C})))$ , which may be called sheaf of filtered modules over  $A$  (on the site  $\mathcal{C}$ ).
- (5) Let  $R$  be any ordinary commutative rings. Then the  $\infty$ -category of  $\mathcal{S}$ -valued presheaves on the category of affine schemes over  $R$  that satisfies fpqc descent will be called the  $\infty$ -category of (higher) stacks over  $R$ . For a commutative group scheme  $G$  over  $R$ , we let  $BG$  denote the classifying stack and  $B^n G$  denote the  $n$ -stack  $K(G, n)$ , which are all examples of (higher) stacks. The notion of  $p$ -adic formal stacks is slightly different and appears in Definition 3.15, which uses the notion of animated rings; the latter being the  $\infty$ -category obtained from the category of simplicial commutative rings by inverting weak equivalences.
- (6) Let  $R$  be an ordinary commutative ring and  $I$  be a finitely generated ideal of  $R$ . In this situation, one can define  $D(R)_I^{\wedge}$  to be the full subcategory of  $D(R)$  of derived  $I$ -complete objects of  $R$  (see [BS15, § 3.4]). The embedding  $D(R)_I^{\wedge} \rightarrow D(R)$  preserves limits, and therefore is a right adjoint. Its left adjoint will be called derived  $I$ -completion functor. For  $M, N \in D(R)_I^{\wedge}$ , we let  $M \widehat{\otimes}_R N$  denote the derived  $I$ -completion of  $M \otimes_R N$ .
- (7) We work with a fixed prime  $p$ . We will let  $R\Gamma_{\text{crys}}(\cdot)$  denote derived crystalline cohomology, which reduces to derived de Rham cohomology  $dR(\cdot)$  modulo  $p$  (see [Bha12]). We freely use the quasisyntomic descent techniques using quasiregular semiperfectoid algebras introduced in [BMS19]. We also freely use the formalism of (absolute) prismatic cohomology developed by Bhatt–Scholze [BS19], as well as the stacky approach to prismatic cohomology developed by Drinfeld [Dri21b] and Bhatt–Lurie [BL22a], [BL22b], [BL]. At the moment, our main reference for working with prismatic  $F$ -gauges is based on Bhatt’s lecture notes [Bha23]. We will give a presentation of this theory in Section 3.2 from a slightly different perspective that is more similar to [FJ13, § 1.4] (see also [Bha23, Rmk. 3.4.4]).
- (8) Let  $S$  be a quasisyntomic ring and let  $(S)_{\text{qsyn}}$  denote the quasisyntomic site of  $S$  (see [BMS19, Variant. 4.33]). For two abelian sheaves  $\mathcal{F}$  and  $\mathcal{G}$  on  $(S)_{\text{qsyn}}$ , we let  $\mathcal{E}xt_{(S)_{\text{qsyn}}}^i(\mathcal{F}, \mathcal{G})$  denote the sheafified Ext groups. If  $Y$  is a stack over  $S$ , we let

$\mathcal{H}_{(S)_{\text{qsyn}}}^i(Y, \mathcal{F})$  be sheafification of the abelian group valued functor that sends

$$(S)_{\text{qsyn}} \ni A \mapsto H_{(S)_{\text{qsyn}}}^i(Y|_{(A)_{\text{qsyn}}}, \mathcal{F}|_{(A)_{\text{qsyn}}}).$$

When  $\mathcal{F} = \Delta_{(\cdot)}$  (see Notation 3.2), we use  $\mathcal{H}_{\Delta}^i(Y)$  to denote  $\mathcal{H}_{(S)_{\text{qsyn}}}^i(Y, \Delta_{(\cdot)})$ .

- (9) Let  $G$  be a  $p$ -divisible group over a quasisyntomic ring  $S$ . In this set up, for  $n \geq 0$ , we can regard the group scheme of  $p^n$ -torsion of  $G$ , denoted by  $G[p^n]$ , as an abelian sheaf on  $(S)_{\text{qsyn}}$ . In this situation, the collection of  $G[p^n]$ 's naturally defines an ind-object, as well as a pro-object in  $(S)_{\text{qsyn}}$ . We will again use  $G$  to denote  $\text{colim } G[p^n]$  as an object of  $(S)_{\text{qsyn}}$ . We will use  $T_p(G)$  to denote  $\lim G[p^n] \in (S)_{\text{qsyn}}$ , and call it the Tate module of  $G$ . In  $(S)_{\text{qsyn}}$ , one has an exact sequence of abelian sheaves  $0 \rightarrow T_p(G) \rightarrow \lim_p G \rightarrow G \rightarrow 0$ . This implies that the derived  $p$ -completion of the abelian sheaf  $G$  is isomorphic to  $T_p(G)[1]$  in the derived category of abelian sheaves on  $(S)_{\text{qsyn}}$ . We let  $\mathbb{Z}_p(1)$  denote the Tate module of of the  $p$ -divisible group  $\mu_{p^\infty}$ .
- (10) All group schemes appearing in this paper are commutative unless otherwise mentioned. We use  $\text{FFG}(S)$  to denote the category of finite, locally free, commutative group schemes of  $p$ -power rank over a ring  $S$ . The category  $\text{FFG}(S)$  is self dual to itself, induced by the functor  $G \mapsto G^\vee$ , where  $G^\vee$  denotes the Cartier dual of  $G$ . We will let  $\mathbb{G}_a$  denote the additive group scheme,  $\mathbb{G}_m$  denote the multiplicative group scheme and  $W$  denote the group scheme underlying  $p$ -typical Witt vectors.
- (11) Typically, we write  $A^*$  to denote a (co)chain complex,  $\text{Fil}^\bullet B$  or  $B^\bullet$  to denote a filtered object, and  $C^{[\bullet]}$  to denote a (co)simplicial object. For a filtered object  $B^\bullet$ , we let  $\text{gr}^\bullet B$  denote the associated graded object.

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## 2. CRYSTALLINE DIEUDONNÉ THEORY

In this section, we prove Theorem 1.2 (see Proposition 2.19) and deduce [BBM82, Thm. 4.2.14] due to Berthelot–Breen–Messing (see Corollary 2.22). One of the key ingredients in our proof of Theorem 1.2 is Construction 2.5 based on the de Rham–Witt complex. Construction 2.5 below extends a construction from [Ill79, § 6] to stacks, which was originally introduced by Illusie to produce torsion in crystalline cohomology by using the de Rham–Witt complex. First we recall some notations and definitions.

**Notation 2.1.** In this section, we fix a prime  $p$  and a perfect field  $k$  of characteristic  $p > 0$ . For a ring  $R$ , we let  $W(R)$  denote the ring of  $p$ -typical Witt vectors and  $W_n(R)$  denote its  $n$ -truncated variant. We let  $\sigma : W(k) \rightarrow W(k)$  denote the Frobenius operator on Witt vectors. We let  $(k/W(k))_{\text{Crys}}$  denote the (big) crystalline topos of  $k$  (for the Zariski topology) and  $(k/W_n(k))_{\text{Crys}}$  denote its  $n$ -truncated variant (see [BBM82, §1.1.1]). For any scheme  $X$  over  $k$ , as in [BBM82, 1.1.4.5], one can canonically associate an object in  $(k/W(k))_{\text{Crys}}$  or  $(k/W_n(k))_{\text{Crys}}$  that we may also denote by  $X$ , when no confusion is likely to arise.

**Notation 2.2.** The association  $A \mapsto W(A)$  determines a contravariant functor from the category of affine schemes over  $k$  to the category of abelian groups, which is represented by an affine group scheme over  $k$  that we denote by  $W$ . We let  $W_n$  denote the  $n$ -truncated variant of the group scheme  $W$ . We refer to [Dem72, §III] for more details on these Witt group schemes.

**Definition 2.3.** In this section, we will work with derived crystalline cohomology (see [Bha12]). In particular, let  $(k)_{\text{qsyn}}$  denote the quasisyntomic site of  $k$  (see [BMS19, Variant. 4.33]). Using [BMS19, Ex. 5.12] (and arguing derived modulo  $p$ ), it follows that the functor that sends  $(k)_{\text{qsyn}} \ni A \mapsto R\Gamma_{\text{crys}}(A)$  sheaf with values in  $D(W(k))_p^\wedge$ . Therefore, it extends to a functor  $R\Gamma_{\text{crys}}(\cdot) : \text{Shv}_{\mathcal{S}}((k)_{\text{qsyn}}) \rightarrow \text{Fil}(D(W(k))_p^\wedge)$ , where  $\text{Shv}_{\mathcal{S}}((k)_{\text{qsyn}})$  denotes the category of sheaves on  $(k)_{\text{qsyn}}$  with values in the  $\infty$ -category  $\mathcal{S}$ . This extends the formalism of derived crystalline cohomology to (higher) stacks over  $k$ , which will be used in this section. For any (higher) stack  $\mathcal{Y}$ , we will call  $R\Gamma_{\text{crys}}(\mathcal{Y})$  the derived crystalline cohomology of  $\mathcal{Y}$ . We let  $R\Gamma_{\text{crys}}(\mathcal{Y}/W_n(k))$  denote  $R\Gamma_{\text{crys}}(\mathcal{Y}) \otimes_{W(k)} W_n(k)$ .

**Remark 2.4.** By [Bha12, Thm. 3.27], for an lci  $k$ -algebra  $A$ , the derived crystalline cohomology of  $A$  agrees with crystalline cohomology computed in the topos  $(k/W(k))_{\text{Crys}}$ .

**Construction 2.5.** We discuss the construction of a map

$$(2.0.1) \quad T : \varinjlim_{n, \mathcal{V}} R\Gamma(\mathcal{Y}, W_n \mathcal{O}) \rightarrow \sigma_* R\Gamma_{\text{crys}}(\mathcal{Y})[1]$$

for a stack  $\mathcal{Y}$  over a perfect field  $k$  of characteristic  $p > 0$ . To this end, suppose that  $R$  is a finitely generated polynomial algebra over  $k$ . Note that we have an exact sequence

$$0 \rightarrow W(R) \xrightarrow{V^n} W(R) \rightarrow W_n(R) \rightarrow 0.$$

Let  $W\Omega_R^*$  denote the de Rham Witt complex of  $R$ . As in [Ill79, §6, 6.1.1] we have another complex

$$W\Omega_R^*(n) := W(R) \xrightarrow{F^n d} W\Omega_R^1 \xrightarrow{d} W\Omega_R^2 \xrightarrow{d} \dots$$

There is a natural map  $W\Omega_R^* \rightarrow W\Omega_R^*(n)$  induced by  $V^n$  (note that  $FdV = d$ ). Therefore, we obtain an exact sequence of complexes

$$(2.0.2) \quad 0 \rightarrow W\Omega_R^* \rightarrow W\Omega_R^*(n) \rightarrow W_n(R) \rightarrow 0.$$

Taking direct limits produce an exact sequence of complexes

$$(2.0.3) \quad 0 \rightarrow W\Omega_R^* \rightarrow \varinjlim_{n, \mathcal{V}} W\Omega_R^*(n) \rightarrow \varinjlim_{n, \mathcal{V}} W_n(R) \rightarrow 0.$$

Passing to the derived category, using the comparison of de Rham–Witt and crystalline cohomology and keeping track of the  $W(k)$ -module structure, we obtain the map

$$\varinjlim_{n, \mathcal{V}} W_n(R) \rightarrow \sigma_* R\Gamma_{\text{crys}}(R)[1].$$

Using [DM23, Example 2.13] and left Kan extension from the category of finitely generated polynomial algebras over  $k$  to all  $k$ -algebras, we obtain a map

$$(2.0.4) \quad \varinjlim_{n, \mathcal{V}} W_n(A) \rightarrow \sigma_* R\Gamma_{\text{crys}}(A)[1]$$

for any  $k$ -algebra  $A$ . Using (2.0.4), for a stack  $\mathcal{Y}$  over  $k$ , we obtain a natural map

$$\varinjlim_{n, \mathcal{V}} R\Gamma(\mathcal{Y}, W_n\mathcal{O}) \rightarrow \sigma_* R\Gamma_{\text{crys}}(\mathcal{Y})[1].$$

This gives the desired map (2.0.1).

**Remark 2.6.** Let us give further explanation regarding Construction 2.5, which is based on the construction of the natural map  $W_n\mathcal{O}_X \rightarrow R\Gamma_{\text{crys}}(X)[1]$ . Note that such a map arises from a map  $W_n\mathcal{O}_X \rightarrow R\Gamma_{\text{crys}}(X)/p^n$  via composition with  $R\Gamma_{\text{crys}}(X)/p^n \rightarrow R\Gamma_{\text{crys}}(X)[1]$ . The desired map  $W_n\mathcal{O}_X \rightarrow R\Gamma_{\text{crys}}(X)/p^n$  can be thought of as simply arising from the more general conjugate filtration  $\text{Fil}_{\text{conj}}^\bullet R\Gamma_{\text{crys}}(X)/p^n$  whose graded pieces can be understood by the  $\varphi^n$ -linear higher Cartier isomorphism

$$(2.0.5) \quad \mathbb{L}W_n\Omega_X^i[-i] \simeq \text{gr}_{\text{conj}}^i(R\Gamma_{\text{crys}}(X)/p^n).$$

Left hand side of (2.0.5) comes from animation of the de Rham–Witt theory as in [DM23, Rmk. Def. 2.11]. The filtration  $\text{Fil}_{\text{conj}}^\bullet R\Gamma_{\text{crys}}(X)/p^n$  is obtained from animating the canonical filtration when  $X$  is smooth and using [IR83, Prop. 1.4], which gives the desired description of the graded pieces. We thank Illusie for pointing out the isomorphism (2.0.5).

We will apply the map  $T$  from Construction 2.5 (2.0.1) in the case  $\mathcal{Y} = BG$ . To this end, let us note a few general remarks about cohomology of  $BG$ . Let  $(k/W(k))_{\text{Crys}}$  denote the big crystalline topos, and  $(k/W_n(k))_{\text{Crys}}$  denote the  $n$ -truncated variant. Let  $\mathcal{F}$  be any object in the derived category of quasisyntomic sheaves. By Čech descent along the effective epimorphism  $* \rightarrow BG$ , we obtain

$$R\Gamma(BG, \mathcal{F}) \simeq \varprojlim_{m \in \Delta} R\Gamma(G^{[m]}, \mathcal{F}),$$

where  $G^{[\bullet]}$  is the Čech nerve of  $* \rightarrow BG$ . Applying this to  $\mathcal{F} = R\Gamma_{\text{crys}}(\cdot)$  (resp.  $R\Gamma_{\text{crys}}(\cdot)/p^n$ ), we obtain

$$R\Gamma_{\text{crys}}(BG, \mathcal{F}) \simeq \varprojlim_{m \in \Delta} R\Gamma_{\text{crys}}(G^{[m]}, \mathcal{F}) \simeq R\text{Hom}_{(k/W(k))_{\text{Crys}}} \left( \varinjlim_{n \in \Delta^{op}} \mathbb{Z}[G^{[n]}], \mathcal{O}^{\text{crys}} \right).$$

Let us define  $\mathbb{Z}[BG] := \varinjlim_{n \in \Delta^{op}} \mathbb{Z}[G^{[m]}]$ . This way, we get a spectral sequence with  $E_2$ -page

$$(2.0.6) \quad E_2^{i,j} = \text{Ext}_{(k/W(k))_{\text{Crys}}}^i(H^{-j}(\mathbb{Z}[BG]), \mathcal{O}^{\text{crys}}) \implies H_{\text{crys}}^{i+j}(BG)$$

(resp. a similar spectral sequence in  $(k/W_n(k))_{\text{Crys}}$  converging to  $H_{\text{crys}}^*(BG/W_n)$ ).

**Lemma 2.7.** *Let  $G$  be a group scheme of order  $p^m$ . Then for any  $i > 0$ , the group  $H_{\text{crys}}^i(BG)$  is killed by a power of  $p$ .*

*Proof.* This is a consequence of the above  $E_2$ -spectral sequence and the fact that an  $n$ -torsion ordinary abelian group  $T$ , the group homology  $H_i(T, \mathbb{Z}) = H_i(\mathbb{Z}[BT])$  is  $n$ -torsion.  $\square$

Note that, by definition, for any stack  $\mathcal{Y}$ , we have an exact sequence

$$(2.0.7) \quad 0 \rightarrow H_{\text{crys}}^i(\mathcal{Y})/p^n \rightarrow H_{\text{crys}}^i(\mathcal{Y}/W_n) \rightarrow H_{\text{crys}}^{i+1}(\mathcal{Y})[p^n] \rightarrow 0.$$

**Lemma 2.8.** *Let  $G$  be a group scheme of order  $p^m$ . Then  $H_{\text{crys}}^1(BG) = 0$*

*Proof.* We choose a large enough  $n$  such that  $H_{\text{crys}}^1(BG)[p^n] = H_{\text{crys}}^1(BG)$  and apply (2.0.7) for  $i = 0$ .  $\square$

**Lemma 2.9.** *Let  $G$  be a finite group scheme of  $p$ -power order. Then for all  $n \gg 0$ , the map  $H_{\text{crys}}^1(BG/W_n) \rightarrow H_{\text{crys}}^2(BG)$  is an isomorphism.*

*Proof.* We just need to choose  $n$  large enough such that  $H_{\text{crys}}^2(BG)$  is killed by  $p^n$  and apply (2.0.7) for  $i = 1$  along with the previous lemma.  $\square$

**Lemma 2.10.** *Let  $k'/k$  be an extension of perfect fields. Then*

$$R\Gamma_{\text{crys}}(BG_{k'}) \simeq R\Gamma_{\text{crys}}(BG) \otimes_{W(k)} W(k').$$

*Proof.* By derived  $p$ -completeness, one can reduce modulo  $p$ . By the de Rham–crystalline comparison, it would be enough to prove that  $R\Gamma_{\text{dR}}(BG_{k'}) \simeq R\Gamma_{\text{dR}}(BG) \otimes_k k'$ . Since  $R\Gamma_{\text{dR}}(G_{k'}) \simeq R\Gamma_{\text{dR}}(G) \otimes_k k'$  the result follows from descent along  $* \rightarrow BG$  and using the fact that totalization of bounded below cochain complexes commute with filtered colimits.  $\square$

**Lemma 2.11.** *Let  $0 \rightarrow G' \rightarrow G \rightarrow G'' \rightarrow 0$  be an exact sequence of finite group schemes over  $k$ . Then we have an exact sequence  $0 \rightarrow H_{\text{crys}}^2(BG'') \rightarrow H_{\text{crys}}^2(BG) \rightarrow H_{\text{crys}}^2(BG')$  of  $W(k)$ -modules.*

*Proof.* We pick an  $n$  large enough so that  $H_{\text{crys}}^2(BG), H_{\text{crys}}^2(BG'), H_{\text{crys}}^2(BG'')$  are all killed by  $p^n$ . Then the desired exactness of the maps follow from Lemma 2.9 and the fact that  $H_{\text{crys}}^1(BH/W_n) = \text{Ext}_{\text{QSyn}}^0(H, R\Gamma(\cdot)/p^n)$ .  $\square$

**Lemma 2.12.** *Let  $G$  be a finite group scheme of  $p$ -power rank over a perfect field  $k$ . Then  $H_{\text{crys}}^2(BG)$  is a finite length  $W(k)$ -module.*

*Proof.* By Lemma 2.10, one may assume that  $k$  is algebraically closed. In that case, one can argue by induction using Lemma 2.11, which reduces us to the statement for the simple group schemes  $\mathbb{Z}/p, \alpha_p$  and  $\mu_p$ . This follows from Proposition 2.13 below.  $\square$

**Proposition 2.13** (cf. [ABM21]).  *$H_{\text{crys}}^2(BH) = k$  when  $H$  is either  $\mathbb{Z}/p, \mu_p$  or  $\alpha_p$ .*

*Proof.* By base change, it is enough to argue when  $k = \mathbb{F}_p$ . First note that for  $n \gg 0$ , we have isomorphisms  $\mathrm{Ext}_{\mathrm{QSyn}}^0(H, R\Gamma(\cdot)/p^n) \simeq H_{\mathrm{crys}}^1(BH/W_n) \simeq H_{\mathrm{crys}}^2(BH)$ . Since  $H$  is  $p$ -torsion, it follows by looking at the left term above that  $H_{\mathrm{crys}}^2(BH)$  is  $p$ -torsion. Therefore, it suffices to show that  $H_{\mathrm{dR}}^1(BH) = k$  as a  $k$ -vector space.

- (1)  $H = \mathbb{Z}/p$  : Using descent along  $* \rightarrow B\mathbb{Z}/p$ , one sees that  $H_{\mathrm{dR}}^*(B\mathbb{Z}/p) \simeq H^*(\mathbb{Z}/p, k)$ , where the latter denotes group cohomology. Thus, the claim follows.
- (2)  $H = \mu_p$  : Since  $\mu_p$  lifts to  $\mathbb{Z}/p^2$  as a group scheme (along with lift of the Frobenius, which is simply the zero map), the conjugate filtration on  $R\Gamma_{\mathrm{dR}}(B\mu_p)$  splits. Since  $\mathbb{L}_{B\mu_p} = \mathcal{O} \oplus \mathcal{O}[-1]$ , our claim follows by noting that  $H^{>0}(B\mu_p, \mathcal{O}) = 0$ .
- (3)  $H = \alpha_p$  : We use the conjugate spectral sequence. Note that  $H^1(B\alpha_p, \mathcal{O}) = \mathrm{Hom}(\alpha_p, \mathbb{G}_a) = k$ . Note once again that  $\mathbb{L}_{B\alpha_p} = \mathcal{O} \oplus \mathcal{O}[-1]$ . It would be enough to prove that the map  $d$  given by the differential  $k = H^0(B\alpha_p, \mathbb{L}_{B\alpha_p}) \rightarrow H^2(B\alpha_p, \mathcal{O})$  is injective. Note that there is an obstruction class

$$c \in \mathrm{Ext}^2(\mathbb{L}_{B\alpha_p}, \mathcal{O}) = H^2(B\alpha_p, \mathcal{O}) \oplus H^3(B\alpha_p, \mathcal{O}),$$

which is nonzero since  $B\alpha_p$  does not lift to  $\mathbb{Z}/p^2$ . To see the latter statement, note that if  $B\alpha_p$  lifted to  $\mathbb{Z}/p^2$ , due to smoothness of  $B\alpha_p$  as a stack, we would be able to lift the map  $* \rightarrow B\alpha_p$  too, thus ultimately, producing a lifting of the group scheme  $\alpha_p$  to  $\mathbb{Z}/p^2$ , which is impossible. Now, the map  $d$  is obtained by applying  $H^0$  to the map  $\mathbb{L}_{B\alpha_p} \rightarrow \mathcal{O}[2]$  parametrizing the obstruction class  $c$ . Using the map  $B\alpha_p \rightarrow B\mathbb{G}_a$ , by functoriality of obstruction class (see [FGI<sup>+</sup>05, 8.5.10]), we know that the obstruction classes to lifting for  $B\alpha_p$  and  $B\mathbb{G}_a$  have the same image in  $H^2(B\alpha_p, \mathcal{O}[1]) = H^3(B\alpha_p, \mathcal{O})$ ; but that must be zero, since  $B\mathbb{G}_a$  is liftable. This implies that projection of  $c$  on  $H^3(B\alpha_p, \mathcal{O})$  is zero. Since  $c$  is nonzero itself, we see that the map  $d$  must be nonzero. This gives the claim that  $H_{\mathrm{dR}}^1(B\alpha_p) = k$ .

This ends the proof.  $\square$

**Proposition 2.14.** *Let  $k$  be a perfect field. Then as  $W(k)$ -modules, we have a canonical isomorphism  $H_{\mathrm{crys}}^2(B\mu_{p^m}) \simeq \sigma^*(W(k)/p^m)$ .*

*Proof.* Chern class of the line bundle corresponding to the map  $B\mu_{p^m} \rightarrow B\mathbb{G}_m$  defines a canonical  $p^m$ -torsion class on  $H_{\mathrm{crys}}^2(B\mu_{p^m})$ . Since  $H_{\mathrm{crys}}^1(B\mu_{p^m}) = 0$ , we obtain a canonical map

$$\sigma^*W(k) \oplus \sigma^*W(k)/p^m[-2] \rightarrow R\Gamma_{\mathrm{crys}}(B\mu_{p^m})$$

in the derived category of  $W(k)$ -modules. Let  $C$  denote the cofiber. It will be enough to prove that  $C \in D^{\geq 3}(W(k))$ . Since  $C$  is derived  $p$ -complete it is enough to prove the same for  $C/p$ . It is therefore enough to prove that the induced map

$$k^{(1)} \oplus k^{(1)}[-1] \oplus k^{(1)}[-2] \rightarrow R\Gamma_{\mathrm{dR}}(B\mu_{p^m})$$

has cofiber in  $D^{\geq 3}(k)$ . To this end, we will use the conjugate spectral sequence. Since  $\mu_{p^m}$  lifts to  $W_2(k)$  as a group scheme along with a lift of the Frobenius (which is just multiplication by  $p$ ), the conjugate filtration splits. The claim now follows from the fact that  $\mathbb{L}_{B\mu_{p^m}} = \mathcal{O} \oplus \mathcal{O}[-1]$  and  $H^{>0}(B\mu_{p^m}, \mathcal{O}) = 0$ , as the map constructed above is seen to induce isomorphism on  $i$ -th cohomology for  $i \leq 2$ .  $\square$

**Construction 2.15.** Now we can use the map  $T$  from Construction 2.5 to obtain a map

$$C : \varinjlim_n H^1(BG, W_n) \rightarrow \sigma_* H_{\mathrm{crys}}^2(BG).$$

Note that  $\varinjlim_n H^1(BG, W_n) = \varinjlim_n \text{Hom}(G, W_n)$ . Therefore, when  $G$  is unipotent, we get a natural map

$$(2.0.8) \quad C^{\text{uni}} : \sigma^* M(G) \rightarrow H_{\text{crys}}^2(BG).$$

Suppose that  $G$  is a local group scheme over  $k$  of order  $p^k$  whose Cartier dual  $G^\vee$  is étale. We will produce a natural map

$$(2.0.9) \quad C^{\text{mult}} : H_{\text{crys}}^2(BG) \rightarrow \sigma^* M(G).$$

To this end, in the remarks below, we recall certain relevant constructions.

**Remark 2.16** (Duality). Let  $\text{Mod}_{W(k)}^{\text{fl}}$  denote the category of finite length  $W(k)$ -modules. The functor that sends  $M \mapsto \text{Hom}_{W(k)}(M, W(k)[\frac{1}{p}]/W(k))$  induces an antiequivalence of  $\text{Mod}_{W(k)}^{\text{fl}}$ . This duality also extends to the set up of Dieudonné modules whose underlying  $W(k)$ -module is finite length.

**Remark 2.17** (Galois descent). Let  $\bar{k}$  be an algebraic closure of  $k$ . Let  $(\text{Mod}_{W(\bar{k})}^{\text{fl}})^{\text{Gal}(\bar{k}/k)}$  denote the category of finite length  $W(\bar{k})$ -modules equipped with a (semilinear) action of  $\text{Gal}(\bar{k}/k)$ . By Galois descent, we obtain an equivalence of categories  $(\text{Mod}_{W(\bar{k})}^{\text{fl}})^{\text{Gal}(\bar{k}/k)} \simeq \text{Mod}_{W(k)}^{\text{fl}}$  induced by the functors that send  $M \mapsto M \otimes_{W(k)} W(\bar{k}) \in (\text{Mod}_{W(\bar{k})}^{\text{fl}})^{\text{Gal}(\bar{k}/k)}$ , and  $N \rightarrow N^{\text{Gal}(\bar{k}/k)} \otimes_{W(k)} W(\bar{k}) \rightarrow N$  is an isomorphism. Suppose first that  $N$  is  $p$ -torsion. Then  $N$  corresponds to a vector bundle in the étale site of  $\text{Spec } k$ , which must be trivial by descent; i.e.,  $N \simeq M \otimes_k \bar{k}$  for some finite dimensional  $k$ -vector space  $N$ , which implies that the desired map is an isomorphism. The case of general  $N$  follows by considering the (finite)  $p$ -adic filtration on  $N$  and using that  $H^{>0}(\text{Gal}(\bar{k}/k), \bar{k}) = 0$ .

Now, let  $G$  be such that  $G^\vee$  is étale. Note that for  $n \gg 0$ , we have

$$M(G_{\bar{k}}^\vee) \simeq \text{Hom}(G_{\bar{k}}^\vee, \mathbb{Z}/p^n) \otimes_{\mathbb{Z}_p} W(\bar{k}) \simeq \text{Hom}(\mu_{p^n}, G_{\bar{k}}) \otimes_{\mathbb{Z}_p} W(\bar{k}),$$

where the last step follows from Cartier duality. By functoriality of crystalline cohomology and Proposition 2.14, we obtain a map

$$\text{Hom}(\mu_{p^n}, G_{\bar{k}}) \otimes_{\mathbb{Z}_p} W(\bar{k}) \rightarrow \text{Hom}_{W(\bar{k})}^{\text{Gal}(\bar{k}/k)}(\sigma_* H_{\text{crys}}^2(BG_{\bar{k}}), W(\bar{k})/p^n);$$

the latter denotes maps taken in  $(\text{Mod}_{W(\bar{k})}^{\text{fl}})^{\text{Gal}(\bar{k}/k)}$ . Taking Galois fixed points, we obtain a map  $M(G^\vee) \rightarrow (\sigma_* H_{\text{crys}}^2(BG))^*$ . Applying duality now produces a map

$$\sigma_* H_{\text{crys}}^2(BG) \rightarrow M(G^\vee)^* \simeq M(G).$$

This constructs  $C^{\text{mult}}$  as desired in (2.0.9).

**Lemma 2.18.** *Let  $G$  be unipotent. The map  $C^{\text{uni}}$  is injective.*

*Proof.* By Construction 2.5, we have the following commutative diagram where the bottom row is exact:

$$\begin{array}{ccccc}
& H^1(BG, \varinjlim W_n) & \longrightarrow & H_{\text{crys}}^2(BG) & \\
& \downarrow & & \downarrow & \\
H^1(BG, \varinjlim_V W) & \longrightarrow & H^1(BG, \varinjlim W_n) & \longrightarrow & H^2(BG, W)
\end{array}$$

Therefore, to prove the injectivity of  $C^{\text{uni}}$ , it suffices to show that  $H^1(BG, \varinjlim_V W) = \text{Hom}(G, \varinjlim_V W) = 0$ . But this follows because  $W$  is  $V$ -torsion free, and  $G$ , being finite and unipotent, is killed by a power of  $V$ .  $\square$

**Proposition 2.19.** *Let  $G$  be a finite commutative  $p$ -power rank group scheme over  $k$ . We have a canonical isomorphism  $\sigma^*M(G) \simeq H_{\text{crys}}^2(BG)$ .*

*Proof.* We use the natural maps  $C^{\text{uni}}$  and  $C^{\text{mult}}$  constructed before and argue separately. To check that these natural maps are isomorphisms, we may assume that  $k$  is algebraically closed (Lemma 2.10). We have an exact sequence  $0 \rightarrow G' \rightarrow G \rightarrow G'' \rightarrow 0$ , where one may assume that  $G'$  is simple. Since  $k$  is algebraically closed,  $G'$  must be either  $\mathbb{Z}/p, \mu_p$  or  $\alpha_p$ . By Lemma 2.11, we have the following diagram where the rows are exact:

$$\begin{array}{ccccccc}
0 & \longrightarrow & H_{\text{crys}}^2(BG'') & \longrightarrow & H_{\text{crys}}^2(BG) & \longrightarrow & H_{\text{crys}}^2(BG') \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \sigma^*M(G'') & \longrightarrow & \sigma^*M(G) & \longrightarrow & \sigma^*M(G')
\end{array}$$

One sees directly that in this case, the map  $\sigma^*M(G) \rightarrow \sigma^*M(G')$  is surjective; thus the claim follows by Proposition 2.13 and induction on the length of  $G$ .  $\square$

**Proposition 2.20.** *Let  $G$  be a finite commutative  $p$ -power rank group scheme over  $k$ . We have a canonical isomorphism  $H_{\text{crys}}^2(BG) \simeq H_{\text{crys}}^3(B^2G)$ .*

*Proof.* Using descent along  $* \rightarrow B^2G$ , we obtain an  $E_1$ -spectral sequence

$$E_1^{i,j} = H_{\text{crys}}^j((BG)^i) \implies H_{\text{crys}}^{i+j}(B^2G).$$

The claim now follows by using this spectral sequence and Lemma 2.8.  $\square$

**Lemma 2.21.** *For any group scheme  $G$  over  $k$ , one has*

$$H_{\text{crys}}^3(B^2G) \simeq \text{Ext}_{(k/W(k))_{\text{Crys}}}^1(G, \mathcal{O}^{\text{crys}}).$$

*Proof.* By applying descent along  $* \rightarrow B^2G$ , similar to (2.0.6), we obtain an  $E_2$ -spectral sequence

$$E_2^{i,j} = \text{Ext}_{(k/W(k))_{\text{Crys}}}^i(H^{-j}(\mathbb{Z}[B^2G]), \mathcal{O}^{\text{crys}}) \implies H_{\text{crys}}^{i+j}(B^2G).$$

The claim now follows from the fact that  $H^{-1}(\mathbb{Z}[B^2G]) = 0$ ,  $H^{-2}(\mathbb{Z}[B^2G]) = G$  (by e.g., Hurewicz theorem) and  $H^{-3}(\mathbb{Z}[B^2G]) = 0$ ; the latter vanishing can be seen by applying the Serre fibration spectral sequence for the fibration  $K(G, n) \rightarrow * \rightarrow K(G, n+1)$ .  $\square$

Combining Proposition 2.19, Proposition 2.20 and Lemma 2.21, we obtain

**Corollary 2.22** (Berthelot–Breen–Messing). *Let  $G$  be a finite commutative group scheme over  $k$  of  $p$ -power rank. Then  $\sigma^*M(G) \simeq \text{Ext}_{(k/W(k))_{\text{Crys}}}^1(G, \mathcal{O}^{\text{crys}})$ .*

3. DIEUDONNÉ THEORY AND PRISMATIC  $F$ -GAUGES

In this section, we prove that the Dieudonné module functor induced by  $G \mapsto H_{\Delta}^2(BG)$  (where the latter denotes absolute prismatic cohomology) give a fully faithful functor. However, one cannot expect this functor to be fully faithful without carefully analyzing what other extra data on  $H_{\Delta}^2(BG)$  one needs to remember. Let us first take a step back to explain our perspective on Dieudonné theory taken in this paper.

By Pontryagin duality, for a finite discrete abelian group  $G$ , the functor  $G \mapsto \text{Hom}(G, \mathbb{S}^1)$  gives an equivalence of categories. Note that in this case, since  $\mathbb{S}^1 = K(\mathbb{Z}, 1)$  and  $G$  is finite, one has a natural isomorphism  $\text{Hom}(G, \mathbb{S}^1) = H^1(BG, \mathbb{Z}[1]) = H^2(BG, \mathbb{Z})$ , where the latter denotes Betti cohomology. Our main point here is that it is possible to reconstruct a group (or rather its Pontryagin dual) from the cohomology of  $BG$ .

Now, let  $G$  be a finite locally free commutative group scheme over a base scheme  $S$ . By Cartier duality, the functor  $G \rightarrow G^{\vee} := \mathcal{H}om(G, \mathbb{G}_m)$  gives an antiequivalence of categories. Note that  $\mathcal{H}om(G, \mathbb{G}_m) = \mathcal{H}^1(BG, \mathbb{G}_m)$ . Further, one may write  $\mathbb{G}_m = \mathbb{Z}(1)[1]$ , where  $\mathbb{Z}(1)$  is the Tate twist. Thus,  $\mathcal{H}^1(BG, \mathbb{G}_m) = \mathcal{H}^2(BG, \mathbb{Z}(1))$ ; the latter recovers the group scheme  $G$  (or rather, its Cartier dual). The notion of Tate twists and other related twists would play a very important role in our approach.

Since the goal of Dieudonné theory is to classify finite locally free or  $p$ -divisible group schemes by linear algebraic data, it is natural to look for a more linearized way to recover the Tate twist  $\mathbb{Z}(1)$ . To this end, we now entirely specialize to the  $p$ -adic set up. We take  $S$  to be a  $p$ -complete quasisyntomic ring. The  $p$ -adic Tate module of  $\mathbb{G}_m$  denoted by  $\mathbb{Z}_p(1)$  can be understood in terms of prismatic cohomology by means of the following formula:

$$(3.0.1) \quad R\Gamma(\mathcal{Y}, \mathbb{Z}_p(1)) \simeq \text{Fib} \left( \text{Fil}_{\text{Nygaard}}^1 R\Gamma_{\Delta}(\mathcal{Y}) \{1\} \xrightarrow{\varphi_1 - \text{can}} R\Gamma_{\Delta}(\mathcal{Y}) \{1\} \right),$$

where the curly brackets denote the Breuil–Kisin twist, and  $\mathcal{Y}$  is a stack over  $S$ .

**Remark 3.1.** We refer to [BS19] and [BL22a] for the theory of absolute prismatic cohomology (equipped with the Nygaard filtration) that will be used in this section. In particular, let  $(S)_{\text{qsyn}}$  denote the quasisyntomic site of  $S$  (see [BMS19, Variant. 4.33]). By [BL22a, Cor. 5.5.21], the functor that sends  $(S)_{\text{qsyn}} \ni A \mapsto \text{Fil}_{\text{Nygaard}}^* R\Gamma_{\Delta}(A)$ , is a sheaf with values in filtered objects of  $D(\mathbb{Z}_p)_p^{\wedge}$ . Therefore, it extends to a functor  $\text{Fil}_{\text{Nygaard}}^{\bullet} R\Gamma_{\Delta}(\cdot) : \text{Shv}_{\mathcal{S}}((S)_{\text{qsyn}}) \rightarrow \text{Fil}(D(\mathbb{Z}_p)_p^{\wedge})$ , where  $\text{Shv}_{\mathcal{S}}((S)_{\text{qsyn}})$  denotes the category of sheaves on  $(S)_{\text{qsyn}}$  with values in the  $\infty$ -category  $\mathcal{S}$ . This extends the formalism of Nygaard prismatic cohomology to (higher) stacks, which will be used in our paper. For any (higher) stack  $\mathcal{Y}$ , we will call  $\text{Fil}_{\text{Nygaard}}^{\bullet} R\Gamma_{\Delta}(\mathcal{Y})$  the Nygaard filtered prismatic cohomology of  $\mathcal{Y}$ .

**Notation 3.2.** The functor that sends  $(S)_{\text{qsyn}} \ni A \mapsto R\Gamma_{\Delta}(A)$  will be denoted as  $\Delta_{(\cdot)}$ . Similarly, the functor that sends  $(S)_{\text{qsyn}} \ni A \mapsto \text{Fil}_{\text{Nygaard}}^* R\Gamma_{\Delta}(A)$  will be denoted as  $\text{Fil}_{\text{Nygaard}}^* \Delta_{(\cdot)}$ .

**3.1. Some calculations of syntomic cohomology.** In this subsection, we record certain calculations by specializing (3.0.1) to  $\mathcal{Y} = BG$ , that will be important later.

**Lemma 3.3.** *Let  $G$  be a  $p$ -divisible group over a quasisyntomic ring  $S$ . We have  $\mathcal{H}_{\Delta}^1(BG) = 0$ .*

*Proof.* Note that any class in prismatic cohomology is killed quasisyntomic locally. Therefore, by applying descent along  $* \rightarrow BG$ , one obtains that  $\mathcal{H}_{\Delta}^1(BG) \simeq \mathcal{H}om_{(S)_{\text{qsyn}}}(G, \Delta_{(\cdot)})$ . Thus it is enough to prove that  $\mathcal{H}om_{(S)_{\text{qsyn}}}(G, \Delta_{(\cdot)}) = 0$ . Since  $G$  is  $p$ -divisible, this follows because  $\Delta_{(\cdot)}$  is derived  $p$ -complete.  $\square$

**Remark 3.4.** Even if we are working over a quasisyntomic base ring  $S$ , for a finite locally group  $G$  scheme over  $S$ , the above vanishing need not hold. One can take  $S$  to be a quasiregular semiperfectoid algebra such that  $\Delta_S$  has  $p$ -torsion. Then  $G = \mathbb{Z}/p\mathbb{Z}_S$  gives such an example. Let us point out that if  $S$  has characteristic  $p > 0$ , this ‘‘pathology’’ does not occur since for any quasiregular semiperfect ring  $S$ , the ring  $\mathbb{A}_{\text{crys}}(S)$  is  $p$ -torsion free. It is precisely to circumvent this pathology that we will be working with  $B^2G$  for finite locally free commutative group schemes, whereas for  $p$ -divisible groups  $G$ , the stack  $BG$  suffices.

**Lemma 3.5.** *Let  $G$  be finite locally free group scheme of  $p$ -power rank over a quasisyntomic ring  $S$ . Then we have the following isomorphisms in  $(S)_{\text{qsyn}}$*

$$\mathcal{H}_{(S)_{\text{qsyn}}}^2(BG, \mathbb{Z}_p(1)) \simeq \mathcal{E}xt_{(S)_{\text{qsyn}}}^1(G, \mathbb{Z}_p(1)) \simeq G^\vee.$$

*Proof.* Since  $G$  is killed by a power of  $p$ , it follows that the natural map  $\mathbb{G}_m \rightarrow \mathbb{Z}_p(1)[1]$  induces an isomorphism  $\mathcal{E}xt^1(G, \mathbb{Z}_p(1)) \simeq \mathcal{H}om(G, \mathbb{G}_m) \simeq G^\vee$ . To see that  $\mathcal{H}^2(BG, \mathbb{Z}_p(1)) = \mathcal{E}xt^1(G, \mathbb{Z}_p(1))$ , we use the  $E_2$ -spectral sequence

$$E_2^{i,j} = \text{Ext}_{(S)_{\text{qsyn}}}^i(H^{-j}(\mathbb{Z}[BG]), \mathbb{Z}_p(1)) \implies H_{\text{qsyn}}^{i+j}(BG, \mathbb{Z}_p(1)).$$

Note that  $H^{-1}(\mathbb{Z}[BG]) \simeq G$  and  $H^{-2}(\mathbb{Z}[BG]) = G \wedge G$ . Since locally in  $(S)_{\text{qsyn}}$ ,  $\mathbb{Z}_p(1)$  has no higher cohomology, it suffices to show that  $\mathcal{H}om(G \wedge G, \mathbb{Z}_p(1)) = 0$ ; the latter follows from the fact that we have an injection

$$\mathcal{H}om(G \wedge G, \mathbb{Z}_p(1)) \rightarrow \mathcal{H}om(G \otimes G, \mathbb{Z}_p(1)) \simeq \mathcal{H}om(G, \mathcal{H}om(G, \mathbb{Z}_p(1)))$$

and  $\mathcal{H}om(G, \mathbb{Z}_p(1)) \simeq \mathcal{E}xt^{-1}(G, \mathbb{G}_m) \simeq 0$ .  $\square$

**Proposition 3.6.** *Let  $G$  be a finite locally free group scheme of  $p$ -power rank over a quasisyntomic ring  $S$ . Then  $\mathcal{H}_{(S)_{\text{qsyn}}}^2(B^2G, \mathbb{Z}_p(1)) = 0$  and  $\mathcal{H}_{(S)_{\text{qsyn}}}^3(B^2G, \mathbb{Z}_p(1)) = G^\vee$ .*

*Proof.* For any quasisyntomic sheaf of abelian groups  $\mathcal{F}$  on  $(S)_{\text{qsyn}}$  such that  $\mathcal{H}^i(*, \mathcal{F}) = 0$  for  $i > 0$ , we have  $\mathcal{H}^2(B^2G, \mathcal{F}) \simeq \mathcal{H}om(G, \mathcal{F})$  and  $\mathcal{H}^3(B^2G, \mathcal{F}) \simeq \mathcal{E}xt^1(G, \mathcal{F})$ .

In order to see the above, we use the  $E_2$ -spectral sequence

$$(3.1.1) \quad E_2^{i,j} = \text{Ext}_{(S')_{\text{qsyn}}}^i(H^{-j}(\mathbb{Z}[B^2G_{S'}]), \mathcal{F}) \implies H_{(S')_{\text{qsyn}}}^{i+j}(B^2G_{S'}, \mathcal{F})$$

for any quasisyntomic algebra  $S'$  over  $S$ . The claims now follow from the fact that  $H^{-2}(\mathbb{Z}[B^2G_{S'}]) = G_{S'}$  and  $H^{-3}(\mathbb{Z}[B^2G_{S'}]) = H^{-1}(\mathbb{Z}[B^2G_{S'}]) = 0$  and  $H^0(\mathbb{Z}[B^2G_{S'}]) \simeq \mathbb{Z}$  (cf. [Mon21, Rmk. 3.17]).

Applying the isomorphisms in the first paragraph to  $\mathcal{F} = \mathbb{Z}_p(1)$ , the conclusion follows from Lemma 3.5.  $\square$

**Lemma 3.7.** *Let  $G$  be a  $p$ -divisible group. Then*

$$H_{(S)_{\text{qsyn}}}^2(BG, \mathbb{Z}_p(1)) \simeq \mathcal{E}xt_{(S)_{\text{qsyn}}}^1(G, \mathbb{Z}_p(1)) \simeq T_p(G^\vee).$$

*Proof.* Follows from Lemma 3.5 by taking inverse limits.  $\square$

**Lemma 3.8.** *Let  $G$  be a finite locally free group scheme or a  $p$ -divisible group. Then  $\mathcal{E}xt_{(S)_{\text{qsyn}}}^2(G, \mathbb{Z}_p(1)) = 0$ .*



*Proof.* By considering inverse limits, the case of  $p$ -divisible group reduces to the finite locally free case. The fiber of the natural map  $\mathbb{G}_m \rightarrow \mathbb{Z}_p(1)[1]$  is uniquely  $p$ -divisible. But since  $G$  is killed by a power of  $p$ , we must have  $\mathcal{E}xt^2(G, \mathbb{Z}_p(1)) = \mathcal{E}xt^1(G, \mathbb{G}_m)$ . To show vanishing of the latter, let  $n$  be an integer that kills  $G$ . Using the exact sequence  $0 \rightarrow \mu_n \rightarrow \mathbb{G}_m \xrightarrow{n} \mathbb{G}_m \rightarrow 0$ , one sees that any class  $u \in \text{Ext}^1(G, \mathbb{G}_m)$  arises from a class  $v \in \text{Ext}^1(G, \mu_n)$ . Let us represent  $v$  by an exact sequence  $0 \rightarrow \mu_n \rightarrow H \rightarrow G \rightarrow 0$ . The class  $u$  can be described via pushout of  $\mu_n \rightarrow H$  along  $\mu_n \rightarrow \mathbb{G}_m$ . The class  $u$  can be killed if there exists a map  $H \rightarrow \mathbb{G}_m$  such that the composition  $\mu_n \rightarrow H \rightarrow \mathbb{G}_m$  is the natural inclusion. However, by Cartier duality, we have a surjection of group schemes  $H^\vee \rightarrow \mu_n^\vee$ ; therefore,  $u$  can be killed syntomic locally.  $\square$

**Proposition 3.9.** *For a finite locally free group scheme  $G$  over a quasisyntomic ring  $S$  for which  $\mathcal{H}_\Delta^1(BG) = 0$ , we have the following exact sequence of abelian sheaves in  $(S)_{\text{qsyn}}$ :*

$$(3.1.2) \quad 0 \rightarrow G^\vee \rightarrow \mathcal{H}_{(S)_{\text{qsyn}}}^2(BG, \text{Fil}_{\text{Nyg}}^1 \Delta(\cdot)) \{1\} \xrightarrow{\varphi_1 - \text{can}} \mathcal{H}_\Delta^2(BG) \{1\} \rightarrow 0.$$

*Proof.* Applying (3.0.1) for  $\mathcal{Y} := BG$  and using Lemma 3.5, we obtain the desired exactness in the left and in the middle. For the surjectivity on the right, we first note that by applying descent along  $* \rightarrow BG$ , we have

$$\mathcal{H}_\Delta^2(BG) \simeq \mathcal{E}xt_{(S)_{\text{qsyn}}}^1(G, \Delta(\cdot)).$$

In order to see the above, we use the  $E_2$ -spectral sequence

$$(3.1.3) \quad E_2^{i,j} = \text{Ext}_{(S')_{\text{qsyn}}}^i(H^{-j}(\mathbb{Z}[BG_{S'}]), \Delta(\cdot)) \implies H_{(S')_{\text{qsyn}}}^{i+j}(BG_{S'}, \Delta(\cdot))$$

for any quasisyntomic algebra  $S'$  over  $S$ . This implies that  $\mathcal{H}_\Delta^1(BG) = \mathcal{H}om_{(S)_{\text{qsyn}}}(G, \Delta(\cdot))$ . By our hypothesis, it follows that  $\mathcal{H}om_{(S)_{\text{qsyn}}}(G, \Delta(\cdot)) = 0$ . To prove that  $\mathcal{H}_\Delta^2(BG) \simeq \mathcal{E}xt^1(G, \Delta(\cdot))$ , by the above spectral sequence and the isomorphism  $H^{-2}(\mathbb{Z}[BG]) \simeq G \wedge G$ , it would be enough to prove that  $\mathcal{H}om_{(S)_{\text{qsyn}}}(G \wedge G, \Delta(\cdot)) = 0$ . This follows from the vanishing  $\mathcal{H}om_{(S)_{\text{qsyn}}}(G, \Delta(\cdot)) = 0$  in a way similar to the proof of Lemma 3.5. Now we claim that

$$\mathcal{H}_{(S)_{\text{qsyn}}}^2(BG, \text{Fil}_{\text{Nyg}}^1 \Delta(\cdot)) \simeq \mathcal{E}xt_{(S)_{\text{qsyn}}}^1(G, \text{Fil}_{\text{Nyg}}^1 \Delta(\cdot)).$$

In order to see this, we may again use an  $E_2$ -spectral sequence

$$(3.1.4) \quad E_2^{i,j} = \text{Ext}_{(S')_{\text{qsyn}}}^i(H^{-j}(\mathbb{Z}[BG_{S'}]), \text{Fil}_{\text{Nyg}}^1 \Delta(\cdot)) \implies H_{(S')_{\text{qsyn}}}^{i+j}(BG_{S'}, \text{Fil}_{\text{Nyg}}^1 \Delta(\cdot))$$

for any quasisyntomic algebra  $S'$  over  $S$ . Note that for a qrsp algebra  $S_0 \in (S)_{\text{qsyn}}$ , we have an injection of discrete abelian groups  $0 \rightarrow \text{Fil}_{\text{Nyg}}^1 \Delta_{S_0} \rightarrow \Delta_{S_0}$ . Since qrps algebras form a basis for  $(S)_{\text{qsyn}}$ , it follows that we have an injection  $\mathcal{H}om_{(S)_{\text{qsyn}}}(G, \text{Fil}_{\text{Nyg}}^1 \Delta(\cdot)) \rightarrow \mathcal{H}om_{(S)_{\text{qsyn}}}(G, \Delta(\cdot)) = 0$ . This implies that  $\mathcal{H}om_{(S)_{\text{qsyn}}}(G, \text{Fil}_{\text{Nyg}}^1 \Delta(\cdot)) = 0$ . To prove that  $\mathcal{H}_{(S)_{\text{qsyn}}}^2(BG, \text{Fil}_{\text{Nyg}}^1 \Delta(\cdot)) \simeq \mathcal{E}xt_{(S)_{\text{qsyn}}}^1(G, \text{Fil}_{\text{Nyg}}^1 \Delta(\cdot))$ , using the spectral sequence, it is enough to prove that  $\mathcal{H}om_{(S)_{\text{qsyn}}}(G \wedge G, \text{Fil}_{\text{Nyg}}^1 \Delta(\cdot)) = 0$ , for which we argue the same way as before. Now the surjectivity in the right of (3.1.2) is equivalent to the surjectivity of  $\mathcal{E}xt_{(S)_{\text{qsyn}}}^1(G, \text{Fil}_{\text{Nyg}}^1 \Delta(\cdot)) \xrightarrow{\varphi_1 - \text{can}} \mathcal{E}xt_{(S)_{\text{qsyn}}}^1(G, \Delta(\cdot))$ . However, this surjectivity follows from  $\mathcal{E}xt_{(S)_{\text{qsyn}}}^2(G, \mathbb{Z}_p(1)) = 0$  (see Lemma 3.8).  $\square$

**Proposition 3.10.** *For a  $p$ -divisible group  $G$  over a quasisyntomic ring  $S$ , we have the following exact sequence of abelian sheaves in  $(S)_{\text{qsyn}}$ :*

$$(3.1.5) \quad 0 \rightarrow T_p(G^\vee) \rightarrow H_{(S)_{\text{qsyn}}}^2(BG, \text{Fil}_{\text{Nyg}}^1 \Delta(\cdot)) \{1\} \xrightarrow{\varphi_1 - \text{can}} \mathcal{H}_{\Delta}^2(BG) \{1\} \rightarrow 0.$$

*Proof.* By Lemma 3.3, in this case we have  $\mathcal{H}_{\Delta}^1(BG) = 0$ . The proof now follows exactly in the same way as Proposition 3.9.  $\square$

**Notation 3.11.** Let  $\tau_{[-3, -2]} R\Gamma(B^2 G(\cdot), \mathbb{Z}_p(1))^\sharp$  denotes the  $D(\mathbb{Z}_p)_p^\wedge$  valued sheaf obtained by sheafifying the presheaf that sends

$$(S)_{\text{qsyn}} \ni T \mapsto \tau_{[-3, -2]} R\Gamma(B^2 G_T, \mathbb{Z}_p(1));$$

similarly for  $\tau_{[-3, -2]} \text{Fil}_{\text{Nyg}}^1 R\Gamma_{\Delta}(B^2 G(\cdot)) \{1\}^\sharp$  and  $\tau_{[-3, -2]} R\Gamma_{\Delta}(B^2 G(\cdot)) \{1\}^\sharp$ . We have the following:

**Proposition 3.12.** *Let  $G$  be a finite locally free group scheme over a quasisyntomic ring  $S$ . We have the following fiber sequence of  $D(\mathbb{Z}_p)_p^\wedge$ -valued sheaves on  $(S)_{\text{qsyn}}$ :*

$$(3.1.6) \quad \tau_{[-3, -2]} R\Gamma(B^2 G(\cdot), \mathbb{Z}_p(1))^\sharp \rightarrow \tau_{[-3, -2]} \text{Fil}_{\text{Nyg}}^1 R\Gamma_{\Delta}(B^2 G(\cdot)) \{1\}^\sharp \xrightarrow{\varphi_1 - \text{can}} \tau_{[-3, -2]} R\Gamma_{\Delta}(B^2 G(\cdot)) \{1\}^\sharp.$$

*Proof.* Using (3.0.1) for  $\mathcal{Y} := B^2 G$ , it would suffice to show that  $\mathcal{H}_{\Delta}^1(B^2 G) = 0$  and the map  $\mathcal{H}_{(S)_{\text{qsyn}}}^3(B^2 G, \text{Fil}_{\text{Nyg}}^1 \Delta(\cdot)) \xrightarrow{\varphi_i - \text{can}} \mathcal{H}_{(S)_{\text{qsyn}}}^3(B^2 G, \Delta(\cdot))$  is surjective.

To this end, we use the  $E_2$ -spectral sequence

$$(3.1.7) \quad E_2^{i,j} = \text{Ext}_{(S')_{\text{qsyn}}}^i(H^{-j}(\mathbb{Z}[B^2 G_{S'}]), \Delta(\cdot)) \implies H_{(S')_{\text{qsyn}}}^{i+j}(B^2 G_{S'}, \Delta(\cdot))$$

for any quasisyntomic algebra  $S'$  over  $S$ . Since  $H^{-1}(\mathbb{Z}[B^2 G]) = 0$ , the above spectral sequence gives  $\mathcal{H}_{\Delta}^1(B^2 G) = 0$ . Using the same spectral sequence and the fact that  $H^{-3}(\mathbb{Z}[B^2 G]) = 0$ ,  $H^{-2}(\mathbb{Z}[B^2 G]) \simeq G$ , and  $H^{-1}(\mathbb{Z}[B^2 G]) = 0$ , it follows that

$$\mathcal{H}_{(S)_{\text{qsyn}}}^3(B^2 G, \Delta(\cdot)) \simeq \mathcal{E}xt_{(S)_{\text{qsyn}}}^1(G, \Delta(\cdot)).$$

By an entirely similar argument, we have

$$\mathcal{H}_{(S)_{\text{qsyn}}}^3(B^2 G, \text{Fil}_{\text{Nyg}}^1 \Delta(\cdot)) \simeq \mathcal{E}xt_{(S)_{\text{qsyn}}}^1(G, \text{Fil}_{\text{Nyg}}^1 \Delta(\cdot)).$$

The desired surjectivity is equivalent to the surjectivity of  $\mathcal{E}xt_{(S)_{\text{qsyn}}}^1(G, \text{Fil}_{\text{Nyg}}^1 \Delta(\cdot)) \xrightarrow{\varphi_1 - \text{can}} \mathcal{E}xt_{(S)_{\text{qsyn}}}^1(G, \Delta(\cdot))$ . However, this surjectivity follows from the vanishing  $\mathcal{E}xt_{(S)_{\text{qsyn}}}^2(G, \mathbb{Z}_p(1)) = 0$  (Lemma 3.8).  $\square$

**Proposition 3.13.** *Let  $G$  be a finite locally free group scheme over a quasisyntomic ring  $S$ . We have the following fiber sequence of  $D(\mathbb{Z}_p)_p^\wedge$ -valued sheaves on  $(S)_{\text{qsyn}}$ :*

$$(3.1.8) \quad R\Gamma_{\text{qsyn}}((\cdot), G^\vee)[-3] \rightarrow \tau_{[-3, -2]} \text{Fil}_{\text{Nyg}}^1 R\Gamma_{\Delta}(B^2 G(\cdot)) \{1\}^\sharp \xrightarrow{\varphi_1 - \text{can}} \tau_{[-3, -2]} R\Gamma_{\Delta}(B^2 G(\cdot)) \{1\}^\sharp.$$

*Proof.* This follows from Proposition 3.6 and Proposition 3.12.  $\square$

**Remark 3.14.** The above exact sequences make it clear that it is possible to reconstruct  $G$  from prismatic cohomology, and what other *extra data* is essential for the purpose of reconstruction. Namely, we use  $\mathcal{H}_{\Delta}^2(BG)$  or  $\tau_{[-2, -3]} R\Gamma_{\Delta}(B^2 G)$ , viewed as a prismatic crystal, as well as the Nygaard filtration, and the *divided* Frobenius. If one wanted to fully

faithfully embed  $p$ -divisible groups and finite locally free group schemes, one naturally looks for a category where all of these data make sense. This is given by the derived category of the stack  $S^{\text{syn}}$ , introduced by Drinfeld and Bhatt–Lurie, which gives the category of coefficients for prismatic cohomology along with the other natural data such as the Nygaard filtration, divided Frobenius, etc. We will see that the formalism of quasi-coherent sheaves on these stacks (which automatically keeps track of the relevant data) gives a very convenient framework for Dieudonné theory.

**3.2. Preliminaries on prismatic  $F$ -gauges.** For the rest of this section, the base ring  $S$  should be assumed to be a  $p$ -complete quasisyntomic ring unless otherwise mentioned. In this set up, one can consider the prismatization stacks  $\text{Spf}(S)^\Delta$ ,  $\text{Spf}(S)^{\text{Nyg}}$  and  $\text{Spf}(S)^{\text{syn}}$  as defined in [Dri21a], [BL22b]; they are also discussed in [Bha23], which will be our main reference. The category of prismatic  $F$ -gauges, by definition (see Definition 3.20), is the derived  $\infty$ -category of quasicoherent sheaves on the stack  $\text{Spf}(S)^{\text{syn}}$ . In Remark 3.32, we give an equivalent description of the category of prismatic  $F$ -gauges by working in the quasisyntomic site of  $S$ , which is similar to the notion of  $F$ -gauges in [FJ13] in spirit. In Proposition 3.45, we explain how to view admissible prismatic Dieudonné modules introduced in [ALB23] in terms of prismatic  $F$ -gauges.

**Definition 3.15** ( $p$ -adic formal stack). A functor  $X$  from the category of  $p$ -nilpotent animated rings to the  $\infty$ -category  $\mathcal{S}$  will be simply called a  $p$ -adic formal prestack. A  $p$ -adic formal prestack  $X$  that satisfies descent (in the sense of [Lur09b]) for the fpqc topology will be called a  $p$ -adic formal stack.

**Example 3.16.** When  $R$  is qrsp, see [Bha23, Rmk. 5.5.3] for the description of  $\text{Spf}(R)^{\text{Nyg}}$  as a functor defined on  $p$ -nilpotent rings.

**Remark 3.17.** By [Bha23, Rmk. 5.5.18], for a quasisyntomic ring  $S$ , one has a natural isomorphism of  $p$ -adic formal stacks

$$\text{colim}_{\substack{S \rightarrow R; \\ R \text{ is qrsp}}} \text{Spf}(R)^{\text{Nyg}} \xrightarrow{\sim} \text{Spf}(S)^{\text{Nyg}}.$$

In particular, let  $S \rightarrow R_0$  be a chosen quasisyntomic cover, where  $R_0$  is qrsp. Then each term in the Čech conerve of  $S \rightarrow R_0$  is a qrsp algebra ([BMS18, Lem. 4.30]). By the finite limit preservation property of the functor  $X \mapsto X^{\text{Nyg}}$  (see [Bha23, Rmk. 5.5.18]), it follows that we have an isomorphism

$$(3.2.1) \quad \text{colim}_{[n] \in \Delta^{\text{op}}} \text{Spf}(R_0^{[n]})^{\text{Nyg}} \simeq \text{Spf}(S)^{\text{Nyg}},$$

where  $R_0^{[n]} := R_0^{\otimes n}$  is a qrsp algebra. Note that one defines

$$(3.2.2) \quad \text{Spf}(S)^{\text{syn}} := \text{coeq} \left( \text{Spf}(S)^\Delta \begin{array}{c} \xrightarrow{\text{can}} \\ \xrightarrow{\varphi} \end{array} \text{Spf}(S)^{\text{Nyg}} \right)$$

Therefore, using the cover  $S \rightarrow R_0$ , we have an isomorphism

$$(3.2.3) \quad \text{colim}_{[n] \in \Delta^{\text{op}}} \text{Spf}(R_0^{[n]})^{\text{syn}} \simeq \text{Spf}(S)^{\text{syn}}.$$

These covers by syntomification stacks associated to qrsp algebras will be used later in certain arguments.

**Remark 3.18.** The stacks  $(S)^\Delta$ ,  $(S)^{\text{Nyg}}$  and  $(S)^{\text{syn}}$  are related to the theory of (absolute) prismatic cohomology in the following way: there is a canonical line bundle on each of these stacks, denoted as  $\mathcal{O}\{i\}$ , and called the Breuil–Kisin twist (see Example 3.24) such that

$$\begin{aligned} R\Gamma(S^\Delta, \mathcal{O}\{i\}) &\simeq \Delta_S\{i\}, \\ R\Gamma(S^{\text{Nyg}}, \mathcal{O}\{i\}) &\simeq \text{Fil}_{\text{Nyg}}^i \Delta_S\{i\}, \end{aligned}$$

and

$$R\Gamma(S^{\text{syn}}, \mathcal{O}\{i\}) \simeq R\Gamma_{\text{syn}}(S, \mathbb{Z}_p(i)),$$

where the latter denotes syntomic cohomology of weight  $n$ . See Remark 3.39 for certain generalizations.

**Definition 3.19** (Quasi-coherent sheaves on  $p$ -adic formal stacks). Let  $X$  be a  $p$ -adic formal prestack. One defines the quasi-coherent derived  $\infty$ -category of  $X$  denoted as

$$D_{\text{qc}}(X) := \varinjlim_{(\text{Spec } T \rightarrow X) \in \mathcal{C}_X} D(T),$$

where  $D(T)$  is the derived category of the  $p$ -nilpotent animated ring  $T$ .

**Definition 3.20** (Prismatic  $F$ -gauges). Let  $S$  be a quasi-syntomic ring. One defines the stable  $\infty$ -category of prismatic  $F$ -gauges over  $S$  to be

$$F\text{-Gauge}_\Delta(S) := D_{\text{qc}}(\text{Spf}(S)^{\text{syn}}).$$

**Remark 3.21.** Note that by Remark 3.17, for a quasisyntomic ring  $S$ , specifying a quasicohherent sheaf on  $\text{Spf}(S)^{\text{syn}}$  is equivalent to producing a quasicohherent sheaf on  $\text{Spf}(R)^{\text{syn}}$  for every map  $S \rightarrow R$ , where  $R$  is qrsp, in a manner that is compatible with base change. Therefore, it follows that

$$F\text{-Gauge}_\Delta(S) \simeq \lim_{\substack{S \rightarrow R; \\ R \text{ is qrsp}}} F\text{-Gauge}_\Delta(R).$$

**Remark 3.22** (Explicit description of  $F$ -gauges). When  $R$  is qrsp, it is possible to describe the category  $F\text{-Gauge}_\Delta(R)$  completely algebraically. Let  $(\Delta_R, I)$  be the prism associated to  $R$ . The Nygaard filtered prismatic cohomology of  $R$  denoted as  $\text{Fil}_{\text{Nyg}}^\bullet \Delta_R$  is naturally a filtered ring. Note that the Frobenius on prismatic cohomology naturally defines a map  $\text{Fil}_{\text{Nyg}}^\bullet \Delta_R \rightarrow I^\bullet \Delta_R$  of filtered rings.

Now,  $F\text{-Gauge}_\Delta(R)$  is the  $\infty$ -category of  $(p, I)$ -complete  $\mathbb{Z}$ -indexed filtered modules  $\text{Fil}^\bullet M$  over  $\text{Fil}_{\text{Nyg}}^\bullet \Delta_R$  equipped with a  $\text{Fil}_{\text{Nyg}}^\bullet \Delta_R$ -linear map

$$\varphi : \text{Fil}^\bullet M \rightarrow I^\bullet \Delta_R \otimes_{\Delta_R} M =: I^\bullet M$$

(where the  $\text{Fil}_{\text{Nyg}}^\bullet \Delta_R$ -module structure on the right hand side is obtained by restriction of scalar and  $M$  is the  $(p, I)$ -completed underlying object of  $\text{Fil}^\bullet M$ ) with the property that the map

$$(3.2.4) \quad \text{Fil}^\bullet M \widehat{\otimes}_{\text{Fil}_{\text{Nyg}}^\bullet \Delta_R} I^\bullet \Delta_R \rightarrow I^\bullet M$$

associated via adjunction is an isomorphism of filtered  $I^\bullet \Delta_R$ -modules.

Combined with Remark 3.21, we obtain a purely algebraic description of  $F\text{-Gauge}_\Delta(S)$  for any quasisyntomic ring  $S$ .

**Example 3.23.** Let  $k$  be a perfect field of characteristic  $p$ . Then the category  $F\text{-Gauge}_{\Delta}(k)$  is equivalent to the category of “ $\varphi$ -gauges” introduced by Fontaine–Jannsen [FJ13, § 1.2].

**Example 3.24** (Breuil–Kisin twists). For any quasisyntomic ring  $S$ , the stack  $\text{Spf}(S)^{\text{syn}}$  is equipped with a line bundle  $\mathcal{O}\{1\}$  arising from the Breuil–Kisin twist in prismatic cohomology. Let  $\mathcal{O}\{n\} := \mathcal{O}\{1\}^{\otimes n}$  for  $n \in \mathbb{Z}$ . We explain an algebraic description of this in the case when  $S$  is qrsp, which glues to define the desired line bundle on  $\text{Spf}(S)^{\text{syn}}$ . For  $n \in \mathbb{Z}$ , the filtered  $\text{Fil}_{\text{Nyg}}^{\bullet} \Delta_S$ -module defined by setting

$$\text{Fil}^i M := \text{Fil}_{\text{Nyg}}^{i-n} \Delta_S \otimes_{\Delta_S} \Delta_S \{-n\}$$

equipped with the natural Frobenius map defines an object of  $F\text{-Gauge}_{\Delta}(S)$  that gives the Breuil–Kisin twist  $\mathcal{O}\{-n\}$ .

**Remark 3.25** (Effective  $F$ -gauges). When  $R$  is qrsp, an object of  $F\text{-Gauge}_{\Delta}(R)$  is called effective if the underlying  $\mathbb{Z}$ -filtered object  $\text{Fil}^{\bullet} M$  has the property that the natural maps  $\text{Fil}^i M \rightarrow \text{Fil}^{i-1} M$  are all isomorphisms for  $i \leq 0$ .

**Example 3.26.** Let  $X$  be a  $p$ -adic formal scheme (or a stack) over the qrsp algebra  $R$ . Then the filtered  $\text{Fil}_{\text{Nyg}}^{\bullet} \Delta_R$ -module  $\text{Fil}_{\text{Nyg}}^{\bullet} R\Gamma_{\Delta}(X)$  equipped with the natural Frobenius map has the structure of an effective  $F$ -gauge.

**Definition 3.27** (Weak  $F$ -gauges). Let  $R$  be a qrsp ring. The category of prismatic weak  $F$ -gauges over  $R$  denoted as  $F\text{-Gauge}_{\Delta}^{\text{w}}(R)$  is the  $\infty$ -category of  $(p, I)$ -complete  $\mathbb{Z}$ -indexed filtered modules  $\text{Fil}^{\bullet} M$  over  $\text{Fil}_{\text{Nyg}}^{\bullet} \Delta_R$  equipped with a  $\text{Fil}_{\text{Nyg}}^{\bullet} \Delta_R$ -linear map

$$\varphi : \text{Fil}^{\bullet} M \rightarrow I^{\bullet} \Delta_R \otimes_{\Delta_R} M =: I^{\bullet} M;$$

as before, the  $\text{Fil}_{\text{Nyg}}^{\bullet} \Delta_R$ -module structure on the right hand side is obtained by restriction of scalar and  $M$  is the  $(p, I)$ -completed underlying object of  $\text{Fil}^{\bullet} M$ . Note that the natural functor  $F\text{-Gauge}_{\Delta}(R) \rightarrow F\text{-Gauge}_{\Delta}^{\text{w}}(R)$  is fully faithful.

**Remark 3.28.** If  $\text{Fil}^{\bullet} M$  is a weak prismatic  $F$ -gauge over a qrsp ring  $R$  such that  $\text{Fil}^n M$  stabilizes for  $n \ll 0$ , then by passing to filtered colimits on both sides of the map

$$\varphi : \text{Fil}^{\bullet} M \rightarrow I^{\bullet} \Delta_R \otimes_{\Delta_R} M =: I^{\bullet} M,$$

we obtain a map

$$\varphi_M : (\varphi_{\Delta_R}^* M)[1/I] \rightarrow M[1/I].$$

If  $\text{Fil}^{\bullet} M$  is further assumed to be a prismatic  $F$ -gauge, then  $\varphi_M$  is an isomorphism.

**Remark 3.29** (Canonical  $t$ -structures). We call a prismatic weak  $F$ -gauge as above effective if the underlying  $\mathbb{Z}$ -filtered object  $\text{Fil}^{\bullet} M$  has the property that the maps  $\text{Fil}^i M \rightarrow \text{Fil}^{i-1} M$  are isomorphisms for  $i \leq 0$ . We note that the category of effective prismatic weak  $F$ -gauges (which is a stable  $\infty$ -category with all small limits and colimits) has the pleasant feature that it is canonically equipped with a  $t$ -structure, where an object is connective (resp. coconnective) if the underlying filtered module  $\text{Fil}^{\bullet} M$  has the property that  $\text{Fil}^i M$  is connective (resp. coconnective) for all  $i \in \mathbb{Z}$ .

**Remark 3.30** (Gauges as filtered crystals on the quasisyntomic site). Suppose that  $S$  is a quasisyntomic ring. In [Bha23], one defines the category of prismatic gauges over  $S$  to be  $D_{\text{qc}}(\text{Spf}(S)^{\text{Nyg}})$ . We will explain how to think of them in the quasisyntomic site of  $S$ . Let

$(S)_{\text{qrsp}}$  denote category of qrsp algebras over  $S$  equipped with the quasisyntomic topology, which, by [BMS19, Lem. 4.27] forms a site (also see [BMS19, Variant. 4.33]).

Note that the association

$$(S)_{\text{qrsp}} \ni R \mapsto \text{Fil}_{\text{Nyg}}^{\bullet} \Delta_R$$

defines a  $D(\mathbb{Z})$ -valued sheaf of  $\mathbb{Z}$ -indexed filtered rings on  $(S)_{\text{qrsp}}$ , which we denote by  $\text{Fil}^{\bullet} \Delta_{(\cdot)}$ . Also, the association  $(S)_{\text{qrsp}} \ni R \mapsto I$ , where  $I$  is the invertible ideal of  $\Delta_R$  defines an invertible sheaf of  $\Delta_{(\cdot)}$ -modules which we will denote by  $\mathcal{I}$ . Let  $\text{Gauge}_{\Delta}(S)$  be the  $\infty$ -category of derived  $(p, \mathcal{I})$ -complete sheaf of filtered modules  $\text{Fil}^{\bullet} \mathcal{M}$  over the sheaf of filtered ring  $\text{Fil}^{\bullet} \Delta_{(\cdot)}$  that satisfies the following ‘‘crystal’’ property: for every map  $R \rightarrow T$  in  $(S)_{\text{qrsp}}$ , the natural map  $\text{Fil}^{\bullet} \mathcal{M}(R) \rightarrow \text{Fil}^{\bullet} \mathcal{M}(T)$  of filtered  $\text{Fil}^{\bullet} \Delta_R$ -modules induces an isomorphism

$$(3.2.5) \quad \text{Fil}^{\bullet} \mathcal{M}(R) \widehat{\otimes}_{\text{Fil}^{\bullet} \Delta_R} \text{Fil}^{\bullet} \Delta_T \xrightarrow{\sim} \text{Fil}^{\bullet} \mathcal{M}(T)$$

of derived  $(p, \mathcal{I}(T))$ -complete  $\text{Fil}^{\bullet} \Delta_T$ -modules.

**Proposition 3.31.** *Let  $S$  be a quasisyntomic ring. Then  $D_{\text{qc}}(\text{Spf}(S)^{\text{Nyg}}) \simeq \text{Gauge}_{\Delta}(S)$ .*

*Proof.* Indeed, by [Bha23, Rmk. 5.5.4], if  $R$  is qrsp, an object of  $D_{\text{qc}}(\text{Spf}(R)^{\text{Nyg}})$  identifies with a derived  $(p, I)$ -complete filtered module over the filtered ring  $\text{Fil}_{\text{Nyg}}^{\bullet} \Delta_R$ . Now let  $\mathcal{F} \in D_{\text{qc}}(\text{Spf}(S)^{\text{Nyg}})$  and let  $R \in (S)_{\text{qrsp}}$ . The pullback of  $\mathcal{F}$  to  $\text{Spf}(R)^{\text{Nyg}}$  can be identified with a derived  $(p, I)$ -complete filtered module over the filtered ring  $\text{Fil}_{\text{Nyg}}^{\bullet} \Delta_R$ , that we denote as  $\text{Fil}^{\bullet} \mathcal{M}_R$ . Using the fact that  $X \mapsto X^{\text{Nyg}}$  preserves finite limits and sends quasisyntomic covers to covers of  $p$ -adic formal stacks (equivalently, one can use (1) and (2) from [GL23, Prop. 2.29]), it follows that the association  $(S)_{\text{qrsp}} \ni R \mapsto \text{Fil}^{\bullet} \mathcal{M}_R$  defines a derived  $(p, \mathcal{I})$ -complete  $D(\mathbb{Z})$ -valued sheaf that satisfies the crystal property. This determines a functor  $D_{\text{qc}}(\text{Spf}(S)^{\text{Nyg}}) \rightarrow \text{Gauge}_{\Delta}(S)$ .

Conversely, given  $\text{Fil}^{\bullet} \mathcal{M} \in \text{Gauge}_{\Delta}(S)$  and  $R \in (S)_{\text{qrsp}}$ , the filtered module  $\text{Fil}^{\bullet} \mathcal{M}(R)$  over the filtered ring  $\text{Fil}_{\text{Nyg}}^{\bullet} \Delta_R$  determines an object  $\mathcal{F}_R$  of  $D_{\text{qc}}(\text{Spf}(R)^{\text{Nyg}})$ . The crystal property of  $\text{Fil}^{\bullet} \mathcal{M}$  ensures that the construction  $R \mapsto \mathcal{F}_R$  is compatible with (completed) base change for maps  $R \rightarrow R'$  in  $(S)_{\text{qrsp}}$ . Since  $D_{\text{qc}}(\text{Spf}(S)^{\text{Nyg}}) \simeq \lim_{R \in (S)_{\text{qrsp}}} D_{\text{qc}}(\text{Spf}(R)^{\text{Nyg}})$  (see [Bha23, Rmk. 5.5.18]), the association  $R \mapsto \mathcal{F}_R$  determines an object of  $D_{\text{qc}}(\text{Spf}(S)^{\text{Nyg}})$ . This determines a functor  $\text{Gauge}_{\Delta}(S) \rightarrow D_{\text{qc}}(\text{Spf}(S)^{\text{Nyg}})$  that is inverse to the functor described in previous paragraph, which proves the proposition.  $\square$

**Remark 3.32** ( $F$ -gauges as filtered Frobenius crystals on the quasisyntomic site). Let  $S$  be a quasisyntomic ring. We work on the site  $(S)_{\text{qrsp}}$  as before. Note that the category of prismatic  $F$ -gauges over  $S$  can be equivalently described as the  $\infty$ -category of derived  $(p, \mathcal{I})$ -complete sheaf of  $\mathbb{Z}$ -indexed filtered modules  $\text{Fil}^{\bullet} \mathcal{N}$  over the sheaf of filtered ring  $\text{Fil}^{\bullet} \Delta_{(\cdot)}$  on  $(S)_{\text{qrsp}}$  equipped with a  $\text{Fil}^{\bullet} \Delta_{(\cdot)}$ -linear Frobenius map

$$\varphi : \text{Fil}^{\bullet} \mathcal{N} \rightarrow \mathcal{I}^{\bullet} \Delta_{(\cdot)} \otimes_{\Delta_{(\cdot)}} \mathcal{N} =: \mathcal{I}^{\bullet} \mathcal{N}$$

(where the  $\text{Fil}^{\bullet} \Delta_{(\cdot)}$ -module structure on the right hand side is obtained by restriction of scalar along the Frobenius map  $\text{Fil}^{\bullet} \Delta_{(\cdot)} \rightarrow \mathcal{I}^{\bullet} \Delta_{(\cdot)}$  and  $\mathcal{N}$  denotes  $(p, \mathcal{I})$ -completed underlying object of  $\mathcal{N}$ ) such that the following two conditions hold:

- (1)  $\text{Fil}^{\bullet} \mathcal{N}$  satisfies the ‘‘crystal’’ property as in Remark 3.30, (3.2.5).
- (2) The natural map  $\text{Fil}^{\bullet} \mathcal{N} \widehat{\otimes}_{\text{Fil}^{\bullet} \Delta_{(\cdot)}} \mathcal{I}^{\bullet} \Delta_{(\cdot)} \rightarrow \mathcal{I}^{\bullet} \mathcal{N}$  associated to  $\varphi$  via adjunction is an isomorphism of  $\mathcal{I}^{\bullet} \Delta_{(\cdot)}$ -modules.

The equivalence of the above description with  $D_{\text{qc}}(S^{\text{syn}})$  follows from Remark 3.17 and Remark 3.22. A prismatic  $F$ -gauge  $\text{Fil}^\bullet \mathcal{N}$  is called effective if the natural maps  $\text{Fil}^i \mathcal{N} \rightarrow \text{Fil}^{i-1} \mathcal{N}$  are isomorphisms for all  $i \leq 0$  and the full subcategory spanned by such objects will be denoted by  $F\text{-Gauge}_\Delta(S)^{\text{eff}}$ .

**Remark 3.33.** We record a variant of the category of prismatic  $F$ -gauges. Let  $S$  be a quasisyntomic ring. We work on the site  $(S)_{\text{qrsp}}$  as before. We define the  $\infty$ -category  $F\text{-Mod}_{\text{Fil}^\bullet \Delta}(S)$  to be the  $\infty$ -category of derived  $(p, \mathcal{I})$ -complete sheaf of  $\mathbb{Z}$ -indexed filtered modules  $\text{Fil}^\bullet \mathcal{N}$  over the sheaf of filtered ring  $\text{Fil}^\bullet \Delta_{(\cdot)}$  on  $(S)_{\text{qrsp}}$  equipped with a  $\text{Fil}^\bullet \Delta_{(\cdot)}$ -linear Frobenius map

$$\varphi : \text{Fil}^\bullet \mathcal{N} \rightarrow \mathcal{I}^\bullet \Delta_{(\cdot)} \otimes_{\Delta_{(\cdot)}} \mathcal{N} =: \mathcal{I}^\bullet \mathcal{N}.$$

By definition, there is a fully faithful functor  $F\text{-Gauge}_\Delta(S) \rightarrow F\text{-Mod}_{\text{Fil}^\bullet \Delta}(S)$ . Let us denote by  $F\text{-Mod}_{\text{Fil}^\bullet \Delta}(S)^{\text{eff}}$  the full subcategory spanned by  $\text{Fil}^\bullet \mathcal{N} \in F\text{-Mod}_{\text{Fil}^\bullet \Delta}(S)$  with the property that the natural maps  $\text{Fil}^i \mathcal{N} \rightarrow \text{Fil}^{i-1} \mathcal{N}$  are isomorphisms for all  $i \leq 0$ . Note that  $F\text{-Mod}_{\text{Fil}^\bullet \Delta}(S)^{\text{eff}}$  is a stable  $\infty$ -category with all small limits and colimits.

**Remark 3.34.** Let  $S$  be a quasisyntomic ring. Let  $\mathcal{F} \in \text{Shv}_{D(\mathbb{Z})}(S_{\text{qrsp}})$ . Then the association

$$\mathcal{F} \mapsto R\mathcal{H}om_{(S)_{\text{qrsp}}}(\mathcal{F}, \text{Fil}^\bullet \Delta_{(\cdot)})$$

determines a functor

$$\text{Shv}_{D(\mathbb{Z})}(S_{\text{qrsp}}) \rightarrow (F\text{-Mod}_{\text{Fil}^\bullet \Delta}(S)^{\text{eff}})^{\text{op}}$$

that preserves colimits. By the adjoint functor theorem, it must have a right adjoint. In this case, the right adjoint is explicitly determined by the functor that sends

$$(F\text{-Mod}_{\text{Fil}^\bullet \Delta}(S)^{\text{eff}})^{\text{op}} \ni \text{Fil}^\bullet \mathcal{N} \mapsto R\mathcal{H}om_{\text{syn}}(\text{Fil}^\bullet \mathcal{N}, \text{Fil}^\bullet \Delta_{(\cdot)}),$$

where  $R\mathcal{H}om_{\text{syn}}(\text{Fil}^\bullet \mathcal{N}, \text{Fil}^\bullet \Delta_{(\cdot)})$  is the  $D(\mathbb{Z})$ -valued sheaf on  $(S)_{\text{qrsp}}$  determined by

$$(S)_{\text{qrsp}} \ni R \mapsto R\mathcal{H}om_{F\text{-Mod}_{\text{Fil}^\bullet \Delta}(R)}(\text{Fil}^\bullet \mathcal{N}|_{(R)_{\text{qrsp}}}, \text{Fil}^\bullet \Delta_{(\cdot)}|_{(R)_{\text{qrsp}}}).$$

**Definition 3.35** (Hodge–Tate weights). Let us suppose that  $S$  is a quasiregular semiperfectoid algebra. We have a natural map of graded rings

$$\bigoplus_{i \in \mathbb{Z}} \text{Fil}_{\text{Nyg}}^i \Delta_S \rightarrow S,$$

where  $S$  is viewed as a graded ring sitting in degree zero, which is obtained by quotienting the left hand side by the graded ideal

$$I := \left( \bigoplus_{i \neq 0} \text{Fil}_{\text{Nyg}}^i \Delta_S \right) \oplus \text{Fil}_{\text{Nyg}}^1 \Delta_S,$$

where the summand  $\text{Fil}_{\text{Nyg}}^1 \Delta_S$  on the right has weight zero. This defines a map

$$\text{Spf}(S) \times B\mathbb{G}_m \rightarrow \text{Spf}(S)^{\text{Nyg}}.$$

By quasisyntomic descent, this defines a map

$$(3.2.6) \quad j : \text{Spf}(S) \times B\mathbb{G}_m \rightarrow \text{Spf}(S)^{\text{Nyg}}$$

for any quasisyntomic ring  $S$  (cf. [Bha23, Remark 5.3.14]). For any  $M \in D_{\text{qc}}(\text{Spf}(S)^{\text{Nyg}})$ , the pullback  $j^* M$  can be identified with  $\bigoplus_{i \in \mathbb{Z}} M_i \in D_{\text{qc}}(\text{Spf}(S))$ . The set of integers  $i$  such

that  $M_i \neq 0$  is called the set of Hodge–Tate weights of  $M$ . For  $M \in D_{\text{qc}}(\text{Spf}(S)^{\text{syn}})$ , the set of Hodge–Tate weights of  $M$  is defined as the set of Hodge–Tate weights of the pullback of  $M$  along  $\text{Spf}(S)^{\text{Nyg}} \rightarrow \text{Spf}(S)^{\text{syn}}$ .

**Remark 3.36.** Let  $S$  be a qrsp algebra. We will describe an explicit way to understand pullback of an object in  $\mathcal{F} \in D_{\text{qc}}(\text{Spf}(S)^{\text{Nyg}})$  along the map  $j$  from (3.2.6) (see [GL23, Cons. 2.41]). By Proposition 3.31, we may identify  $\mathcal{F}$  with a filtered module  $\text{Fil}^\bullet M$  over the filtered ring  $\text{Fil}_{\text{Nyg}}^\bullet \Delta_S$ . Passing to graded pieces, we obtain a graded module  $\text{gr}^\bullet M$  over the graded ring  $\text{gr}_{\text{Nyg}}^\bullet \Delta_S$ . Using the isomorphism  $\text{gr}_{\text{Nyg}}^0 \Delta_S \simeq S$ , we obtain a natural map  $\text{gr}_{\text{Nyg}}^\bullet \Delta_S \rightarrow S$  of graded rings. In this situation, by construction, it follows that  $j^* \mathcal{F}$  corresponds to the graded  $S$ -module

$$(3.2.7) \quad \text{gr}^\bullet M \otimes_{\text{gr}_{\text{Nyg}}^\bullet \Delta_S} S.$$

**Remark 3.37.** By Definition 3.35, the Breuil–Kisin twist  $\mathcal{O}\{-n\}$  has Hodge–Tate weight  $n$ .

**Remark 3.38.** Suppose that  $S$  is a quasiregular semiperfectoid algebra. Let  $\text{Fil}^\bullet M$  (equipped with a Frobenius) denote a prismatic  $F$ -gauge over  $S$  in the sense of Remark 3.22. Then the filtered module underlying the prismatic  $F$ -gauge  $\text{Fil}^\bullet M\{n\} := \text{Fil}^\bullet M \otimes \mathcal{O}\{n\}$  is given by

$$\text{Fil}^i M\{n\} \simeq \Delta_S\{n\} \otimes_{\Delta_S} \text{Fil}^{i+n} M.$$

**Remark 3.39** (Syntomic cohomology of coefficients). Suppose that  $S$  is a quasiregular semiperfectoid algebra. Let  $\text{Fil}^\bullet M$  (equipped with a Frobenius) denote a prismatic  $F$ -gauge over  $S$  in the sense of Remark 3.22 and let  $\mathcal{M}$  be the object of  $D_{\text{qc}}(\text{Spf}(S)^{\text{syn}})$  corresponding to it. Let  $\mathcal{M}\{n\} := \mathcal{M} \otimes \mathcal{O}\{n\}$ .

For simplicity, let us assume that  $\text{Fil}^\bullet M$  is effective. Let  $\mathcal{M}'\{n\}$  denote the pullback of  $\mathcal{M}\{n\}$  to  $\text{Spf}(S)^{\text{Nyg}}$ . Then, by Remark 3.38,

$$(3.2.8) \quad R\Gamma(\text{Spf}(S)^{\text{Nyg}}, \mathcal{M}'\{n\}) \simeq \Delta_S\{n\} \otimes_{\Delta_S} \text{Fil}^n M =: \text{Fil}^n M\{n\}.$$

Also, we have  $R\Gamma(\text{Spf}(S)^\Delta, \text{can}^* \mathcal{M}'\{n\}) \simeq \text{Fil}^0 M\{n\}$ . Moreover, we have an induced map  $i : \text{Fil}^n M\{n\} \rightarrow \text{Fil}^0 M\{n\}$ , which is simply the canonical map arising from the filtration. Now, we also have the map  $\varphi : \text{Spf}(S)^\Delta \rightarrow \text{Spf}(S)^{\text{Nyg}}$ , which induces a map

$$(3.2.9) \quad R\Gamma(\text{Spf}(S)^{\text{Nyg}}, \mathcal{M}'\{n\}) \rightarrow R\Gamma(\text{Spf}(S)^\Delta, \varphi^* \mathcal{M}'\{n\}).$$

However, since  $\mathcal{M}'\{n\}$  by construction descends to  $\text{Spf}(S)^{\text{syn}}$ , we have an isomorphism  $\varphi^* \mathcal{M}'\{n\} \simeq \text{can}^* \mathcal{M}'\{n\}$ . Combining this with (3.2.8) and (3.2.9), we obtain a map

$$(3.2.10) \quad \nu_n : \text{Fil}^n M\{n\} \rightarrow \text{Fil}^0 M\{n\}.$$

One may think of the above map as a certain “divided Frobenius” for coefficients. In this set up, by (3.2.2), we have

$$(3.2.11) \quad R\Gamma(\text{Spf}(S)^{\text{syn}}, \mathcal{M}\{n\}) \simeq \text{fib} \left( \text{Fil}^n M\{n\} \xrightarrow{\nu_n} \text{Fil}^0 M\{n\} \right).$$

We now move on to discussing some examples and a concrete description of prismatic  $F$ -gauges with Hodge–Tate weights on  $[0, 1]$  (see Proposition 3.45, cf. [GL23, Thm. 2.54], [GMM24, § 5.27]).



**Example 3.40.** Let  $S$  be a quasiregular semiperfectoid ring. In this case, the filtered object  $H^2(\mathrm{Fil}_{\mathrm{Nyg}}^\bullet R\Gamma_\Delta(B\mathbb{G}_m))$  defines an effective prismatic weak  $F$ -gauge over  $S$ . We will prove that the latter prismatic weak  $F$ -gauge is isomorphic to the prismatic  $F$ -gauge  $\mathcal{O}\{-1\}$ . Let us also choose a map  $R \rightarrow S$ , where  $R$  is a perfectoid ring mapping onto  $S$ . It would suffice to prove that the natural map

$$\bigoplus_{i \in \mathbb{Z}} \mathrm{Fil}_{\mathrm{Nyg}}^{i-1} \Delta_S \{i-1\} \rightarrow \bigoplus_{i \in \mathbb{Z}} H^2(\mathrm{Fil}_{\mathrm{Nyg}}^i R\Gamma_\Delta(B\mathbb{G}_m) \{i\})$$

of graded of modules over the graded ring  $\bigoplus_{i \in \mathbb{Z}} \mathrm{Fil}_{\mathrm{Nyg}}^i \Delta_S \{i\}$  induced by the tautological Chern class  $c_1 \in H^2(\mathrm{Fil}_{\mathrm{Nyg}}^1 R\Gamma_\Delta(B\mathbb{G}_m) \{1\})$  is an isomorphism. It suffices to prove that the map on each component

$$\mathrm{Fil}_{\mathrm{Nyg}}^{i-1} \Delta_S \{i-1\} \rightarrow H^2(\mathrm{Fil}_{\mathrm{Nyg}}^i R\Gamma_\Delta(B\mathbb{G}_m) \{i\})$$

is an isomorphism. By reducing to Hodge–Tate cohomology and using the fact that  $\mathbb{L}_{B\mathbb{G}_m/S} = \mathcal{O}[-1]$ , we can deduce that the above map is an isomorphism for  $i = 0$ . From now on in this proof, we omit the Breuil–Kisin twists. By using the fact  $\mathrm{gr}_{\mathrm{Nyg}}^i R\Gamma_\Delta(B\mathbb{G}_m) \simeq \mathrm{Fil}_{\mathrm{conj}}^i R\Gamma_{\overline{\Delta}}(B\mathbb{G}_m)$ , where the latter denotes  $i$ -th conjugate conjugate filtration on absolute Hodge–Tate cohomology. Using the Nygaard filtration and induction on  $i$ , we would be done if we prove that the induced map

$$\mathrm{Fil}_{\mathrm{conj}}^{i-1} \overline{\Delta}_S \rightarrow H^2(\mathrm{Fil}_{\mathrm{conj}}^i R\Gamma_{\overline{\Delta}}(B\mathbb{G}_m))$$

is an isomorphism. To check this, we use the fact that  $\mathrm{gr}_{\mathrm{conj}}^i R\Gamma_{\overline{\Delta}}(B\mathbb{G}_m) \simeq \wedge^i \mathbb{L}_{B\mathbb{G}_m/R}[-i]$ , and the isomorphism  $\mathbb{L}_{B\mathbb{G}_m/R} \simeq \mathbb{L}_{S/R} \oplus \mathcal{O}[-1]$ . The latter implies that

$$\wedge^i \mathbb{L}_{B\mathbb{G}_m/R}[-i] \simeq \bigoplus_{m \geq 0} \wedge^{i-m} \mathbb{L}_{S/R}[m-i] [-2m] \simeq \bigoplus_{m \geq 0} \mathrm{gr}_{\mathrm{conj}}^{i-m} \overline{\Delta}_S[-2m],$$

which yields the desired claim.

**Example 3.41** (Breuil–Kisin modules). Let  $R$  be a qrsp ring and let  $(\Delta_R, \varphi, I)$  be the associated prism. Let  $M$  be a finite rank projective module over  $\Delta_R$  equipped with an isomorphism

$$\varphi_M : (\varphi^* M)[1/I] \xrightarrow{\sim} M[1/I]$$

or  $\Delta_R$ -modules. This data is also called a Breuil–Kisin module over  $\Delta_R$ . We will denote the category of Breuil–Kisin modules by  $\mathrm{BK}(\Delta_R)$ .

For  $i \in \mathbb{Z}$ , let us define

$$\mathrm{Fil}^i M := \{m \in M \mid \varphi_M(m) \in I^i M\}.$$

Then  $\mathrm{Fil}^\bullet M$  is a filtered module over  $\mathrm{Fil}_{\mathrm{Nyg}}^\bullet \Delta_R$ . Since  $M$  is projective of finite rank, it follows that there exists some  $n \in \mathbb{Z}$  such that  $\varphi_M(M) \subseteq I^n M$ . In particular,  $\mathrm{Fil}^i M$  stabilizes for  $i \ll 0$  and the underlying object of  $M^\bullet$  is naturally isomorphic to  $M$ . Further, we have a  $\mathrm{Fil}_{\mathrm{Nyg}}^\bullet \Delta_R$ -linear map

$$\tilde{\varphi}_M : \mathrm{Fil}^\bullet M \rightarrow I^\bullet M.$$

This gives a fully faithful functor (see Remark 3.28)

$$(3.2.12) \quad \mathrm{BK}(\Delta_R) \rightarrow F\text{-Gauge}_\Delta^w(R)$$

In general, this functor does not factor through the inclusion  $F\text{-Gauge}_\Delta(R) \rightarrow F\text{-Gauge}_\Delta^w(R)$ .

**Example 3.42.** Let  $R$  be a qrsp ring. Let us denote by  $F\text{-Gauge}_{\Delta}^{\text{vect}}(R)$  the category  $\text{Vect}(\text{Spf}(R)^{\text{syn}})$  of vector bundles on  $\text{Spf}(R)^{\text{syn}}$ . By Remark 3.28, one obtains a functor

$$(3.2.13) \quad \text{BK} : F\text{-Gauge}_{\Delta}^{\text{vect}}(R) \rightarrow \text{BK}(\Delta_R).$$

By [GL23, Cor. 2.53], this functor is fully faithful.

**Remark 3.43.** Let  $R$  be a qrsp ring and let  $\text{DM}^{\text{adm}}(R) \subset \text{BK}(\Delta_R)$  be the full subcategory of admissible prismatic Dieudonné modules defined in [ALB23, Def. 1.3.5] which we recall. An object  $(M, \varphi_M) \in \text{BK}(\Delta_R)$  is an admissible prismatic Dieudonné module if the following two conditions are satisfied.

- (1) The map  $\varphi_M|_{\varphi^*M}$  has its image contained in  $M$  and cokernel of the induced map  $\varphi^*M \rightarrow M$  is killed by  $I$ .
- (2) The composition  $M \xrightarrow{\varphi_M|_M} M \rightarrow M/IM$  is a finite locally free  $\Delta_R/\text{Fil}_{\text{Nyg}}^1\Delta_R \simeq R$ -module  $F_M$  such that the induced map  $\bar{\Delta}_R \otimes_R F_M \rightarrow M/IM$  is a monomorphism.

Let  $\text{Fil}^\bullet M$  be the prismatic weak  $F$ -gauge associated to  $(M, \varphi_M) \in \text{DM}^{\text{adm}}(R)$  by the functor (3.2.12). In this situation, it follows that one can write  $M \simeq W \oplus T$  for two projective  $\Delta_R$ -modules  $W$  and  $T$  such that we have  $\text{Fil}^1 M \simeq (\text{Fil}_{\text{Nyg}}^1\Delta_R \otimes W) \oplus T$  (see proof of [ALB23, Lem. 4.1.23]). One may define a filtration

$$(3.2.14) \quad F^i M := (\text{Fil}_{\text{Nyg}}^i\Delta_R \otimes W) \oplus (\text{Fil}_{\text{Nyg}}^{i-1}\Delta_R \otimes T),$$

which can be naturally equipped with the structure of a prismatic weak  $F$ -gauge denoted as  $F^\bullet M$ .

We will now check that the prismatic weak  $F$ -gauge  $F^\bullet M$  is actually a prismatic  $F$ -gauge. We need to check that the Frobenius map

$$F^\bullet M \widehat{\otimes}_{\text{Fil}_{\text{Nyg}}^\bullet\Delta_R} I^\bullet\Delta_R \rightarrow I^\bullet M$$

is an isomorphism. In our case, one can directly compute the filtered tensor product appearing on the left hand side of the above map and see that this amounts to checking that the map

$$\theta_n : (W \otimes_{\Delta_R, \varphi} I^n\Delta_R) \oplus (T \otimes_{\Delta_R, \varphi} I^{n-1}\Delta_R) \rightarrow I^n M$$

is an isomorphism for all  $n \in \mathbb{Z}$ . Note that these maps are injective by construction, so it suffices to check that they are surjective. Since  $I$  is a principal ideal generated by a nonzerodivisor, by scaling, it suffices to show the surjectivity  $\theta_1$ . Note that the natural map  $u : \text{Fil}^1 M \otimes_{\Delta_R, \varphi} \Delta_R \rightarrow IM$  factors canonically through  $\theta_1$ . Thus, it suffices to show that  $u$  is surjective. But that follows from [ALB23, Rmk. 4.1.7] since  $(M, \varphi_M)$  was assumed to be admissible.

Note that there are natural injective maps  $F^i M \rightarrow \text{Fil}^i M$ , which are isomorphisms. Indeed, to see this isomorphism, it suffices to show that the induced map  $F^i M/F^{i+1} M \rightarrow I^i M/I^{i+1} M$  is injective, which follows from the fact that  $W$  and  $T$  are projective modules (cf. [GL23, Cor. 2.53]).

In summary, we see that Remark 3.28 refines to a fully faithful functor

$$(3.2.15) \quad G'_\Delta : \text{DM}^{\text{adm}}(R) \rightarrow F\text{-Gauge}_{\Delta}^{\text{vect}}(R).$$

**Remark 3.44.** Let  $M \in \text{DM}^{\text{adm}}(R)$ . Using the explicit description in (3.2.14) of  $G'_\Delta(M)$  as a filtered module over  $\text{Fil}^\bullet\Delta_S$ , and Remark 3.36 (3.2.7), one sees that the pullback  $j^*G'_\Delta(M)$  identifies with the graded module  $W_0 \oplus T_0$ , where  $W_0 := W \otimes_{\Delta_R} R$  and  $T_0 := T \otimes_{\Delta_R} R$ ,

and  $W_0$  (resp.  $T_0$ ) is in degree 0 (resp. degree 1). Therefore,  $G'_\Delta(M)$  has Hodge–Tate weights in  $[0, 1]$  and it follows that  $G'_\Delta$  refines to a functor

$$(3.2.16) \quad G_\Delta : \mathrm{DM}^{\mathrm{adm}}(R) \rightarrow F\text{-Gauge}_{[0,1]}^{\mathrm{vect}}(R),$$

where the latter denotes the full subcategory of  $F\text{-Gauge}_{[0,1]}^{\mathrm{vect}}(R)$  spanned by objects of Hodge–Tate weights in  $[0, 1]$ .

**Proposition 3.45.** *The functor  $G_\Delta : \mathrm{DM}^{\mathrm{adm}}(R) \rightarrow F\text{-Gauge}_{[0,1]}^{\mathrm{vect}}(R)$  from (3.2.16) is an equivalence.*

*Proof.* It suffices to show that for  $M \in F\text{-Gauge}_{[0,1]}^{\mathrm{vect}}(R)$ , the Breuil–Kisin module  $Q := \mathrm{BK}(M)$  (3.2.13) is an object of  $\mathrm{DM}^{\mathrm{adm}}(R)$ . Using Remark 3.36 (3.2.7), it follows that since  $M$  has Hodge–Tate weights in  $\geq 0$ , the  $F$ -gauge  $M$  must be effective. This implies that the Frobenius  $\varphi_Q$  on  $Q$  refines to a map  $\varphi^*Q \rightarrow Q$ . Further, since  $M$  has Hodge–Tate weights in  $[0, 1]$ , the pullback  $j^*M$  identifies with a graded module  $W_0 \oplus T_0$ , where  $W_0, T_0$  are projective  $R$ -modules in degree 0 and 1, respectively. Let us use  $\mathrm{Fil}^\bullet M$  to denote the filtered  $\mathrm{Fil}_{\mathrm{Nyg}}^\bullet \Delta_R$  module underlying  $M$ . We set  $N^\bullet := \mathrm{gr}^\bullet M$ , which is a (locally free) graded module over  $\mathrm{gr}_{\mathrm{Nyg}}^\bullet \Delta_R \simeq \mathrm{Fil}_{\mathrm{conj}}^\bullet \bar{\Delta}_R \{\bullet\}$ . By Remark 3.36,

$$(3.2.17) \quad \mathrm{gr}^\bullet M \otimes_{\mathrm{gr}_{\mathrm{Nyg}}^\bullet \Delta_R} R \simeq W_0 \oplus T_0.$$

One can view  $P^\bullet := N^\bullet \{-\bullet\}$  as a filtered object, and because  $N^\bullet$  is locally free as a graded module over  $\mathrm{gr}_{\mathrm{Nyg}}^\bullet \Delta_R$ , it follows that the canonical maps  $P^i \rightarrow P^{i+1}$  are injective. Further, the underlying object of  $P^\bullet$ , denoted by  $P^\infty := \mathrm{colim} P_i$  can be viewed as a projective  $\bar{\Delta}_R$ -module. By (3.2.17), and the fact that  $T_0$  is locally free, one can choose maps

$$(3.2.18) \quad (\mathrm{gr}_{\mathrm{Nyg}}^i \Delta_R \otimes_R W_0) \oplus (\mathrm{gr}_{\mathrm{Nyg}}^{i-1} \Delta_R \otimes_R T_0) \rightarrow N^i.$$

By (3.2.17), it also follows that these maps are surjective. Passing to colimit, we obtain a surjective map

$$(3.2.19) \quad (\bar{\Delta}_R \otimes_R W_0) \oplus (\bar{\Delta}_R \{-1\} \otimes_R T_0) \rightarrow P^\infty.$$

Since both sides are projective modules of the same rank, the map (3.2.19) must be an isomorphism. This implies that the maps (3.2.18) are also isomorphisms.

Let  $M^u$  be the underlying object of the  $F$ -gauge  $M$ . Note that we have  $M^u = \mathrm{Fil}^0 M$ , since we have shown that  $M$  is effective. There is a natural surjective map  $M^u \otimes_{\Delta_R} R \rightarrow \mathrm{gr}^1 M \simeq W_0$ , whose kernel is a projective module surjecting onto  $T_0$ ; since the kernel and  $T_0$  must have the same rank, we obtain an exact sequence

$$0 \rightarrow T_0 \rightarrow M^u \otimes_{\Delta_R} R \rightarrow W_0 \rightarrow 0.$$

By projectivity, we may split this sequence as  $M^u \otimes_{\Delta_R} R \simeq W_0 \oplus T_0$ . Since  $\Delta_R \rightarrow R$  is henselian (see [ALB23, Lem. 4.1.28]), we can lift it to an isomorphism  $M^u \simeq W \oplus T$ , where  $W$  and  $T$  are projective  $\Delta_R$ -modules lifting  $W_0$  and  $T_0$ . This implies that  $\mathrm{Fil}^1 M = (\mathrm{Fil}_{\mathrm{Nyg}}^1 \Delta_R \otimes_{\Delta_R} W) \oplus T$ . Therefore, we can define filtered maps

$$(3.2.20) \quad (\mathrm{Fil}_{\mathrm{Nyg}}^i \Delta_R \otimes_{\Delta_R} W) \oplus (\mathrm{Fil}_{\mathrm{Nyg}}^{i-1} \Delta_R \otimes_{\Delta_R} T) \rightarrow \mathrm{Fil}^i M.$$

Since the above maps induce isomorphism on the underlying objects and the maps in (3.2.18) are isomorphisms, it follows that the maps (3.2.20) are isomorphisms too. The

condition (3.2.4) of  $M$  being an  $F$ -gauge now translates to the map

$$\theta_n : (W \otimes_{\Delta_R, \varphi} I^n \Delta_R) \oplus (T \otimes_{\Delta_R, \varphi} I^{n-1} \Delta_R) \rightarrow I^n M$$

being isomorphism for all  $n \in \mathbb{Z}$ . In particular,  $\theta_1$  is surjective. This implies that cokernel of the map  $\varphi^* Q \rightarrow N$  is killed by  $I$ , where  $Q = \text{BK}(M)$ . To check that  $Q$  is an admissible prismatic Dieudonné module, one needs to further verify that the image of the composition  $Q \xrightarrow{\varphi_Q} Q \rightarrow Q/IQ$  is a finite locally free  $R \simeq \text{gr}_{\text{Nyg}}^0 \Delta_R$ -module  $F_Q$  and the induced map  $\bar{\Delta}_R \otimes_R F_Q \rightarrow Q/IQ$  is an injection. Both of these follow from the fact that  $F_Q$  identifies with  $\text{gr}^0 M = N^0 \simeq W_0$  as an  $R$ -module, and the map (3.2.19) is an isomorphism, where  $P^\infty$  further identifies with  $M^u/IM^u \simeq Q/IQ$ .  $\square$

**3.3. Prismatic Dieudonné theory.** In this section, we begin by explaining how to reinterpret the work of [ALB23] in terms of prismatic  $F$ -gauges. More specifically, we define the prismatic Dieudonné  $F$ -gauge of a  $p$ -divisible group over a quasisyntomic ring. After that, we will define the prismatic Dieudonné  $F$ -gauge of a finite locally free group scheme of  $p$ -power rank and prove fully faithfulness. Let us recall the following definition:

**Definition 3.46.** For a  $p$ -divisible group  $G = \{G_n\}$ , the classifying stack of  $G$  denoted as  $BG$  is defined to be  $\varinjlim BG_n$  taken in the category of stacks.

Our main construction in this section can be stated in terms of the following definition:

**Definition 3.47.** Let  $S$  be a quasiregular semiperfectoid ring. Let  $G$  be a  $p$ -divisible group over  $S$ . We define  $\mathcal{M}(G)$  to be the effective prismatic weak  $F$ -gauge over  $S$  obtained from  $H^2(\mathrm{Fil}_{\mathrm{Nyg}}^\bullet R\Gamma_{\Delta}(BG))$  (see Remark 3.29).

We will prove that  $\mathcal{M}(G)$  is in fact a prismatic  $F$ -gauge and moreover, a vector bundle when viewed as a quasi-coherent sheaf on  $\mathrm{Spf}(S)^{\mathrm{syn}}$ . To this end, we will begin by an understanding of the cotangent complex of the classifying stack of a  $p$ -divisible group (cf. [Ill85]). At first, we assume that  $p^N S = 0$ , and we consider an  $n$ -truncated Barsotti–Tate group  $G$  for  $n \geq N$  over  $S$ . Let  $\ell_G$  denote the co-Lie complex of  $G$ , which is a perfect complex of  $S$ -modules with Tor amplitude in  $[0, 1]$ . Its dual  $\ell_G^\vee$  is called the lie complex. We let  $\omega_G := H^0(\ell_G)$ , which is a finite locally free  $S$ -module, whose rank is called the dimension of  $G$ .

By a result of Grothendieck [MM74, App.], one has  $\ell_G^\vee = \tau_{\geq -1} R\mathcal{H}om_{\mathbb{Z}}(G^\vee, \mathbb{G}_a)$ . Note that there is a natural map  $\phi_n : \mathcal{E}xt^1(G^\vee, \mathbb{G}_a) \rightarrow \mathcal{H}om(G^\vee, \mathbb{G}_a)$  obtained as follows (cf. [Ill85, 2.2.3, 2.2.4]): an element of  $\mathcal{E}xt^1(G^\vee, \mathbb{G}_a)$  determines a map  $u : G^\vee \rightarrow \mathbb{G}_a[1]$ . Applying  $(\cdot) \otimes_{\mathbb{Z}}^L \mathbb{Z}/p^n$  and noting that  $G^\vee$  is killed by  $p^n$ , we obtain a map  $G^\vee[1] \otimes G^\vee \rightarrow \mathbb{G}_a[2] \oplus \mathbb{G}_a[1]$ ; applying  $\pi_1$  gives the desired map  $\phi_n$ . If  $G$  is an  $n$ -truncated Barsotti–Tate group, then  $\phi_n$  is an isomorphism. Suppose now that  $G = \{G_n\}$  is a  $p$ -divisible group over  $S$ . Let  $f : G_{n+1} \rightarrow G_n$  be any map of group schemes. By construction, we have a commutative diagram

$$\begin{array}{ccc} \mathcal{E}xt^1(G_n, \mathbb{G}_a) & \xrightarrow{\phi_n} & \mathcal{H}om(G_n, \mathbb{G}_a) \\ \downarrow f & & \downarrow pf \\ \mathcal{E}xt^1(G_{n+1}, \mathbb{G}_a) & \xrightarrow{\phi_{n+1}} & \mathcal{H}om(G_{n+1}, \mathbb{G}_a). \end{array}$$

**Proposition 3.48.** *Let  $S$  be a bounded  $p^\infty$ -torsion,  $p$ -complete ring and  $G = \{G_n\}$  be a  $p$ -divisible group over  $S$ . Then, we have a natural isomorphism*

$$\mathbb{L}_{BG/S} := \varprojlim (L_{BG_n/S})_p^\wedge \simeq \omega_G[-1],$$

where  $\omega_G$  is a locally free  $S$ -module of finite rank.

*Proof.* First, we argue over  $S/p^N S$ . The above commutative diagram shows that the map “ $p$ ” :  $G_{n+1}^* \rightarrow G_n^*$  induces an ind-object  $\mathcal{E}xt^1(G_n^*, \mathbb{G}_a)$  that is equivalent to zero (cf. [Ill85, 2.2.1]). By duality, this implies that the pro-object  $H^{-1}(\ell_G)$  is zero. Since  $\mathbb{L}_{BG_n} \simeq \ell_{G_n}[-1]$ , we see that  $\mathbb{L}_{BG_n}$  is naturally pro-isomorphic to  $\omega_{G_n}[-1]$ ; since  $\omega_G := \varprojlim \omega_{G_n}$  is locally free, this gives the claim in the case over  $S/p^N S$ . Since  $S$  has bounded  $p$ -power torsion, the pro-objects (in the category of animated rings)  $S \otimes_{\mathbb{Z}}^L \mathbb{Z}/p^k$  and  $S/p^k S$  are pro-isomorphic, and our claim in the  $p$ -complete case follows from base change properties of cotangent complex, using [SP24, Tag 0D4B], and taking limits.  $\square$

**Remark 3.49.** Note that for any map  $g : G_n \rightarrow G_{n+1}$ , one has a similar diagram

$$\begin{array}{ccc} \mathcal{E}xt^1(G_{n+1}, \mathbb{G}_a) & \xrightarrow{\phi_{n+1}} & \mathcal{H}om(G_{n+1}, \mathbb{G}_a) \\ \downarrow pg & & \downarrow g \\ \mathcal{E}xt^1(G_n, \mathbb{G}_a) & \xrightarrow{\phi_n} & \mathcal{H}om(G_n, \mathbb{G}_a). \end{array}$$

Let  $S$  be such that  $p^N S = 0$ . For a  $p$ -divisible group  $G = \{G_n\}$  over  $S$ , the above diagram implies that the structure maps  $i : G_n \rightarrow G_{n+1}$  induces a pro-system  $\mathcal{H}om(G_n, \mathbb{G}_a)$  that is pro-zero. We have already seen in the above proof that the maps “ $p$ ” :  $G_{n+1} \rightarrow G_n$  induces a direct system  $\mathcal{E}xt^1(G_n, \mathbb{G}_a)$  that is equivalent to zero. However, the pro-system  $\mathcal{E}xt^1(G_n, \mathbb{G}_a)$  induced by the maps  $i : G_n \rightarrow G_{n+1}$  is nonzero; in fact, we have  $\mathcal{E}xt^1(G, \mathbb{G}_a) \simeq \omega_{G^\vee}^\vee$ , the latter will be denoted by  $t_{G^\vee}$ .

**Proposition 3.50.** *Let  $R \rightarrow S$  be a perfectoid ring surjecting onto a quasiregular semiperfectoid algebra  $S$ . Let  $G$  be a  $p$ -divisible group over  $S$ . Then we have*

$$\mathbb{L}_{BG/R} \simeq \mathbb{L}_{S/R} \oplus \omega_G[-1].$$

*Proof.* Follows from the transitivity fiber sequence of cotangent complex associated to the maps  $R \rightarrow S \rightarrow BG$  and the fact that  $\omega_G$  is locally free.  $\square$

Using the above proposition, one can fully describe the conjugate filtration on absolute Hodge–Tate cohomology  $H_{\Delta}^2(BG)$ . Below, let  $t_{G^\vee} := H^2(BG, \mathcal{O})$  which is a locally free  $S$ -module.

**Proposition 3.51** (Hodge–Tate sequence). *Let  $G$  be a  $p$ -divisible group over a quasiregular semiperfectoid ring  $S$ . Then there is a canonical exact sequence*

$$0 \rightarrow \overline{\Delta}_S \otimes_S t_{G^\vee} \rightarrow H_{\Delta}^2(BG) \rightarrow (\overline{\Delta}_S \otimes_S \omega_G) \{-1\} \rightarrow 0.$$

*Proof.* Note that  $H_{\Delta}^2(BG)$  can be computed as relative prismatic cohomology of  $BG_{\overline{\Delta}_S}$  with respect to the canonical prism  $(\Delta_S, I)$  associated to  $S$ . The claim now follows from the relative Hodge–Tate filtration and Proposition 3.48.  $\square$

**Remark 3.52.** The above sequence can be split non-canonically. It also implies that  $H_{\Delta}^2(BG)$  is a locally free  $\Delta_S$ -module of rank  $\text{height}(G)$ , as  $\text{height}(G) = \dim(G) + \dim(G^\vee)$ . Using the fact that  $H^3(BG, \mathcal{O}) = 0$  (resp.  $H^1(BG, \mathcal{O}) = 0$ ) and Proposition 3.48, one can obtain that  $H_{\Delta}^3(BG) = 0$  (resp.  $H_{\Delta}^1(BG) = 0$ ).

**Proposition 3.53.** *Let  $G$  be a  $p$ -divisible group over a quasiregular semiperfectoid ring  $S$ . Then  $H_{\Delta}^2(BG)$  is a locally free  $\Delta_S$ -module of rank equal to  $\text{height}(G)$ .*

*Proof.* Let  $(\Delta_S, I)$  be the prism associated to  $I$ . In this situation,  $I = (d)$  for some element  $d$ . Using the fact that  $R\Gamma_{\Delta}(BG)$  is derived  $d$ -complete, by a limit argument, one deduces that  $H_{\Delta}^3(BG) = 0$ . Indeed, since  $H_{\Delta}^3(BG) = 0$  (Remark 3.52), by induction, it follows that  $H^3(R\Gamma_{\Delta}(BG) \otimes_{\Delta_S} (\Delta_S/d^k)) = 0$ . By derived  $d$ -completeness,  $R\Gamma_{\Delta}(BG) \simeq \varprojlim R\Gamma_{\Delta}(BG) \otimes_{\Delta_S} (\Delta_S/d^k)$ . Using Milnor sequences, it suffices to prove that  $R^1 \varprojlim H^2(R\Gamma_{\Delta}(BG) \otimes_{\Delta_S} (\Delta_S/d^k)) = 0$ . Note that we have a fiber sequence

$$R\Gamma_{\Delta}(BG) \otimes_{\Delta_S} (\Delta_S/d) \rightarrow R\Gamma_{\Delta}(BG) \otimes_{\Delta_S} (\Delta_S/d^k) \rightarrow R\Gamma_{\Delta}(BG) \otimes_{\Delta_S} (\Delta_S/d^{k-1}).$$

This implies that the map  $H^2(R\Gamma_{\Delta}(BG) \otimes_{\Delta_S} (\Delta_S/d^k)) \rightarrow H^2(R\Gamma_{\Delta}(BG) \otimes_{\Delta_S} (\Delta_S/d^{k-1}))$  is a surjection, which gives the claim.

Now, using the fact that  $H^1_{\Delta}(BG) = 0$ ,  $H^3_{\Delta}(BG) = 0$ , and the universal coefficient theorem, one see that  $H^2_{\Delta}(BG) \otimes_{\Delta_S}^L (\Delta_S/d) = H^2_{\Delta}(BG)$ . Invoking Proposition 3.51 now gives the claim.  $\square$

**Notation 3.54.** Let  $G$  be a  $p$ -divisible group over a quasiregular semiperfectoid ring  $S$ . Let  $R\Gamma_{\Delta}(BG)$  denote the absolute prismatic cohomology. Below, we use the following notations:  $F_N^k := H^2(\mathrm{Fil}_{\mathrm{Nyg}}^k R\Gamma_{\Delta}(BG))$  and  $F_{\mathrm{conj}}^k := H^2(\mathrm{Fil}_{\mathrm{conj}}^k R\Gamma_{\Delta}(BG))$ .

**Lemma 3.55.** *Let  $R \rightarrow S$  be a perfectoid ring mapping surjectively onto the quasiregular semiperfectoid algebra  $S$ . In the above notations, the natural maps  $F_{\mathrm{conj}}^{k-1} \rightarrow F_{\mathrm{conj}}^k$  are injective and*

$$F_{\mathrm{conj}}^k / F_{\mathrm{conj}}^{k-1} \simeq (\wedge^k \mathbb{L}_{S/R}[-k] \otimes_S t_{G^\vee}) \oplus (\wedge^{k-1} \mathbb{L}_{S/R}[-k+1] \otimes_S \omega_G).$$

*Proof.* We have a fiber sequence

$$(3.3.1) \quad \mathrm{Fil}_{\mathrm{conj}}^{k-1} R\Gamma_{\Delta}(BG) \rightarrow \mathrm{Fil}_{\mathrm{conj}}^k R\Gamma_{\Delta}(BG) \rightarrow \mathrm{gr}_{\mathrm{conj}}^k R\Gamma_{\Delta}(BG).$$

Note that

$$\mathrm{gr}_{\mathrm{conj}}^k R\Gamma_{\Delta}(BG) \simeq \wedge^k \mathbb{L}_{BG/R}[-k] \simeq \bigoplus_{u+v=k} \wedge^u \mathbb{L}_{S/R}[-u] \otimes_S \mathrm{Sym}^v \omega_G[-2v].$$

It follows that  $H^2(\mathrm{gr}_{\mathrm{conj}}^k R\Gamma_{\Delta}(BG)) \simeq (\wedge^k \mathbb{L}_{S/R}[-k] \otimes_S t_{G^\vee}) \oplus (\wedge^{k-1} \mathbb{L}_{S/R}[-k+1] \otimes_S \omega_G)$ . We also note that  $H^1(\mathrm{gr}_{\mathrm{conj}}^k R\Gamma_{\Delta}(BG)) = 0$ . Moreover, using the vanishing  $H^3(BG, \mathcal{O}) = 0$ , it follows that  $H^3(\mathrm{gr}_{\mathrm{conj}}^k R\Gamma_{\Delta}(BG)) = 0$ . Inductively, we obtain that  $H^3(\mathrm{Fil}_{\mathrm{conj}}^k R\Gamma_{\Delta}(BG)) = 0$  for all  $k$ . Applying  $H^2(\cdot)$  to (3.3.1) now gives the desired claim.  $\square$

**Proposition 3.56** (Dualizability). *Let  $S$  be a quasiregular semiperfectoid algebra. Let  $G$  be a  $p$ -divisible group over  $S$  of height  $h$ . Then  $\mathcal{M}(G)$  (see Definition 3.47) is a prismatic  $F$ -gauge on  $S$  and is a vector bundle of rank  $h$  when viewed as a quasicohherent sheaf on  $\mathrm{Spf}(S)^{\mathrm{syn}}$ .*

*Proof.* The proof will proceed via an explicit understanding of  $H^2(\mathrm{Fil}_{\mathrm{Nyg}}^\bullet R\Gamma_{\Delta}(BG))$ . To this end, we first note that we have a short exact sequence  $0 \rightarrow F_N^1 \rightarrow H^2_{\Delta}(BG) \rightarrow t_{G^\vee} \rightarrow 0$ . This gives a natural surjection  $H^2_{\Delta}(BG) \otimes_{\Delta_S} S \rightarrow t_{G^\vee}$ . Further, one sees that the kernel admits a surjection to  $H^2(BG, \mathbb{L}_{BG/S}[-1]) = \omega_G$ . By Proposition 3.53, this gives a natural short exact sequence  $0 \rightarrow \omega_G \rightarrow H^2_{\Delta}(BG) \otimes_{\Delta_S} S \rightarrow t_{G^\vee} \rightarrow 0$ . We recall that the modules  $\omega_G$  and  $t_{G^\vee}$  are locally free. Let us choose a splitting  $H^2_{\Delta}(BG) \otimes_{\Delta_S} S \simeq \omega_G \oplus t_{G^\vee}$ . Since the surjection  $\Delta_S \rightarrow S$  is henselian (see [ALB23, Lem. 4.1.28]), it is possible to choose an isomorphism  $H^2_{\Delta}(BG) \simeq W \oplus T$ , such that  $W \otimes_{\Delta_S} S \simeq \omega_G$ ,  $T \otimes_{\Delta_S} S \simeq t_{G^\vee}$ , and lifting the isomorphism  $H^2_{\Delta}(BG) \otimes_{\Delta_S} S \simeq \omega_G \oplus t_{G^\vee}$ . It follows that under these identifications,  $F_N^1 = (\mathrm{Fil}_{\mathrm{Nyg}}^1 \Delta_S \otimes_{\Delta_S} T) \oplus W$ . Let us define

$$G_N^k := (\mathrm{Fil}_{\mathrm{Nyg}}^k \Delta_S \otimes_{\Delta_S} T) \oplus (\mathrm{Fil}_{\mathrm{Nyg}}^{k-1} \Delta_S \otimes_{\Delta_S} W).$$

We have a natural map  $G_N^k \rightarrow F_N^k$  of (decreasing) filtered objects. We will show that it is an isomorphism. Since the underlying objects are isomorphic, we need to check that it

induces isomorphism on graded pieces. The graded pieces for the left hand side are given by

$$(3.3.2) \quad T^k := \mathrm{gr}^k(G_N^\bullet) \simeq (\mathrm{Fil}_{\mathrm{conj}}^k \overline{\Delta}_S \{k\} \otimes_S t_{G^\vee}) \oplus (\mathrm{Fil}_{\mathrm{conj}}^{k-1} \overline{\Delta}_S \{k-1\} \otimes_S \omega_G).$$

Note that we have a fiber sequence

$$(3.3.3) \quad \mathrm{Fil}_{\mathrm{Nyg}}^{k+1} R\Gamma_{\Delta}(BG) \rightarrow \mathrm{Fil}_{\mathrm{Nyg}}^k R\Gamma_{\Delta}(BG) \rightarrow \mathrm{Fil}_{\mathrm{conj}}^k R\Gamma_{\overline{\Delta}}(BG) \{k\}.$$

Since  $H^1(\mathrm{Fil}_{\mathrm{conj}}^k R\Gamma_{\overline{\Delta}}(BG)) = 0$ , it follows that  $F_N^{k+1} \rightarrow F_N^k$  is injective. This gives a map  $T^k \{-k\} \rightarrow F_{\mathrm{conj}}^k$  of (increasing) filtered objects. By Proposition 3.51, the underlying objects are isomorphic. Thus, to prove that  $T^k \{-k\} \rightarrow F_{\mathrm{conj}}^k$  is an isomorphism, we are reduced to checking isomorphism on graded pieces. However, this follows from Lemma 3.55. Now, we have maps

$$T^k \rightarrow F_N^k / F_N^{k+1} \rightarrow F_{\mathrm{conj}}^k \{k\}$$

such that the composition is an isomorphism. From (3.3.3), we see that the map in the right is injective, and thus must be an isomorphism. Therefore,  $T^k = \mathrm{gr}^k(G_N^\bullet) \simeq \mathrm{gr}^k(F_N^\bullet)$ , which implies that  $G_N^k \rightarrow F_N^k$  is an isomorphism; i.e.,

$$(3.3.4) \quad F_N^k \simeq (\mathrm{Fil}_{\mathrm{Nyg}}^k \Delta_S \otimes_{\Delta_S} T) \oplus (\mathrm{Fil}_{\mathrm{Nyg}}^{k-1} \Delta_S \otimes_{\Delta_S} W).$$

We will now verify that the prismatic weak  $F$ -gauge given by  $H^2(\mathrm{Fil}_{\mathrm{Nyg}}^\bullet R\Gamma_{\Delta}(BG))$  is indeed a prismatic  $F$ -gauge. To do so, we need to verify the condition (3.2.4). In our situation, using (3.3.4), checking (3.2.4) amounts to checking that the canonical map of filtered objects

$$(3.3.5) \quad \theta_n : (T \otimes_{\Delta_S, \varphi} I^n \Delta_S) \oplus (W \otimes_{\Delta_S, \varphi} I^{n-1} \Delta_S) \rightarrow I^n H_{\Delta}^2(BG)$$

are isomorphisms. Note that since the Frobenius on prismatic cohomology  $\varphi^* H_{\Delta}^2(BG) \rightarrow H_{\Delta}^2(BG)$  is an isomorphism, the filtered map  $\theta_\bullet$  induces an isomorphism on underlying objects. By Lemma 3.57, it suffices to check that the induced map on associated graded objects are isomorphic. However, that follows from the isomorphism  $T^k \{-k\} \rightarrow F_{\mathrm{conj}}^k$  shown above.

Since we have shown  $\mathcal{M}(G)$  is a prismatic  $F$ -gauge over  $S$ , the fact that it is a vector bundle of height  $h$  as a quasicoherent sheaf on  $\mathrm{Spf}(S)^{\mathrm{syn}}$  can be checked by pulling back along  $\mathrm{Spf}(S)^{\mathrm{Nyg}} \rightarrow \mathrm{Spf}(S)^{\mathrm{syn}}$ ; but that follows from (3.3.4).  $\square$

The following lemma was used in the above proof.

**Lemma 3.57.** *Let  $\mathrm{Fil}^\bullet A$  be any  $\mathbb{Z}$ -indexed filtered object in a stable  $\infty$ -category  $\mathcal{C}$  such that the underlying object and the associated graded object of  $\mathrm{Fil}^\bullet A$  are both zero. Then  $\mathrm{Fil}^\bullet A \simeq 0$ .*

*Proof.* Since the associated graded object is zero, the maps in the diagram  $\dots \rightarrow \mathrm{Fil}^n A \rightarrow \mathrm{Fil}^{n-1} A \rightarrow \dots$  are all isomorphisms. The underlying object being zero implies that colimit of the above diagram is zero. This implies that  $\mathrm{Fil}^n A = 0$ .  $\square$

**Remark 3.58.** As a consequence of the proof of Proposition 3.56, when  $S$  is a quasiregular semiperfectoid algebra, using (3.3.3), we see that the map  $F_N^k \rightarrow F_{\mathrm{conj}}^k$  is surjective. Using the fact that  $H_{\Delta}^3(BG) = 0$  (see proof of Proposition 3.53), we inductively obtain that



$H^3(\mathrm{Fil}_{\mathrm{Nyg}}^k R\Gamma_{\Delta}(BG)) = 0$  for all  $k$ . Using the  $E_2$ -spectral sequence (where the Ext-groups are computed in the quasisyntomic topos)

$$E_2^{i,j} = \mathrm{Ext}^i(H^{-j}(\mathbb{Z}[BG]), \mathrm{Fil}_{\mathrm{Nyg}}^k \Delta(\cdot)) \implies H^{i+j}(\mathrm{Fil}_{\mathrm{Nyg}}^k(R\Gamma_{\Delta}(BG))),$$

the vanishing  $H^3(\mathrm{Fil}_{\mathrm{Nyg}}^k R\Gamma_{\Delta}(BG)) = 0$  implies that

$$(3.3.6) \quad \mathrm{Ext}^2(G, \mathrm{Fil}_{\mathrm{Nyg}}^k \Delta(\cdot)) = 0.$$

Similarly, using  $H^1(\mathrm{Fil}_{\mathrm{Nyg}}^k R\Gamma_{\Delta}(BG)) = 0$ , we also obtain  $\mathrm{Hom}(G, \mathrm{Fil}_{\mathrm{Nyg}}^k \Delta(\cdot)) = 0$ . Further, one also has

$$(3.3.7) \quad H^2(\mathrm{Fil}_{\mathrm{Nyg}}^{\bullet} R\Gamma_{\Delta}(BG)) \simeq \mathrm{Ext}^1(G, \mathrm{Fil}_{\mathrm{Nyg}}^{\bullet} \Delta(\cdot)),$$

as prismatic  $F$ -gauges.

**Remark 3.59** (Base change). Let  $S \rightarrow S'$  be a map of qrsp algebras. Let  $G$  be a  $p$ -divisible group over  $S$  and let  $G_{S'}$  be the  $p$ -divisible group over  $S'$  obtained by base change. Let  $f : \mathrm{Spf}(S')^{\mathrm{syn}} \rightarrow \mathrm{Spf}(S)^{\mathrm{syn}}$  denote the natural map. By the proof of Proposition 3.56, one obtains that the natural map  $f^* \mathcal{M}(G) \rightarrow \mathcal{M}(G_{S'})$  must be an isomorphism.

Finally, we can define the prismatic Dieudonné  $F$ -gauge associated to a  $p$ -divisible group.

**Definition 3.60** (Dieudonné  $F$ -gauge of a  $p$ -divisible group). Let  $T$  be a quasisyntomic ring and  $G$  be a  $p$ -divisible group over  $T$ . By Remark 3.59, for every qrsp algebra  $S$  over  $T$ , we obtain a vector bundle  $\mathcal{M}(G_S)$  that is compatible under pullback for every map  $S \rightarrow S'$  of qrsp algebras over  $T$ . By Remark 3.21, this data descends to a prismatic  $F$ -gauge over  $T$ , which we denote as  $\mathcal{M}(G)$  and call it the (prismatic) Dieudonné  $F$ -gauge of  $G$ . By Proposition 3.56, it follows that  $\mathcal{M}(G)$  is naturally a vector bundle over  $\mathrm{Spf}(T)^{\mathrm{syn}}$  of rank equal to the height of  $G$ .

**Remark 3.61.** By the above definition, if  $T$  is a quasisyntomic ring and  $S$  is a qrsp algebra over  $T$ , then the pullback of  $\mathcal{M}(G)$  to  $\mathrm{Spf}(S)^{\mathrm{syn}}$  is canonically isomorphic to  $\mathcal{M}(G_S)$ . By the first paragraph in the proof of Proposition 3.31, it follows that the functor determined by

$$(T)_{\mathrm{qrsp}} \ni S \mapsto H^2(\mathrm{Fil}_{\mathrm{Nyg}}^{\bullet} R\Gamma_{\Delta}(BG_S))$$

is a sheaf of filtered modules over  $\mathrm{Fil}^{\bullet} \Delta(\cdot)$ .

**Remark 3.62** (Compatibility with [ALB23]). As explained in Remark 3.93, for a  $p$ -divisible group  $G$  over a qrsp algebra, the prismatic  $F$ -gauge  $\mathcal{M}(G)$  has Hodge–Tate weights in  $[0, 1]$ . By Proposition 3.45, one can associate an admissible prismatic Dieudonné module to  $\mathcal{M}(G)$ . By [Mon21, Thm. 1.6] this is isomorphic to the admissible prismatic Dieudonné module associated to  $G$  by [ALB23].

**Example 3.63.** Let  $T$  be a quasisyntomic ring and let us consider the  $p$ -divisible group  $\mathbb{Q}_p/\mathbb{Z}_p$  over  $T$ . The prismatic Dieudonné  $F$ -gauge of  $\mathbb{Q}_p/\mathbb{Z}_p$  is naturally isomorphic to the structure sheaf  $\mathcal{O}_{T^{\mathrm{syn}}}$ . This follows from (3.3.7).

**Example 3.64.** Let  $T$  be a quasisyntomic ring and let us consider the  $p$ -divisible group  $\mu_{p^\infty}$  over  $T$ . The prismatic Dieudonné  $F$ -gauge of  $\mu_{p^\infty}$  is naturally isomorphic to the Breuil–Kisin twist  $\mathcal{O}\{-1\}$ . See [Mon22].

**Remark 3.65** (Dieudonné  $F$ -gauge of abelian schemes). Let  $S$  be a qrsp algebra. Let  $A$  be an abelian scheme over  $S$ . We define  $\mathcal{M}(A)$  to be the effective prismatic weak  $F$ -gauge over  $S$  obtained from  $H^2(\mathrm{Fil}_{\mathrm{Nyg}}^\bullet R\Gamma_\Delta(BA))$  (see Remark 3.29). By the same steps in the proof of Proposition 3.56 (where one replaces  $\omega_G$  by the locally free  $S$ -module  $\Omega_A^1$ ), it follows that  $\mathcal{M}(A)$  is a prismatic  $F$ -gauge and a vector bundle of rank  $2 \dim A$  over  $\mathrm{Spf}(S)^{\mathrm{syn}}$ . If  $S \rightarrow S'$  is a map of qrsp algebras, and  $A_{S'}$  is the abelian scheme over  $S'$  obtained by base change, it follows again that the natural map  $f^*\mathcal{M}(A) \rightarrow \mathcal{M}(A_{S'})$  is an isomorphism, where  $f : \mathrm{Spf}(S')^{\mathrm{syn}} \rightarrow \mathrm{Spf}(S)^{\mathrm{syn}}$  denotes the natural map. As a consequence, if  $T$  is a quasisyntomic ring and  $A$  is an abelian scheme over  $T$ , by Remark 3.21, one obtains a vector bundle on  $\mathrm{Spf}(T)^{\mathrm{syn}}$  of rank  $2 \dim A$  which we denote as  $\mathcal{M}(A)$ .

**Remark 3.66.** Similar to Remark 3.61, by the above definition, if  $T$  is a quasisyntomic ring and  $S$  is a qrsp algebra over  $T$ , then the pullback of  $\mathcal{M}(A)$  to  $\mathrm{Spf}(S)^{\mathrm{syn}}$  is canonically isomorphic to  $\mathcal{M}(A_S)$ . By the first paragraph in the proof of Proposition 3.31, it follows that the functor determined by

$$(3.3.8) \quad (T)_{\mathrm{qrsp}} \ni S \mapsto H^2(\mathrm{Fil}_{\mathrm{Nyg}}^\bullet R\Gamma_\Delta(BA_S))$$

is a sheaf of filtered modules over  $\mathrm{Fil}^\bullet \Delta_{(\cdot)}$ .

**Remark 3.67.** Let  $A$  be an abelian scheme over a qrsp algebra  $S$ . All the Ext-groups below are computed in the quasisyntomic topos of  $S$ . Similar to Remark 3.58, we have  $H^1(\mathrm{Fil}_{\mathrm{Nyg}}^\bullet R\Gamma_\Delta(BA)) = 0$ , which implies  $\mathrm{Hom}(A, \mathrm{Fil}_{\mathrm{Nyg}}^k \Delta_{(\cdot)}) = 0$ . One also has

$$H^3(\mathrm{Fil}_{\mathrm{Nyg}}^\bullet R\Gamma_\Delta(BA)) = 0,$$

which implies by the  $E_2$  spectral sequence as in Remark 3.58 that we have

$$(3.3.9) \quad \mathrm{Ext}^2(A, \mathrm{Fil}_{\mathrm{Nyg}}^k \Delta_{(\cdot)}) = 0.$$

for all  $k$ . Further, one also has

$$(3.3.10) \quad H^2(\mathrm{Fil}_{\mathrm{Nyg}}^\bullet R\Gamma_\Delta(BA)) = \mathrm{Ext}^1(A, \mathrm{Fil}_{\mathrm{Nyg}}^\bullet \Delta_{(\cdot)}),$$

as prismatic  $F$ -gauges.

Let  $S$  be a qrsp algebra. If  $A$  is an abelian scheme over  $S$ , one may also consider the effective prismatic weak  $F$ -gauge obtained from  $H^1(\mathrm{Fil}_{\mathrm{Nyg}}^\bullet R\Gamma_\Delta(A))$ , which we denote as  $\mathcal{M}'(A)$ . We note the following.

**Proposition 3.68.** *Let  $A$  be an abelian scheme over a qrsp algebra  $S$ . Then  $\mathcal{M}(A) \simeq \mathcal{M}'(A)$  as prismatic weak  $F$ -gauges over  $S$ .*

*Proof.* First we note that  $\mathcal{M}'(A \times A) \simeq \mathcal{M}'(A) \oplus \mathcal{M}'(A)$ . To check this, it would suffice to show that  $H^1(\mathrm{Fil}_{\mathrm{Nyg}}^i R\Gamma_\Delta(A)) \oplus H^1(\mathrm{Fil}_{\mathrm{Nyg}}^i R\Gamma_\Delta(A)) \simeq H^1(\mathrm{Fil}_{\mathrm{Nyg}}^i R\Gamma_\Delta(A \times A))$ . The latter follows by induction on  $i$ , starting from the case  $i = 0$ , using the description of graded pieces of the Nygaard filtration in terms of conjugate filtered Hodge–Tate cohomology, where we may use the fact that for any perfectoid ring  $R$  mapping onto  $S$ , we have  $\mathbb{L}_{A/R} = \mathbb{L}_{S/R} \oplus \Omega_A$ .

Now in order to prove  $\mathcal{M}(A) \simeq \mathcal{M}'(A)$ , we will compute  $H^2(\mathrm{Fil}_{\mathrm{Nyg}}^\bullet R\Gamma_\Delta(BA))$  by descent along the smooth map  $* \rightarrow BA$ . Applying descent gives an  $E_1$ -spectral sequence

$$E_1^{i,j} = H^j(\mathrm{Fil}_{\mathrm{Nyg}}^\bullet R\Gamma_\Delta(A^{\times i})) \implies H^{i+j}(\mathrm{Fil}_{\mathrm{Nyg}}^\bullet R\Gamma_\Delta(BA)).$$

The spectral sequence implies that  $H^2(\mathrm{Fil}_{\mathrm{Nyg}}^\bullet R\Gamma_\Delta(BA)) \simeq E_2^{1,1}$ . Since the differential  $d_1^{0,1} : E_1^{0,1} \rightarrow E_1^{1,1}$  is zero, it follows that  $E_2^{1,1} \simeq \mathrm{Ker}(d_1^{1,1})$ . Let  $\mathrm{pr}_1, \mathrm{pr}_2$  denote the natural projection maps  $A \times A \rightarrow A$  and let  $m$  denote the multiplication map  $m : A \times A \rightarrow A$ . It follows that the map  $d_1^{1,1} : H^1(\mathrm{Fil}_{\mathrm{Nyg}}^\bullet R\Gamma_\Delta(A)) \rightarrow H^1(\mathrm{Fil}_{\mathrm{Nyg}}^\bullet R\Gamma_\Delta(A \times A))$  is the same as  $\mathrm{pr}_1^* - m^* + \mathrm{pr}_2^*$ . Using the identification,  $H^1(\mathrm{Fil}_{\mathrm{Nyg}}^\bullet R\Gamma_\Delta(A)) \oplus H^1(\mathrm{Fil}_{\mathrm{Nyg}}^\bullet R\Gamma_\Delta(A)) \simeq H^1(\mathrm{Fil}_{\mathrm{Nyg}}^\bullet R\Gamma_\Delta(A \times A))$ , we see that  $d_1^{1,1} = 0$ . This implies that

$$H^2(\mathrm{Fil}_{\mathrm{Nyg}}^\bullet R\Gamma_\Delta(BA)) \simeq E_2^{1,1} \simeq E_1^{1,1} \simeq H^1(\mathrm{Fil}_{\mathrm{Nyg}}^\bullet R\Gamma_\Delta(A)),$$

implying  $\mathcal{M}(A) \simeq \mathcal{M}'(A)$ , as desired.  $\square$

**Proposition 3.69** (Duality compatibility for abelian schemes). *Let  $A$  be an abelian scheme over a quasisyntomic ring  $T$ . Let  $A^\vee$  denote the dual abelian scheme. Then we have a natural isomorphism*

$$\mathcal{M}(A)^* \{-1\} \simeq \mathcal{M}(A^\vee)$$

of prismatic  $F$ -gauges over  $T$ .

*Proof.* By descent, it is enough to argue in the case when  $T$  is qrsp. Note that there is a canonical map  $A \times A^\vee \rightarrow BG_m$  classifying the universal line bundle on  $A \times A^\vee$ . By Example 3.40, we obtain a natural map  $\mathcal{O}\{-1\} \rightarrow H^2(\mathrm{Fil}_{\mathrm{Nyg}}^\bullet R\Gamma_\Delta(A \times A^\vee))$  of prismatic weak  $F$ -gauges over  $T$ . By Kunnetth formula, we have a natural map  $\mathcal{O}\{-1\} \rightarrow \mathcal{M}'(A) \otimes \mathcal{M}'(A^\vee)$  of prismatic  $F$ -gauges over  $T$ . Using Remark 3.65 and Proposition 3.68, we obtain a natural map  $\mathcal{M}(A)^* \{-1\} \rightarrow \mathcal{M}(A^\vee)$ , which as one can check (e.g., one may use the calculation appearing in Proposition 3.56), is an isomorphism.  $\square$

**Construction 3.70.** Let  $T$  be a quasisyntomic ring and let  $\mathrm{FFG}(T)$  denote the category of finite locally free commutative group schemes over  $S$  of  $p$ -power rank. Let  $G \in \mathrm{FFG}(T)$ . The functor determined by

$$(3.3.11) \quad (T)_{\mathrm{qrsp}} \ni S \mapsto (\tau_{[-2, -3]} \mathrm{Fil}_{\mathrm{Nyg}}^\bullet R\Gamma_\Delta(B^2 G_S))[3]$$

defines a presheaf of filtered modules over the filtered ring  $\mathrm{Fil}^\bullet \Delta_{(\cdot)}$ . We let  $\mathcal{M}(G)$  denote the sheafification of this functor for the quasisyntomic topology on  $(T)_{\mathrm{qrsp}}$  as a sheaf of derived  $(p, \mathcal{I})$ -complete filtered module over the filtered ring  $\mathrm{Fil}^\bullet \Delta_{(\cdot)}$ . Let  $\mathcal{M}(G)^u$  denote the underlying object of the filtered object  $\mathcal{M}(G)$ . There is a natural map  $\varphi : \mathcal{M}(G) \rightarrow \mathcal{I}^\bullet \otimes_{\Delta_{(\cdot)}} \mathcal{M}(G)^u$  of filtered  $\mathrm{Fil}^\bullet \Delta_{(\cdot)}$ -modules induced from the natural Frobenius map defined at the level of presheaves in (3.3.11). We will prove in Proposition 3.76 that  $(\mathcal{M}(G), \varphi)$  satisfies conditions (1) and (2) from Remark 3.32 and therefore, defines a prismatic  $F$ -gauge, which we would again denote as  $\mathcal{M}(G)$ .

Before that, we will need the following preparation.

**Proposition 3.71.** *Let  $S$  be a quasiregular semiperfectoid algebra and let  $G \in \mathrm{FFG}(S)$ . Then we have a natural isomorphism of prismatic weak  $F$ -gauges over  $S$*

$$\tau_{\geq 0} R\mathrm{Hom}_{(S)_{\mathrm{qrsp}}}(G, \mathrm{Fil}^\bullet \Delta_{(\cdot)}[1]) \simeq (\tau_{[-2, -3]} \mathrm{Fil}_{\mathrm{Nyg}}^\bullet R\Gamma_\Delta(B^2 G))[3].$$

*Proof.* Let  $P$  be an ordinary abelian group. We have a natural map  $\mathbb{Z} \rightarrow \mathbb{Z}[B^2 P]$  whose cofiber will be denoted as  $\mathbb{Z}^{\mathrm{red}}[B^2 P]$ . There is a natural map  $\mathbb{Z}^{\mathrm{red}}[B^2 P] \rightarrow P[2]$ , which is

the 2-truncation. This induces a natural map  $\mathbb{Z}[B^2P] \rightarrow P[2]$ . Applying this to  $G$  regarded as an abelian group object of the topos associated to  $(S)_{\text{qrsp}}$ , we obtain a natural map

$$R\text{Hom}_{(S)_{\text{qrsp}}}(G, \text{Fil}^\bullet \Delta_{(\cdot)}[1]) \rightarrow R\text{Hom}_{(S)_{\text{qrsp}}}(\mathbb{Z}[B^2G], \text{Fil}^\bullet \Delta_{(\cdot)})[3].$$

This induces a natural map

$$\tau_{\geq 0} R\text{Hom}_{(S)_{\text{qrsp}}}(G, \text{Fil}^\bullet \Delta_{(\cdot)}[1]) \rightarrow (\tau_{[-2, -3]} \text{Fil}_{\text{Nyg}}^\bullet R\Gamma_{\Delta}(B^2G))[3].$$

To show that it is an isomorphism, it suffices to show that

$$H^2(\text{Fil}_{\text{Nyg}}^\bullet R\Gamma_{\Delta}(B^2G)) \simeq \text{Hom}_{(S)_{\text{qrsp}}}(G, \text{Fil}^\bullet \Delta_{(\cdot)})$$

and

$$H^3(\text{Fil}_{\text{Nyg}}^\bullet R\Gamma_{\Delta}(B^2G)) \simeq \text{Ext}_{(S)_{\text{qrsp}}}^1(G, \text{Fil}^\bullet \Delta_{(\cdot)}).$$

Both of these isomorphisms follow from the  $E_2$ -spectral sequence

$$E_2^{i,j} = \text{Ext}_{(S)_{\text{qrsp}}}^i(H^{-j}(\mathbb{Z}[B^2G]), \text{Fil}^\bullet \Delta_{(\cdot)}) \implies H^{i+j}(\text{Fil}_{\text{Nyg}}^\bullet R\Gamma_{\Delta}(B^2G))$$

as in the proof of Proposition 3.12.  $\square$

**Remark 3.72.** Let  $G$  be a  $p$ -divisible group over a quasisyntomic ring  $T$ . Let  $S \in (T)_{\text{qrsp}}$ . An argument similar to above shows that

$$H^2(\text{Fil}_{\text{Nyg}}^\bullet R\Gamma_{\Delta}(B^2G_S)) \simeq \text{Hom}_{(S)_{\text{qrsp}}}(G, \text{Fil}^\bullet \Delta_{(\cdot)})$$

and

$$H^3(\text{Fil}_{\text{Nyg}}^\bullet R\Gamma_{\Delta}(B^2G_S)) \simeq \text{Ext}_{(S)_{\text{qrsp}}}^1(G, \text{Fil}^\bullet \Delta_{(\cdot)}).$$

By Remark 3.58, it follows that there is an isomorphism

$$(\tau_{[-2, -3]} \text{Fil}_{\text{Nyg}}^\bullet R\Gamma_{\Delta}(B^2G_S))[3] \simeq \mathcal{M}(G_S)$$

of prismatic  $F$ -gauges over  $S$ .

**Remark 3.73.** Similar to the above remark, for an abelian scheme  $A$  over a quasisyntomic ring  $T$  and  $S \in (T)_{\text{qrsp}}$ , it follows from Remark 3.67 that

$$\tau_{\geq 0} R\text{Hom}_{(S)_{\text{qrsp}}}(A_S, \text{Fil}^\bullet \Delta_{(\cdot)}[1]) \simeq (\tau_{[-2, -3]} \text{Fil}_{\text{Nyg}}^\bullet R\Gamma_{\Delta}(B^2A_S))[3] \simeq \mathcal{M}(A_S)$$

as prismatic  $F$ -gauges over  $S$ .

**Lemma 3.74.** *Let  $T$  be a quasisyntomic ring and  $G \in \text{FFG}(T)$ . Assume that there is a short exact sequence*

$$0 \rightarrow G \rightarrow A' \rightarrow A \rightarrow 0,$$

where  $A, A'$  are abelian schemes over  $S$ . Then  $\mathcal{M}(G)$  is a prismatic  $F$ -gauge over  $T$  and we have a fiber sequence

$$\mathcal{M}(A) \rightarrow \mathcal{M}(A') \rightarrow \mathcal{M}(G)$$

of prismatic  $F$ -gauges over  $T$ .

*Proof.* Let  $S \in (T)_{\text{qrsp}}$ . We have a fiber sequence of filtered  $\text{Fil}_{\text{Nyg}}^\bullet \Delta_S$ -modules

$$R\text{Hom}_{(S)_{\text{qrsp}}}(A, \text{Fil}^\bullet \Delta_{(\cdot)}[1]) \rightarrow R\text{Hom}_{(S)_{\text{qrsp}}}(A', \text{Fil}^\bullet \Delta_{(\cdot)}[1]) \rightarrow R\text{Hom}_{(S)_{\text{qrsp}}}(G, \text{Fil}^\bullet \Delta_{(\cdot)}[1]).$$

By the vanishing in (3.3.9), we have a fiber sequence

(3.3.12)

$$\tau_{\geq 0} R\text{Hom}_{(S)_{\text{qrsp}}}(A, \text{Fil}^\bullet \Delta_{(\cdot)}[1]) \rightarrow \tau_{\geq 0} R\text{Hom}_{(S)_{\text{qrsp}}}(A', \text{Fil}^\bullet \Delta_{(\cdot)}[1]) \rightarrow \tau_{\geq 0} R\text{Hom}_{(S)_{\text{qrsp}}}(G, \text{Fil}^\bullet \Delta_{(\cdot)}[1]).$$

By Remark 3.65 and Remark 3.73, the presheaf of filtered  $\mathrm{Fil}^\bullet \Delta_{(\cdot)}$ -modules on  $(T)_{\mathrm{qrsp}}$  (equipped with the natural Frobenius) determined by the association

$$(T)_{\mathrm{qrsp}} \ni S \mapsto \tau_{\geq 0} \mathrm{RHom}_{(S)_{\mathrm{qrsp}}}(A_S, \mathrm{Fil}^\bullet \Delta_{(\cdot)}[1])$$

is a prismatic  $F$ -gauge (similarly for  $A'$ ), it follows from (3.3.12) that the presheaf of filtered  $\mathrm{Fil}^\bullet \Delta_{(\cdot)}$ -modules on  $(T)_{\mathrm{qrsp}}$  (equipped with the natural Frobenius) determined by the association

$$(T)_{\mathrm{qrsp}} \ni S \mapsto \tau_{\geq 0} \mathrm{RHom}_{(S)_{\mathrm{qrsp}}}(G_S, \mathrm{Fil}^\bullet \Delta_{(\cdot)}[1])$$

is a prismatic  $F$ -gauge. By Proposition 3.71, we conclude that  $\mathcal{M}(G)$  is a prismatic  $F$ -gauge and one also has a fiber sequence as desired.  $\square$

**Construction 3.75.** Let  $T$  be a quasisyntomic ring and let  $G \in \mathrm{FFG}(T)$ . Let us consider the full subcategory of  $(T)_{\mathrm{qrsp}}$  spanned by  $S \in (T)_{\mathrm{qrsp}}$  with the property that there exists a short exact sequence

$$0 \rightarrow G_S \rightarrow A' \rightarrow A \rightarrow 0,$$

where  $A, A'$  are abelian schemes over  $S$ . We denote this full subcategory by  $(T)_{\mathrm{qrsp}}^G$ . If  $S \in (T)_{\mathrm{qrsp}}$  and  $S \rightarrow S_1$  is a quasisyntomic cover with  $S_1 \in (T)_{\mathrm{qrsp}}^G$  and  $S \rightarrow S_2$  is any map in  $(T)_{\mathrm{qrsp}}$ , then the ( $p$ -completed) pushout is again in  $(T)_{\mathrm{qrsp}}^G$  by the proof of [BMS19, Lem. 4.27].

In particular, equipped with the quasisyntomic topology, we see that  $(T)_{\mathrm{qrsp}}^G$  forms a site. Using a theorem of Raynaud [BBM82, Thm. 3.1.1] combined with [BMS19, Lem. 4.28], it follows that for any  $S \in (T)_{\mathrm{qrsp}}$ , there is a quasisyntomic cover  $S \rightarrow S'$ , such that  $S' \in (T)_{\mathrm{qrsp}}^G$ . Also, by [BMS19, Lem. 4.30], we have that if  $S \rightarrow S'$  is a quasisyntomic cover with  $S \in (T)_{\mathrm{qrsp}}$  and  $S' \in (T)_{\mathrm{qrsp}}^G$ , then all terms of the Čech nerve  $S'^\bullet \in (T)_{\mathrm{qrsp}}^G$ . In particular,  $(T)_{\mathrm{qrsp}}^G$  forms a basis for the site  $(T)_{\mathrm{qrsp}}$ .

**Proposition 3.76.** *Let  $T$  be a quasisyntomic ring and  $G \in \mathrm{FFG}(T)$ . Then  $(\mathcal{M}(G), \varphi)$  from Construction 3.70 defines a prismatic  $F$ -gauge over  $T$ .*

*Proof.* Note that the association

$$(3.3.13) \quad (T)_{\mathrm{qrsp}}^G \ni S \mapsto (\tau_{[-2, -3]} \mathrm{Fil}_{\mathrm{Nyg}}^\bullet \mathrm{R}\Gamma_{\Delta}(B^2 G_S))[3]$$

determines a sheaf of  $(p, \mathcal{I})$ -complete filtered modules  $\mathrm{Fil}^\bullet \mathcal{N}$  over the filtered ring  $\mathrm{Fil}^\bullet \Delta_{(\cdot)}$  equipped with a  $\mathrm{Fil}^\bullet \Delta_{(\cdot)}$ -linear Frobenius map

$$\varphi : \mathrm{Fil}^\bullet \mathcal{N} \rightarrow \mathcal{I}^\bullet \mathcal{N}$$

that satisfies natural analogues of (1) and (2) from Remark 3.32. This follows from the fiber sequence (3.3.12), Remark 3.66, the base change property for Dieudonné  $F$ -gauges of abelian schemes in (3.3.12), and Lemma 3.74. In the above, we abuse notation slightly and use  $\mathcal{I}$  and  $\mathrm{Fil}^\bullet \Delta_{(\cdot)}$  to denote their restriction to  $(T)_{\mathrm{qrsp}}^G$ .

By Construction 3.70,  $\mathcal{M}(G)$  (along with the Frobenius) is obtained by sheafifying the presheaf on  $(T)_{\mathrm{qrsp}}$  described (3.3.11). By the previous paragraph, it follows that the latter presheaf restricted to  $(T)_{\mathrm{qrsp}}^G$  is a sheaf, and for  $S \in (T)_{\mathrm{qrsp}}^G$ , we have  $\mathcal{M}(G)(S) \simeq (\tau_{[-2, -3]} \mathrm{Fil}_{\mathrm{Nyg}}^\bullet \mathrm{R}\Gamma_{\Delta}(B^2 G_S))[3]$  is a prismatic  $F$ -gauge over  $S$ .

Further, as in Remark 3.17, by descent, we have

$$(3.3.14) \quad F\text{-Gauge}_{\Delta}(T) \simeq \lim_{S \in (T)_{\mathrm{qrsp}}^G} F\text{-Gauge}_{\Delta}(S).$$

Therefore, by the previous paragraph, the association in (3.3.13) determines a prismatic  $F$ -gauge, which we denote as  $\mathcal{M}'(G)$  and view it as as in Remark 3.32. Note that by construction, there is a natural map  $\mathcal{M}(G) \rightarrow \mathcal{M}'(G)$  of sheaf of filtered modules over  $\mathrm{Fil}^\bullet \Delta_{(\cdot)}$  on  $(T)_{\mathrm{qrsp}}$ , compatibly with the Frobenius. Since this map is an isomorphism when restricted to  $(T)_{\mathrm{qrsp}}^G$ , it must be an isomorphism. Since  $\mathcal{M}'(G)$  is a prismatic  $F$ -gauge,  $\mathcal{M}(G)$  satisfies conditions (1) and (2) from Remark 3.32 and therefore, is a prismatic  $F$ -gauge.  $\square$

**Definition 3.77** (Dieudonné  $F$ -gauge of a finite locally free group scheme). Let  $T$  be quasisyntomic ring and let  $G \in \mathrm{FFG}(T)$ . We call the prismatic  $F$ -gauge over  $T$  given by  $\mathcal{M}(G)$  from Proposition 3.76 the (prismatic) Dieudonné  $F$ -gauge of  $G$ .

**Remark 3.78.** We can view the prismatic  $F$ -gauge  $\mathcal{M}(G)$  as an object of  $D_{\mathrm{qc}}(\mathrm{Spf}(T)^{\mathrm{syn}})$  or in the sense of Remark 3.32. If  $T' \rightarrow T$  is a map of quasisyntomic rings inducing a map  $f : \mathrm{Spf}(T')^{\mathrm{syn}} \rightarrow \mathrm{Spf}(T)^{\mathrm{syn}}$ , then by construction we have  $f^* \mathcal{M}(G) \simeq \mathcal{M}(G_{T'})$ .

**Remark 3.79.** Let  $T$  be a quasisyntomic ring and  $G \in \mathrm{FFG}(T)$ . As a consequence of the proof of Proposition 3.76, we see that the prismatic  $F$ -gauge  $\mathcal{M}(G)$ , when viewed as a sheaf of filtered modules (equipped with a Frobenius) on  $(T)_{\mathrm{qrsp}}^G$ , is given by the formula (3.3.13). Using Proposition 3.71, one may rewrite the latter as

$$(3.3.15) \quad (T)_{\mathrm{qrsp}}^G \ni S \mapsto \tau_{\geq 0} \mathrm{RHom}_{(S)_{\mathrm{qrsp}}} (G, \mathrm{Fil}^\bullet \Delta_{(\cdot)}[1]).$$

**Remark 3.80.** Although, logically, it does not play any direct role in our paper, we give another description of the prismatic Dieudonné  $F$ -gauge functor. Let  $S$  be a qrsp ring and  $G \in \mathrm{FFG}(S)$ . Then  $\mathcal{M}(G)$  as in Construction 3.70, can also be written as  $(\tau_{[-2, -3]} \mathrm{R}v_* \mathcal{O})[3]$ , where  $v : B^2 G^{\mathrm{syn}} \rightarrow S^{\mathrm{syn}}$  denotes the natural map. Similar to the proof of Proposition 3.71, we obtain a natural map

$$(3.3.16) \quad \mathrm{RHom}_{S^{\mathrm{syn}}} (G^{\mathrm{syn}}[2], \mathbb{G}_a) \rightarrow \mathrm{RHom}_{S^{\mathrm{syn}}} (\mathbb{Z}[B^2 G^{\mathrm{syn}}], \mathbb{G}_a) \simeq \mathrm{R}v_* \mathcal{O}.$$

Note that  $\mathcal{E}xt^3(\mathbb{Z}[B^2 G^{\mathrm{syn}}], \mathbb{G}_a) \simeq \pi_0 \mathcal{M}aps_*(B^2 G^{\mathrm{syn}}, K(\mathbb{G}_a, 3))$ , where the latter denotes internal mapping space as pointed objects. By delooping twice, it is equivalent to  $\pi_0 \mathcal{M}aps_{\mathbb{E}_2}(G^{\mathrm{syn}}, K(\mathbb{G}_a, 1))$ , where the latter denote maps as  $\mathbb{E}_2$ -group objects. Since the group objects themselves are 1-truncated, we see that it is isomorphic to  $\mathcal{E}xt^1(G^{\mathrm{syn}}, \mathbb{G}_a)$ . By the same argument, we obtain  $\mathcal{E}xt^2(\mathbb{Z}[B^2 G^{\mathrm{syn}}], \mathbb{G}_a) \simeq \mathcal{H}om(G^{\mathrm{syn}}, \mathbb{G}_a)$ . Therefore, (3.3.16) induces an isomorphism

$$(3.3.17) \quad \tau_{[-2, -3]} \mathrm{RHom}_{S^{\mathrm{syn}}} (G^{\mathrm{syn}}[2], \mathbb{G}_a) \simeq \tau_{[-2, -3]} \mathrm{R}v_* \mathcal{O}.$$

This implies that for a quasisyntomic ring  $T$ , and  $G \in \mathrm{FFG}(T)$ , we have

$$(3.3.18) \quad \mathcal{M}(G) \simeq \tau_{\geq 0} \mathrm{RHom}_{S^{\mathrm{syn}}} (G^{\mathrm{syn}}, \mathbb{G}_a[1]).$$

**Remark 3.81.** Let  $S$  be a quasisyntomic ring and  $G \in \mathrm{BT}(S)$ . Let  $G_n := G[p^n]$ . Then the fiber sequence  $G_n \rightarrow G \xrightarrow{p^n} G$  induces a fiber sequence

$$(3.3.19) \quad \mathcal{M}(G) \xrightarrow{p^n} \mathcal{M}(G) \rightarrow \mathcal{M}(G_n)$$

of prismatic  $F$ -gauges over  $S$ . In order to prove this, one may work locally on  $S$ . In particular, we may work in the site  $(S)_{\mathrm{qrsp}}^{G_n}$ . Using the description Remark 3.79, the desired fiber sequence (3.3.19) follows from the vanishing (3.3.6) (cf. (3.3.12)).

**Remark 3.82.** Let  $S$  be a quasisyntomic ring and  $G \in \mathrm{BT}(S)$ . Then the natural map  $\mathcal{M}(G) \rightarrow \varinjlim \mathcal{M}(G_n)$  is an isomorphism of prismatic  $F$ -gauges. This follows from (3.3.19) since  $\mathcal{M}(G)$  is derived  $p$ -complete.

**Remark 3.83.** Let  $S$  be a quasisyntomic ring and  $G, G' \in \mathrm{BT}(S)$ . Assume that we have a fiber sequence  $H \rightarrow G' \rightarrow G$ , where  $H \in \mathrm{FFG}(S)$ . By the same argument as in Remark 3.81, we have a fiber sequence  $\mathcal{M}(G) \rightarrow \mathcal{M}(G') \rightarrow \mathcal{M}(H)$  or prismatic  $F$ -gauges over  $S$ .

We will now explain how one can recover cohomology with coefficients in  $G \in \mathrm{FFG}(T)$  in terms of quasicoherent cohomology of the prismatic Dieudonné  $F$ -gauge of the Cartier dual  $G^\vee$ .

**Proposition 3.84** (Cohomology with coefficients in group schemes). *Let  $T$  be a quasisyntomic ring and let  $G \in \mathrm{FFG}(T)$ . There is a natural isomorphism*

$$R\Gamma_{\mathrm{qsyn}}(T, G) \simeq R\Gamma(\mathrm{Spf}(T)^{\mathrm{syn}}, \mathcal{M}(G^\vee) \{1\}).$$

*Proof.* Note that both sides satisfy quasisyntomic descent, viewed as a functor on quasisyntomic rings. Therefore, it suffices to produce a natural isomorphism when  $T$  is qrsp. By the construction of  $\mathcal{M}(G)$  via sheafification in Construction 3.70 and (3.2.11) for  $n = 1$ , we have a natural map (see (3.1.8) for the notation of the source) of quasisyntomic sheaves on  $(T)_{\mathrm{qrsp}}$  as below.

$$(3.3.20) \quad \begin{array}{c} \mathrm{Fib} \left( (\tau_{[-3, -2]} \mathrm{Fil}_{\mathrm{Nyg}}^1 R\Gamma_{\Delta}(B^2 G_{(\cdot)}^\vee) \{1\}^\sharp) [3] \xrightarrow{\varphi_{1-\mathrm{can}}} (\tau_{[-3, -2]} R\Gamma_{\Delta}(B^2 G_{(\cdot)}^\vee) \{1\}^\sharp) [3] \right) \\ \downarrow \\ R\Gamma(\mathrm{Spf}(T)^{\mathrm{syn}}, \mathcal{M}(G^\vee) \{1\}) \end{array}$$

By (3.1.8), this constructs a natural map of quasisyntomic sheaves on  $(T)_{\mathrm{qrsp}}$

$$(3.3.21) \quad R\Gamma_{\mathrm{qsyn}}(S, G) \rightarrow R\Gamma(\mathrm{Spf}(S)^{\mathrm{syn}}, \mathcal{M}(G^\vee) \{1\}),$$

where  $S \in (T)_{\mathrm{qrsp}}$ . To prove that (3.3.21) is an isomorphism, by quasisyntomic descent, it is enough to prove it for  $S \in (T)_{\mathrm{qrsp}}^G$  (Construction 3.75). But then by the sheafiness of the functor in (3.3.13), it follows that the downward vertical map (3.3.20) is an isomorphism. Therefore, by (3.1.8), the map (3.3.21) is an isomorphism. This finishes the proof.  $\square$

**Proposition 3.85** (Cohomology with coefficients in Tate modules). *Let  $S$  be a quasisyntomic ring and let  $G \in \mathrm{BT}(S)$ . There is a natural isomorphism*

$$R\Gamma_{\mathrm{qsyn}}(S, T_p(G)) \simeq R\Gamma(\mathrm{Spf}(S)^{\mathrm{syn}}, \mathcal{M}(G^\vee) \{1\}).$$

*Proof.* Similar to the above proof, it suffices to produce a natural isomorphism when  $S$  is quasisyntomic. In this case, it follows from an analogous argument by using (3.1.5) and sheafiness of the functor in (3.3.8). Alternatively, one may directly deduce it from Remark 3.81, Remark 3.82 and Proposition 3.84.  $\square$

**Construction 3.86.** Let  $T$  be a quasisyntomic ring and let  $\mathcal{M}, \mathcal{N}$  be two objects of  $D_{\mathrm{qc}}(\mathrm{Spf}(T)^{\mathrm{syn}})$ . Then the functor  $(T)_{\mathrm{qrsp}} \rightarrow D(\mathbb{Z})$  determined by the association

$$(T)_{\mathrm{qrsp}} \ni S \mapsto R\mathrm{Hom}_{D_{\mathrm{qc}}(\mathrm{Spf}(S)^{\mathrm{syn}})}(\mathcal{M}|_{\mathrm{Spf}(S)^{\mathrm{syn}}}, \mathcal{N}|_{\mathrm{Spf}(S)^{\mathrm{syn}}})$$

is a  $D(\mathbb{Z})$ -valued quasisyntomic sheaf. We will denote this sheaf by  $R\mathcal{H}om_{\mathrm{qsyn}}(\mathcal{M}, \mathcal{N})$ .

**Proposition 3.87** (Duality compatibility for finite flat group schemes). *Let  $T$  be a quasisyntomic ring and  $G \in \text{FFG}(T)$ . Then  $\mathcal{M}(G)$  is a dualizable object of  $F$ -Gauge $_{\Delta}(T)$ . Further, if  $G^{\vee}$  is the Cartier dual of  $G$ , then we have a natural isomorphism*

$$\mathcal{M}(G)^* \{-1\} [1] \simeq \mathcal{M}(G^{\vee}).$$

*Proof.* Dualizability of  $\mathcal{M}(G)$  can be checked locally and follows from Lemma 3.74 and Remark 3.65 (cf. Construction 3.75 and (3.3.14)).

For the latter part, since both sides satisfy quasisyntomic descent, it is enough to produce a natural isomorphism when  $T$  is qrsp. Note that the functor determined by

$$(T)_{\text{qrsp}} \ni S \mapsto \text{Fil}^{\bullet} \Delta_S$$

is a sheaf with values in filtered objects of  $D(\mathbb{Z})$ . Therefore, we have a natural isomorphism

$$R\text{Hom}_{(T)_{\text{qrsp}}} (R\mathcal{H}om_{(T)_{\text{qrsp}}}(\mathbb{Z}, G^{\vee}), \text{Fil}^{\bullet} \Delta_{(\cdot)}[1]) \simeq R\text{Hom}_{(T)_{\text{qrsp}}}(G^{\vee}, \text{Fil}^{\bullet} \Delta_{(\cdot)}[1]),$$

where the  $D(\mathbb{Z})$ -valued sheaf  $R\mathcal{H}om_{(T)_{\text{qrsp}}}(\mathbb{Z}, G^{\vee})$  is determined by the association

$$(T)_{\text{qrsp}} \ni S \mapsto R\Gamma_{\text{qsyn}}(S, G^{\vee}).$$

By Proposition 3.84, we have an isomorphism

$$R\text{Hom}_{(T)_{\text{qrsp}}}(R\mathcal{H}om_{\text{syn}}(\mathcal{O}, \mathcal{M}(G) \{1\}), \text{Fil}^{\bullet} \Delta_{(\cdot)}[1]) \simeq R\text{Hom}_{(T)_{\text{qrsp}}}(R\mathcal{H}om_{(T)_{\text{qrsp}}}(\mathbb{Z}, G^{\vee}), \text{Fil}^{\bullet} \Delta_{(\cdot)}[1]).$$

By dualizability of  $\mathcal{M}(G)$ , we have a natural isomorphism

$$R\mathcal{H}om_{\text{syn}}(\mathcal{O}, \mathcal{M}(G) \{1\}) \simeq R\mathcal{H}om_{\text{syn}}(\mathcal{M}(G)^* \{-1\}, \mathcal{O}).$$

Note that we have a natural evaluation map (the counit of the adjunction in Remark 3.34)

$$\mathcal{M}(G)^* \{-1\} [1] \rightarrow R\text{Hom}_{(T)_{\text{qrsp}}}(R\mathcal{H}om_{\text{syn}}(\mathcal{M}(G)^* \{-1\}, \mathcal{O}), \text{Fil}^{\bullet} \Delta_{(\cdot)}[1])$$

of prismatic weak  $F$ -gauges over  $T$  (Definition 3.27). Combining with the above isomorphisms, we have a natural map  $\mathcal{M}(G)^* \{-1\} [1] \rightarrow R\text{Hom}_{(T)_{\text{qrsp}}}(G^{\vee}, \text{Fil}^{\bullet} \Delta_{(\cdot)}[1])$  of prismatic weak  $F$ -gauges over  $T$ . Repeating the same argument for any  $S \in (T)_{\text{qrsp}}$ , we obtain a natural map

$$(3.3.22) \quad \mathcal{M}(G_S)^* \{-1\} [1] \rightarrow R\text{Hom}_{(S)_{\text{qrsp}}}(G_S^{\vee}, \text{Fil}^{\bullet} \Delta_{(\cdot)}[1])$$

of prismatic weak  $F$ -gauges over  $S$ . Note that by Proposition 3.69, Lemma 3.74 and Construction 3.75, the prismatic  $F$ -gauge  $\mathcal{M}(G)^* \{-1\} [1]$  viewed as a sheaf on  $(T)_{\text{qrsp}}$  of filtered module over  $\text{Fil}^{\bullet} \Delta_{(\cdot)}$  equipped with a Frobenius map is sheafification of the functor determined by

$$(T)_{\text{qrsp}} \ni S \mapsto \tau_{\geq 0}(\mathcal{M}(G_S)^* \{-1\} [1]).$$

By Construction 3.70 and Proposition 3.71, the prismatic  $F$ -gauge  $\mathcal{M}(G^{\vee})$  is obtained from sheafification of the functor determined by

$$(T)_{\text{qrsp}} \ni S \mapsto \tau_{\geq 0} R\text{Hom}_{(S)_{\text{qrsp}}}(G_S^{\vee}, \text{Fil}^{\bullet} \Delta_{(\cdot)}[1])$$

of prismatic  $F$ -gauges over  $T$ . Thus, (3.3.22) constructs a natural map

$$\mathcal{M}(G)^* \{-1\} [1] \rightarrow \mathcal{M}(G^{\vee}).$$

Since one can locally embed  $G$  into abelian schemes, by using Proposition 3.69, Lemma 3.74 and Construction 3.75, we see that the above map must be an isomorphism. This finishes the proof.  $\square$



**Proposition 3.88** (Duality compatibility for  $p$ -divisible groups). *Let  $G$  be a  $p$ -divisible group of height  $h$  over a quasisyntomic ring  $T$ . If  $G^\vee$  is the Cartier dual of  $G$ , then we have a natural isomorphism*

$$\mathcal{M}(G^\vee) \simeq \mathcal{M}(G)^* \{-1\}$$

of prismatic  $F$ -gauges over  $T$ .

*Proof.* Let  $G_n = G[p^n]$ . We have a fiber sequence  $G_n \rightarrow G \xrightarrow{p^n} G$ . By (3.3.19), we have a fiber sequence

$$\mathcal{M}(G) \xrightarrow{p^n} \mathcal{M}(G) \rightarrow \mathcal{M}(G_n).$$

Dualizing, we obtain that  $\mathcal{M}(G_n)^* \simeq (\mathcal{M}(G)^*/p^n)[-1]$ . Now, we have

$$\mathcal{M}(G^\vee) \simeq \varprojlim \mathcal{M}(G_n^\vee) \simeq \varprojlim \mathcal{M}(G_n)^* \{-1\} [1],$$

where the last step follows from Proposition 3.87. Further,

$$\varprojlim \mathcal{M}(G_n)^* \{-1\} [1] \simeq \varprojlim (\mathcal{M}(G)^* \{-1\})/p^n \simeq \mathcal{M}(G)^* \{-1\}.$$

This finishes the proof.  $\square$

**Proposition 3.89.** *Let  $T$  be a quasisyntomic ring. The prismatic Dieudonné  $F$ -gauge functor*

$$\mathcal{M} : \text{FFG}(T)^{\text{op}} \rightarrow F\text{-Gauge}_\Delta(T)$$

is fully faithful.

*Proof.* We use  $\text{Hom}$  to denote the connective truncation of  $R\text{Hom}$  (i.e.,  $\tau_{\geq 0}R\text{Hom}$ ) and refer to it as the mapping space. We would like to prove that the natural map

$$(3.3.23) \quad \text{Hom}_{\text{FFG}(T)}(H, G) \rightarrow \text{Hom}_{F\text{-Gauge}_\Delta(T)}(\mathcal{M}(G), \mathcal{M}(H))$$

is an isomorphism. For notational simplicity, we will sometimes omit the subscript indicating the category where maps are being taken. Note that for any  $G \in \text{FFG}(S)$ , the prismatic  $F$ -gauge  $\mathcal{M}(G)$  is effective by construction. It follows that the mapping spaces on the right hand side of (3.3.23) can also be computed in the category  $F\text{-Mod}_{\text{Fil}^\bullet \Delta}(T)^{\text{eff}}$  (Remark 3.33). Note that by Remark 3.34,  $R\mathcal{H}om_{(S)_{\text{qrsp}}}(H, \text{Fil}^\bullet \Delta_{(\cdot)})$  can naturally be viewed as an object of  $F\text{-Mod}_{\text{Fil}^\bullet \Delta}(T)^{\text{eff}}$ .  $\square$

**Lemma 3.90.** *We have an isomorphism*

$$\text{Hom}_{F\text{-Mod}_{\text{Fil}^\bullet \Delta}(T)^{\text{eff}}}(\mathcal{M}(G), \mathcal{M}(H)) \simeq \text{Hom}_{F\text{-Mod}_{\text{Fil}^\bullet \Delta}(T)^{\text{eff}}}(\mathcal{M}(G), R\mathcal{H}om_{(T)_{\text{qrsp}}}(H, \text{Fil}^\bullet \Delta_{(\cdot)}[1])).$$

*Proof.* We will introduce some notations for the proof of the lemma. Note that the site  $(T)_{\text{qrsp}}^{G \times H}$  (Construction 3.75) forms a basis for  $(T)_{\text{qrsp}}$ . Let us define  $\mathcal{C}$  to be the  $\infty$ -category of sheaves on  $(T)_{\text{qrsp}}^{G \times H}$  of derived  $(p, \mathcal{I})$ -complete  $\mathbb{Z}$ -indexed filtered modules  $\text{Fil}^\bullet \mathcal{N}$  over the sheaf of filtered ring  $\text{Fil}^\bullet \Delta_{(\cdot)}$  on  $(T)_{\text{qrsp}}^{G \times H}$  equipped with a  $\text{Fil}^\bullet \Delta_{(\cdot)}$ -linear Frobenius map

$$\varphi : \text{Fil}^\bullet \mathcal{N} \rightarrow \mathcal{I}^\bullet \Delta_{(\cdot)} \otimes_{\Delta_{(\cdot)}} \mathcal{N} =: \mathcal{I}^\bullet \mathcal{N},$$

with the property that the natural maps  $\text{Fil}^i \mathcal{N} \rightarrow \text{Fil}^{i-1} \mathcal{N}$  are isomorphisms for all  $i \leq 0$ . The natural restriction functor induces an equivalence  $u^{-1} : F\text{-Mod}_{\text{Fil}^\bullet \Delta}(T)^{\text{eff}} \simeq \mathcal{C}$ .

Let us define  $\mathcal{D}$  to be the analogous presheaf category; there is a fully faithful functor  $\mathcal{C} \rightarrow \mathcal{D}$ . The left hand side of the lemma is then equivalent to  $\text{Hom}_{\mathcal{D}}(u^{-1} \mathcal{M}(G), u^{-1} \mathcal{M}(H))$ . Note that by Remark 3.79, the object  $u^{-1} \mathcal{M}(H)$  is given by the following association

$$(T)_{\text{qrsp}}^{G \times H} \ni S \mapsto \tau_{\geq 0} R\mathcal{H}om_{(S)_{\text{qrsp}}}(H, \text{Fil}^\bullet \Delta_{(\cdot)}[1]),$$

and similarly for  $u^{-1}\mathcal{M}(G)$ . Further, the association

$$(T)_{\text{qrsp}}^{G \times H} \ni S \mapsto R\text{Hom}_{(S)_{\text{qrsp}}}(H, \text{Fil}^\bullet \Delta_{(\cdot)}[1]),$$

defines an object of  $\mathcal{C}$ , which is simply  $u^{-1}R\mathcal{H}om_{(T)_{\text{qrsp}}}(H, \text{Fil}^\bullet \Delta_{(\cdot)})$  from Remark 3.34. Thus, we have

$$\begin{aligned} \text{Hom}_{F\text{-Mod}_{\text{Fil}^\bullet \Delta}(T)^{\text{eff}}}(\mathcal{M}(G), \mathcal{M}(H)) &\xrightarrow{\sim} \text{Hom}_{\mathcal{D}}(u^{-1}\mathcal{M}(G), u^{-1}\mathcal{M}(H)) \\ &\xrightarrow{\sim} \text{Hom}_{\mathcal{D}}(u^{-1}\mathcal{M}(G), u^{-1}R\mathcal{H}om(H, \text{Fil}^\bullet \Delta_{(\cdot)}[1])) \\ &\xrightarrow{\sim} \text{Hom}_{\mathcal{C}}(u^{-1}\mathcal{M}(G), u^{-1}R\mathcal{H}om(H, \text{Fil}^\bullet \Delta_{(\cdot)}[1])) \\ &\xrightarrow{\sim} \text{Hom}_{F\text{-Mod}_{\text{Fil}^\bullet \Delta}(T)^{\text{eff}}}(\mathcal{M}(G), R\mathcal{H}om_{(T)_{\text{qrsp}}}(H, \text{Fil}^\bullet \Delta_{(\cdot)}[1])). \end{aligned}$$

In the above, second isomorphism follows because  $u^{-1}\mathcal{M}(G)$  is a (pre)sheaf of connective filtered modules; the rest follows from the equivalence  $u^{-1} : F\text{-Mod}_{\text{Fil}^\bullet \Delta}(T)^{\text{eff}} \simeq \mathcal{C}$ .  $\square$

Back to proof of the proposition, we have

$$\begin{aligned} \text{Hom}(\mathcal{M}(G), R\mathcal{H}om_{(T)_{\text{qrsp}}}(H, \text{Fil}^\bullet \Delta_{(\cdot)}[1])) &\xrightarrow{\sim} \text{Hom}_{(T)_{\text{qrsp}}}(H, R\mathcal{H}om_{\text{syn}}(\mathcal{M}(G), \text{Fil}^\bullet \Delta_{(\cdot)}[1])) \\ &\xrightarrow{\sim} \text{Hom}_{(T)_{\text{qrsp}}}(H, R\mathcal{H}om_{\text{syn}}(\mathcal{O}, \mathcal{M}(G)^*[1])) \\ &\xrightarrow{\sim} \text{Hom}_{(T)_{\text{qrsp}}}(H, R\mathcal{H}om_{\text{syn}}(\mathcal{O}, \mathcal{M}(G^\vee)\{1\})) \\ &\xrightarrow{\sim} \text{Hom}_{(T)_{\text{qrsp}}}(H, R\mathcal{H}om_{(T)_{\text{qrsp}}}(\mathbb{Z}, G)) \\ &\xrightarrow{\sim} \text{Hom}_{(T)_{\text{qrsp}}}(H, G). \end{aligned}$$

Here, the first isomorphism follows from the adjunction in Remark 3.34, the second isomorphism uses dualizability of  $\mathcal{M}(G)$ , the third isomorphism uses the duality compatibility formula in Proposition 3.87, the fourth isomorphism uses the formula for cohomology with coefficients in  $G$  (Proposition 3.84), and the final isomorphism uses the fact that  $H$  is connective as a presheaf of abelian groups and  $\tau_{\geq 0}R\mathcal{H}om(\mathbb{Z}, G) \simeq G$ , as presheaves on  $(T)_{\text{qrsp}}$ . These chain of isomorphisms and Lemma 3.90 imply that (3.3.23) is an isomorphism, which finishes the proof.

**Remark 3.91.** Although it is not used in our paper, let us remark that the arguments used in the proof of Proposition 3.89 can also prove the following statement. Let  $S$  be a quasisyntomic ring. For  $M \in D_{\text{perf}}(\text{Spf}(S)^{\text{syn}})$ , let  $T(M)(1) := R\mathcal{H}om_{\text{syn}}(\mathcal{O}, M\{1\})$  (see Construction 3.86) and  $T^0(M)(1)$  be the connective truncation of  $T(M)(1)$ . Let  $D_{\text{perf}}(S^{\text{syn}})_{\leq 1}^c$  denote the full subcategory of dualizable  $\mathcal{O}_{S^{\text{syn}}}$ -modules  $M$  with Tor amplitude in homological degrees  $\leq 1$  for which  $T(M)(1)$  is coconnective and  $T^0M(1)$  is representable by an object of  $\text{FFG}(S)$ . Then the functor  $\mathcal{M}^\vee : \text{FFG}(S) \rightarrow D_{\text{perf}}(S^{\text{syn}})_{\leq 1}^c$  that is determined by  $G \mapsto \mathcal{M}(G^\vee)$  is left adjoint to  $T^0(M)(1)$ . Using Remark 3.80, one can argue as sketched below. In what follows, the first isomorphism is a consequence of dualizability as objects of  $D(S^{\text{syn}}, \mathcal{O})$ , the second one follows from the Cartier duality compatibility from Proposition 3.87, the third one comes from (3.3.18), the fourth one is a consequence of adjunction, and the final one follows from Yoneda lemma.

$$\begin{aligned}
\mathrm{Hom}_{D(S^{\mathrm{syn}}, \mathcal{O})}(\mathcal{M}(G^\vee), M) &\xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}}(M^*[1], \mathcal{M}(G^\vee)^*[1]) \\
&\xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}}(M^*[1], \mathcal{M}(G)\{1\}) \\
&\xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}}(M^*[1], R\mathcal{H}om_{D(S^{\mathrm{syn}}, \mathbb{Z})}(G^{\mathrm{syn}}, \mathcal{O}[1])\{1\}) \\
&\xrightarrow{\sim} \mathrm{Hom}_{\mathbb{Z}}(G^{\mathrm{syn}}, R\mathcal{H}om_{\mathcal{O}}(\mathcal{O}, M\{1\})) \\
&\xrightarrow{\sim} \mathrm{Hom}_{D_{\mathrm{qsyn}}(S, \mathbb{Z})}(G, T^0(M)(1)).
\end{aligned}$$

**Proposition 3.92.** *Let  $S$  be a quasisyntomic ring. The prismatic Dieudonné  $F$ -gauge functor*

$$\mathcal{M} : \mathrm{BT}(S)^{\mathrm{op}} \rightarrow F\text{-Gauge}_{\Delta}^{\mathrm{vect}}(S)$$

*is fully faithful.*

*Proof.* This is proven similarly to the proof of Proposition 3.89. Indeed, let  $G, H \in \mathrm{BT}(S)$ . As before, let  $\mathrm{Hom}$  denote the connective truncation of  $R\mathrm{Hom}$ . We have

$$\begin{aligned}
\mathrm{Hom}(\mathcal{M}(G), \mathcal{M}(H)) &\xrightarrow{\sim} \mathrm{Hom}(\mathcal{M}(G), R\mathcal{H}om_{(S)_{\mathrm{qrsp}}}(H, \mathrm{Fil}^\bullet \Delta_{(\cdot)}[1])) \\
&\xrightarrow{\sim} \mathrm{Hom}_{(S)_{\mathrm{qrsp}}}(H, R\mathcal{H}om_{\mathrm{syn}}(\mathcal{O}, \mathcal{M}(G)^*[1])) \\
&\xrightarrow{\sim} \mathrm{Hom}_{(S)_{\mathrm{qrsp}}}(H, R\mathcal{H}om_{\mathrm{syn}}(\mathcal{O}, \mathcal{M}(G^\vee)\{1\}[1])) \\
&\xrightarrow{\sim} \mathrm{Hom}_{(S)_{\mathrm{qrsp}}}(H[-1], R\mathcal{H}om_{(S)_{\mathrm{qrsp}}}(\mathbb{Z}, T_p(G))),
\end{aligned}$$

where the first isomorphism follows similar to Lemma 3.90, the second isomorphism uses the adjunction from Remark 3.34 and dualizability of  $\mathcal{M}(G)$ , the third isomorphism uses the duality compatibility from Proposition 3.88 and the fourth one uses Proposition 3.85.

Note that we have a fiber sequence  $T_p(H) \rightarrow \varprojlim_p H \rightarrow H$ . Using the fact that multiplication by  $p$  is an isomorphism on  $\varprojlim_p H$  and  $R\mathcal{H}om_{(S)_{\mathrm{qrsp}}}(\mathbb{Z}, T_p(G))$  is a derived  $p$ -complete sheaf on  $(S)_{\mathrm{qrsp}}$ , it further follows that

$$\begin{aligned}
\mathrm{Hom}_{(S)_{\mathrm{qrsp}}}(H[-1], R\mathcal{H}om_{(S)_{\mathrm{qrsp}}}(\mathbb{Z}, T_p(G))) &\xrightarrow{\sim} \mathrm{Hom}_{(S)_{\mathrm{qrsp}}}(T_p(H), R\mathcal{H}om_{(S)_{\mathrm{qrsp}}}(\mathbb{Z}, T_p(G))) \\
&\xrightarrow{\sim} \mathrm{Hom}_{(S)_{\mathrm{qrsp}}}(T_p(H), T_p(G)) \\
&\xrightarrow{\sim} \mathrm{Hom}_{\mathrm{BT}(S)}(H, G).
\end{aligned}$$

This finishes the proof.  $\square$

**Remark 3.93** (Essential image of  $p$ -divisible groups, cf. [Kis06]). Let  $S$  be a qrsp algebra and let  $G$  be a  $p$ -divisible group over  $S$ . By Proposition 3.56, we know that  $\mathcal{M}(G)$  is a vector bundle on  $S^{\mathrm{syn}}$ . Using the description from Proposition 3.56 (3.3.2), it follows that if denote the underlying filtered  $\mathrm{Fil}_{\mathrm{Nyg}}^\bullet \Delta_R$ -module of  $\mathcal{M}(G)$  as  $\mathrm{Fil}^\bullet M$ , then

$$(3.3.24) \quad \mathrm{gr}^\bullet M \otimes_{\mathrm{gr}_{\mathrm{Nyg}}^\bullet \Delta_S} S \simeq t_{G^\vee} \oplus \omega_G,$$

where as a graded  $S$ -module,  $t_{G^\vee}$  is in weight 0 and  $\omega_G$  is in weight 1. By Remark 3.36, we conclude that  $\mathcal{M}(G)$  has Hodge–Tate weights in  $[0, 1]$ . By quasisyntomic descent, if  $G$  is a  $p$ -divisible group over a quasisyntomic ring  $T$ , it follows that  $\mathcal{M}(G)$  has Hodge–Tate weights in  $[0, 1]$ . Therefore the functor from Proposition 3.92 refines to a fully faithful functor

$$(3.3.25) \quad \mathcal{M} : \mathrm{BT}(T)^{\mathrm{op}} \rightarrow F\text{-Gauge}_{[0,1]}^{\mathrm{vect}}(T),$$

where the latter denotes the category of vector bundles on  $\mathrm{Spf}(T)^{\mathrm{syn}}$  of Hodge–Tate weights in  $[0, 1]$ . The functor in (3.3.25) is an *equivalence* of categories. Indeed, in order to check that the functor is essentially surjective, we can reduce to the case when  $T$  is a qrsp algebra. In that case, under the equivalence in Proposition 3.45, using the argument in the second paragraph of the proof of [ALB23, Thm. 4.9.5], one can further reduce to the case when  $T$  is perfectoid, when the claim follows from [SW20, Theorem 17.5.2].

**Remark 3.94.** Let  $A$  be an abelian scheme over a quasisyntomic ring  $S$ . Similar to Remark 3.93, it follows that  $\mathcal{M}(A)$  is a vector bundle over  $\mathrm{Spf}(S)^{\mathrm{syn}}$  with Hodge–Tate weights in  $[0, 1]$ .

**Remark 3.95** (Essential image of finite locally free  $p$ -power rank group schemes). Let  $S$  be a quasisyntomic ring. Let  $D_{\mathrm{perf}}(\mathrm{Spf}(S)^{\mathrm{syn}})$  denote the full subcategory of dualizable objects of  $D_{\mathrm{qc}}(\mathrm{Spf}(S)^{\mathrm{syn}})$ . We will determine the full subcategory of  $D_{\mathrm{perf}}(\mathrm{Spf}(S)^{\mathrm{syn}})$  spanned by the essential image of finite locally free commutative group schemes of  $p$ -power rank over  $S$ . Let  $\mathrm{Vect}_{[0,1]}^{\mathrm{iso}}(S^{\mathrm{syn}})$  be the full subcategory of  $D_{\mathrm{perf}}(S^{\mathrm{syn}})$  spanned by objects  $M$  satisfying the following properties:

- (1) There exists a quasisyntomic cover  $(S_i)_{i \in I}$  of  $S$  such that  $M_i := M|_{S_i^{\mathrm{syn}}}$  is isomorphic to  $\mathrm{cofib}(V_i \xrightarrow{f_i} V'_i)$ , for some  $V_i, V'_i \in \mathrm{Vect}(S^{\mathrm{syn}})$  and  $f_i : V_i \rightarrow V'_i$ .
- (2) The vector bundles  $V_i, V'_i$  appearing above have Hodge–Tate weights in  $[0, 1]$  for all  $i \in I$ .
- (3) The map  $f_i$  appearing above has the property that it is an isomorphism when viewed in the category  $\mathrm{Vect}(S^{\mathrm{syn}})_{[p]}^{\frac{1}{p}}$ .

Let  $G \in \mathrm{FFG}(S)$ . By a result of Raynaud [BBM82, Thm. 3.1.1], one can Zariski locally realize  $G$  as a kernel of an isogeny  $A' \rightarrow A$  of abelian varieties. By Lemma 3.74, it follows that (locally) we have a fiber sequence

$$\mathcal{M}(A) \rightarrow \mathcal{M}(A') \rightarrow \mathcal{M}(G).$$

Since (Remark 3.94),  $\mathcal{M}(A)$  and  $\mathcal{M}(A')$  are vector bundles of Hodge–Tate weights in  $[0, 1]$ , it follows that  $\mathcal{M}(G)$  indeed lies in  $\mathrm{Vect}_{[0,1]}^{\mathrm{iso}}(S^{\mathrm{syn}})$ . Now we can formulate the following.

**Proposition 3.96.** *The functor*

$$(3.3.26) \quad \mathcal{M} : \mathrm{FFG}(S)^{\mathrm{op}} \rightarrow \mathrm{Vect}_{[0,1]}^{\mathrm{iso}}(S^{\mathrm{syn}})$$

*induces an equivalence of categories.*

*Proof.* Let us introduce some notations for the proof. For  $M \in \mathrm{Vect}_{[0,1]}^{\mathrm{iso}}(S^{\mathrm{syn}})$ , we define  $T(M)$  to be the  $D(\mathbb{Z})$ -valued sheaf on  $(S)_{\mathrm{qsyn}}$  given by  $R\mathrm{Hom}_{\mathrm{syn}}(\mathcal{O}, M\{1\})$  (see Construction 3.86). We will denote  $T^0(M) := \tau_{\geq 0}T(M)$ , which is a sheaf of abelian groups on  $(S)_{\mathrm{qsyn}}$ . For  $G \in \mathrm{FFG}(S)$ , we denote  $\mathcal{M}^\vee(G) := \mathcal{M}(G^\vee)$ .

It would be enough to prove that if  $M \in \mathrm{Vect}_{[0,1]}^{\mathrm{iso}}(S^{\mathrm{syn}})$ , then the quasi-syntomic sheaf  $T^0(M)$  is representable by a group scheme and lies in  $\mathrm{FFG}(S)$ , and  $\mathcal{M}^\vee(T^0(M)) \simeq M$ . To this end, we may work locally and assume without loss of generality that there exists  $V, V' \in \mathrm{Vect}(S^{\mathrm{syn}})$  with Hodge–Tate weights in  $[0, 1]$  and a map  $f : V \rightarrow V'$  such that we have a fiber sequence  $V \rightarrow V' \rightarrow M$  and  $f$  is an isomorphism when viewed in the category  $\mathrm{Vect}(S^{\mathrm{syn}})_{[p]}^{\frac{1}{p}}$ . By construction, we have a fiber sequence

$$(3.3.27) \quad T(V) \rightarrow T(V') \rightarrow T(M)$$

By Remark 3.93, the map  $f$  corresponds to an isogeny  $\underline{f} : G' \rightarrow G$  of  $p$ -divisible groups such that  $\mathcal{M}(G')$ ,  $\mathcal{M}(G)$  identify with  $V'$ ,  $V$  respectively. We have a fiber sequence

$$(3.3.28) \quad H^\vee \rightarrow G^\vee \rightarrow G'^\vee$$

where  $H$  is a finite locally free commutative group scheme of  $p$ -power rank. In particular, it follows that  $H^\vee$  is killed by a power of  $p$ . Applying derived  $p$ -completion to (3.3.28), we obtain a fiber sequence

$$H^\vee \rightarrow T_p(G^\vee)[1] \rightarrow T_p(G'^\vee)[1],$$

which maybe rewritten as a fiber sequence

$$(3.3.29) \quad T_p(G^\vee) \rightarrow T_p(G'^\vee) \rightarrow H^\vee$$

of quasisyntomic sheaves of abelian groups on  $S$ .

Using (3.3.27), (3.3.29) and Proposition 3.85, it follows that the  $D(\mathbb{Z})$ -valued sheaf on  $(S)_{\text{qsyn}}$  determined by

$$(S)_{\text{qsyn}} \ni A \mapsto R\Gamma_{\text{qsyn}}(A, H^\vee)$$

is naturally isomorphic to  $T(M)$ . This shows that  $T^0(M) \simeq H^\vee$  as quasisyntomic sheaf of abelian groups. Now,  $\mathcal{M}^\vee(T^0(M)) \simeq \mathcal{M}^\vee(H^\vee) \simeq \mathcal{M}(H) \simeq \text{cofib}(\mathcal{M}(G) \rightarrow \mathcal{M}(G'))$ , where the last isomorphism follows from the fiber sequence  $H \rightarrow G' \rightarrow G$  and Remark 3.83. Since  $\mathcal{M}(G')$ ,  $\mathcal{M}(G)$  naturally identify with  $V'$ ,  $V$  respectively, and we have a fiber sequence  $V \rightarrow V' \rightarrow M$ , we see that  $\mathcal{M}(H) \simeq M$ . Thus, we obtain  $\mathcal{M}^\vee(T^0(M)) \simeq M$ , which finishes the proof.  $\square$

**Remark 3.97.** In a future joint work with Madapusi, we will show that the natural functor from the category  $\text{Vect}_{[0,1]}^{\text{iso}}(S^{\text{syn}})$  to the category of  $p$ -power torsion perfect complexes on  $\text{Spf}(S)^{\text{syn}}$  with Tor-amplitude in homological degrees  $[0, 1]$  and with Hodge–Tate weights in  $[0, 1]$  is an equivalence.

**Remark 3.98** (Étale realizations of prismatic Dieudonné  $F$ -gauges). We now discuss how to recover local systems (on the generic fiber) associated to  $p$ -divisible groups or finite locally free commutative group schemes of  $p$ -power rank  $G$  from their prismatic Dieudonné  $F$ -gauge  $\mathcal{M}(G)$ .

Let  $S$  be a quasisyntomic ring. As constructed in [Bha23, Cons. 6.3.1], there is a symmetric monoidal étale realization functor

$$T_{\text{ét}} : \text{Perf}(\text{Spf}(S)^{\text{syn}}) \rightarrow D_{\text{lisse}}^b(\text{Spa}(S[1/p], S), \mathbb{Z}_p),$$

where we think of  $\text{Spa}(S[1/p], S)$  as a presheaf on affinoid perfectoid spaces over  $\mathbb{Q}_p$  and  $D_{\text{lisse}}^b(\text{Spa}(S[1/p], S), \mathbb{Z}_p)$  is the full subcategory of  $D_{\text{pro-ét}}^b(\text{Spa}(S[1/p], S), \mathbb{Z}_p)$  with lisse cohomology (see [BS23, 3.1]). Note that any  $\mathcal{F} \in D_{\text{lisse}}^b(\text{Spa}(S[1/p], S), \mathbb{Z}_p)$  automatically satisfies  $v$ -descent (in the derived sense). Let us denote  $\text{Spf}(S)_\eta := \text{Spa}(S[1/p], S)$ . For any  $G \in \text{FFG}(S)$ , we will identify  $T_{\text{ét}}(\mathcal{M}(G)\{1\})$  with the generic fiber  $G_\eta^\vee$  of the Cartier dual  $G^\vee$ .

**Proposition 3.99.** *Let  $S$  be a quasisyntomic ring and  $G \in \text{FFG}(S)$ . In the above set up, we have an isomorphism*

$$T_{\text{ét}}(\mathcal{M}(G)\{1\}) \simeq G_\eta^\vee.$$

*Proof.* Let  $D((S)_{\text{qrsp}}, \mathbb{Z}_p)$  denote the category of  $p$ -complete objects in the derived category of  $\mathbb{Z}_p$ -modules on the site  $(S)_{\text{qrsp}}$  and let  $D_v(\text{Spf}(S)_\eta, \mathbb{Z}_p)$  denote the category of  $p$ -complete objects in the derived category of  $\mathbb{Z}_p$ -modules on the  $v$ -site of  $\text{Spf}(S)_\eta$ . There is a canonical functor  $u^* : D((S)_{\text{qrsp}}, \mathbb{Z}_p) \rightarrow D_v(\text{Spf}(S)_\eta, \mathbb{Z}_p)$  determined by (sheafifying)  $\mathcal{F} \mapsto \mathcal{G} := (\text{Spa}(R, R^+) \mapsto \mathcal{F}(R^+))$ . We note that  $u^*$  has a right adjoint  $u_* : D_v(\text{Spf}(S)_\eta, \mathbb{Z}_p) \rightarrow D((S)_{\text{qrsp}}, \mathbb{Z}_p)$  determined by

$$\mathcal{G} \mapsto \mathcal{F} := ((S)_{\text{qrsp}} \ni R \mapsto \mathcal{F}(\text{Spa}(R_{\text{perfd}}[1/p], R_{\text{perfd}}))).$$

In the above,  $R_{\text{perfd}}$  is the universal perfectoid ring that receives a map from  $R$  (see [BS19, Thm. 1.12]). Note that by construction,  $u^*u_*$  is naturally isomorphic to identity. By construction of the functor  $T_{\text{ét}}$  (see [Bha23, Cons. 6.3.1, 6.3.2], [BS19, Cor. 3.7]), it follows that

$$(3.3.30) \quad T_{\text{ét}}(\mathcal{M}(G)) \simeq u^*(\mathcal{M}(G)[1/\mathcal{I}]_p^\wedge)^{\varphi=1}$$

Using the Artin–Schreier sequence for the tilt of the structure sheaf of  $\text{Spf}(S)_\eta$  along with [Sch17, Prop. 8.5, Prop. 8.8], it follows that  $u_*\mathbb{Z}/p^n \simeq \mu_*\mathbb{Z}/p^n$ , where the latter denotes (quasisyntomic sheafification of) the derived pushforward along the map  $\mu : \text{Spf}(S)_{\eta, \text{ét}} \rightarrow \text{Spf}(S)_{\text{ét}}$  of topoi. By [BS19, Thm. 9.1], we have an isomorphism  $u_*\mathbb{Z}/p^n \simeq (\Delta_{(\cdot)}[1/\mathcal{I}]/p^n)^{\varphi=1}$ . Passing to limits, we have

$$(3.3.31) \quad u_*\mathbb{Z}_p \simeq (\Delta_{(\cdot)}[1/\mathcal{I}]_p^\wedge)^{\varphi=1}.$$

Note that we have  $\mathcal{E}xt^2(G_\eta, \mathbb{Z}_p) = 0$ ; in fact, since every finite locally free commutative group scheme of  $p$ -power rank over a base where  $p$  is invertible must be finite étale, by the proof of Lemma 3.8, for any affinoid perfectoid space  $X$  over  $\text{Spf}(S)_\eta$ , a class in  $\text{Ext}^2(G_{\eta, X}, \mathbb{Z}_p)$  is killed by a finite étale cover of  $X$ , which must again be affinoid perfectoid by the almost purity theorem [Sch12, Thm. 1.10]. Via adjunction, this implies that  $\mathcal{H}^2(u^*R\mathcal{H}om(G, u_*\mathbb{Z}_p)) = 0$ . Therefore, we have a fiber sequence of (derived) sheaves

$$\tau_{\geq 0}u^*R\mathcal{H}om(G, u_*\mathbb{Z}_p[1]) \rightarrow \tau_{\geq 0}u^*R\mathcal{H}om(G, \Delta_{(\cdot)}[1/\mathcal{I}]_p^\wedge[1]) \xrightarrow{\varphi=1} \tau_{\geq 0}u^*R\mathcal{H}om(G, \Delta_{(\cdot)}[1/\mathcal{I}]_p^\wedge[1]).$$

By construction of  $\mathcal{M}(G)$ , and the above fiber sequence, it follows that

$$u^*(\mathcal{M}(G)[1/\mathcal{I}]_p^\wedge)^{\varphi=1} \simeq \tau_{\geq 0}u^*R\mathcal{H}om(G, u_*\mathbb{Z}_p[1]).$$

Combining with (3.3.30), and using adjunction, this implies that

$$(3.3.32) \quad T_{\text{ét}}(\mathcal{M}(G)) \simeq \tau_{\geq 0}R\mathcal{H}om(G_\eta, \mathbb{Z}_p[1]).$$

By (3.3.31),  $T_{\text{ét}}(\mathcal{O}) \simeq \mathbb{Z}_p$ . Applying (3.3.32) to  $G = \mu_{p^n}$  and taking limits, we obtain  $T_{\text{ét}}(\mathcal{O}\{-1\}) \simeq \mathbb{Z}_p(-1)$ . Since  $T_{\text{ét}}$  is symmetric monoidal, we have  $T_{\text{ét}}(\mathcal{O}\{1\}) \simeq \mathbb{Z}_p(1)$ . Using the fact that  $T_{\text{ét}}$  is symmetric monoidal again, we have an isomorphism

$$(3.3.33) \quad T_{\text{ét}}(\mathcal{M}(G)\{1\}) \simeq \tau_{\geq 0}R\mathcal{H}om(G_\eta, \mathbb{Z}_p(1)[1]).$$

Further, since  $G_\eta$  is killed by a power of  $p$ , and  $\mathbb{Z}_p$  is  $p$ -torsion free, we have an isomorphism of sheaves  $\tau_{\geq 0}R\mathcal{H}om_{\text{Spf}(S)_{\eta, v}}(G_\eta, \mathbb{Z}_p[1]) \simeq \mathcal{E}xt_{\text{Spf}(S)_{\eta, v}}^1(G_\eta, \mathbb{Z}_p)$ . Finally, since  $G_\eta^\vee \simeq \mathcal{E}xt_{\text{Spf}(S)_{\eta, v}}^1(G_\eta, \mathbb{Z}_p(1))$  (cf. Lemma 3.5), we obtain the desired statement.  $\square$

**Remark 3.100.** We end this paper by noting some consequences of Proposition 3.99. By taking limits, we obtain that (cf. [ALB23, § 5.4]) for a quasisyntomic ring  $S$ , and a  $G \in \text{BT}(S)$ , we have an isomorphism

$$T_{\text{ét}}(\mathcal{M}(G)\{1\}) \simeq T_p(G_\eta^\vee).$$

Further, note that using [BS23, Prop. 2.14, Cor. 3.7], for  $G \in \text{FFG}(S)$ , we obtain

$$R\Gamma_{\text{qsyn}}(\text{Spf}(S), \mathcal{M}(G) \{1\} [1/\mathcal{I}]_p^\wedge)^{\varphi=1} \simeq R\Gamma_{\text{pro-ét}}(\text{Spf}(S)_\eta, G_\eta^\vee),$$

and similarly, for  $G \in \text{BT}(S)$ , we obtain

$$R\Gamma_{\text{qsyn}}(\text{Spf}(S), \mathcal{M}(G) \{1\} [1/\mathcal{I}]_p^\wedge)^{\varphi=1} \simeq R\Gamma_{\text{pro-ét}}(\text{Spf}(S)_\eta, T_p(G_\eta^\vee)).$$

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